**METHODS FOR ESTIMATING THE PARAMETERS OF THE POWER FUNCTION DISTRIBUTION.**

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**Abstract**

In this paper, we present some methods for estimating the parameters of the two parameter Power function distribution. We used the least squares method (LSM), relative least squares method (RELS) and ridge regression method (RR). Sampling behavior of the estimates is indicated by a Monte Carlo simulation. The objective of identifying the best estimator amongst them we use the Total Deviation (T.D) and Mean Square Error (M.S.E) as performance index. We determined the best method for estimation using different values for the parameters and different sample sizes.

**Key Words:**

Power function distributions, parameter estimation, least squares method, relative least squares, ridge regression, Monte Carlo study, total deviation, mean square error.

**Introduction:**

Numerous parametric models are used in the analysis of lifetime data and in problems related to the modeling of failure processes. Among univariate models, a few particular distributions occupy a central role because of their demonstrated usefulness in a wide range of situations. Foremost in this category are the Exponential, Weibul , Gamma and Lognormal distributions.

The Power function distribution is also a flexible life time distribution model that may offer a good fit to some sets of failure data. Theoretically Power function distribution is a special case of Pareto distribution. Power function distribution is the best distribution to check the reliability of any electrical component. Meniconi and Barry (1995) discussed the application of Power function distribution. They proved that the Power function distribution is the best distribution to check the reliability of any electrical component. They used Exponential distribution, Lognornal distribution and Weibull distribution and showed from reliability and hazard function that Power function distribution is the best distribution.

The probability distribution of Power function distribution is

 (1)

With shape parameter and scale parameter, the interval (0,)

Rider (1964) derived distributions of the product and quotients of the order statistics from a Power function distribution. Moments of order statistics for a Power function distribution were calculated by Malik (1967). Lwin (1972), Arnold and Press (1983) discussed Bayesian estimation for the scale parameter of the pareto distribution using a Power function Prior. Ahsanullah and Kabir (1975) discussed the Estimation of the location and scale parameters of a Power function distribution. Vinod and Ullah (1981) showed that a ridge estimator having smaller MSE than OLS estimator.

Cohen and Whitten (1982) used the moment and modified moment estimators for the Weibull distribution. Samia and Mohamed (1993) used five modifications of moments to estimate the parameters of the Pareto distribution. Lalitha and Anand (1996) used modified maximum likelihood to estimate the scale parameter of the Rayleigh distribution. Rafiq et al. (1996) discussed the parameters of the Gamma distribution. Rafiq (1999) dicussed the method of fractional moments to estimate the parameters of Weibull distribution. Kang and Young (1997) estimated the parameters of a Pareto distribution by jackknife and bootstrap methods. A.V and Khalid (2005) discussed the application of ridge regression. Marks (2005) estimated the parameters of Weibull distribution with the help of percentiles. He called it Common Percentile Method. Razali et al. (2009) studied the estimation accuracy of three parameter Weibull distribution.

In this paper, we use the least squares method, relative least squares and ridge regression to estimate the two parameter of the Power function distribution. The present paper introduces the ridge regression estimators by taking different values of “K”. Also, we compared between these methods using two parameter Power function distribution to find the most accurate method.

 **Methodology**

1. **Least squares method (LSM):**

For the estimation of probability distribution parameters, the least square method (LSM) is extensively used in reliability engineering and mathematics problems.

 The cumulative distribution function of Power function distribution is given by

 (2)

 To get a linear relation between the two parameters taking the natural logarithm of above

 equation as follows

(3)

After simplification, we get

=

 Where = 1,2,…… n and n is the sample size.

Let be a random sample of and is estimated and replaced by the median rank method as follows:

Because of the mean rank method

May be a larger value for smaller and a smaller value for larger

Thus, equation (3) is a linear equation and is expressed as

 = a+d

To compute a and d by simple linear regression we proceed as follows

 Let

Differentiating S w.r.t a and d then equate to zero, we obtain the following two normal equations

Solving the above two equations for a and d, we obtain the least square estimates (LSE) of a and d as:

Therefore (4)

(5)

 Where i = 1, ……., n

**2) Relative Least squares Method (RELS):**

The relative least squares estimators of a and d can be obtained by minimizing the sum of squares of the relative residuals, Pablo and Bruce (1992), w.r.t. a and d as follows

 Where  ,

 Differentiating w.r.t, a and d then equate to zero

 After simplification, we get

 ,

=

(6)

(7)

**3) Ridge regression method (RR):**

The ridge regression estimators are given by

Where 0<k<1 is the ridge coefficient, is the p\*p identity matrix and p is the number of parameters.

=

 After simplification we get

Where

=

and

 (9)

 Where 0<k<1 is the ridge coefficient the readers may see Ronald and Raymond

 (1978) if k=0, we obtain the least square estimates.

***Performance Indices: Goodness of Fit Analysis:***

Some methods of goodness of fit analysis are employed here. Mean square error MSE and total deviation TD are two measurements that give an indication of the accuracy of parameter estimation. AL-Fawzan (2000) referred to the use of the procedure of MSE and TD.

***a) Mean Square Errors (MSE):***

The MSE can be calculated as below

Where is the value of the cumulative distribution function of the 2 parameter power function distribution using the estimated parameters, and

Also Standard bias, Bias = E (**) –** and M.S.E () =

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1. ***Total Derivation (TD)***

The total derivation TD, calculated for each method is as follows

Where and are the known parameters, and nare the estimated parameters by any method. These techniques are used to measure the variability of parameter estimates for each simulation. These are used to determine the overall “best” parameter estimation method.

**Application:**

A simulation study is used in order to compare the performance of the proposed estimation methods. We carry out this comparison taking the samples of sizes as n = 20, 60 and 100 with pairs of **( ,)** = {(1,2), (3,2),(4,3)}. We generated random samples of different sizes by observing that if R is uniform (0, 1), then is the random number generation of power function distribution with (, ) parameters. All results are based on 10,000 replications.

Such generated data have been used to obtain estimates of the unknown parameters. The results obtained from parameters estimation of the 2-parameters power function distribution using different sample sizes and different values of parameters with mean square error MSE and total deviation TD.

 **Table#1**

**(Estimates for the parameters β and γ of Power function distribution under the sample size 20)**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| METHODS  |  True Values  | Estimated Values | M.S.E | T.D  |
| β | γ |  |  |  |  |
| L.S.E | 1 | 2 | 1.001218 | 2.062543 | 0.006737 | 0.309703 | 0.032489 |
| 4 | 3 | 3.999863 | 3.063222 | 0.051887 | 0.643331 | 0.021108 |
| 3 | 2 | 2.996574 | 1.997344 | 0.098993 | 0.266866 | 0.00247 |
| R.L.S.E | 1 | 2 | 0.997928 | 2.4512 | 0.000267 | 1.600062 | 0.227672 |
| 4 | 3 | 3.913205 | 3.176421 | 0.239967 | 0.867286 | 0.080506 |
| 3 | 2 | 2.636301 | 2.602738 | 0.490666 | 1.77547 | 0.422602 |
| R.R(.1) | 1 | 2 | 0.984061 | 2.135137 | 0.006174 | 0.344699 | 0.083507 |
| 4 | 3 | 3.895059 | 3.265332 | 0.055787 | 0.837077 | 0.114679 |
| 3 | 2 | 2.908783 | 2.098611 | 0.094425 | 0.310977 | 0.079711 |
| R.R(.2) | 1 | 2 | 0.968673 | 2.206826 | 0.006247 | 0.390202 | 0.13474 |
| 4 | 3 | 3.80006 | 3.476352 | 0.079031 | 1.144886 | 0.208769 |
| 3 | 2 | 2.829907 | 2.20134 | 0.104646 | 0.379475 | 0.157368 |
| R.R(.3) | 1 | 2 | 0.954808 | 2.277641 | 0.006778 | 0.445823 | 0.184012 |
| 4 | 3 | 3.713498 | 3.696877 | 0.116492 | 1.583523 | 0.303918 |
| 3 | 2 | 2.758647 | 2.305551 | 0.125486 | 0.473517 | 0.233227 |
| R.R(.4) | 1 | 2 | 0.942266 | 2.34761 | 0.007638 | 0.511195 | 0.231539 |
| 4 | 3 | 3.634248 | 3.927562 | 0.164373 | 2.172441 | 0.400625 |
| 3 | 2 | 2.693942 | 2.411268 | 0.153914 | 0.594307 | 0.307654 |
| R.R(.5) | 1 | 2 | 0.930878 | 2.416761 | 0.008729 | 0.585972 | 0.277503 |
| 4 | 3 | 3.561378 | 4.169131 | 0.219839 | 2.934324 | 0.499366 |
| 3 | 2 | 2.634917 | 2.518513 | 0.187708 | 0.743099 | 0.380951 |
| R.R(.6) | 1 | 2 | 0.920504 | 2.485121 | 0.009979 | 0.669827 | 0.322057 |
| 4 | 3 | 3.494104 | 4.422391 | 0.280754 | 3.895759 | 0.600604 |
| 3 | 2 | 2.580845 | 2.627308 | 0.225224 | 0.921199 | 0.453373 |
| R.R(.7) | 1 | 2 | 0.911024 | 2.552716 | 0.011334 | 0.762452 | 0.365334 |
| 4 | 3 | 3.43177 | 4.688242 | 0.345506 | 5.088094 | 0.704805 |
| 3 | 2 | 2.531117 | 2.737679 | 0.265245 | 1.129968 | 0.525134 |
| R.R(.8) | 1 | 2 | 0.902337 | 2.619569 | 0.012754 | 0.863556 | 0.407448 |
| 4 | 3 | 3.373814 | 4.967692 | 0.412866 | 6.548536 | 0.812444 |
| 3 | 2 | 2.485221 | 2.849651 | 0.30686 | 1.370824 | 0.596418 |
| R.R(.9) | 1 | 2 | 0.894356 | 2.685705 | 0.014207 | 0.972861 | 0.448496 |
| 4 | 3 | 3.31976 | 5.261882 | 0.481898 | 8.321574 | 0.924021 |
| 3 | 2 | 2.442719 | 2.963247 | 0.349391 | 1.645248 | 0.667384 |

**Table#2 (Estimation for the parameters β and γ of Power Function Distribution for sample size 60):**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| METHODS  |  True Values  | Estimated Values | M.S.E | T.D  |
| β | γ |  |  |  |  |
| L.S.E | 1 | 2 | 0.994648 | 2.019416 | 0.003304 | 0.111938 | 0.01506 |
| 4 | 3 | 4.006399 | 2.992553 | 0.021194 | 0.240086 | 0.004082 |
| 3 | 2 | 3.033886 | 1.96236 | 0.034403 | 0.103488 | 0.030115 |
| R.L.S.E | 1 | 2 | 0.998754 | 2.136961 | 6.98E-05 | 0.186667 | 0.069727 |
| 4 | 3 | 3.844167 | 3.278693 | 0.287488 | 1.42537 | 0.131856 |
| 3 | 2 | 2.661028 | 2.466177 | 0.339682 | 1.405301 | 0.346079 |
| R.R(.1) | 1 | 2 | 0.989673 | 2.039686 | 0.003258 | 0.115163 | 0.03017 |
| 4 | 3 | 3.9736 | 3.046982 | 0.020863 | 0.254506 | 0.022261 |
| 3 | 2 | 3.006583 | 1.990433 | 0.031844 | 0.105801 | 0.006977 |
| R.R(.2) | 1 | 2 | 0.984856 | 2.059879 | 0.003264 | 0.119207 | 0.045084 |
| 4 | 3 | 3.941755 | 3.102009 | 0.022636 | 0.275542 | 0.048564 |
| 3 | 2 | 2.980138 | 2.018611 | 0.030834 | 0.109792 | 0.015926 |
| R.R(.3) | 1 | 2 | 0.980189 | 2.079997 | 0.003317 | 0.124061 | 0.059809 |
| 4 | 3 | 3.910822 | 3.157644 | 0.026331 | 0.30342 | 0.074843 |
| 3 | 2 | 2.954511 | 2.046892 | 0.031233 | 0.115478 | 0.038609 |
| R.R(.4) | 1 | 2 | 0.975667 | 2.10004 | 0.003413 | 0.129717 | 0.074353 |
| 4 | 3 | 3.88076 | 3.213895 | 0.031786 | 0.338374 | 0.101108 |
| 3 | 2 | 2.929664 | 2.075277 | 0.032914 | 0.12288 | 0.061084 |
| R.R(.5) | 1 | 2 | 0.971283 | 2.120009 | 0.003547 | 0.136165 | 0.088721 |
| 4 | 3 | 3.851532 | 3.270771 | 0.03885 | 0.380645 | 0.127374 |
| 3 | 2 | 2.905563 | 2.103766 | 0.035761 | 0.132017 | 0.083362 |
| R.R(.6) | 1 | 2 | 0.967032 | 2.139905 | 0.003716 | 0.143396 | 0.102921 |
| 4 | 3 | 3.823102 | 3.328282 | 0.047385 | 0.430481 | 0.153652 |
| 3 | 2 | 2.882174 | 2.132359 | 0.039669 | 0.142908 | 0.105455 |
| R.R(.7) | 1 | 2 | 0.962907 | 2.159728 | 0.003917 | 0.151402 | 0.116957 |
| 4 | 3 | 3.795437 | 3.386436 | 0.057267 | 0.488139 | 0.179953 |
| 3 | 2 | 2.859465 | 2.161057 | 0.044542 | 0.155575 | 0.127373 |
| R.R(.8) | 1 | 2 | 0.958904 | 2.179479 | 0.004146 | 0.160175 | 0.130836 |
| 4 | 3 | 3.768505 | 3.445243 | 0.06838 | 0.553885 | 0.206288 |
| 3 | 2 | 2.837408 | 2.18986 | 0.050292 | 0.170037 | 0.149127 |
| R.R(.9) | 1 | 2 | 0.955018 | 2.19916 | 0.004402 | 0.169705 | 0.144562 |
| 4 | 3 | 3.742275 | 3.504713 | 0.080617 | 0.627991 | 0.232669 |
| 3 | 2 | 2.815975 | 2.218769 | 0.05684 | 0.186315 | 0.170726 |

**Table # 3 (Estimation for the parameters β and γ of Power Function Distribution for sample size 100)**:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| METHODS  |  True Values  | Estimated Values | M.S.E | T.D  |
| β | γ |  |  |  |  |
| L.S.E | 1 | 2 | 1.000677 | 2.023245 | 0.001921 | 0.071615 | 0.0123 |
| 4 | 3 | 4.018766 | 2.973905 | 0.017037 | 0.196505 | 0.01339 |
| 3 | 2 | 3.02700 | 1.98070 | 0.0174251 | 0.0590387 | 0.0186514 |
| R.L.S.E | 1 | 2 | 0.999534 | 2.126007 | 1.8E-05 | 0.108957 | 0.063469 |
| 4 | 3 | 3.779907 | 3.278657 | 0.325124 | 1.191472 | 0.147909 |
| 3 | 2 | 2.78427 | 2.26152 | 0.168653 | 0.360695 | 0.202670 |
| R.R(.1) | 1 | 2 | 0.997791 | 2.034987 | 0.001883 | 0.07305 | 0.019702 |
| 4 | 3 | 3.999364 | 3.004927 | 0.016253 | 0.201713 | 0.001801 |
| 3 | 2 | 3.01122 | 1.99702 | 0.0163858 | 0.0598607 | 0.0052325 |
| R.R(.2) | 1 | 2 | 0.994958 | 2.046704 | 0.001864 | 0.07476 | 0.028394 |
| 4 | 3 | 3.980293 | 3.036141 | 0.016225 | 0.209021 | 0.016974 |
| 3 | 2 | 2.99573 | 2.01337 | 0.0158587 | 0.0612337 | 0.0081072 |
| R.R(.3) | 1 | 2 | 0.992177 | 2.058395 | 0.00186 | 0.076743 | 0.03702 |
| 4 | 3 | 3.961544 | 3.06755 | 0.016915 | 0.218469 | 0.032131 |
| 3 | 2 | 2.98052 | 2.02975 | 0.0158166 | 0.0631612 | 0.0213702 |
| R.R(.4) | 1 | 2 | 0.989447 | 2.070061 | 0.001873 | 0.078998 | 0.045584 |
| 4 | 3 | 3.943109 | 3.099156 | 0.018287 | 0.230099 | 0.047275 |
| 3 | 2 | 2.96558 | 2.04617 | 0.0162336 | 0.0656468 | 0.0345591 |
| R.R(.5) | 1 | 2 | 0.986766 | 2.081703 | 0.0019 | 0.081522 | 0.054085 |
| 4 | 3 | 3.92498 | 3.130959 | 0.020306 | 0.243954 | 0.062408 |
| 3 | 2 | 2.95090 | 2.06262 | 0.0170851 | 0.0686940 | 0.0476764 |
| R.R(.6) | 1 | 2 | 0.984133 | 2.093319 | 0.001942 | 0.084315 | 0.062526 |
| 4 | 3 | 3.907149 | 3.162962 | 0.022941 | 0.260076 | 0.077533 |
| 3 | 2 | 2.93649 | 2.07911 | 0.0183481 | 0.0723064 | 0.0607244 |
| R.R(.7) | 1 | 2 | 0.981547 | 2.104911 | 0.001996 | 0.087373 | 0.070908 |
| 4 | 3 | 3.889608 | 3.195165 | 0.026161 | 0.278508 | 0.092653 |
| 3 | 2 | 2.92233 | 2.09563 | 0.0200005 | 0.0764875 | 0.0737054 |
| R.R(.8) | 1 | 2 | 0.979007 | 2.116478 | 0.002063 | 0.090696 | 0.079232 |
| 4 | 3 | 3.87235 | 3.227571 | 0.029935 | 0.299296 | 0.10777 |
| 3 | 2 | 2.90841 | 2.11218 | 0.0220215 | 0.0812411 | 0.0866217 |
| R.R(.9) | 1 | 2 | 0.976512 | 2.128021 | 0.002142 | 0.094282 | 0.087499 |
| 4 | 3 | 3.855367 | 3.260181 | 0.034237 | 0.322485 | 0.122885 |
| 3 | 2 | 2.89473 | 2.12877 | 0.0243914 | 0.0865706 | 0.0994754 |

**Results and Conclusion:**

All the results are listed in table (1), (2) and (3). From these tables, we see that the L.S.E estimates of parameters are too close to the true values, and the values of MSE and TD are very small. The parameter estimates from R.L.S.E, R.R method are close to the true values but not as L.S.E estimates, because the values of MSE and TD are greater than the corresponding values from L.S.E.

From all of these observations it is concluded that the decrease in the value of “K” in ridge regression method it will gives us the accurate result in estimated values.

It was observed that M.S.Es and T.Ds of all estimators of scale parameter as well as all the estimators of shape parameter is decreasing with the increase of sample size.

Consequently, we recommend using the L.S.E method for the parameters estimation of the Power function distribution. After L.S.E, the R.R (0.1) and R.L.S.E method are best for estimation of scale and shape parameters of the Power function distribution.

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