On a New Class of Univariate Continuous Distributions that are Closed Under Inversion

Saleha Naghmi Habibullah
Kinnaird College for Women
Lahore, Pakistan
Email: salehahabibullah@hotmail.com

Munir Ahmad
National College of Business Administration
& Economics, Lahore, Pakistan
Email: drmunir@brain.net.pk

Abstract

Inverted probability distributions find applications in various real – life situations including econometrics, survey sampling, biological sciences and *life – testing*. Closure under inversion implies that the reciprocal of a continuous random variable X has the *same* probability function as the original random variable, allowing for a possible change in parameter values. To date, only a very *few* probability distributions have been found to possess the closure property.

In this paper, an attempt has been made to generate a *class* of distributions that are closed under inversion, and to develop some statistical properties of this class of distributions.

Keywords: Inverted probability distribution, Closure under inversion

1. Introduction

Inverted probability distributions find applications in various real – life situations including econometrics, survey sampling, biological sciences, engineering sciences, and , most prominently, in *life – testing*. As such, various authors have derived a variety of inverted distributions, and have developed their statistical properties.

Closure under inversion implies that the reciprocal of a continuous random variable X has the same probability function as the original random variable, *allowing* for a possible change in parameter values. In case the parameter values are *identical* to those of the original distribution, the random variable X (and its reciprocal) will be said to be *Strictly Closed Under Inversion.*

To date, only a very *few* probability distributions have been found to possess the closure property. For example, the Cauchy (0, 1) distribution is closed under inversion in the strict sense, whereas the $F(\nu_1, \nu_2)$ distribution is closed in the generalized sense.

In this paper, an attempt has been made to generate a *class* of distributions that are closed under inversion, and to establish some of the fundamental properties of this particular class of distributions.

2. Closure Under Inversion

Definition

A probability density function f(x) will be said to be closed under inversion if the *form* of the probability density function of 1 / X is the same as that of f(x). In case the parameters of the inverted distribution are *identical* to those of the original distribution, the random variable X (and its reciprocal) will be said to be *strictly closed under inversion.*

3. A Class of Distributions that are Strictly Closed Under Inversion

With reference to the development of a class of distributions that are Strictly Closed Under Inversion, we present the following theorem:

Theorem 3.1

Every function of the form

$$f(x) = k[\, M[w(x).w(1/x)]\,]/x \qquad (3.1)$$

defined on (a, 1/a), 0<a<1

where

(i) k is a non – zero real number,
(ii) w represents an algebraic function of x &
(iii) M represents an algebraic function of w(x).w(1/x)

such that

a) $f(x) \geq 0$ over its domain, and

(b) $\int_{a}^{1/a} f(x)dx = 1$)

represents a continuous probability distribution that is *Strictly Closed Under Inversion.*

Proof

Let Y = 1 / X;

Then

modulus of $dx / dy = 1 / y^2$

When x -> a, y -> 1 / a
When x -> 1 /a, y -> a

Hence

$h(y) = k[M[w(1/y).w(y)]]y. 1/y^2 = k[M[w(y).w(1/y)]]/y, \quad a < y < 1/a$
------- exactly the same form as that of f(x)

Hence the function given by (3.1) is *Strictly Closed Under Inversion.*

As far as the point regarding f(x) being a proper pdf is concerned, any function of the form given by (3.1) will be a proper pdf as long as f(x) > 0 over its domain, and the integral of the function over its domain is convergent.

Alternative Proof

If the function f(x) is Strictly Closed Under Inversion, then it satisfies the functional equation

$$f(x) = f(1/x) / x^2$$

$$\text{or} \quad xf(x) = f(1/x) /x$$

Now eq (3.1) can be written as

$$xf(x) = k[M[w(x).w(1/x)]]$$

Replacing x by 1/x, we obtain

$$f(1/x) / x = k[M[w(1/x).w(x)]]$$

$$= k[M[w(x).w(1/x)]]$$

Hence we have xf(x) = f(1/x) / x

$$\text{or} \quad f(x) = f(1/x) / x^2$$

Hence the function f(x) given by (3.1) is Strictly Closed Under Inversion.

Remarks

1. In the above, if we let a -> 0^+ then the domain (a, 1/a) tends to (0, infinity).
2. If we let a -> 1, then 1/a -> 1 and f(x) is degenerate. (In other words, if a -> 1, f(x) is a one – point distribution located on x = 1.)

Examples

The class of SCUI distributions given by (3.1) encompasses a *variety* of probability density functions, some of which are presented in Table 1:

Table 1: Some distributions belonging to the class of Probability Distributions Given by Equation (3.1)

S #	Distribution	Remarks
1.	f(x) = 1/2ax $e^{-a} < x < e^{a}, a > 0$	Here k = 1 / 2a w(x) could be ***any*** function of x & M = [w(x).w(1/x)]^0 = 1
2.	$g(y) = \left(\frac{1}{y}\right)\left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln y}{\sigma}\right]^2}\right],$ $0 < y < \infty, \sigma > 0$	Here k = 1 / sigma [sq rt of 2pi] w(y) = ln y so that w(1/y) = - ln y and w(y)w(1/y) = - (ln y)^2 & M = exp{ [w(y).w(1/y)/ 2 sigma^2]
3.	The well – known F distribution with v1 = v2 = v i.e. $g(x) = \frac{\Gamma(\nu)}{\left[\Gamma\left(\frac{\nu}{2}\right)\right]^2} x^{\nu/2 - 1}(1+x)^{-\nu}, \quad 0 < x < \infty$	Here $K = \frac{\Gamma(\nu)}{\left[\Gamma\left(\nu/2\right)\right]^2}$ $w(x) = x^{\nu/4}(1+x)^{-\nu/2}$ & M = [w(x).w(1/x)]

4. Some Properties Pertaining to the Quantiles of *Scui* distributions given by (3.1)

Theorem

For every pdf belonging to the class of SCUI distributions given by (3.1) above, the (1 – q)th quantile is the reciprocal of the qth quantile
i.e.

$X_{1-q} = 1 / X_q$ where $0 < q < 1$

Proof

Let f(x) be a continuous SCUI distribution defined on [a, 1/a], 0 < a < 1.
Then f(x) satisfies the functional equation

$$f(x) = f(1/x) / x^2$$

for all x belonging to [a, 1/a].

Hence

$$q = \int_a^{X_q} f(x)dx = \int_a^{X_q} f(1/x) / x^2 \, dx \qquad \ldots\ldots\ldots. (2)$$

Now, in the RHS of eq. (2), let Y = 1/X so that X = 1/Y.
When x = a, y = 1/a
When x = X_q, y = $1/X_q$
$dx = - dy / y^2$

Hence, eq (2) becomes

$$q = \int_a^{X_q} f(x)dx = - \int_{1/a}^{1/X_q} [\, y^2 f(y) / y^2 \,] \, dy$$

$$= \int_{1/X_q}^{1/a} f(y)dy = \int_{1/X_q}^{1/a} f(x)dx \quad \text{(dummy variable)}$$

But

$$q = \int_a^{X_q} f(x)dx \quad = \int_{1/X_q}^{1/a} f(x)dx$$

means that

$$1 / X_q = X_{1-q}$$

Hence proved.

Corollary No. 1

For every pdf belonging to the class of SCUI distributions given by eq. (3.1) above, the median is equal to unity
i.e.

$$\tilde{X} = X_{0.5} = 1$$

Proof

We know that, for every pdf belonging to the class of SCUI distributions given by eq. (3.1) above,

$$X_{1-q} = 1 / X_q$$
where $< q < 1$

Putting q = 0.5, we obtain

$$X_{0.5} = 1/X_{0.5}$$
$$=> [X_{0.5}]^2 = 1$$
$$=> X_{0.5} = 1$$
(since X a positive random variable).

Remark

The converse of this result is not generally true. There *do* exist continuous distributions that are defined on (0, infinity) and the median of which is unity but which are *not* Strictly Closed Under Inversion.

Corollary No. 2

For every pdf belonging to the class of SCUI distributions given by eq. (3.1) above, the area under the curve between X_q and $1/X_q$ is equal to 1 – 2q
i.e.

$$P[\, X_q < X < 1/X_q \,] = 1-2q$$

Proof:

The proof is simple.

Remark

This result is somewhat comparable with the well – known result that, for a normal distribution with mean Mu and standard deviation Sigma:

P[Mu - Sigma < X < Mu + Sigma] = 0.6826

P[Mu – 2 Sigma < X < Mu + 2 Sigma] = 0.9544

P[Mu – 3 Sigma < X < Mu +3 Sigma] = 0. 9973

5. Some General Remarks

Remark No. 1

All of the above is true under *Regularity Conditions* such as absolute continuity, absolute differentiability, etc.

Remark No. 2

The class of distributions of non – negative random variables given in this paper is *not exhaustive*. There exists *at least one* probability density function that extends *from – infinity to infinity* and is *closed* under inversion i.e. the Cauchy (0, 1) distribution.

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