**Estimation Accuracy of Exponential Distribution Parameters**

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**Abstract**

The exponential distribution is commonly used to model the behavior of units that have a constant failure rate. The two-parameter exponential distribution provides a simple but nevertheless useful model for the analysis of lifetimes, especially when investigating reliability of technical equipment.

This paper is concerned with estimation of parameters of the two parameter (location and scale) exponential distribution. We used the least squares method (LSM), relative least squares method (RELS), ridge regression method (RR), moment estimators (ME), modified moment estimators (MME), maximum likelihood estimators (MLE) and modified maximum likelihood estimators (MMLE). We used the mean square error MSE, and total deviation TD, as measurement for the comparison between these methods. We determined the best method for estimation using different values for the parameters and different sample sizes.

**Key Words:**

Exponential distribution, parameter estimation, least squares method, relative least squares, ridge regression, moment estimators, modified moment estimators, maximum likelihood estimators, modified maximum likelihood estimators, total deviation, mean square error.

**Introduction:**

Exponential distribution plays an important part in the field of reliability engineering and life testing. The exponential distribution would be an appropriate choice for an experimenter, if failure rate appears to be more or less constant. “This is a distribution of the time to an event when the probability of the event occurring in the next small time interval does not vary through time. It is used in the theory of waiting lines or queues, which are found in many situations: from the gates at the entrance to toll roads through the time taken for an answer to a telephone enquiry, to the time taken for an ambulance to arrive at the scene of an accident” Evans et al; (2000). Exponential distribution is a special case of the Weibull distribution where =1.

The exponential distribution can be found with one or two parameters; scale and location parameters. The values of these parameters must be more than zero except for the location parameter which can be greater than or equal to zero.

The probability density function of two parameter exponential distribution is given by

t≥; β>0 < t < ∞ (1)

The parameters and β are interpreted as measure of guarantee and failure rate respectively.

Maguire, Pearson and Wynn (1952) studied mine accidents and showed that time intervals between industrial accidents follow exponential distribution. Cohen and Helm (1973) used (BLUE), (MLE), (ME), (MVUE) and MME to estimate the parameters of the exponential distribution. Peter (1974) used robust M-estimation method for the scale parameter, with application to the exponential distribution. Cohen and Whitten (1982) used the moment and modified moment estimators for the Weibull distribution. Samia and Mohamed (1993) used five modifications of moments to estimate the parameters of the Pareto distribution. Lalitha and Anand (1996) used modified maximum likelihood to estimate the scale parameter of the Rayleigh distribution. Kang and Young (1997) estimated the parameters of a Pareto distribution by jackknife and bootstrap methods. Razali et al. (2009) studied the estimation accuracy of three parameter Weibull distribution.

In this paper, we use the least squares method, relative least squares, ridge regression, moment and modified moment estimators, maximum and modified maximum likelihood estimators to estimate the two parameter of the exponential distribution. The present paper introduces third modified moment estimators and first modified maximum likelihood estimators. Also, we compared between these methods using two parameter exponential distributions to find the most accurate method.

**Methodology:**

**1) Least squares method (LSM):**

The cumulative distribution function of (1) is given by

Taking log on both sides

After simplification, we get

= + [ln (1F ())]

The last equation can be represented by

= a+b (2)

Where = , a = , b = , = [-ln (1-F ())] = 1,2,…… *n* and *n* is the sample size. Let



Differentiating S w.r.t a and b then equate to zero, we obtain the following two normal equations





Solving the above two equations for a and b, we obtain the least square estimates (LSE) of a and b as:

(3)

, *i* = *1*, …, *n* (4)

**2) Relative Least squares Method (RELS):**

The relative least squares estimators of a and b can be obtained by minimizing the sum of squares of the relative residuals, Pablo and Bruce (1992), w.r.t. a and b as follows



 (5)

Differentiating w.r.t, a and b then equate to zero





After simplification, we get





Where  and 

**3) Ridge regression method (RR):**

The ridge regression (RR) estimates of A and B can be obtained by minimizing the error sum of squares for the model

= a+b

Subject to the single constraint that a2+b2= ρ where ρ is a finite positive constant.

The method of Lagrange’s multiplier requires the differentiation of

w.r.t a and b. when these derivatives are equated to zero, we obtain the following two equations

Solving above two equations for a and b we get

For two parameter exponential distribution we know that = , a = , b =

= [-ln(1-F())] = 1,2,…… n

(8)

(9)

Where 0<λ<1 is the ridge coefficient the readers may see Ronald and Raymond (1978) if λ=0, we obtain the least square estimates.

**4) Moment Estimators (ME):**

For the 2 parameter exponential distribution, the first two moments of exponential distribution are given by

*μ1* = E(x) = + (10)

 = E(x2) = + 2 + 2 (11)

The variance of x is given by

var(x) = 

= + 2 +2 – ( + )2

= (12)

By equating the first two population moments with sample moments

 (13)

And  (14)

(15)

(16)

From above equation the estimate of is given by.

By substitute about into (15) the estimate of is given by .

**5) First Modified Moment Estimators (MME-I):**

In this modification of the moment estimators, the second moment of two parameter exponential distribution is replaced by

E [F (t (1))] = F (t (1))

Where t (1) is the first order statistic

(17)

We know that the first sample standard moment

m1 = +

= m1 – (18)

The solution of (18) and (17) is given the estimate of as

(19)

where

From (18) and (19) we can obtain the estimate of as

(20)

**6) Second Modified Moment Estimators (MME-II):**

In this case, the second standard moment of two parameter exponential distribution is replaced by

E(t(1))= t(1)

That is

Solution of (18) and (21) is given the estimate of

(22)

From equations (21) and (22) we can obtain the estimate of as

(23)

**7) Third Modified Moment estimators (MME-III):**

In this modification the second moment is replaced by Met = tme, where Met is the

population median and tme is the sample median.

The median of two parameter exponential distribution is given by

+ (ln2)

Thus, we have

+ (ln2) = tme (24)

From equation (18) and (24) we obtain

(25)

From equation (26) and (25), we obtain

(26)

**8) Maximum likelihood Method (MLM):**

The likelihood function can be shown as follows,

(27)

The maximum likelihood estimator MLE of the parameter is the value of the parameter that maximizes L and MLM for 2 parameter exponential distribution can be obtained by solving the equations resulting from setting the two partial derivatives of L(,) to zero;

(28)

(29)

It can be seen that L ( , tm) is monotonically increasing with tm i.e. the greater the value of tm the greater the value of likelihood function.

Hence since t ≥ tm we conclude

(30)

(31)

**9) First Modified Maximum Likelihood estimator:**

The first equation of MLM is replaced by

E()=

(32)

By solving equation (32) and (31) simultaneously, we obtain the parameter estimates

(33)

**Goodness of Fit Analysis:**

Some methods of goodness of fit analysis are employed here. Mean square error MSE and total deviation TD are two measurements that give an indication of the accuracy of parameter estimation. AL-Fawzan (2000) referred to the use of the procedure of MSE and TD.

**a) Mean Square Errors (MSE):**

The MSE can be calculated as below

Where is the value of the cumulative distribution function of the 2 parameter exponential distribution using the estimated parameters, and

1. **Total Derivation (TD)**

The total derivation TD, calculated for each method is as follows

Where and are the known parameters, and are the estimated parameters by any method.

**Application:**

A simulation study is used in order to compare the performance of the proposed estimation methods. According to limitations of the computer time, we carry out this comparison taking the sample sizes as n = 20, 40, 60, 80 and 100 with pairs of ( , ) = (1,1), (1,2), (2,3) we generated random samples of different sizes by observing that if U is uniform (0,1), then

T = + [−ln (1−U)] is exponential of ( , )

Such generated data have been used to obtain estimates of the unknown parameters. The results obtained from parameter estimation of the 2-parameter exponential distribution using different sample sizes and different values of parameters with mean square error MSE and total deviation TD, are given in tables (1), (2), (3), (4) and (5).

**Table (1): The estimates for n=20**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Method | True values | | Estimated VALUES | | MSE | TD |
|  |  |  |  |
| LSM | 1 | 1 | 0.999997763 | 0.9999864 | 3.83633×10-11 | 0.0000158 |
| 1 | 2 | 1.00002381 | 1.99998 | 6.45067×10-10 | 0.0000333 |
| 2 | 3 | 2.00000623 | 3.00000825 | 6.0299×10-11 | 0.0000059 |
| RELS | 1 | 1 | 0.95797 | 1.0000018 | 0.0114253 | 0.0420285 |
| 1 | 2 | 0.9999 | 2.18104 | 0.0097119 | 0.0905306 |
| 2 | 3 | 1.99999 | 2.999998 | 1.64×10-12 | 0.000001 |
| RR | 1 | 1 | 0.973875 | 1.00162 | 0.00432129 | 0.027745 |
| 1 | 2 | 0.999202 | 1.97792 | 0.0001753 | 0.011838 |
| 2 | 3 | 1.97308 | 2.97954 | 0.0008347 | 0.02028 |
| ME | 1 | 1 | 1.06899 | 1.02620 | 0.0408251 | 0.09519 |
| 1 | 2 | 1.13799 | 2.0524 | 0.0408306 | 0.16419 |
| 2 | 3 | 2.206977 | 3.0786 | 0.0408276 | 0.129689 |
| (MME-I) | 1 | 1 | 0.944557 | 1.15063 | 0.0150322 | 0.0206073 |
| 1 | 2 | 0.88912 | 2.30126 | 0.0150319 | 0.26151 |
| 2 | 3 | 1.83367 | 3.45190 | 0.0150327 | 0.233798 |
| (MME-II) | 1 | 1 | 0.943095 | 1.15209 | 0.0153931 | 0.208995 |
| 1 | 2 | 0.88618 | 2.3042 | 0.0153951 | 0.26592 |
| 2 | 3 | 1.82927 | 3.45630 | 0.0153951 | 0.237465 |
| (MME-III) | 1 | 1 | 1.00664 | 1.08855 | 0.0596031 | 0.09519 |
| 1 | 2 | 1.1771021 | 2.0132779 | 0.0596043 | 0.183741 |
| 2 | 3 | 2.26565 | 3.01993 | 0.0596040 | 0.139468 |
| MLE | 1 | 1 | 1.00070 | 1.09449 | 0.0107806 | 0.09519 |
| 1 | 2 | 1.00139 | 2.18899 | 0.0107800 | 0.095885 |
| 2 | 3 | 2.00209 | 3.283485 | 0.0107805 | 0.09554 |
| MMLE-I | 1 | 1 | 0.9459755 | 1.09449 | 0.0098298 | 0.148515 |
| 1 | 2 | 0.8919405 | 2.18899 | 0.0098315 | 0.202555 |
| 2 | 3 | 1.837915 | 3.283485 | 0.0098310 | 0.175538 |

**Table (2):** **The estimates for n=40**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Method | True values | | Estimated VALUES | | MSE | TD |
|  |  |  |  |
| LSM | 1 | 1 | 0.999996 | 0.999924 | 2.041x 10-8 | 0.0000789 |
| 1 | 2 | 1.004568 | 1.99696 | 0.0000482 | 0.0060837 |
| 2 | 3 | 2.00000185 | 2.99999 | 4.1 x 10-12 | 0.0000012 |
| RELS | 1 | 1 | 0.99999 | 0.99999746 | 4.0544 x 10-11 | 0.0000032 |
| 1 | 2 | 0.99992 | 2.0227 | 0.0003941 | 0.0113576 |
| 2 | 3 | 2.0001945 | 2.999844 | 3.567 x 10-8 | 0.0001493 |
| RR | 1 | 1 | 0.992288 | 0.994180 | 0.0013807 | 0.013532 |
| 1 | 2 | 1.01501 | 1.96564 | 0.0004904 | 0.03219 |
| 2 | 3 | 2.00730 | 2.95982 | 0.0003316 | 0.0170433 |
| ME | 1 | 1 | 1.15174 | 0.654301 | 0.414524 | 0.497439 |
| 1 | 2 | 1.30347 | 1.30860 | 0.414503 | 0.64917 |
| 2 | 3 | 2.45521 | 1.962901 | 0.414511 | 0.573305 |
| (MME-I) | 1 | 1 | 0.982780 | 0.823257 | 0.157346 | 0.193965 |
| 1 | 2 | 0.965560 | 1.646513 | 0.157347 | 0.211184 |
| 2 | 3 | 1.94834 | 2.46977 | 0.157347 | 0.202573 |
| (MME-II) | 1 | 1 | 0.982520 | 0.823516 | 0.157587 | 0.193964 |
| 1 | 2 | 0.965040 | 1.647032 | 0.157587 | 0.211444 |
| 2 | 3 | 1.947560 | 2.470548 | 0.157857 | 0.202704 |
| (MME-III) | 1 | 1 | 1.00950 | 0.796538 | 0.138321 | 0.212962 |
| 1 | 2 | 1.01900 | 1.59308 | 0.138317 | 0.22246 |
| 2 | 3 | 2.02849 | 2.38962 | 0.138324 | 0.217705 |
| MLE | 1 | 1 | 1.00311 | 0.802928 | 0.141786 | 0.200182 |
| 1 | 2 | 1.00622 | 1.60586 | 0.141783 | 0.20329 |
| 2 | 3 | 2.00932 | 2.40878 | 0.141796 | 0.201733 |
| MMLE-I | 1 | 1 | 0.983035 | 0.802928 | 0.193612 | 0.214037 |
| 1 | 2 | 0.966070 | 1.60586 | 0.193608 | 0.231 |
| 2 | 3 | 1.94911 | 2.40878 | 0.193610 | 0.222518 |

**Table** **(3): The estimates for n=60**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Method | True values | | Estimated VALUES | | MSE | TD |
|  |  |  |  |
| LSM | 1 | 1 | 0.9999982 | 0.9999975 | 6.5×10-10 | 0.0000094 |
| 1 | 2 | 1.00005263 | 2.0004348 | 0.0000003 | 0.00027 |
| 2 | 3 | 2.0000054 | 2.999996 | 3.66×10-11 | 1.0000016 |
| RELS | 1 | 1 | 1.0000094 | 1.0000038 | 2.1×10-9 | 0.0000132 |
| 1 | 2 | 1.0000174 | 1.999886 | 9.68×10-9 | 0.0000741 |
| 2 | 3 | 1.999963 | 2.99995 | 6.4×10-9 | 0.000034 |
| RR | 1 | 1 | 0.990869 | 1.00078 | 0.0013624 | 0.009911 |
| 1 | 2 | 0.998391 | 1.99403 | 0.0000859 | 0.004594 |
| 2 | 3 | 1.98926 | 2.99481 | 0.0003236 | 0.0071 |
| ME | 1 | 1 | 1.01216 | 1.11234 | 0.0719734 | 0.1245 |
| 1 | 2 | 1.02432 | 2.22468 | 0.0719734 | 0.13666 |
| 2 | 3 | 2.03648 | 3.337022 | 0.0719741 | 0.130581 |
| (MME-I) | 1 | 1 | 0.996175 | 1.12833 | 0.0623616 | 0.132155 |
| 1 | 2 | 0.992349 | 2.25665 | 0.0623561 | 0.135976 |
| 2 | 3 | 1.988523 | 3.384975 | 0.0623559 | 0.134063 |
| (MME-II) | 1 | 1 | 0.996017 | 1.12848 | 0.0622806 | 0.132463 |
| 1 | 2 | 0.992034 | 2.25697 | 0.0622854 | 0.136451 |
| 2 | 3 | 1.988051 | 3.385448 | 0.0622831 | 0.134457 |
| (MME-III) | 1 | 1 | 1.04236 | 1.08214 | 0.103420 | 0.1245 |
| 1 | 2 | 1.08471 | 2.16429 | 0.103414 | 0.166855 |
| 2 | 3 | 2.12707 | 3.24643 | 0.103416 | 0.145678 |
| MLE | 1 | 1 | 1.01483 | 1.10967 | 0.0740230 | 0.1245 |
| 1 | 2 | 1.02965 | 2.219349 | 0.0740185 | 0.139324 |
| 2 | 3 | 2.04448 | 3.3290 | 0.0740204 | 0.131913 |
| MMLE-I | 1 | 1 | 0.996330 | 1.10967 | 0.0458705 | 0.11334 |
| 1 | 2 | 0.992660 | 2.219349 | 0.0458743 | 0.117014 |
| 2 | 3 | 1.98899 | 3.32902 | 0.0458733 | 0.115178 |

**Table (4):** **The estimates for n=80**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Method | | True values | | Estimated VALUES | | MSE | TD |
|  |  |  |  |
| LSM | 1 | 1 | 0.999998 | 0.9999977 | 1.84312×10-10 | 0.0000039 |
| 1 | 2 | 0.99999802 | 1.999998 | 4.0589×10-11 | 0.0000025 |
| 2 | 3 | 1.9999955 | 2.999996 | 1.14738×10-10 | 0.0000036 |
| RELS | 1 | 1 | 1.0000176 | 1.0000132 | 0.0000001 | 0.0000898 |
| 1 | 2 | 1.0000082 | 2.0000144 | 1.355×10-9 | 0.0000153 |
| 2 | 3 | 1.999994 | 2.99997 | 7.9446×10-10 | 0.0000109 |
| RR | 1 | 1 | 0.995201 | 0.998303 | 0.0008376 | 0.006496 |
| 1 | 2 | 1.00363 | 1.98818 | 0.0001032 | 0.00954 |
| 2 | 3 | 1.99883 | 2.98649 | 0.0001617 | 0.0050883 |
| ME | 1 | 1 | 1.053605 | 0.783958 | 0.193896 | 0.269647 |
| 1 | 2 | 1.0721 | 1.56792 | 0.223028 | 0.28814 |
| 2 | 3 | 2.16082 | 2.35187 | 0.193897 | 0.296453 |
| (MME-I) | 1 | 1 | 0.989691 | 0.847872 | 0.205316 | 0.162437 |
| 1 | 2 | 0.979383 | 1.69574 | 0.205309 | 0.172742 |
| 2 | 3 | 1.96907 | 2.54362 | 0.205317 | 0.167592 |
| (MME-II) | 1 | 1 | 0.989625 | 0.847938 | 0.205412 | 0.162437 |
| 1 | 2 | 0.979250 | 1.69588 | 0.2054069 | 0.17281 |
| 2 | 3 | 1.96887 | 2.54382 | 0.205414 | 0.167625 |
| (MME-III) | 1 | 1 | 0.959586 | 0.877977 | 0.262005 | 0.162437 |
| 1 | 2 | 0.919172 | 1.75595 | 0.262011 | 0.202583 |
| 2 | 3 | 1.87876 | 2.63393 | 0.2620024 | 0.182643 |
| MLE | 1 | 1 | 1.00022 | 0.837339 | 0.191808 | 0.162881 |
| 1 | 2 | 1.00045 | 1.67468 | 0.191787 | 0.16311 |
| 2 | 3 | 2.00067 | 2.51202 | 0.191792 | 0.162995 |
| MMLE-I | 1 | 1 | 0.989757 | 0.837339 | 0.234187 | 0.172904 |
| 1 | 2 | 0.979514 | 1.67468 | 0.234184 | 0.183146 |
| 2 | 3 | 1.969271 | 2.51202 | 0.234184 | 0.178024 |

**Table (5):** **the estimates for n=100**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Method | True values | | Estimated VALUES | | MSE | TD |
|  |  |  |  |
| LSM | 1 | 1 | 1.00000202 | 1.0000017 | 2.58×10-10 | 0.0000037 |
| 1 | 2 | 1.00000213 | 2.0000022 | 7.884×10-11 | 0.0000032 |
| 2 | 3 | 2.00000312 | 2.999967 | 6.105×10-10 | 0.0000127 |
| RELS | 1 | 1 | 1.00000113 | 1.00000111 | 8.72×10-11 | 1.000002 |
| 1 | 2 | 0.9999394 | 2.0000044 | 3.52×10-8 | 0.000067 |
| 2 | 3 | 2.0000157 | 2.999997 | 9.81×10-10 | 0.0000089 |
| RR | 1 | 1 | 0.995823 | 0.999050 | 0.0007896 | 0.005127 |
| 1 | 2 | 1.00085 | 1.99307 | 0.0000546 | 0.004315 |
| 2 | 3 | 1.99667 | 2.992125 | 0.0001602 | 0.00429 |
| ME | 1 | 1 | 0.930781 | 0.914596 | 0.394559 | 0.154623 |
| 1 | 2 | 0.861562 | 1.82919 | 0.394562 | 0.223843 |
| 2 | 3 | 1.79234 | 2.74379 | 0.394564 | 0.189233 |
| (MME-I) | 1 | 1 | 0.995422 | 0.849955 | 0.196354 | 0.15458 |
| 1 | 2 | 0.990844 | 1.69991 | 0.196344 | 0.159201 |
| 2 | 3 | 1.98627 | 2.54987 | 0.196344 | 0.156908 |
| (MME-II) | 1 | 1 | 0.995380 | 0.849998 | 0.196423 | 0.154622 |
| 1 | 2 | 0.990760 | 1.699995 | 0.196424 | 0.159243 |
| 2 | 3 | 1.98614 | 2.54999 | 0.196426 | 0.156933 |
| (MME-III) | 1 | 1 | 0.6126 | 1.23278 | 2.18395 | 0.62018 |
| 1 | 2 | 0.225200 | 2.46555 | 2.18398 | 1.00758 |
| 2 | 3 | 0.837800 | 3.69833 | 2.18397 | 0.813877 |
| MLE | 1 | 1 | 1.00388 | 0.841498 | 0.184035 | 0.162382 |
| 1 | 2 | 1.00776 | 1.68300 | 0.184029 | 0.16626 |
| 2 | 3 | 2.01164 | 2.52449 | 0.184038 | 0.164323 |
| MMLE-I | 1 | 1 | 0.995465 | 0.841498 | 0.219908 | 0.163037 |
| 1 | 2 | 0.990930 | 1.68300 | 0.219903 | 0.16757 |
| 2 | 3 | 1.98639 | 2.52449 | 0.219921 | 0.165308 |

**Conclusion:**

The results are listed in the tables (1), (2), (3), (4), and (5). From the computations, we note that, the estimates of parameters from the LSM are too close to the true values, and the values of MSE and TD are very small. The parameter estimates from RELS, RR method are close to the true values but not as LS estimates, because the values of MSE and TD are greater than the corresponding values from LSM.

In this article, our proposed third modified moment estimators (MME-III) is given better accuracy as compared to previously defined traditional method of moments (ME) and first modified moment estimators (MME-I) for small samples of size n=20, 40. For the samples of large size n=100, the values of MSE and TD obtained from (MME-111) are too large.

Another modification which we made in this article is the first modified maximum likelihood estimators (MMLE-I).First modified maximum likelihood estimators (MMLE-I) is given better accurate estimates as compared to previously defined traditional method of moments (ME) and maximum likelihood method for the samples of size n=20, 60, because they gave the least value for the mean square error.

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