Fuzzy Programming with Quadratic Membership Functions for Multi-objective Transportation Problem

A. Satyanarayana Murthy
Department of Mathematics, GITAM University, Hyderabad, India
asnmurthynitw@gmail.com

Abstract

In the present paper, a fuzzy programming model with quadratic membership functions has been developed for the solution of a Multi-Objective Transportation problem. In literature, several fuzzy programming approaches exist with various types of membership functions such as linear, exponential, hyperbolic etc. These membership functions are defined, by taking the lower and upper values of the objective functions into account. In some cases, these methods fail to obtain an integer compromise optimal solution. In the present method, two coefficients of the quadratic membership functions are determined by the lower and upper values of the objective functions. The other coefficient is taken as a variable in the fuzzy programming approach. This means that the membership curve is fixed at the two end points and set free in between. Application of the method on numerical examples proved that the approach could generate integer compromise optimal solutions.

Keywords: Fuzzy Programming, Quadratic membership functions, Compromise optimal solution.

1. Introduction

In the classical transportation problem, unit quantities of a homogeneous product are to be transported from m sources to n destinations in such a way that the total transportation cost is a minimum. A variable x_{ij} represents the unknown quantity to be transported from the i^{th} source to the j^{th} destination. In addition, there is a penalty c_{ij} associated with transporting a unit of the product from the i^{th} source to the j^{th} destination. The penalty may be cost or delivery time or safety of delivery etc. In the real world situations, the transportation problem usually involves multiple, incommensurable and conflicting objectives. This kind of problem is called multi-objective transportation problem (MOTP). The mathematical model of the MOTP is written as follows.

$$\text{Minimize } Z_k(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad \text{for} \quad k=1,2,\ldots,K$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \ (i=1,2,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \ (j=1,2,\ldots,n) \qquad (1)$$

$x_{ij} \geq 0$ and are integer, i = 1,2,…,m, j=1,2,…,n

The concept of fuzzy set theory, first introduced by Zadeh[12], is used for solving different types of linear programming problems. Zimmermann[14] applied the fuzzy set theory with some suitable membership functions to solve linear programming problem with several objective functions. Efficient solutions, as well as an optimal compromise solution for MOTP was obtained by Bit.et.al[4] using fuzzy programming technique with linear membership functions. Bit.et.al[6], have also presented an additive fuzzy programming model that considers weights and priorities for all non-equivalent objectives for the MOTP problem. A fuzzy compromise programming method to obtain a non-dominated compromise solution to the MOTP was developed by Li and Lai[10]. Fuzzy programming techniques with hyperbolic and exponential membership functions to obtain optimal compromise solutions of the MOTP were introduced by Verma et.al[11]. A fuzzy programming approach with the linear membership functions to determine the optimal compromise solution of the MOTP was presented by Abd El-wahed [1]. An interactive fuzzy goal programming for multi-objective transportation problems was developed by Abd El-wahed and Lee[2].

2. Solution of the problem

Let L_k and U_k be the aspired level of achievement and the highest acceptable level of achievement for the k-th objective function, respectively.

We assume the membership function of the k-th objective function as

$$\mu_k(z) = a_{k1}z^2 + a_{k2}z + a_3 \tag{2}$$

Setting the membership values of the k-th objective function at the aspired level and highest acceptable level, 1 and 0 respectively, we have the following equations

$$\mu_k(L_k) = a_{k1}L_k^2 + a_{k2}L_k + a_{k3} = 1$$
$$\mu_k(U_k) = a_{k1}U_k^2 + a_{k2}U_k + a_{k3} = 0$$

By solving the above system of linear equations, a_{k2} and a_{k3} are expressed in terms of a_{k1}. Thus, the membership function of the k-th objective function takes the following form

$$\mu_k(Z) = \frac{U_k - Z_k}{U_k - L_k} + a_{k1}Z_k^2 - a_{k1}(L_k + U_k)Z_k + a_{k1}L_kU_k$$

We, now describe the proposed approach for the solution of the MOTP.

2.1. Fuzzy programming approach

STEP 1. Solve the multi-objective transportation problem as a single objective transportation problem, taking each time only one objective as objective function and ignoring all others.

STEP 2. Evaluate each objective function at each solution derived in STEP 1. For each objective function, determine it's lower and upper bounds (L_k and U_k) according to the set of optimal solutions.

STEP 3. Define the membership function of the k-th objective as

$$\mu_k(Z) = \frac{U_k - Z_k}{U_k - L_k} + a_{k1}Z_k^2 - a_{k1}(L_k + U_k)Z_k + a_{k1}L_kU_k$$

STEP 4. Adopting the fuzzy decision of Bellmann and Zadeh[3], together with the Quadratic membership function , a fuzzy optimization model of the MOTP can be written as follows

$$\text{Maximize } \lambda$$

$$\text{Subject to } \lambda \le \frac{U_k - Z_k}{U_k - L_k} + a_{k1}Z_k^2 - a_{k1}(L_k + U_k)Z_k + a_{k1}L_kU_k \text{ ,k=1,2,…,K}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ , (i=1,2,…,m)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \text{ (j=1,2,…,n)} \qquad (3)$$

$$x_{ij} \ge 0 \text{ and are integer, i = 1,2,…,m, j=1,2,…,n}$$

The above model can be simplified as

$$\text{Maximize } \lambda$$

Subject to

$$-a_{k1}Z_k^2 + \lambda + Z_k\left(\frac{1}{U_k - L_k} + a_{k1}(L_k + U_k)\right) - a_{k1}L_kU_k \le \frac{U_k}{U_k - L_k}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ , (i=1,2,…,m)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \text{ (j=1,2,…,n)} \qquad (4)$$

$$x_{ij} \ge 0 \text{ and are integer, i = 1,2,…,m, j=1,2,…,n}$$

STEP 4. Solve (4) using a nonlinear programming technique to get an integer compromise optimal solution Verma et al[11] presented the following three fuzzy optimization models using hyperbolic, exponential and linear membership functions respectively

$$\text{Maximize } x_{mn+1}$$

$$\text{Subject to } \alpha_k Z_k(x) + x_{mn+1} \le \alpha_k(L_k + U_k)/2, \text{ k=1,2,…,K}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ , (i=1,2,…,m)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \text{ (j=1,2,…,n)} \qquad (5)$$

$$x_{ij} \ge 0 \text{ for i=1,2,…,m, j=1,2,…,n and } x_{mn+1} \ge 0$$

Where $\alpha_k = 6/(U_k - L_k)$

Maximize λ

Subject to $\exp(-s\Psi_k(x)) - (1 - \exp(-s))\,\lambda \geq \exp(-s)$, for k=1,2,…,K

$$\sum_{j=1}^{n} x_{ij} = a_i\,, (i=1,2,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ (j=1,2,\ldots,n) \qquad (6)$$

$$x_{ij} \geq 0 \text{ for } i=1,2,\ldots,m,\ j=1,2,\ldots,n \text{ and } \lambda \geq 0$$

Where $\Psi_k(x) = (Z_k(x) - L_k)/(U_k - L_k)$ and s is a non-zero parameter prescribed by the decision maker.

Maximize λ

Subject to $Z_k + (U_k - L_k)\,\lambda \leq U_k$, for k=1,2,…,K

$$\sum_{j=1}^{n} x_{ij} = a_i\,, (i=1,2,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ (j=1,2,\ldots,n) \qquad (7)$$

$$x_{ij} \geq 0 \text{ for } i=1,2,\ldots,m,\ j=1,2,\ldots,n \text{ and } \lambda \geq 0.$$

We, now define the family of distance functions of a compromise solution from the ideal solution as

$L_p(\lambda,k) = [\sum_{k=1}^{K} \lambda_k^p(1-d_k)^p]^{1/p}$ where d_k is the degree of closeness of the compromise solution vector to the ideal solution vector with respect to the k-th objective. For a minimization objective function,

$$d_k = (\text{The ideal value of } Z_k)/(\text{The compromise value of } Z_k)$$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ are the weights attached to the objective functions. The power p represents a distance parameter $1 \leq p \leq \infty$. Assuming that the sum of all the weights is unity, we define L_1, L_2 and $L\infty$ as follows

$$L_1(\lambda,k) = 1- \sum_{k=1}^{K} \lambda_k d_k$$

$$L_2(\lambda,k) = [\sum_{k=1}^{K} \lambda_k^2(1-d_k)^2]^{1/2}$$

$$L\infty(\lambda,k) = \max_k\{\lambda_k(1-d_k)\}$$

The approach which gives a compromise solution close to the ideal solution, is better than the other if Min $L_p(\lambda,k)$ is achieved for it's solution with respect to some p. In the next section, we consider three numerical examples, calculate the distance functions, and carry out a comparison of the proposed method with the three existing methods.

3. Numerical examples

Example 1.

Minimize $x_{11}+2x_{12}+7x_{13}+7x_{14}+x_{21}+9x_{22}+3x_{23}+4x_{24}+8x_{31}+9x_{32}+4x_{33}+6x_{34}$

Minimize $4x_{11}+4x_{12}+3x_{13}+4x_{14}+5x_{21}+8x_{22}+9x_{23}+10x_{24}+6x_{31}+2x_{32}+5x_{33}+x_{34}$

Subject to $x_{11} + x_{12} + x_{13} + x_{14} = 8$

$x_{21} + x_{22} + x_{23} + x_{24} = 19$

$x_{31} + x_{32} + x_{33} + x_{34} = 17$

$x_{11} + x_{21} + x_{31} = 11$ (8)

$x_{12} + x_{22} + x_{32} = 3$

$x_{13} + x_{23} + x_{33} = 14$

$x_{14} + x_{24} + x_{34} = 16$

$x_{ij} \geq 0$ and are integer for i=1,2,3,j=1,2,3,4

First, the two objective functions are, minimized separately with respect to the same set of constraints. The lower bounds of the objective functions are $L_1 = 143$, $L_2 = 167$ and the upper bounds are $U_1 = 208$, $U_2 = 265$. Using (4), we construct the following linear programming model

Maximize λ

Subject to $x_{11} + x_{12} + x_{13} + x_{14} = 8$

$x_{21} + x_{22} + x_{23} + x_{24} = 19$

$x_{31} + x_{32} + x_{33} + x_{34} = 17$

$x_{11} + x_{21} + x_{31} = 11$

$x_{12} + x_{22} + x_{32} = 3$

$x_{13} + x_{23} + x_{33} = 14$

$x_{14} + x_{24} + x_{34} = 1$

$$-a_{11}z_1^2 + \lambda + z_1\left(\frac{1}{65} + 345a_{11}\right) - (143)(208)a_{11} \leq \frac{208}{65}$$

$$-a_{21}z_2^2 + \lambda + z_2\left(\frac{1}{98} + 432a_{21}\right) - (167)(265)a_{21} \leq \frac{265}{98}$$

$z_1 = x_{11}+2x_{12}+7x_{13}+7x_{14}+x_{21}+9x_{22}+3x_{23}+4x_{24}+8x_{31}+9x_{32}+4x_{33}+6x_{34}$

$z_2 = 4x_{11}+4x_{12}+3x_{13}+4x_{14}+5x_{21}+8x_{22}+9x_{23}+10x_{24}+6x_{31}+2x_{32}+5x_{33}+x_{34}$

$x_{ij} \geq 0$ and are integer for i=1,2,3,j=1,2,3,4.

Solving by LINGO 14.0, the optimal compromise solution of problem (8) is obtained as

$x_{11} = 4, x_{12} = 3$, $x_{13} = 1$, $x_{21} = 7$, $x_{23} = 12$, $x_{33} = 1$, $x_{34} = 16$

Example 2.

Minimize $16x_{11}+19x_{12}+12x_{13}+22x_{21}+13x_{22}+19x_{23}+14x_{31}+28x_{32}+8x_{33}$

Minimize $9x_{11}+14x_{12}+12x_{13}+16x_{21}+10x_{22}+14x_{23}+8x_{31}+20x_{32}+6x_{33}$

$$
\begin{aligned}
\text{Subject to}\quad & x_{11} + x_{12} + x_{13} = 14 \\
& x_{21} + x_{22} + x_{23} = 16 \\
& x_{31} + x_{32} + x_{33} = 12 \\
& x_{11} + x_{21} + x_{31} = 10 \\
& x_{12} + x_{22} + x_{32} = 15 \\
& x_{13} + x_{23} + x_{33} = 17
\end{aligned} \tag{9}
$$

$x_{ij} \geq 0$ and are integer for i=1,2,3,j=1,2,3.

First, the two objective functions are, minimized separately with respect to the same set of constraints. The lower bounds of the objective functions are $L_1 = 517$, $L_2 = 374$ and the upper bounds are $U_1 = 518$, $U_2 = 379$. Using (4), we construct the following linear programming model

Maximize λ

$$
\begin{aligned}
\text{Subject to}\quad & x_{11} + x_{12} + x_{13} = 14 \\
& x_{21} + x_{22} + x_{23} = 16 \\
& x_{31} + x_{32} + x_{33} = 12 \\
& x_{11} + x_{21} + x_{31} = 10 \\
& x_{12} + x_{22} + x_{32} = 15 \\
& x_{13} + x_{23} + x_{33} = 17
\end{aligned}
$$

$$-a_{11}z_1^2 + \lambda + z_1(1+1035a_{11}) - (517)(518)a_{11} \leq 518$$

$$-a_{21}z_2^2 + \lambda + z_2(\frac{1}{5} + 753a_{21}) - (374)(379)a_{21} \leq \frac{379}{5}$$

$z_1=16x_{11}+19x_{12}+12x_{13}+22x_{21}+13x_{22}+19x_{23}+14x_{31}+28x_{32}+8x_{33}$

$z_2=9x_{11}+14x_{12}+12x_{13}+16x_{21}+10x_{22}+14x_{23}+8x_{31}+20x_{32}+6x_{33}$

$x_{ij} \geq 0$ and are integer for i=1,2,3,j=1,2,3.

Solving by LINGO 14.0, the optimal compromise solution of problem (9) is obtained as

$x_{11} = 10$, $x_{13} = 4$, $x_{22} = 15$, $x_{23} = 1$, $x_{33} = 12$

Example 3.

Minimize

$9x_{11}+12x_{12}+9x_{13}+6x_{14}+9x_{15}+7x_{21}+3x_{22}+7x_{23}+7x_{24}+5x_{25}+6x_{31}+5x_{32}+9x_{33}+11x_{34}+3x_{35}+$

$6x_{41}+8x_{42}+11x_{43}+2x_{44}+2x_{45}$

Minimize

$2x_{11}+9x_{12}+8x_{13}+x_{14}+4x_{15}+x_{21}+9x_{22}+9x_{23}+5x_{24}+2x_{25}+8x_{31}+x_{32}+8x_{33}+4x_{34}+5x_{35}+$
$2x_{41}+8x_{42}+6x_{43}+9x_{44}+8x_{45}$

Minimize

$2x_{11}+4x_{12}+6x_{13}+3x_{14}+6x_{15}+4x_{21}+8x_{22}+4x_{23}+9x_{24}+2x_{25}+5x_{31}+3x_{32}+5x_{33}+3x_{34}+6x_{35}+$
$6x_{41}+9x_{42}+6x_{43}+3x_{44}+x_{45}$

Subject to $x_{11}+x_{12}+x_{13}+x_{14}+x_{15} = 5$

$$x_{21}+x_{22}+x_{23}+x_{24}+x_{25} = 4$$
$$x_{31}+x_{32}+x_{33}+x_{34}+x_{35} = 2$$
$$x_{41}+x_{42}+x_{43}+x_{44}+x_{45} = 9$$
$$x_{11}+x_{21}+x_{31}+x_{41} = 4 \qquad (10)$$
$$x_{12}+x_{22}+x_{32}+x_{42} = 4$$
$$x_{13}+x_{23}+x_{33}+x_{43} = 6$$
$$x_{14}+x_{24}+x_{34}+x_{44} = 2$$
$$x_{15}+x_{25}+x_{35}+x_{45} = 4$$
$$x_{ij} \geq 0 \text{ for } i=1,2,3,4, j=1,2,3,4,5.$$

First, the three objective functions are, minimized separately with respect to the same set of constraints. The lower bounds of the objective functions are $L_1 = 102$, $L_2 =72$, $L_3 = 64$ and the upper bounds are $U_1 = 157$, $U_2 = 148$, $U_3 = 100$. Using (4), we construct the following linear programming model

Maximize λ

Subject to $x_{11}+x_{12}+x_{13}+x_{14}+x_{15} = 5$

$$x_{21}+x_{22}+x_{23}+x_{24}+x_{25} = 4$$
$$x_{31}+x_{32}+x_{33}+x_{34}+x_{35} = 2$$
$$x_{41}+x_{42}+x_{43}+x_{44}+x_{45} = 9$$
$$x_{11}+x_{21}+x_{31}+x_{41} = 4$$
$$x_{12}+x_{22}+x_{32}+x_{42} = 4$$
$$x_{13}+x_{23}+x_{33}+x_{43} = 6$$
$$x_{14}+x_{24}+x_{34}+x_{44} = 2$$
$$x_{15}+x_{25}+x_{35}+x_{45} = 4$$

$$-a_{11}z_1^2 + \lambda + z_1(\frac{1}{55} + 259a_{11}) - (102)(157)a_{11} \leq \frac{157}{55}$$

$$-a_{21}z_2^2 + \lambda + z_2(\frac{1}{76} + 220a_{21}) - (72)(148)a_{21} \leq \frac{148}{76}$$

$$-a_{31}z_3^2+\lambda+z_3(\frac{1}{36}+164a_{31})-(64)(100)a_{31}\leq\frac{100}{36}$$

$z_1=9x_{11}+12x_{12}+9x_{13}+6x_{14}+9x_{15}+7x_{21}+3x_{22}+7x_{23}+7x_{24}+5x_{25}+6x_{31}+5x_{32}+9x_{33}+11x_{34}+3x_{35}+6x_{41}+8x_{42}+11x_{43}+2x_{44}+2x_{45}$

$z_2=2x_{11}+9x_{12}+8x_{13}+x_{14}+4x_{15}+x_{21}+9x_{22}+9x_{23}+5x_{24}+2x_{25}+8x_{31}+x_{32}+8x_{33}+4x_{34}+5x_{35}+2x_{41}+8x_{42}+6x_{43}+9x_{44}+8x_{45}$

$z_3=2x_{11}+4x_{12}+6x_{13}+3x_{14}+6x_{15}+4x_{21}+8x_{22}+4x_{23}+9x_{24}+2x_{25}+5x_{31}+3x_{32}+5x_{33}+3x_{34}+6x_{35}+6x_{41}+9x_{42}+6x_{43}+3x_{44}+x_{45}$

$x_{ij}\geq 0$ for i=1,2,3,4,j=1,2,3,4,5

Solving by LINGO 14.0, the optimal compromise solution of problem (10) is

$x_{11}=2$, $x_{13}=1$,$x_{14}=2$,$x_{22}=2$, $x_{23}=2$, $x_{32}=2$,$x_{41}=2$, $x_{43}=3$, $x_{45}=4$

We compare the results obtained by the proposed method with those obtained by the fuzzy approaches with linear, hyperbolic and exponential membership functions in the next section (We call these as the hyperbolic, exponential and linear approaches respectively).

4. Analysis of the results

The distances L_1, L_2 and $L\infty$ from the ideal solutions, for all the approaches are computed, for all the considered examples and are tabulated as follows. We have assumed equal weights to all the objective functions, in all the three examples.

For the first example, the hyperbolic and the proposed approaches produce the same compromise solution, which is nearer to the ideal solution with respect to all the distances L_1, L_2 and $L\infty$ than the solution produced by the linear and the exponential approaches.

In case of the second example, the hyperbolic approach does not give an integer compromise optimal solution. It, however produces a non-integer compromise optimal solution. The distances L_1, L_2 and $L\infty$ are calculated using the non-integer solution in this case. All the other methods produce integer compromise optimal solutions. It can be observed that the proposed method gives a compromise optimal solution, which is nearer to the ideal solution with respect to all the distances L_1, L_2 and $L\infty$ than the solutions produced by all the other methods.

However, in the case of the third example, where there are a greater number of objective functions and constraints, the linear and the exponential approaches produce a compromise optimal solution nearer to the ideal solution, than the solutions produced by the hyperbolic and the proposed approaches.

DISTANCES FROM IDEAL SOLUTIONS

EXAMPLE NO.		LINEAR APPROACH	HYPERBOLIC APPROACH	EXPONENTIAL APPROACH	QUADRATIC APPROACH
1	**(Z_1,Z_2)**	(170,190)	(160,195)	(170,190)	(160,195)
	L_1	0.13993808	0.12491871	0.13993808	0.12491871
	L_2	0.099848	0.08931276	0.099848	0.08931276
	$L\infty$	0.0794	0.07179	0.0794	0.07179
2	**(Z_1,Z_2)**	(517,379)	(517.5,376.5)	(517,379)	(518,374)
	L_1	0.0065963	0.003803	0.0065963	0.00096525
	L_2	0.0065963	0.003355	0.0065963	0.00096525
	$L\infty$	0.0065963	0.00332	0.0065963	0.00096525
3	**(Z_1,Z_2,Z_3)**	(112,106,80)	(122,106,80)	(112,106,80)	(122,106,80)
	L_1	0.2033468	0.228229714	0.2033468	0.228229714
	L_2	0.129467	0.13733903	0.129467	0.13733903
	$L\infty$	0.106918	0.106918	0.106918	0.106918

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6. References

1. W.F. Abd El-wahed (2001). A multi-objective transportation problem under fuzziness, *Fuzzy Sets and Systems* 117, 27-33.
2. W.F. Abd El-wahed and S.M. Lee (2006). Intearctive fuzzy goal programming for multi-objective transportation problems, *Omega* 34, 158-166.
3. R.R. Bellman and L.A. Zadeh (1970). Decision making in a fuzzy environment, *Management Sci.* 17, 141-164.
4. A.K. Bit, M.P. Biswal and S.S. Alam (1992). Fuzzy programming approach to multicriteria decision making transportation problem, *Fuzzy Sets and Systems* 50, 135-141
5. A.K. Bit, M.P. Biswal and S.S. Alam (1993). Fuzzy programming approach to multiobjective solid transportation problem, *Fuzzy Sets and Systems* 57, 183-194.
6. A.K. Bit, M.P. Biswal and S.S. Alam (1993). An additive fuzzy programming model for multiobjective transportation problem, *Fuzzy Sets and Systems* 57, 313-319.
7. S. Chanas, W. Kolodziejczyk and A. Machaj (1984). A fuzzy approach to the transportation problem, *Fuzzy Sets and Systems* 13, 211-221.

8. S. Chanas and D. Kuchta (1996). A concept of the optimal solution of the transportation problem with fuzzy cost coefficients, *Fuzzy Sets and Systems* 82, 299-305.

9. S. Chanas, D. Kuchta (1998). Fuzzy integer transportation problem, *Fuzzy Sets and Systems* 98, 291-298.

10. L.Li and K.K. Lai (2000). A fuzzy approach to the multiobjective transportation problem, *Computers and Operations Research* 28, 43-57.

11. R. Verma, M.P. Biswal and A. Biswas (1997). Fuzzy programming technique to solve mutiobjective transportation problem with some non-linear membership functions, *Fuzzy Sets and Systems* 91, 37-43.

12. L.A. Zadeh (1965). Fuzzy Sets, *Inform. And Control* 8, 338-353.

13. M. Zangiabadi and H.R. Maleki (2007). Fuzzy goal programming for multiobjective transportation problems, *Journal of Applied Mathematics and Computing* 24,449-460.

14. H.J. Zimmermann (1978). Fuzzy programming and linear programming with several objective functions, *Fuzzy sets and Systems* 1, 141-164.