

An Efficient Class of Estimators of Population Mean in Two-Phase Sampling Using Two Auxiliary Variables

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Abstract

This study is devoted to obtaining an efficient estimator for population mean in two-phase sampling using two auxiliary variables following Vishwakarma and Kumar (2014). Expression for bias and mean squared error (MSE) are obtained to the first order of approximation. The new proposed estimator is compared with some competitor estimators both theoretically and numerically using eight different data sets. It has been shown that the new proposed estimator gives more efficient results as compared to its competitor estimators.

Keywords: Two-phase sampling, Auxiliary variable, Bias, Mean squared error, Efficiency.

1. Introduction

In sample surveys, it is common practice to use the auxiliary information either at selection stage or on estimation stage, or at both stages, to improve precision of the estimators of the population means. Several methods of using the auxiliary variable at the estimation stage are described, which include linear regression estimator, ratio estimator and product estimator. According to Cochran (1940), ratio method of estimation is most preferred when the study variable is positively correlated with the auxiliary variable. Robson (1957) and Murthy (1964) proposed product method of estimation in case when there is negative correlation between the study variable and the auxiliary variable.

The use of ratio and product strategies in sample survey solely depends upon the knowledge of population mean of the auxiliary variable x . In case when the population mean of the auxiliary variable x is not known in advance, then in such situation a first phase sample of size n' is selected from the population of size N and only the auxiliary variable (x) is measured to get an estimate of population mean (\bar{X}). Then in second phase a sample of a sample of size n is drawn from the first phase sample of size n' on which both the study variable y and auxiliary variable x are measured. This technique of selecting the samples from given population is known as two-phase or double sampling. Sukhatme (1962), Hidiroglou and Sarndal (1998), Singh and Vishwakarma (2007) and Sahoo et al. (2010) have suggested some improved ratio, product and regression type estimators in two-phase sampling.

The chain regression type estimator was first introduced by Swain (1970). Chand (1975), Sukhatme and Chand (1977) and Kiregyera (1980) suggested some chain ratio and regression type estimators based on two auxiliary variables in two-phase sampling. Prasad et al. (1996), Singh and Espejo (2007), Singh and Choudhury (2012), Vishwakarma and Gangele (2014) and several authors have proposed some improved chain ratio, product and regression type estimators in two-phase sampling based on two auxiliary variables.

Following Vishwakarma and Kumar (2014), we suggest a class of estimators for the population mean in two-phase sampling using two auxiliary variables. It is shown that the proposed class of estimators outperforms as compared to the Vishwakarma and Kumar (2014) and several other competitors. Also, some special cases of the proposed class are considered in Table 2 (see Appendix).

2. Notations and some existing estimates

To obtain the bias and *MSE* of the estimators, we define:

$$\bar{y} = \bar{Y}(1 + e_0), \quad \bar{x} = \bar{X}(1 + e_1), \quad \bar{x}' = \bar{X}(1 + e_1'), \quad \bar{z}' = \bar{Z}(1 + e_2'),$$

Such that

$$E(e_0) = E(e_1) = E(e_1') = E(e_2') = 0, \text{ and}$$

$$E(e_0^2) = f_1 C_y^2, \quad E(e_1^2) = f_1 C_x^2, \quad E(e_1'^2) = E(e_1 e_1') = f_2 C_x^2,$$

$$E(e_2'^2) = E(e_2 e_2') = f_2 C_z^2, \quad E(e_0 e_1) = f_1 \rho_{yx} C_y C_x, \quad E(e_0 e_1') = f_2 \rho_{yx} C_y C_x,$$

$$E(e_0 e_2') = f_2 \rho_{yz} C_y C_z, \quad E(e_1 e_2') = E(e_1' e_2') = f_2 \rho_{yz} C_y C_z.$$

Where $f_1 = \left(\frac{1}{n} - \frac{1}{N}\right)$, $f_2 = \left(\frac{1}{n'} - \frac{1}{N}\right)$, $f_3 = \left(\frac{1}{n} - \frac{1}{n'}\right)$,

$$C_y^2 = \frac{S_y^2}{\bar{Y}^2}, \quad C_x^2 = \frac{S_x^2}{\bar{X}^2}, \quad C_z^2 = \frac{S_z^2}{\bar{Z}^2}, \quad \rho_{yx} = \frac{S_{yx}}{S_y S_x}, \quad \rho_{yz} = \frac{S_{yz}}{S_y S_z}, \quad \rho_{xz} = \frac{S_{xz}}{S_x S_z},$$

$$S_y^2 = \frac{1}{(N-1)} \sum_{i=1}^N (y_i - \bar{Y})^2, \quad S_x^2 = \frac{1}{(N-1)} \sum_{i=1}^N (x_i - \bar{X})^2, \quad S_z^2 = \frac{1}{(N-1)} \sum_{i=1}^N (z_i - \bar{Z})^2,$$

$$S_{yx} = \frac{1}{(N-1)} \sum_{i=1}^N (y_i - \bar{Y})(x_i - \bar{X}), \quad S_{yz} = \frac{1}{(N-1)} \sum_{i=1}^N (y_i - \bar{Y})(z_i - \bar{Z}) \text{ and}$$

$$S_{xz} = \frac{1}{(N-1)} \sum_{i=1}^N (x_i - \bar{X})(z_i - \bar{Z}).$$

The variance of the usual sample mean \bar{y} is given by

$$\text{Var}(\bar{y}) = f_1 \bar{Y}^2 C_y^2. \tag{1}$$

The usual chain type ratio estimator of \bar{Y} under two-phase sampling scheme using two auxiliary variables x and z , is given by

$$\bar{y}_R^d = \bar{y} \frac{\bar{x}' \bar{Z}}{\bar{x} \bar{z}'}, \tag{2}$$

where \bar{x}' and \bar{z}' are the sample means based on the first-phase sample of size n' . Also \bar{y} and \bar{x} are the sample means based on the second phase sample of size n .

The bias and MSE of \bar{y}_R^d , up to the first degree of approximation, are given by

$$Bias(\bar{y}_R^d) \cong \bar{Y} [f_3 C_x^2 + f_2 C_z^2 - f_3 \rho_{yx} C_y C_x - f_2 \rho_{yz} C_y C_z], \tag{3}$$

and

$$MSE(\bar{y}_R^d) \cong \bar{Y}^2 [f_1 C_y^2 + f_3 C_x^2 + f_2 C_z^2 - 2f_3 \rho_{yx} C_y C_x - 2f_2 \rho_{yz} C_y C_z]. \tag{4}$$

The chain type product estimator of \bar{Y} under two-phase sampling scheme using two auxiliary variables x and z , is given by

$$\bar{y}_P^d = \bar{y} \frac{\bar{x} \bar{z}'}{\bar{x}' \bar{Z}}. \tag{5}$$

The bias and MSE of \bar{y}_P^d , up to the first degree of approximation, are given by

$$Bias(\bar{y}_P^d) \cong \bar{Y} [f_3 \rho_{yx} C_y C_x + f_2 \rho_{yz} C_y C_z], \tag{6}$$

and

$$MSE(\bar{y}_P^d) \cong \bar{Y}^2 [f_1 C_y^2 + f_3 C_x^2 + f_2 C_z^2 + 2f_3 \rho_{yx} C_y C_x + 2f_2 \rho_{yz} C_y C_z]. \tag{7}$$

Singh and Choudhury (2012) suggested exponential chain type ratio estimator of \bar{Y} under two-phase sampling scheme using two auxiliary variables x and z , is given by

$$\bar{y}_{Re}^d = \bar{y} \exp \left\{ \frac{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} - \bar{x}}{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} + \bar{x}} \right\}. \tag{8}$$

The bias and MSE of \bar{y}_{Re}^d , up to the first degree of approximation, are given by

$$Bias(\bar{y}_{Re}^d) \cong \bar{Y} \left[\frac{3}{8} (f_3 C_x^2 + f_2 C_z^2) - \frac{1}{2} (f_3 \rho_{yx} C_y C_x + f_2 \rho_{yz} C_y C_z) \right], \tag{9}$$

and

$$MSE(\bar{y}_{Re}^d) \cong \bar{Y}^2 \left[f_1 C_y^2 + \frac{1}{4} (f_3 C_x^2 + f_2 C_z^2) - (f_3 \rho_{yx} C_y C_x + f_2 \rho_{yz} C_y C_z) \right]. \tag{10}$$

Singh and Choudhury (2012) suggested exponential chain type product estimator of \bar{Y} under two-phase sampling scheme using two auxiliary variables x and z , is given by

$$\bar{y}_{Pe}^d = \bar{y} \exp \left\{ \frac{\bar{x} - \left(\frac{\bar{x}'}{\bar{z}'}\right) \bar{Z}}{\bar{x} + \left(\frac{\bar{x}'}{\bar{z}'}\right) \bar{Z}} \right\}. \tag{11}$$

The bias and *MSE* of \bar{y}_{Pe}^d , up to the first degree of approximation, are given by

$$Bias(\bar{y}_{Pe}^d) \cong \bar{Y} \left[-\frac{1}{8} (f_3 C_x^2 + f_2 C_z^2) + \frac{1}{2} (f_3 \rho_{yx} C_y C_x + f_2 \rho_{yz} C_y C_z) \right], \tag{12}$$

and

$$MSE(\bar{y}_{Pe}^d) \cong \bar{Y}^2 \left[f_1 C_y^2 + \frac{1}{4} (f_3 C_x^2 + f_2 C_z^2) + (f_3 \rho_{yx} C_y C_x + f_2 \rho_{yz} C_y C_z) \right]. \tag{13}$$

Singh and Espejo (2007) suggested the ratio-product type estimator of \bar{Y} under two-phase sampling scheme using single auxiliary variable x as

$$\bar{y}_{RP}^d = \bar{y} \left\{ \alpha \frac{\bar{x}'}{\bar{x}} + (1-\alpha) \frac{\bar{x}}{\bar{x}'} \right\}, \tag{14}$$

where α is a real constant.

The minimum bias and *MSE* of \bar{y}_{RP}^d at optimum value of α i.e.

$$\alpha_{(opt)} = \frac{1}{2} \left(1 + \rho_{yx} \frac{C_y}{C_x} \right),$$

are given by

$$Bias(\bar{y}_{RP}^d)_{min} \cong \frac{1}{2} \bar{Y} f_3 [C_x^2 + \rho_{yx} C_y C_x - 2\rho_{yx}^2 C_y^2], \tag{15}$$

and

$$MSE(\bar{y}_{RP}^d)_{min} \cong \bar{Y}^2 [f_1 C_y^2 - f_3 \rho_{yx}^2 C_y^2]. \tag{16}$$

Vishwakarma and Kumar (2014) suggested exponential chain ratio-product type estimator of \bar{Y} under two-phase sampling scheme using two auxiliary variables x and z , is given by

$$\bar{y}_{RPe}^d = \bar{y} \left[\delta \exp \left\{ \frac{\left(\frac{\bar{x}'}{\bar{z}'}\right) \bar{Z} - \bar{x}}{\left(\frac{\bar{x}'}{\bar{z}'}\right) \bar{Z} + \bar{x}} \right\} + (1-\delta) \exp \left\{ \frac{\bar{x} - \left(\frac{\bar{x}'}{\bar{z}'}\right) \bar{Z}}{\bar{x} + \left(\frac{\bar{x}'}{\bar{z}'}\right) \bar{Z}} \right\} \right], \tag{17}$$

where δ is a real constant.

The minimum bias and MSE of \bar{y}_{RPe}^d at optimum value of δ i.e.

$$\delta_{(opt)} = \frac{f_3(C_x^2 + 2\rho_{yx}C_yC_x) + f_2(C_z^2 + 2\rho_{yz}C_yC_z)}{2(f_3C_x^2 + f_2C_z^2)},$$

are given by

$$Bias(\bar{y}_{RPe}^d)_{min} \cong \bar{Y} \left[\frac{\{(f_3C_x^2 + f_2C_z^2) + 2C_y(f_3\rho_{yx}C_x + f_2\rho_{yz}C_z)\}^2 - 12C_y^2(f_3\rho_{yx}C_x + f_2\rho_{yz}C_z)^2}{8(f_3C_x^2 + f_2C_z^2)} \right], \quad (18)$$

and

$$MSE(\bar{y}_{RPe}^d)_{min} \cong \bar{Y}^2 \left[f_1C_y^2 - \frac{(f_3\rho_{yx}C_yC_x + f_2\rho_{yz}C_yC_z)^2}{(f_3C_x^2 + f_2C_z^2)} \right]. \quad (19)$$

3. Proposed class of estimators

Following Vishwakarma and Kumar (2014), we propose a class of estimators of the population mean \bar{Y} under two-phase sampling scheme using two auxiliary variables x and z , is given by

$$\bar{y}_M^d = \{ \bar{y} + \alpha_1(\bar{x}' - \bar{x}) \} \left[\alpha_2 \exp \left\{ \frac{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} - \bar{x}}{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} + \bar{x}} \right\} + (1 - \alpha_2) \exp \left\{ \frac{\bar{x} - \left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z}}{\bar{x} + \left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z}} \right\} \right], \quad (20)$$

where α_1 and α_2 are constants, whose values are to be determined.

The proposed estimator \bar{y}_M^d can be written in terms of e 's as

$$\begin{aligned} (\bar{y}_M^d - \bar{Y}) = \bar{Y} & \left[e_0 - \frac{\alpha_2}{2}(e_1'^2 - e_2'^2 - e_1^2) + \alpha_2(e_1' - e_2' - e_1 + e_0e_1' - e_0e_2' - e_0e_1) + \frac{1}{8}(e_1^2 + e_1'^2 + e_2'^2) \right] \\ & + \frac{1}{4}(e_1'^2 - e_2'^2 - e_1^2 + e_1e_2' - e_1e_1' - e_1'e_2') - \frac{1}{2}(e_1' - e_2' - e_1 + e_0e_1' - e_0e_2' - e_0e_1) \\ & - \alpha_1\bar{X} \left[(e_1 - e_1') + \frac{1}{2}(e_1^2 + e_1'^2 - 2e_1e_1' - e_1'e_2' - e_1e_2') - \alpha_2(e_1^2 + e_1'^2 - 2e_1e_1' - e_1'e_2' - e_1e_2') \right]. \quad (21) \end{aligned}$$

Taking the expectation of both sides of the above equation, we get bias of \bar{y}_M^d , is given by

$$\begin{aligned} Bias(\bar{y}_M^d) = \bar{Y} & \left[\frac{(4\alpha_2 - 1)}{8} \{ f_3C_x^2 + f_2C_z^2 \} - \frac{(2\alpha_2 - 1)}{2} \{ f_3\rho_{yx}C_yC_x + f_2\rho_{yz}C_yC_z \} \right] \\ & + \alpha_1\bar{X} \left\{ \frac{(2\alpha_2 - 1)}{2} f_3C_x^2 + 2f_2\rho_{yz}C_yC_z \right\}. \quad (22) \end{aligned}$$

Squaring both sides of Equation (21) and ignoring higher order terms of e 's, we have

$$\begin{aligned} (\bar{y}_M^d - \bar{Y})^2 = & \bar{Y}^2 \left[e_0^2 + \frac{(2\alpha_2 - 1)^2}{2} \{e_1'^2 - e_2'^2 + e_1^2 - 2e_1e_1' + 2e_1e_2' - 2e_1'e_2'\} + (2\alpha_2 - 1)(e_0e_1' - e_0e_2' - e_0e_1) \right] \\ & + \alpha_1^2 \bar{X}^2 (e_1^2 + e_1'^2 - 2e_1e_1') - 2\alpha_1 \bar{X}\bar{Y} \left[(e_0e_1 - e_0e_1') + \frac{(2\alpha_2 - 1)}{2} \{e_1'^2 + e_1^2 - 2e_1e_1' + e_1e_2' - e_1'e_2'\} \right]. \end{aligned}$$

Taking the expectation of both sides of the above equation, we obtain the *MSE* of \bar{y}_M^d , as

$$\begin{aligned} MSE(\bar{y}_M^d) = & \bar{Y}^2 \left[f_1 C_y^2 + \frac{(2\alpha_2 - 1)^2}{4} \{f_3 C_x^2 + f_2 C_z^2\} - (2\alpha_2 - 1) \{f_3 \rho_{yx} C_y C_x + f_2 \rho_{yz} C_y C_z\} \right] \\ & + \alpha_1^2 \bar{X}^2 f_3 C_x^2 - 2\alpha_1 \bar{X}\bar{Y} \left\{ f_3 \rho_{yx} C_y C_x - \frac{(2\alpha_2 - 1)}{2} f_3 C_x^2 \right\}. \end{aligned} \tag{23}$$

From Eq (23), the optimum values of α_1 and α_2 are given by

$$\alpha_{1(opt)} = \frac{\bar{Y} (C_z^2 \rho_{yx} C_y C_x - C_x^2 \rho_{yz} C_y C_z)}{\bar{X} C_x^2 C_z^2} \quad \text{and} \quad \alpha_{2(opt)} = \frac{2\rho_{yz} C_y + C_z}{2C_z}.$$

The minimum bias and *MSE* of \bar{y}_M^d , up to the first degree of approximation, are given by

$$Bias(\bar{y}_M^d)_{\min} \cong \frac{\bar{Y}}{8C_z^2} \left[(f_3 C_x^2 + f_2 C_z^2)(C_z^2 + 4\rho_{yz} C_y C_z - 8\rho_{yz} C_y^2) + 16f_2 C_y C_z^2 \rho_{xz} (C_z \rho_{yx} - C_x \rho_{yz}) \right], \tag{24}$$

and

$$MSE(\bar{y}_M^d)_{\min} \cong \bar{Y}^2 C_y^2 [f_1 - f_2 \rho_{yz}^2 - f_3 \rho_{yx}^2]. \tag{25}$$

4. Efficiency comparison

We have obtained the conditions under which the proposed class of estimators is more efficient than its competitor estimators.

(i). By Eq (1) and Eq (25)

$$\begin{aligned} MSE(\bar{y}_M^d)_{\min} & < Var(\bar{y}), \text{ if} \\ (f_2 \rho_{yz}^2 + f_3 \rho_{yx}^2) & > 0. \end{aligned}$$

(ii). By Eq (4) and Eq (25)

$$\begin{aligned} MSE(\bar{y}_M^d)_{\min} & < MSE(\bar{y}_R^d), \text{ if} \\ f_2 (\rho_{yz} C_y - C_z)^2 + f_3 (\rho_{yx} C_y - C_x)^2 & > 0. \end{aligned}$$

(iii). By Eq (7) and Eq (25)

$$MSE(\bar{y}_M^d)_{\min} < MSE(\bar{y}_P^d), \text{ if}$$

$$f_2(\rho_{yz}C_y + C_z)^2 + f_3(\rho_{yx}C_y + C_x)^2 > 0.$$

(iv). By Eq (10) and Eq (25)

$$MSE(\bar{y}_M^d)_{\min} < MSE(\bar{y}_{Re}^d), \text{ if}$$

$$f_2(2\rho_{yz}C_y - C_z)^2 + f_3(2\rho_{yx}C_y - C_x)^2 > 0.$$

(v). By Eq (13) and Eq (25)

$$MSE(\bar{y}_M^d)_{\min} < MSE(\bar{y}_{Pe}^d), \text{ if}$$

$$f_2(2\rho_{yz}C_y + C_z)^2 + f_3(2\rho_{yx}C_y + C_x)^2 > 0.$$

(vi). By Eq (16) and Eq (25)

$$MSE(\bar{y}_M^d)_{\min} < MSE(\bar{y}_{RP}^d)_{\min}, \text{ if}$$

$$f_2\rho_{yz}^2 > 0.$$

(vii). By Eq (19) and Eq (25)

$$MSE(\bar{y}_M^d)_{\min} < MSE(\bar{y}_{RPe}^d)_{\min}, \text{ if}$$

$$(\rho_{yz}C_x - \rho_{yx}C_z)^2 > 0.$$

Note: All above conditions are obviously true.

5. Empirical study

We have taken eight natural populations for numerical study.

Population 1 (source: Cochran (2007))

y: Number of placebo children,

x: Number of paralytic polio cases in the placebo group,

z: Number of paralytic polio cases in the not inoculated group.

$$N = 34$$

$$n = 15$$

$$n' = 10$$

$$\bar{Y} = 4.92$$

$$\bar{X} = 2.59$$

$$\bar{Z} = 2.91$$

$$\rho_{yx} = 0.9801$$

$$\rho_{yz} = 0.9043$$

$$\rho_{xz} = 0.6837$$

$$C_y^2 = 1.0248$$

$$C_x^2 = 1.5175$$

$$C_z^2 = 1.1492$$

Population 2 (source: Murthy (1967))

y: Area under wheat in 1964,

x: Area under wheat in 1963,

z: Cultivated area in 1961.

$N = 34$	$n = 10$	$n' = 7$
$\bar{Y} = 199.44$	$\bar{X} = 208.89$	$\bar{Z} = 747.59$
$\rho_{yx} = 0.9801$	$\rho_{yz} = 0.9043$	$\rho_{xz} = 0.9097$
$C_y^2 = 0.5673$	$C_x^2 = 0.5191$	$C_z^2 = 0.3527$

Population 3 (source: Anderson (1958))

y : Head length of second son,
 x : Head length of first son,
 z : Head breadth of first son.

$N = 25$	$n = 10$	$n' = 7$
$\bar{Y} = 183.84$	$\bar{X} = 185.72$	$\bar{Z} = 151.12$
$\rho_{yx} = 0.7168$	$\rho_{yz} = 0.6932$	$\rho_{xz} = 0.7346$
$C_y^2 = 0.0546$	$C_x^2 = 0.0526$	$C_z^2 = 0.0488$

Population 4 (source: Singh (1967))

y : Number of females employed,
 x : Number of females in service,
 z : Number of educated females.

$N = 61$	$n = 20$	$n' = 10$
$\bar{Y} = 7.46$	$\bar{X} = 5.31$	$\bar{Z} = 179$
$\rho_{yx} = 0.7373$	$\rho_{yz} = -0.2070$	$\rho_{xz} = -0.0033$
$C_y^2 = 0.7103$	$C_x^2 = 0.7587$	$C_z^2 = 0.2515$

Population 5 (source: Murthy (1967))

y : Output,
 x : Number of workers,
 z : Fixed capital.

$N = 80$	$n = 25$	$n' = 10$
$\bar{Y} = 5782.638$	$\bar{X} = 283.875$	$\bar{Z} = 1126.00$
$\rho_{yx} = 0.9136$	$\rho_{yz} = 0.9413$	$\rho_{xz} = 0.9859$
$C_y^2 = 0.3520$	$C_x^2 = 0.9430$	$C_z^2 = 0.7460$

Population 6 (source: Sukhatme and Chand (1977))

y : Apple trees of bearing age in 1964,
 x : Bushels of apple harvested in 1964,
 z : Bushels of apple harvested in 1959.

$N = 200$	$n = 30$	$n' = 20$
$\bar{Y} = 1031.82$	$\bar{X} = 2934.58$	$\bar{Z} = 3651.49$
$\rho_{yx} = 0.93$	$\rho_{yz} = 0.77$	$\rho_{xz} = 0.84$
$C_y^2 = 2.5528$	$C_x^2 = 4.0250$	$C_z^2 = 2.9379$

Population 7 (source: Srivastava et al. (1989))

y : The measurement of weight of children,
 x : Mid arm circumference of children,
 z : Skull circumference of children.

$N = 82$	$n = 43$	$n' = 25$
$\bar{Y} = 5.60$	$\bar{X} = 11.90$	$\bar{Z} = 39.80$
$\rho_{yx} = 0.09$	$\rho_{yz} = 0.12$	$\rho_{xz} = 0.86$
$C_y^2 = 0.0107$	$C_x^2 = 0.0052$	$C_z^2 = 0.0008$

Population 8 (source: Srivastava et al. (1989))

y : The measurement of weight of children,
 x : Mid arm circumference of children,
 z : Skull circumference of children.

$N = 55$	$n = 30$	$n' = 18$
$\bar{Y} = 17.08$	$\bar{X} = 16.92$	$\bar{Z} = 50.44$
$\rho_{yx} = 0.54$	$\rho_{yz} = 0.51$	$\rho_{xz} = -0.08$
$C_y^2 = 0.0161$	$C_x^2 = 0.0049$	$C_z^2 = 0.0007$

We obtain the percent relative efficiency (PRE) of ratio estimator $\bar{y}_R^d = \hat{\theta}_1$ (say), the product estimator $\bar{y}_P^d = \hat{\theta}_2$, exponential chain type ratio estimator $\bar{y}_{Re}^d = \hat{\theta}_3$, exponential chain type product estimator $\bar{y}_{Pe}^d = \hat{\theta}_4$, Singh and Espejo (2007) ratio-product type estimator $\bar{y}_{RP}^d = \hat{\theta}_5$, Vishwakarma and Kumar (2014) exponential chain ratio-product type estimator $\bar{y}_{RPe}^d = \hat{\theta}_6$ and proposed estimator $\bar{y}_M^d = \hat{\theta}_7$ with respect to conventional estimator $\bar{y} = \hat{\theta}_0$ (say). We use the following expression for comparison.

$$PRE(\hat{\theta}_0, \hat{\theta}_j) = \frac{MSE(\hat{\theta}_0)}{MSE(\hat{\theta}_j)} \times 100, \quad j = 1, 2, \dots, 7. \tag{26}$$

The results, based on populations 1 to 8 are given in Table 1.

Table 1: Percentage Relative Efficiencies (PREs) of different estimators with respect to \bar{y}

Estimator	Pop. 1	Pop. 2	Pop. 3	Pop. 4	Pop. 5	Pop. 6	Pop. 7	Pop. 8
\bar{y}	100	100	100	100	100	100	100	100
\bar{y}_R^d	136.91	730.81	81.91	131.91	36.73	178.82	279.93	124.50
\bar{y}_P^d	25.96	30.05	70.22	61.01	8.37	31.78	26.02	37.56
\bar{y}_{Re}^d	47.55	50.48	88.38	78.75	20.45	53.77	46.58	60.68
\bar{y}_{Pe}^d	184.35	259.55	97.11	120.57	359.76	176.54	247.82	139.20
\bar{y}_{RP}^d	133.95	156.96	100.49	120.97	233.83	126.66	147.13	155.76
\bar{y}_{RPe}^d	189.26	763.30	100.81	132.32	621.68	196.28	322.94	141.80
\bar{y}_M^d	189.80	779.54	101.07	138.66	670.30	197.40	326.41	160.06

6. Conclusion

In this paper, we have proposed a class of estimators in two-phase sampling using two auxiliary variables. The bias and mean squared error (MSE) are obtained up to the first order of approximation. Theoretically, it has been shown that the proposed estimator is more efficient than the conventional estimator \bar{y} , two-phase ratio estimator \bar{y}_R^d , two-phase product estimator \bar{y}_{RP}^d , exponential chain type ratio estimator \bar{y}_{Re}^d , exponential chain type product estimator \bar{y}_{Pe}^d , Singh and Espejo (2007) ratio-product type estimator \bar{y}_{RP}^d , Vishwakarma and Kumar (2014) exponential chain ratio-product type estimator \bar{y}_{RPe}^d . The numerical results clearly show that the proposed estimator \bar{y}_M^d performs much better than Vishwakarma and Kumar (2014) and all other competitor estimators. Also, \bar{y}_P^d and \bar{y}_{Re}^d show the poor performances in all data sets.

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Appendix

Table 2: Some special cases of the proposed class of estimators

α_1	α_2	Estimator	Remarks
0	δ	$\bar{y}_{RPe}^d = \bar{y} \left[\delta \exp \left\{ \frac{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} - \bar{x}}{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} + \bar{x}} \right\} + (1-\delta) \exp \left\{ \frac{\bar{x} - \left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z}}{\bar{x} + \left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z}} \right\} \right]$	Vishwakarma and Kumar (2014) estimator
0	0	$\bar{y}_{Pe}^d = \bar{y} \exp \left\{ \frac{\bar{x} - \left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z}}{\bar{x} + \left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z}} \right\}$	Singh and Choudhury (2012) product estimator
0	1	$\bar{y}_{Re}^d = \bar{y} \exp \left\{ \frac{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} - \bar{x}}{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} + \bar{x}} \right\}$	Singh and Choudhury (2012) ratio estimator
d	0	$[\bar{y} + d(\bar{x}' - \bar{x})] \exp \left\{ \frac{\bar{x} - \left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z}}{\bar{x} + \left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z}} \right\}$	Regression exponential type product estimator
d	1	$[\bar{y} + d(\bar{x}' - \bar{x})] \exp \left\{ \frac{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} - \bar{x}}{\left(\frac{\bar{x}'}{\bar{z}'} \right) \bar{Z} + \bar{x}} \right\}$	Regression exponential type ratio estimator

From Eq (25), we get

$$\begin{aligned}
 MSE(\bar{y}_M^d)_{\min} &\cong \bar{Y}^2 C_y^2 [f_1 - f_2 \rho_{yz}^2 - f_3 \rho_{yx}^2] \\
 MSE(\bar{y}_M^d)_{\min} &\cong \bar{Y}^2 C_y^2 [f_1 - f_3 \rho_{yx}^2] - \bar{Y}^2 C_y^2 f_2 \rho_{yz}^2 \\
 MSE(\bar{y}_M^d)_{\min} &\cong MSE(\bar{y}_{RP}^d)_{\min} - \bar{Y}^2 C_y^2 f_2 \rho_{yz}^2
 \end{aligned}$$

Since, $\bar{Y}^2 C_y^2 f_2 \rho_{yz}^2$ is a positive quantity, therefore

$$MSE(\bar{y}_M^d)_{\min} < MSE(\bar{y}_{RP}^d)_{\min}$$

Comparison (vi): By Eq (16) and Eq (25)

$$\begin{aligned}
 &MSE(\bar{y}_M^d)_{\min} < MSE(\bar{y}_{RP}^d)_{\min}, \text{ if} \\
 &\bar{Y}^2 C_y^2 [f_1 - f_2 \rho_{yz}^2 - f_3 \rho_{yx}^2] < \bar{Y}^2 [f_1 C_y^2 - f_3 \rho_{yx}^2 C_y^2] \\
 &\bar{Y}^2 C_y^2 [f_1 - f_2 \rho_{yz}^2 - f_3 \rho_{yx}^2] - \bar{Y}^2 C_y^2 [f_1 - f_3 \rho_{yx}^2] < 0 \\
 &\quad -f_2 \rho_{yz}^2 < 0 \\
 \text{or} &\quad f_2 \rho_{yz}^2 > 0
 \end{aligned}$$

Which is always true.