The New Kumaraswamy Kumaraswamy Family of Generalized Distributions with Application

Mahmoud R. Mahmoud
Institute of Statistical Studies & Research
Cairo University, Egypt

El-Sayed A. El-Sherpieny
Institute of Statistical Studies & Research
Cairo University, Egypt

Mohamed A. Ahmed
Institute of Statistical Studies & Research
Cairo University, Egypt
mrmohamedali2005@ yahoo.com

Abstract

Finding the best fitted distribution for data set becomes practically an important problem in world of data sets so that it is useful to use new families of distributions to fit more cases or get better fits than before. In this paper, a new generating family of generalized distributions so called the Kumaraswamy - Kumaraswamy (KW-KW) family is presented. Four important common families of distributions are illustrated as special cases from the KW KW family. Moments, probability weighted moments, moment generating function, quantile function, median, mean deviation, order statistics and moments of order statistics are obtained. Parameters estimation and variance covariance matrix are computed using maximum likelihood method. A real data set is used to illustrate the potentiality of the KW KW weibull distribution (which derived from the kw kw family) compared with other distributions.

Keywords: Kumaraswamy Kumaraswamy Distribution, Moments, Order Statistics, quantile function, Maximum Likelihood Estimation.

1. Introduction

The main idea of this paper is based on generating new families of generalized distributions, see Wahed (2006), to derive more generalized distributions from the following integration

$$F(x;T,W) = \int_0^{G_1(x;W)} g_2(\mathrm{t};\mathrm{T})\,dt ; 0 < t < 1; -\infty < x < \infty \tag{1}$$

Where $G_1(x;W)$ and $g_1(x;W)$ are the cdf and pdf of the baseline distribution, $G_2(t;T)$ and $g_2(t;T)$ are the cdf and the pdf of the generator distribution, T is the parameters vector of the generator distribution and W is the parameters vector of the baseline distribution.

The contributions of this paper are four parts. First, we present the pdf and cdf of the new KW KW family of generalized distributions (section 2). Second, we calculate some properties of KW KW family like Moments, probability weighted moments, moment generating function, quantile function, median, mean deviation, order statistics and moments of order statistics (section 3). Third, we estimate KW KW family parameters using the maximum likelihood method and calculate the variance covariance matrix (section 4). Finally, we present a numerical example using real data on the KW KW Weibull and it gives the best fit between other distributions (section 5).

2. The New Kumaraswamy Kumaraswamy Family

El-Sherpieny and Ahmed (2014) presented the KW KW distribution where the cumulative distribution function (cdf) and the probability density function (pdf) respectively are:

$$P(t;a,b,\alpha,\beta) = 1-\left\{1-\left[1-\left(1-t^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b} \; ; 0<t<1\; ;\; a,b,\alpha,\beta>0$$

and

$$p(t;a,b,\alpha,\beta) = ab\,\alpha\,\beta t^{\alpha-1}(1-t^{\alpha})^{\beta-1}\left[1-(1-t^{\alpha})^{\beta}\right]^{a-1}\left\{1-\left[1-(1-t^{\alpha})^{\beta}\right]^{a}\right\}^{b-1} ;0<t<1;a,b,\alpha,\beta>0 \quad (2)$$

We will consider $p(t;a,b,\alpha,\beta)$ as the pdf of the generator distribution $g_2(t;T)$ in equation (1), then the cdf of the KW-KW family can be given by substituting equation (2) into equation (1) as follows:

$$F(x;T,W) = ab\,\alpha\,\beta \int_0^{G_1(x;W)} t^{\alpha-1}(1-t^{\alpha})^{\beta-1}\left[1-(1-t^{\alpha})^{\beta}\right]^{a-1}\left\{1-\left[1-(1-t^{\alpha})^{\beta}\right]^{a}\right\}^{b-1} dt$$

Where $T=a,b,\alpha,\beta$ is the parameters vector of the generator distribution, W is the parameters vector of the baseline distribution and $G_1(x;W)$ is the cdf of the baseline distribution.

Then,

$$F(x;T,W) = 1-\left\{1-\left[1-\left(1-G_1^{\alpha}\left(x;W\right)\right)^{\beta}\right]^{a}\right\}^{b} ; -\infty<x<\infty\; ;\; a,b,\alpha,\beta>0 \quad (3)$$

Differentiating equation (3) yields the pdf of the family:

$$f(x;T,W) = ab\,\alpha\,\beta\; g_1(x;W)\; G_1^{\alpha-1}(x;W)\;(1-G_1^{\alpha}(x;W))^{\beta-1}\left[1-(1-G_1^{\alpha}(x;W))^{\beta}\right]^{a-1}$$

$$\times\left\{1-\left[1-(1-G_1^{\alpha}(x;W))^{\beta}\right]^{a}\right\}^{b-1} ; -\infty<x<\infty\; ;\; a,b,\alpha,\beta>0 \quad (4)$$

where a, b, α and β are five shape parameters

Note that: if b=1, then the family of KW KW reduces to the family of exponentiated KW see Cordeiro *et al.* (2013a), if a=1 and b=1, then it reduces to the family of KW see Cordeiro and Castro (2010), if b=1and α=1, then it reduces to the family of Exponentiated Generalized see Cordeiro *et al.* (2013b) and if a=1, b=1 and β=1, then it reduces to the exponentiated family.

An expansion of the density function (for simplicity) will be obtained as follows:
Using binomial expansion in equation (2), where *b* is a real non integer, yields

$$f(x;T,W) = ab\,\alpha\,\beta \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} g_1(x;W)\, G_1^{\alpha-1}(x;W)\,(1-G_1^{\alpha}(x;W))^{\beta-1} \times \left[1-(1-G_1^{\alpha}(x;W))^{\beta}\right]^{a(i+1)-1}$$

Then, using binomial expansion again in the last equation, where *a* is real non integer, leads to:

$$f(x;T,W) = ab\,\alpha\,\beta \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{b-1}{i}\binom{a(i+1)-1}{j} g_1(x;W)\, G_1^{\alpha-1}(x;W)\,(1-G_1^{\alpha}(x;W))^{\beta(j+1)-1}$$

and using binomial expansion more time in the last equation, where β is real non integer, yields:

$$f(x;T,W) = ab\,\alpha\,\beta \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k} g_1(x;W)\, G_1^{\alpha(k+1)-1}(x;W)$$

when *a* , *b* and β are integers *i* stops at *b-1*, *j* stops at *a(j+1)-1* and *k* stops at *β(j+1)-1*

Condition for the pdf expansion:

Since,

$$ab\,\alpha\,\beta \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k} \int_{-\infty}^{\infty} g_1(x;W)\, G_1^{\alpha(k+1)-1}(x;W)\,dx = 1$$

then,

$$ab\alpha\beta \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k} \left[\frac{G_1^{\alpha(k+1)}}{\alpha(k+1)}\right]_{-\infty}^{\infty} = 1$$

Hence,

$$\frac{ab\beta}{k+1} \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k} = 1$$

So the form of the expansion of the pdf with its condition is

$$f(x;T,W) = \sum_{i,j,k=0}^{\infty} w_{i,j,k}\, g_1(x;W)\, G_1^{\alpha(k+1)-1}(x;W) \qquad (5)$$

Where,

$$w_{i,j,k} = ab\alpha\beta\,(-1)^{i+j+k} \binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k}$$

and

$$\frac{1}{\alpha(k+1)} \sum_{i,j,k=0}^{\infty} w_{i,j,k} = 1$$

Another form can be yielded when α is real non integer by adding and subtracting 1 to $G_1^{\alpha(k+1)-1}(x,W)$ into equation (3) as follows

$$f(x;T,W)=ab\alpha\beta\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k}g_1(x;W)\left[1-(1-G_1(x;W))\right]^{\alpha(k+1)-1}$$

Using binomial expansion yields:

$$f(x;T,W)=ab\alpha\beta\sum_{i,j,k,l=0}^{\infty}(-1)^{i+j+k+l}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k}\binom{\alpha(k+1)-1}{l}$$

$$\times g_1(x;W)\left(1-G_1(x;W)\right)^{l}$$

Using binomial expansion more time yields:

$$f(x;T,W)=ab\alpha\beta\sum_{i,j,k,l=0}^{\infty}(-1)^{i+j+k+l}\sum_{m=0}^{l}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k}\binom{\alpha(k+1)-1}{l}\binom{l}{m}$$

$$\times g_1(x;W)\,G_1^m(x;W)$$

Condition for the pdf expansion:

Since,

$$ab\alpha\beta\sum_{i,j,k,l=0}^{\infty}(-1)^{i+j+k+l}\sum_{m=0}^{l}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k}\binom{\alpha(k+1)-1}{l}\binom{l}{m}$$

$$\times\int_{=\infty}^{\infty}g_1(x;W)\,G_1^m(x;W)\,dx=1$$

Then,

$$ab\alpha\beta\sum_{i,j,k,l=0}^{\infty}(-1)^{i+j+k+l}\sum_{m=0}^{l}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k}\binom{\alpha(k+1)-1}{l}\binom{l}{m}\left[\frac{G_1^{m+1}}{m+1}\right]_{-\infty}^{\infty}=1$$

Hence,

$$\frac{ab\alpha\beta}{m+1}\sum_{i,j,k,l=0}^{\infty}(-1)^{i+j+k+l}\sum_{m=0}^{l}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k}\binom{\alpha(k+1)-1}{l}\binom{l}{m}=1$$

So the another form of the expansion of the pdf with its condition is

$$f(x;T,W)=\sum_{i,j,k,l=0}^{\infty}\sum_{m=0}^{l}w_{i,j,k,l,m}\;g_1(x;W)\,G_1^m(x;W) \qquad (6)$$

Where,

$$w_{i,j,k,l,m}=ab\alpha\beta(-1)^{i+j+k+l}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\beta(j+1)-1}{k}\binom{\alpha(k+1)-1}{l}\binom{l}{m}$$

and

$$\frac{1}{m+1}\sum_{i,j,k,l=0}^{\infty}\sum_{m=0}^{l} w_{i,j,k,l,m} = 1$$

An expansion for the cumulative function:

Using binomial expansion for $[F(x)]^h$, where h is integer, leads to:

Since,

$$\left[F(x)\right]^h = \left\langle 1-\left\{1-\left[1-\left(1-G_1^{\alpha}(x;W)\right)^{\beta}\right]^{a}\right\}^{b}\right\rangle^{h}$$

Then,

$$[F(x)]^h = \sum_{g=0}^{h}\binom{h}{g}(-1)^g\left\{1-\left[1-\left(1-G_1^{\alpha}(x,W)\right)^{\beta}\right]^{a}\right\}^{bg}$$

Using binomial expansion another time, where b is real non integer, leads to

$$\left[F(x)\right]^h = \sum_{g=0}^{h}\sum_{p=0}^{\infty}\binom{h}{g}\binom{bg}{p}(-1)^{g+p}\left[1-\left(1-G_1^{\alpha}(x;W)\right)^{\beta}\right]^{ap}$$

Using binomial expansion again, where *a* is real non integer, yields

$$\left[F(x)\right]^h = \sum_{g=0}^{h}\sum_{p,q=0}^{\infty}\binom{h}{g}\binom{bg}{p}\binom{ap}{q}(-1)^{g+p+q}\left(1-G_1^{\alpha}(x;W)\right)^{\beta q}$$

Using binomial expansion again, where β is real non integer, leads to

$$\left[F(x)\right]^h = \sum_{g=0}^{h}\sum_{p,q,t=0}^{\infty}\binom{h}{g}\binom{bg}{p}\binom{ap}{q}\binom{\beta q}{t}(-1)^{g+p+q+t}G_1^{\alpha t}(x;W)$$

Adding and subtracting 1 to $G_1(x,W)$ into last equation yields

$$\left[F(x)\right]^h = \sum_{g=0}^{h}\sum_{p,q,t=0}^{\infty}\binom{h}{g}\binom{bg}{p}\binom{ap}{q}\binom{\beta q}{t}(-1)^{g+p+q+t}\left[1-\left(1-G_1(x;W)\right)\right]^{\alpha t}$$

Using binomial expansion again leads to

$$\left[F(x)\right]^h = \sum_{g=0}^{h}\sum_{p,q,t,f=0}^{\infty}\binom{h}{g}\binom{bg}{p}\binom{ap}{q}\binom{\beta q}{t}\binom{\alpha t}{f}(-1)^{g+p+q+t+f}\left(1-G_1(x;W)\right)^{f}$$

Using binomial expansion again, where f integer leads to

$$\left[F(x)\right]^h = \sum_{g=0}^{h}\sum_{p,q,t,f=0}^{\infty}\sum_{z}^{f}\binom{h}{g}\binom{bg}{p}\binom{ap}{q}\binom{\beta q}{t}\binom{\alpha t}{f}\binom{f}{z}(-1)^{g+p+q+t+f+z}G_1^{z}(x;W)$$

Replacing $\sum_{f=0}^{\infty}\sum_{z=0}^{f}$ with $\sum_{z=0}^{\infty}\sum_{f=z}^{\infty}$ yielding

$$\left[F(x)\right]^{h}=\sum_{g=0}^{h}\sum_{p,q,t=0}^{\infty}\sum_{f=z}^{\infty}\sum_{z=0}^{\infty}\binom{h}{g}\binom{bg}{p}\binom{ap}{q}\binom{\beta q}{t}\binom{\alpha t}{f}\binom{f}{z}(-1)^{g+p+q+t+f+z}\,G_1^{z}(x;W)$$

Finally,

$$\left[F(x)\right]^{h}=\sum_{z=0}^{\infty}s_z\;G_1^{z}(x;W) \qquad (7)$$

Where,

$$s_z=\sum_{g=0}^{h}\sum_{p,q,t=0}^{\infty}\sum_{f=z}^{\infty}\binom{h}{g}\binom{bg}{p}\binom{ap}{q}\binom{\beta q}{t}\binom{\alpha t}{f}\binom{f}{z}(-1)^{g+p+q+t+f+z}$$

3. Some properties of the Kumaraswamy Kumaraswamy Family

In this section some properties of the kw- kw family for any KW KW generalized distribution will be obtained as follows:

3.1 Moments

If X has the pdf (6), then its r[th] moments can be given from

$$E\left(x^{r}\right)=\int_{-\infty}^{\infty}x^{r}f(x)\,dx \qquad (8)$$

First, when α is an integer, substituting equation (5) into equation (8) yields:

$$E\left(x^{r}\right)=\sum_{i,j,k=0}^{\infty}w_{i,j,k}\int_{-\infty}^{\infty}x^{r}\,g_{(x;w)}\,G_1^{\alpha(k+1)-1}(x;W)\,dx$$

Then,

$$E\left(x^{r}\right)=\sum_{i,j,k=}^{\infty}w_{i,j,k}\;\tau_{r,\alpha(k+1)-1,0}$$

Where, τ is the probability weighted moments (PWM) of the baseline distribution.

Second, when α is a real non integer, substituting equation (6) into equation (8) yields:

$$E\left(x^{r}\right)=\sum_{i,j,k,l=0}^{\infty}\sum_{m=0}^{l}w_{i,j,k,l,m}\int_{-\infty}^{\infty}x^{r}g_{(x;w)}G_1^{m}(x;W)\,dx$$

Then,

$$E\left(x^{r}\right)=\sum_{i,j,k,l=0}^{\infty}\sum_{m=0}^{l}w_{i,j,k,l,m}\;\tau_{r,m,0}$$

Another form Based on the parent quantile function:

let,

$G(x;w)=u$

First, when α is an integer yields

$$E\left(x^{r}\right)=\sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_{0}^{1}\left(Q_{G}(\mathrm{u})\right)^{r} u^{\alpha(k+1)-1} du$$

Second, when α is a real non integer yields

$$E\left(x^{r}\right)=\sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} w_{i,j,k,l,m} \int_{0}^{1}\left(Q_{G}(\mathrm{u})\right)^{r} u^{m} du$$

3.2 The Probability Weighted Moments

The probability weighted moments (PWM) can be given from

$$\tau_{s,r,0}=E\left\{x^{s} F^{r}(x)\right\}$$

Then,

$$\tau_{s,r}=\int_{-\infty}^{\infty} x^{s} f(x) F^{r}(x)\, dx \qquad (9)$$

First when α is an integer

Substituting equation (5) & (7) into equation (9), Replacing h with r, leads to:

$$\tau_{s,r,0}=\sum_{i,j,k=0}^{\infty} \sum_{z=0}^{\infty} s_{z} w_{i,j,k} \int_{-\infty}^{\infty} x^{s} g_{1}(x;W) G_{1}^{z}(x;W)\, dx$$

Then,

$$\tau_{s,r,0}=\sum_{i,j,k=0}^{\infty} \sum_{z=0}^{\infty} s_{z} w_{i,j,k} \tau_{s,\alpha(k+1)+z,0}$$

Second, when α is a real non integer, Substituting equation (6) & (7) into equation (9), Replacing h with r, leads to:

$$\tau_{s,r,0}=\sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} \sum_{z=0}^{\infty} s_{z} w_{i,j,k} \int_{-\infty}^{\infty} x^{s} g_{1}(x;W) G_{1}^{m+z}(x;W)\, dx$$

Then,

$$\tau_{s,r,0}=\sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} \sum_{z=0}^{\infty} s_{z} w_{i,j,k} \tau_{s,m+z,0}$$

Another form Based on the parent quantile function:

First, when α is integer yields

$$\tau_{s,r,0}=\sum_{i,j,k=0}^{\infty} \sum_{z=0}^{\infty} s_{z} w_{i,j,k} \int_{0}^{1}\left(Q_{G}(\mathrm{u})\right)^{s} u^{z} du$$

Second, when α is real non integer yields

$$\tau_{s,r,0} = \sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} \sum_{z=0}^{\infty} s_z w_{i,j,k} \int_0^1 (\mathrm{Q_G}(\mathrm{u}))^s \, u^{m+z} \, du$$

3.3 The Moment Generating Function

Generally, the moment generating function is:

$$M_x(t) = E(e^{tx})$$

and using expansion at the last equation leads to

$$M_x(t) = E(\sum_{r=0}^{\infty} \frac{x^r t^r}{r!})$$

So,

$$M_x(t) = \sum_{r=0}^{\infty} \frac{E(x^r) t^r}{r!}$$

Hence,

$$M_x(t) = E(x^r) \sum_{r=0}^{\infty} \frac{t^r}{r!}$$

Another form Based on the parent quantile function:

First, when α is integer

$$M_u(t) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_0^1 \exp(\mathrm{t\,Q_G}(\mathrm{u})) u^{\alpha(k+1)-1} du$$

Second, when α is real non integer

$$M_u(t) = \sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} w_{i,j,k,l,m} \int_0^1 \exp(\mathrm{t\,Q_G}(\mathrm{u})) u^m \, du$$

3.4 Quantile Function and The Median

The quantile function of the KW KW family can be gotten from the cdf of the KW KW family.

As follows:

Let, $u = 1 - \left\{ 1 - \left[1 - \left(1 - G_1^{\alpha}(x;W) \right)^{\beta} \right]^{a} \right\}^{b}; 0 < u < 1, x \in R$

Then,

$$Q_G \left[1 - \left[1 - \left[1 - (1-u)^{\frac{1}{b}} \right]^{\frac{1}{a}} \right]^{\frac{1}{\beta}} \right]^{\frac{1}{\alpha}} = x$$

Since, median can be yielded by substituting u with 0.5

Hence,

$$Median = Q_G\left[1-\left[1-\left[1-\left(\frac{1}{2}\right)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right]^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}$$

3.5 The Mean Deviation

Basically, the mean deviation is a measure for the amount of scatter in X.

Generally, the mean deviation is expressed by:

$$\delta_1(x) = \int_{-\infty}^{\infty} |x-\mu|\; f(x)\,dx \quad \text{and} \quad \delta_2(x) = \int_{-\infty}^{\infty} |x-M|\; f(x)\,dx$$

where, µ is mean and M is median.

These measures can be expressed easily as:

$$\delta_1(X) = 2\mu\, F(\mu) - 2\,T(\mu) \quad and \quad \delta_2(X) = \mu - 2\,T(M)$$

where,

$$T(q) = \int_{-\infty}^{q} x\, f(x)\,dx$$

Which is the first incomplete moment

Another form Based on the parent quantile function:

First, when α is integer yields

$$T\,(q) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_{0}^{G(q)} Q_G(\mathrm{u})\, u^{\alpha(k+1)-1}\, du$$

Second, when α is real non integer yields

$$T\,(q) = \sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} w_{i,j,k,l,m} \int_{0}^{G(q)} Q_G(\mathrm{u})\, u^{m}\, du$$

3.6 Order Statistics

The pdf of the u th order statistics can be given from, see Arnold *et al*. (1992)

$$f_{(u:n)}(x_{u:n}) = \frac{f\,(x_u)}{B\,(u, n-u+1)} F^{u-1}(x_u)\left\{1-F\,(x_u)\right\}^{n-u} \qquad (10)$$

Then, we can use binomial expansion into $F^{u-1}(x_u)\left\{1-F(x_u)\right\}^{n-u}$ as follows

$$F^{u-1}(x_u)\left\{1-F(x_u)\right\}^{n-u} = \sum_{w=0}^{n-u} \binom{n-u}{w} (-1)^w\, F^{u+w-1}(x)$$

Substituting equation (7) into last equation, Replacing *u+w-1* with *r*, leads to:

$$F^{u-1}(x_u)\left\{1-F\left(x_u\right)\right\}^{n-u}=\sum_{w=0}^{n-u}\binom{n-u}{w}(-1)^w\sum_{z=0}^{\infty}s_z\;G_1^z\left(x;W\right)$$

Then,

$$F^{u-1}(x_u)\left\{1-F\left(x_u\right)\right\}^{n-u}=\sum_{w=0}^{n-u}p_{z,w}\;G_1^z\left(x;W\right) \qquad (11)$$

Where,

$$p_{z,w}=\sum_{z=0}^{\infty}s_z\binom{n-u}{w}(-1)^w$$

First: when α is integer

Substituting equation (5) and (11) into (10) leads to

$$f_{(u:n)}\left(x_{u:n}\right)=\frac{g_1\left(x_u;W\right)}{B\left(u,n-u+1\right)}\sum_{i,j,k=0}^{\infty}\sum_{w=0}^{n-u}w_{i,j,k}\;p_{z,w}\;G_1^{z+\alpha(k+1)-1}\left(x_u;W\right) \qquad (12)$$

Second, when α is real non integer

Substituting (6) and (11) into (10) leads to

$$f_{(u:n)}\left(x_{u:n}\right)=\frac{g_1\left(x_u;W\right)}{B\left(u,n-u+1\right)}\sum_{i,j,k,l,m=0}^{\infty}\sum_{m=0}^{l}\sum_{w=0}^{n-u}w_{i,j,k,l,m}\;p_{z,w}\;G_1^{m+z}\left(x_u;W\right) \qquad (13)$$

Moments of Order Statistics

Moments of order statistics is defined by

$$E_{(u:n)}\left(x_u^r\right)=\int_{-\infty}^{\infty}x_u^r\,f_{(u:n)}\left(x_u\right)dx_u \qquad (14)$$

First, when α is integer, substituting equation (12) into equation (14) leads to

$$E_{(u:n)}\left(x_u^r\right)=\frac{1}{B\left(u,n-u+1\right)}\sum_{i,j,k=0}^{\infty}\sum_{w=0}^{n-u}p_{z,w}\,w_{i,j,k}\int_{-\infty}^{\infty}x_u^r\;g_1\left(x_u;W\right)G_1^{z+\alpha(k+1)-1}\left(x_u;W\right)dx_u$$

Then,

$$E_{(u:n)}\left(x_u^r\right)=\frac{1}{B(u,n-u+1)}\sum_{i,j,k=0}^{\infty}\sum_{w=0}^{n-u}p_{z,w}\,w_{i,j,k}\;\tau_{r,z+\alpha(k+1)-1,0}$$

Second, when α is real non integer, substituting equation (13) into equation (14) leads to

$$E_{(u:n)}\left(x_u^r\right)=\frac{1}{B\left(u,n-u+1\right)}\sum_{i,j,k,l,m=0}^{\infty}\sum_{m=0}^{l}\sum_{w=0}^{n-u}w_{i,j,k,l,m}\;p_{z,w}\int_{-\infty}^{\infty}x_u^r\;g_1\left(x_u;W\right)G_1^{m+z}\left(x_u;W\right)dx_u$$

Then,

$$E_{(u:n)}\left(x_u^r\right)=\frac{1}{B(u,n-u+1)}\sum_{i,j,k,l,m=0}^{\infty}\sum_{m=0}^{l}\sum_{w=0}^{n-u}w_{i,j,k,l,m}\;p_{z,w}\,\tau_{r,m+z}$$

Since the L-moments are linear functions of expected order statistics, see Hosking (1990):

$$\lambda_{s+1} = (s+1)^{-1} \sum_{y=0}^{s} (-1)^y \binom{s}{y} E(x_{s+1-y:s+1}) \text{ , } s = 0,1,....$$

So the first four L-moments can be calculated from

$$\lambda_1 = E(X_{1:1})$$

$$\lambda_2 = \frac{1}{2} E(X_{2:2} - 2X_{1:2})$$

$$\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3})$$

$$\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$$

4. Maximum Likelihood Estimators

let X_1, X_2… Xn be the iid random variables from the KW-KW (*a*, b, α, β) distribution and $T = a,b,\alpha,\beta$ is the parameters vector of the generator distribution and W is the parameters vector of the baseline distribution, then The likelihood function is

$$L(x;T,W) = \left(ab\,\alpha\,\beta\right)^n \prod_{i=1}^{n} \left\{ 1 - \left[1-(1-G_1^{\alpha}(x_i;W))^{\beta} \right]^{a} \right\}^{b-1} \left[1-(1-G_1^{\alpha}(x_i;W))^{\beta} \right]^{a-1}$$

$$\times \prod_{i=0}^{n} (1-G_1^{\alpha}(x_i;W))^{\beta-1}\, G_1^{\alpha-1}(x_i;W) \;\; g_1(x_i;W)$$

and the log likelihood is

$$l(x;T,W) = n\left[\log a + \log b + \log\alpha + \log\beta\right] + \left(b-1\right)\sum_{i=1}^{n}\log\left\{ 1 - \left[1-(1-G_1^{\alpha}(x_i;W))^{\beta} \right]^{a} \right\}$$

$$+\left(a-1\right)\sum_{i=1}^{n}\log\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta} \right] + \left(\beta-1\right)\sum_{i=1}^{n}\log(1-G_1^{\alpha}(x_i;W))$$

$$+\left(\alpha-1\right)\sum_{i=1}^{n}\log G_1(x_i;W) + \sum_{i=1}^{n}\log g_1(x_i;W)$$

Partial differentiating with respect to *a* yields:

$$\frac{\partial\, l(x;T,W)}{\partial\, a} = \frac{n}{a} - \left(b-1\right)\sum_{i=1}^{n} \frac{\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta} \right]^{a} \log\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta} \right]}{\left\{ 1 - \left[1-(1-G_1^{\alpha}(x_i,W))^{\beta} \right]^{a} \right\}}$$

$$+\sum_{i=1}^{n}\log\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta} \right]$$

Then.

$$\frac{\partial\, l\,(x;T,W)}{\partial\, a}=\frac{n}{a}+\sum_{i=1}^{n}\log\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]\left\{1-(b-1)\sum_{i=1}^{n}\frac{\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}}{\left\{1-\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}\right\}}\right\}$$

Partial differentiating with respect to *b* yields:

$$\frac{\partial\, l\,(x;T,W)}{\partial\, b}=\frac{n}{b}+\sum_{i=1}^{n}\log\left\{1-\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}\right\}$$

Partial differentiating with respect to α yields:

$$\frac{\partial\, l\,(x;T,W)}{\partial\, \alpha}=\frac{n}{\alpha}-a\,\beta(b-1)\sum_{i=1}^{n}\frac{\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a-1}}{\left\{1-\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}\right\}}(1-G_1^{\alpha}(x_i;W))^{\beta-1}$$

$$\times\, G_1^{\alpha}(x_i;W)\,\log G_1(x_i;W)+\beta(a-1)\sum_{i=1}^{n}\frac{(1-G_1^{\alpha}(x_i;W))^{\beta-1}}{1-(1-G_1^{\alpha}(x_i;W))^{\beta}}G_1^{\alpha}(x_i;W)\,\log G_1(x_i;W)$$

$$-(\beta-1)\sum_{i=1}^{n}\frac{G_1^{\alpha}(x_i;W)\,\log G_1(x_i;W)}{(1-G_1^{\alpha}(x_i;W))}+\sum_{i=1}^{n}\log G_1(x_i;W)$$

then,

$$\frac{\partial\, l\,(x;T,W)}{\partial\, \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}(1-G_1^{\alpha}(x_i;W))^{\beta-1}\,G_1^{\alpha}(x_i;W)\,\log G_1(x_i;W)\left\{\frac{\beta(a-1)}{1-(1-G_1^{\alpha}(x_i;W))^{\beta}}\right.$$

$$\left.-\frac{a\,\beta(b-1)\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a-1}}{\left\{1-\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}\right\}}-\frac{(\beta-1)}{(1-G_1^{\alpha}(x_i;W))^{\beta}}\right\}+\sum_{i=1}^{n}G_1(x_i;W)$$

Partial differentiating with respect to β yields:

$$\frac{\partial\, l\,(x;T,W)}{\partial\, \beta}=\frac{n}{\beta}+a(b-1)\sum_{i=1}^{n}\frac{\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a-1}}{\left\{1-\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}\right\}}(1-G_1^{\alpha}(x_i;W))^{\beta}\,\log\,(1-G_1^{\alpha}(x_i;W))$$

$$-(a-1)\sum_{i=1}^{n}\frac{(1-G_1^{\alpha}(x_i;W))^{\beta}}{1-(1-G_1^{\alpha}(x_i;W))^{\beta}}\quad\log(1-G_1^{\alpha}(x_i;W))+\sum_{i=1}^{n}\log\,(1-G_1^{\alpha}(x_i;W))$$

Then,

$$\frac{\partial\, l\,(x;T,W)}{\partial\,\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}(1-G_1^{\alpha}(x_i;W))^{\beta}\ \log\,(1-G_1^{\alpha}(x_i;W)\left\{\frac{a(b-1)\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a-1}}{\left\{1-\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}\right\}}\right.$$

$$\left.-\frac{(a-1)}{1-(1-G_1^{\alpha}(x_i;W))^{\beta}}+\frac{1}{(1-G_1^{\alpha}(x_i;W))^{\beta}}\right\}$$

Partial differentiating with respect to W_i yields:

$$\frac{\partial\, l\,(x;T,W)}{\partial W_j}=-\alpha\,\beta\, a\,(b-1)\sum_{i=1}^{n}\frac{\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a-1}}{\left\{1-\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}\right\}}(1-G_1^{\alpha}(x_i;W))^{\beta-1}\,G_1^{\alpha-1}(x_i;W)\frac{\partial G_1\,(x_i;W)}{\partial w_j}$$

$$+\alpha\,\beta\,(a-1)\sum_{i=1}^{n}\frac{(1-G_1^{\alpha}(x_i;W))^{\beta-1}}{1-(1-G_1^{\alpha}(x_i;W))^{\beta}}\,G_1^{\alpha-1}(x_i;W)\ \frac{\partial G_1\,(x_i;W)}{\partial w_j}$$

$$-\ \alpha\ (\beta-1)\sum_{i=1}^{n}\frac{G_1^{\alpha-1}(x_i;W)}{1-G_1^{\alpha}(x_i;W)}\ \frac{\partial G_1\,(x_i;W)}{\partial w_j}+(\alpha-1)\sum_{i=1}^{n}\frac{1}{G_1\,(x_i;W)}\ \frac{\partial G_1\,(x_i;W)}{\partial w_j}$$

$$+\sum_{i=1}^{n}\frac{1}{g_1\,(x_i;W)}\ \frac{\partial\, g_1\,(x_i;W)}{\partial w_j}$$

Then,

$$\frac{\partial\, l\,(x;T,W)}{\partial W_j}=\sum_{i=1}^{n}\frac{G_1^{\alpha-1}(x_i;W)}{G_1\,(x_i;W)}\ \frac{\partial G_1\,(x_i;W)}{\partial w_j}\left\{(\alpha-1)G_1^{1-\alpha}(x_i;W)\right.$$

$$-\alpha\,\beta\, a\,(b-1)\sum_{i=1}^{n}\frac{\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a-1}}{\left\{1-\left[1-(1-G_1^{\alpha}(x_i;W))^{\beta}\right]^{a}\right\}}(1-G_1^{\alpha}(x_i;W))^{\beta-1}\,G_1\,(x_i;W)$$

$$\left.+\alpha\,\beta\,(a-1)\sum_{i=1}^{n}\frac{(1-G_1^{\alpha}(x_i;W))^{\beta-1}}{1-(1-G_1^{\alpha}(x_i;W))^{\beta}}\,G_1\,(x_i;W)-\ \alpha\ (\beta-1)\sum_{i=1}^{n}\frac{G_1\,(x_i;W)}{1-G_1^{\alpha}(x_i;W)}\right\}$$

$$+\sum_{i=1}^{n}\frac{1}{g_1\,(x_i;W)}\ \frac{\partial\, g_1\,(x_i;W)}{\partial w_j}$$

Equating these partial derivatives to zero, yields a system of non linear equations that needs to be solved numerically to obtain parameters estimation, Numerical maximization of the log-likelihood above is accomplished by using the RS method see Rigby and Stasinopoulos (2005).

Let θ is the vector of the unknown parameters (a,b,α,β,w_j), then the element of the 5 x 5 information matrix I (a,b,α,β,w_j) can be approximated by:

$$I_{ij}(\hat{\theta}) = E\left[-\frac{\partial^2 \ell(\theta)}{\partial\theta_i\,\partial\theta_j}\Big|_{\theta=\hat{\theta}}\right]$$

$I^{-1}(a,b,\alpha,\beta,w_j)$ is the variance covariance matrix of the unknown parameters (a,b,α,β,w_j) and the asymptotic distributions of the MLE parameters are

$$\sqrt{n}\,(\hat{\theta}_i - \theta_i) \approx N_5(0, I^{-1}(\hat{\theta}_i))\ ,\ i = 1,\ldots,5$$

The approximation 100 (1- α) % confidence intervals for the unknown parameters, based on the asymptotic distributions of the KW-KW family distributions, are determined, respectively, as

$$\hat{\theta}_i \pm z_{\frac{\alpha}{2}}\sqrt{I^{-1}(\hat{\theta}_i)}\ ,\ i = 1,\ldots,5$$

where, $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ th percentile of a standard normal distribution.

5. Application

In this section we give a real data to illustrate an example for one distribution of the new family of KW-KW distributions so called the the Kumaraswamy - Kumaraswamy - Weibull distribution to see how the new model works practically and we will use the Mathematica package version 10 to do that. In our example, We used different distributions like the kumaraswamy - kumaraswamy - Weibull (KW-KW-W) distribution [derived from the KW KW family], the exponentiated kumaraswamy weibull (E-KW-W) distribution [derived from the E KW family], the kumaraswamy weibull (KW-W) distribution [derived from the KW family], the exponentiated generalized weibull (EG-W) distribution [derived from the E G family], the exponentiated weibull (E-W) distribution [derived from the E family] and the weibull (W) distribution.

The following data represents the strengths of 1.5 cm glass fibres for 27 devices, originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper, see http://www.npl.co.uk/.

0.17, 0.13, 0.16, 0.14, 0.20, 0.15, 0.13, 0.11, 0.15, 0.12, 0.12, 0.15, 0.12, 0.16, 0.21, 0.20, 0.23, 0.16, 0.12, 0.10, 0.32, 0.33, 0.33, 0.36, 0.38, 0.20, 0.26

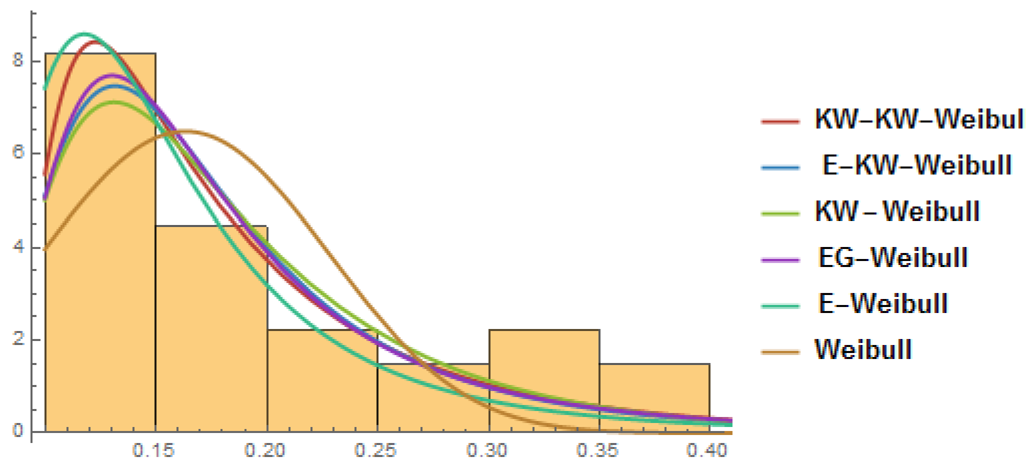

Figure 1: Probability density functions for different distributions

Table 1: The MLE of the parameter(s) and the associated AIC and BIC values

The Model	MLE of parameters	K.S	p-value	AIC	BIC
KW - KW – W (a, b, α, β, θ, λ)	(26.99, 0.17, 13, 3.58, 0.77, 45.11)	0.161	0.381	-74.026	-65.2193
E - KW - W (a, α, β, θ, λ)	(152.84, 4.89, 4.91, 0.28, 172.13)	0.168	0.182	-69.1345	-61.5859
KW – W (α, β, θ, λ)	(37.75, 0.35, 1.09, 29.49)	0.17	0.191	-67.986	-61.6956
EG – W (a, β, θ, λ)	(168336, 10.74, 0.238, 11.374)	0.159	0.124	-63.419	-50.4372
E - W (α, θ, λ)	(20880.9, 0.29, 21634.8)	0.2	0.095	-70.6536	-63.6212
W(θ, λ)	(3.08, 5.39)	0.227	0.054	-51.7742	-47.9999

In Table (1) we compute the MLE of distributions parameters, Kolmogorov-Smirnov (K.S) test statistic, AIC and BIC for every distribution. We find from K.S test statistic that at level of significance 0.05 we can not reject that the data fits all earlier distributions but it fits more the KW - KW - W (a, b, α, β, θ, λ) distribution. We see that the KW - KW - W (a, b, α, β, θ, λ) distribution has the smallest AIC and BIC so the KW - KW - W (a, b, α, β, θ, λ) distribution can be the best fitted distribution compared with earlier distributions

Table 2: The log-likelihood function, The likelihood ratio test statistic and p-values

The Model	H_0	ℓ (log likelihood)	Λ (The likelihood ratio test statistic)	df (degrees of freedom)	P-value
E - KW – W (a,α,β,θ,λ)	b=0	31.5136	2.921	1	0.0238
KW - W (α, β, θ, λ)	a=0, b=0	31.0791	2.869	2	0.0871
EG - W (a, β, θ, λ)	α =0, b=0	30.5947	3.589	2	0.00514
E - W (α, θ, λ)	a=0, b=0, β=0	29.1843	6.6586	3	0.00842
W(θ, λ)	a=0, b=0,α=0,β=0	10.4911	44.045	4	6.279×10^{-9}

In Table (2) and based on the likelihood ratio test, where the KW- KW-W (a, b, α, β, θ, λ) distribution generalizes the E-KW-W (a, α, β, θ, λ) distribution, the KW -W (α, β, θ, λ) distribution, the EG -W (a, β, θ, λ) distribution, the E-W (α, θ, λ) distribution, the E-W (α, θ, λ) distribution and the W(θ, λ) distribution, we find from the p-values that we can reject all null hypotheses when the level of significance is 0.1.

6. Conclusions

The new Kumaraswamy Kumaraswamy family of generalized distributions can be useful in world of data sets because of its flexible properties and its generalization of some important families of distributions like the families of Exponentiated Kumaraswamy, Kumaraswamy, Exponentiated Generalized and the Exponentiated. It is clear from the application that the new model so called the Kumaraswamy Kumaraswamy Weibull distribution which was derived from the Kumaraswamy Kumaraswamy family practically gave the best fit compared with other distributions. We encourage researchers to do more researches and applications on other distributions of the Kumaraswamy Kumaraswamy family in univariate and multivariate cases.

7. Appendix

The elements of the observed information matrix I(a,b,α,β,w_j) for the parameters are:

Let:

$$A_i = \left\{1-\left[1-\left(1-G_1^{\alpha}\left(x_i;\mathrm{w}\right)\right)^{\beta}\right]^{\alpha}\right\}^2, \quad B_i = \left[1-\left(1-G_1^{\alpha}\left(x_i;\mathrm{w}\right)\right)^{\beta}\right]^{\alpha}, \quad C_i = \left(1-G_1^{\alpha}\left(x_i;\mathrm{w}\right)\right)^{\beta},$$

$D_i = G_1^{\alpha}\left(x_i;\mathrm{w}\right)$, $E_i = G_1\left(x_i,\mathrm{w}\right)$ and $H_i = g_1\left(x_i;\mathrm{w}\right)$

Then,

$$\frac{\partial^2 l(x;T,W)}{\partial a^2} = \frac{-n}{a^2} - (b-1)\sum_{i=1}^{n}\left\{\frac{(1-B_i)\log(1-C_i)B_i\log(1-C_i)+B_i\log(1-C_i)B_i\log(1-C_i)}{A_i}\right\},$$

$$\frac{\partial^2 l(x;T,W)}{\partial a\partial b} = -\sum_{i=1}^{n}\frac{\mathrm{B}_i\log(1-C_i)}{1-\mathrm{B}_i},$$

$$\frac{\partial^2 l(x;T,W)}{\partial a\partial\alpha} = -(b-1)\sum_{i=1}^{n}\left\{\frac{(1-B_i)B_i\,\dfrac{\beta(1-D_i)^{\beta-1}D_i\log E_i}{1-C_i}}{A_i}\right.$$

$$+\frac{(1-B_i)\log(1-C_i)a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}D_i\log E_i}{A_i}$$

$$\left.+\frac{B_i\log(1-C_i)a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}D_i\log E_i}{A_i}\right\}$$

$$+\sum_{i=1}^{n}\frac{\beta(1-D_i)^{\beta-1}D_i\log E_i}{(1-C_i)},$$

$$\frac{\partial^2 l(x;T,W)}{\partial a\partial\beta} = -(b-1)\sum_{i=1}^{n}\left\{\frac{-(1-B_i)\log(1-C_i)a(1-C_i)^{a-1}C_i\log(1-D_i)}{A_i}\right.$$

$$\left.-\frac{(1-B_i)B_i\dfrac{C_i\log(1-D_i)}{(1-C_i)}-B_i\log(1-C_i)a(1-B_i)^{a-1}C_i\log(1-D_i)}{A_i}\right\}$$

$$-\sum_{i=1}^{n}\frac{C_i\log(1-D_i)}{(1-C_i)},$$

$$\frac{\partial^2 l(x;T,W)}{\partial a\partial W_j} = -(b-1)\sum_{i=1}^{n}\left\{\frac{(1-B_i)B_i\dfrac{\beta(1-D_i)^{\beta-1}\alpha(E_i)^{\alpha-1}\dfrac{\partial E_i}{\partial W_j}}{(1-C_i)}}{A_i}\right.$$

$$+\frac{(1-B_i)\{\log(1-C_i)\}a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}\alpha(E_i)^{\alpha-1}\dfrac{\partial E_i}{\partial W_j}}{A_i}$$

$$\left.+\frac{B_i\{\log(1-C_i)\}a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}\alpha(E_i)^{\alpha-1}\dfrac{\partial E_i}{\partial W_j}}{A_i}\right\}$$

$$+\sum_{i=1}^{n}\frac{\beta(1-D_i)^{\beta-1}\alpha(E_i)^{\alpha-1}\dfrac{\partial E_i}{\partial W_j}}{(1-C_i)},$$

$$\frac{\partial^2 l(x;T,W)}{\partial b^2} = \frac{-n}{b^2},$$

$$\frac{\partial^2 l(x;T,W)}{\partial b\partial\alpha} = -\sum_{i=1}^{n}\left\{\frac{a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}D_i\log E_i}{(1-B_i)}\right\},$$

$$\frac{\partial^2 l(x;T,W)}{\partial b\partial\beta} = \sum_{i=1}^{n}\left\{\frac{a(1-C_i)^{a-1}C_i\log(1-D_i)}{(1-B_i)}\right\},$$

$$\frac{\partial^2 l(x;T,W)}{\partial b \partial W_i} = -\sum_{i=1}^{n}\left\{\frac{a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}\alpha(E_i)^{\alpha-1}\frac{\partial E_i}{\partial W_j}}{(1-B_i)}\right\},$$

$$\frac{\partial^2 l(x;T,W)}{\partial\alpha\partial\alpha} = \frac{-n}{\alpha^2} - a\beta(b-1)\sum_{i=1}^{n}\left\{\frac{(1-C_i)^{a-1}}{(1-B_i)}(1-D_i)^{\beta-1}D_i[\log E_i]^2\right.$$

$$-\frac{(1-C_i)^{a-1}}{(1-B_i)}D_i(\log E_i)(B-1)(1-D_i)^{\beta-2}D_i(\log E_i)$$

$$+(1-D_i)^{\beta-1}D_i(\log E_i)\frac{(1-B_i)(a-1)(1-C_i)^{a-2}\beta(1-D_i)^{\beta-1}D_i(\log E_i)}{A_i}$$

$$\left.+\frac{(1-C_i)^{a-1}a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}D_i(\log E_i)}{A_i}\right\}$$

$$+\beta(a-1)\sum_{i=1}^{n}\left\{\frac{(1-D_i)^{\beta-1}}{(1-C_i)}D_i[\log E_i]^2\right.$$

$$\left.-D_i\log E_i\frac{(1-C_i)(\beta-1)(1-D_i)^{\beta-2}D_i(\log E_i)-(1-D_i)^{\beta-1}\beta(1-D_i)^{\beta-1}D_i\log E_i}{(1-C_i)^2}\right\}$$

$$+(\beta-1)\sum_{i=1}^{n}\left\{\frac{(1-D_i)(\log E_i)D_i(\log E_i)+D_i(\log E_i)D_i(\log E_i)}{(1-D_i)^2}\right\},$$

$$\frac{\partial^2 l(x;T,W)}{\partial\alpha\partial\beta} = a(b-1)\sum_{i=1}^{n}\left\{\frac{(1-C_i)^{a-1}}{(1-B_i)}(1-D_i)^{\beta-1}D_i(\log E_i)\right.$$

$$+a\beta(b-1)(1-D_i)^{\beta-1}\log(1-D_i)D_i(\log E_i)\frac{(1-C_i)^{a-1}}{(1-B_i)}$$

$$+(1-D_i)^{\beta-1}D_i(\log E_i)\frac{(1-B_i)(a-1)(1-C_i)^{a-2}C_i\log(1-D_i)}{A_i}$$

$$\left.-\frac{(1-C_i)^{a-1}a(1-C_i)^{a-1}C_i\log(1-D_i)}{A_i}\right\}$$

$$+(a-1)\sum_{i=1}^{n}\frac{(1-D_i)^{\beta-1}}{(1-C_i)}D_i(\log E_i)+\beta(a-1)\sum_{i=1}^{n}\left\{D_i(\log E_i)\right.$$

$$\left.\times\frac{(1-C_i)(1-D_i)^{\beta-1}\log(1-D_i)}{(1-C_i)^2}+\frac{(1-D_i)^{\beta-1}(1-D_i)^{\beta}\log(1-D_i)}{(1-C_i)^2}\right\}$$

$$-\sum_{i=1}^{n}\frac{D_i(\log E_i)}{(1-D_i)},$$

$$\frac{\partial^2 l(x;T,W)}{\partial\beta\partial\beta} = \frac{-n}{\beta^2} + a(b-1)\sum_{i=1}^{n}\Bigg\{C_i\left(\log(1-D_i)\right)^2\frac{(1-C_i)^{a-1}}{(1-B_i)}$$
$$-C_i\log(1-D_i)\frac{(1-B_i)(a-1)(1-C_i)^{a-2}C_i\left(\log(1-D_i)\right)}{A_i}$$
$$-\frac{(1-C_i)^{a-1}a(1-C_i)^{a-1}\mathrm{C_i}\log(1-D_i)}{A_i}\Bigg\}$$
$$-(a-1)\sum_{i=1}^{n}\Bigg\{\log(1-D_i)\frac{(1-C_i)C_i\log(1-D_i)}{(1-C_i)^2}$$
$$+\frac{C_i^{\,2}\log(1-D_i)}{(1-C_i)^2},$$

$$\frac{\partial^2 l(x;T,W)}{\partial\alpha\partial w_j} = -a\beta(b-1)\sum_{i=1}^{n}\Bigg\{\frac{(1-C_i)^{a-1}}{(1-B_i)}(1-D_i)^{\beta-1}\frac{D_i}{E_i}\frac{\partial E_i}{\partial w_j}$$
$$+\frac{(1-C_i)^{a-1}}{(1-B_i)}(1-D_i)^{\beta-1}\alpha E_i^{\,\alpha-1}\log E_i\frac{\partial E_i}{\partial w_j}$$
$$-\frac{(1-C_i)^{a-1}}{(1-B_i)}D_i\log E_i(B-1)(1-D_i)^{\beta-2}\alpha E_i^{\,\alpha-1}\frac{\partial E_i}{\partial w_j}$$
$$+(1-D_i)^{\beta-1}D_i\left(\log E_i\right)\frac{(1-B_i)(a-1)(1-C_i)^{a-2}\beta(1-D_i)^{\beta-1}\alpha E_i^{\,\alpha-1}\frac{\partial E_i}{\partial w_j}}{A_i}$$
$$+\frac{(1-C_i)^{a-1}a(1-C_i)^{a-1}+\beta(1-D_i)^{\beta-1}\alpha E_i^{\,\alpha-1}\frac{\partial E_i}{\partial w_j}}{A_i}\Bigg\}$$
$$+\beta(a-1)\sum_{i=1}^{n}\Bigg\{\frac{(1-D_i)^{\beta-1}}{(1-C_i)}\frac{D_i}{E_i}\frac{\partial E_i}{\partial w_j}+\frac{(1-D_i)^{\beta-1}}{(1-C_i)}\left(\log E_i\right)\alpha E_i^{\,\alpha-1}\frac{\partial E_i}{\partial w_j}$$
$$+D_i\left(\log E_i\right)\frac{(1-C_i)(\beta-1)(1-D_i)^{\beta-2}\alpha E_i^{\,\alpha-1}\frac{\partial E_i}{\partial w_j}}{(1-C_i)^2}$$
$$-\frac{(1-D_i)^{\beta-1}\beta(1-D_i)^{\beta-1}\alpha E_i^{\,\alpha-1}\frac{\partial E_i}{\partial w_j}}{(1-C_i)^2}\Bigg\}-(B-1)\sum_{i=1}^{n}\Bigg\{\frac{D_i}{(1-D_i)}\frac{1}{E_i}\frac{\partial E_i}{\partial w_j}$$
$$+\frac{\left(\log E_i\right)\alpha E_i^{\,\alpha-1}}{(1-D_i)}\frac{\partial E_i}{\partial w_j}+D_i\left(\log E_i\right)\frac{\alpha E_i^{\,\alpha-1}\frac{\partial E_i}{\partial w_j}}{(1-D_i)^2}\Bigg\}+\sum_{i=1}^{n}\frac{1}{E_i}\frac{\partial E_i}{\partial w_j},$$

$$\frac{\partial^2 l(x;T,W)}{\partial\beta\partial W_i}=a(b-1)\sum_{i=1}^{n}\left\{-C_i\frac{\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}}{(1-D_i)}\frac{(1-C_i)^{a-1}}{(1-B_i)}\right.$$

$$-\log(1-D_i)\beta(1-D_i)^{\beta-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}\frac{(1-C_i)^{a-1}}{(1-B_i)}$$

$$+C_i\log(1-D_i)\frac{(1-B_i)(a-1)(1-C_i)^{a-2}\beta(1-D_i)^{\beta-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}}{A_i}$$

$$\left.+\frac{(1-C_i)^{a-1}a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}}{A_i}\right\}$$

$$+(a-1)\sum_{i=1}^{n}\left\{\frac{C_i\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}}{(1-C_i)(1-D_i)}-\log(1-D_i)\frac{(1-C_i)\beta(1-D_i)^{\beta-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}}{(1-C_i)^2}\right.$$

$$\left.-\frac{C_i\beta(1-D_i)^{\beta-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}}{(1-C_i)^2}\right\}-\sum_{i=1}^{n}\frac{\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}}{(1-D_i)},$$

and

$$
\begin{aligned}
\frac{\partial^2 l(x;T,W)}{\partial W_j\,\partial W_k} = &-\alpha\beta a(b-1)\sum_{i=1}^{n}\left\{\frac{(1-C_i)^{a-1}}{(1-B_i)}(1-D_i)^{\beta-1}E_i^{\alpha-1}\frac{\partial^2 E_i}{\partial w_j\partial w_k}\right.\\
&+\frac{(1-C_i)^{a-1}}{(1-B_i)}(1-D_i)^{\beta-1}(\alpha-1)E_i^{\alpha-2}\frac{\partial E_i}{\partial w_k}\frac{\partial E_i}{\partial w_j}\\
&-\frac{(1-C_i)^{a-1}}{(1-B_i)}E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}(\beta-1)(1-D_i)^{\beta-2}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_k}\\
&+(1-D_i)^{\beta-1}E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}\frac{(1-B_i)(a-1)(1-C_i)^{a-2}\beta(1-D_i)^{\beta-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_k}}{A_i}\\
&\left.+\frac{(1-C_i)^{a-1}a(1-C_i)^{a-1}\beta(1-D_i)^{\beta-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_k}}{A_i}\right\}\\
&+\alpha\beta(a-1)\sum_{i=1}^{n}\left\{\frac{(1-D_i)^{\beta-1}}{(1-C_i)}E_i^{\alpha-1}\frac{\partial^2 E_i}{\partial w_j\partial w_k}+\frac{(1-D_i)^{\beta-1}}{(1-C_i)}\frac{\partial E_i}{\partial w_j}(\alpha-1)E_i^{\alpha-2}\frac{\partial E_i}{\partial w_k}\right.\\
&-E_i^{\alpha-1}\frac{\partial E_i}{\partial w_j}\frac{(1-C_i)(\beta-1)(1-D_i)^{\beta-2}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_k}}{(1-C_i)^2}\\
&\left.-\frac{(1-D_i)^{\beta-1}\beta(1-D_i)^{\beta-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_k}}{(1-C_i)^2}\right\}\\
&-\alpha(\beta-1)\sum_{i=1}^{n}\left\{\frac{E_i^{\alpha-1}}{(1-D_i)}\frac{\partial^2 E_i}{\partial w_j\partial w_k}\right.\\
&\left.+\frac{\partial E_i}{\partial w_j}\frac{(1-D_i)(\alpha-1)E_i^{\alpha-2}\frac{\partial E_i}{\partial w_k}+E_i^{\alpha-1}\alpha E_i^{\alpha-1}\frac{\partial E_i}{\partial w_k}}{(1-D_i)^2}\right\}\\
&+(\alpha-1)\sum_{i=1}^{n}\left\{\frac{1}{E_i}\frac{\partial^2 E_i}{\partial w_j\partial w_k}-\frac{\partial E_i}{\partial w_j}\frac{\frac{\partial E_i}{\partial w_k}}{E_i^2}\right\}\\
&+\sum_{i=1}^{n}\left\{\frac{1}{H_i}\frac{\partial^2 H_i}{\partial w_j\partial w_k}-\frac{1}{H_i^2}\frac{\partial H_i}{\partial w_j}\frac{\partial H_i}{\partial w_k}\right\}
\end{aligned}
$$

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9. References

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