Exact Distribution of the Ratio of Gamma and Rayleigh Random Variables

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Abstract

The distributions of the ratio of two independent random variables arise in many applied problems and have been extensively studied by many researchers. This article derives the distributions of the ratio Z=$\left|\frac{X}{Y}\right|$, when X and Y are gamma and Rayleigh random variables respectively and are distributed independently of each other. The associated pdf, cdf, and moments have been given in terms of different special functions, for examples, confluent hypergeometric function, parabolic-cylinder function and beta functions. Some plots for the cdf and pdf associated with the distribution of the ratio have been provided.

Keywords: Gamma distribution, hypergeometric function, Parabolic-cylinder, Ratios, Rayleigh distribution.

1. Introduction

The distributions of the ratio of two independent random variables arise in many applied problems of biology, economics, engineering, genetics, hydrology, medicine, number theory, order statistics, physics, psychology, etc, (see, for example, [4], [6], and [8], among others, and references therein). These have been extensively studied by many researchers when the two independent random variables belong to the same family, among them [9], [10], [11], [12], [14], [16], and [17] are notable. In recent years, there has been a great interest in the study of the above kind when X and Y belong to different families, (see, for example, [13], and [15], among others). This paper discusses the distributions of the ratio $Z = \left|\frac{X}{Y}\right|$, when X and Y are gamma and Rayleigh random variables and are distributed independently of each other. The organization of this paper is as follows. In Section 2, the derivation of the cdf of the ratio Z and associated plots of the cdf's are given. The pdfs and their plots have been given in Section 3. The moments are discussed in Section 4. Finally, some concluding remarks are given in Section 5.

The derivations of the associated pdf, cdf, and moments in this paper involve some special functions, which are defined as follows (see, for example, [1], [2], [3], [5], [7], and [18], among others, for details). The integrals $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$, and $\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t}\, dt, \alpha > 0$, are called (complete) gamma and incomplete gamma functions respectively, whereas the integral $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt, \alpha > 0$ is called complementary incomplete gamma function. For negative values, gamma function can be defined as $\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n\, 2^n \sqrt{\pi}}{1.3.5.\ldots(2n-1)}$, where $n \geq 0$ is an integer. The function defined by $B(p, q) = \int_0^1 t^p\, (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, $p > 0, q > 0,$ is known as beta function (or Euler's function of the first kind). The functions defined by $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$, and $erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = 1 - erf(x)$ are called error and complementary error functions respectively. The following series

$$ {}_pF_q\left(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z\right) = \sum_{k=0}^\infty \left\{ \frac{(\alpha_1)_k\, (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k\, (\beta_2)_k \cdots (\beta_q)_k} \frac{z^k}{k!} \right\} $$

is called a generalized hypergeometric series of order (p, q), where $(\alpha)_k$ and $(\beta)_k$ denote Pochhammer symbols. For $p = 1$ and $q = 2$, we have generalized hypergeometric function ${}_1F_2$ of order $(1, 2)$, given by ${}_1F_2(\alpha_1; \beta_1, \beta_2; z) = \sum_{k=0}^\infty \left\{ \frac{(\alpha_1)_k\, z^k}{(\beta_1)_k\, (\beta_2)_k\, k!} \right\}$. For $p = 2$ and $q = 2$, we have generalized hypergeometric function ${}_2F_2$ of order $(2, 2)$. For $p = 2$ and $q = 1$, we have generalized hypergeometric function ${}_2F_1$ of order $(2, 1)$, given by ${}_2F_1(\alpha, \beta; \gamma; z) \equiv F(\alpha, \beta; \gamma; z) \equiv F(\beta, \alpha; \gamma; z) = \sum_{k=0}^\infty \left\{ \frac{(\alpha)_k (\beta)_k\, z^k}{(\gamma)_k\, k!} \right\}$. The following series ${}_1F_1(\alpha; \beta; z) = \sum_{k=0}^\infty \left\{ \frac{(\alpha)_k\, z^k}{(\beta)_k\, k!} \right\}$, (where $|z| < \infty$; $\beta \neq 0, -1, -2, \ldots$), is known as degenerate hypergeometric function or confluent hypergeometric function of Kummer. The confluent hypergeometric function ${}_1F_1(\alpha; \beta; z)$ is a degenerate form of the generalized hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ of order $(2, 1)$ which arises as a solution the confluent hypergeometric differential equation. Also, ${}_1F_1(\alpha, \beta; z) = e^z\, {}_1F_1(\beta - \alpha, \beta; -z)$, known as Kummer Transformation, and $F(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} F\left(\beta, \gamma - \alpha; \gamma; \frac{z}{z-1}\right)$. The series $\Psi(\alpha, \beta; z) = \Gamma\begin{bmatrix} 1-\beta \\ \alpha - \beta + 1 \end{bmatrix}$,

$_1F_1(\alpha;\beta;z)+\Gamma\begin{bmatrix}\beta-1\\ \alpha\end{bmatrix} z^{1-\beta}\ _1F_1(\alpha-\beta+1;2-\beta;z)$ (where $|z|<\infty$; $\beta\neq 0,\pm1,\pm2,...$), is known as confluent hypergeometric function of Tricomi. Note that $\Gamma(\alpha,z)=\Gamma(\alpha)-\gamma(\alpha,z)=\int_z^\infty t^{\alpha-1}e^{-t}\,dt$, $\gamma(\alpha,z)=\frac{z^\alpha}{\alpha}\ _1F_1(\alpha;\alpha+1;-z)$, and $\Gamma(\alpha,z)=e^{-z}\,\Psi(1-\alpha;1-\alpha;z)$. The parabolic-cylinder function $D_\nu(z)$ is defined as follows:

$$D_\nu(z)=2^{\nu/2}\,e^{-z^2/4}\,\Psi\left(-\frac{\nu}{2},\frac{1}{2};\frac{z^2}{2}\right)$$

$$=2^{(\nu-1)/2}\,e^{-z^2/4}\,z\,\Psi\left(\frac{1-\nu}{2},\frac{3}{2};\frac{z^2}{2}\right)$$

$$=2^{\nu/2}\,e^{-z^2/4}\left[\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}\ _1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right)-\frac{\sqrt{2\pi}}{\Gamma\left(\frac{-\nu}{2}\right)}\,z\ _1F_1\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right)\right]$$

The following four Lemmas will also be needed to complete the derivations.

LEMMA 1 (Prudnikov et al. (1986), Volume 2, [18], Equations (2.10.3.9), Page 151).

For $\mathrm{Re}(c)>0$, $\mathrm{Re}(p)>0$, $\mathrm{Re}(\nu)>0$, and $\mathrm{Re}(\alpha+\nu)>0$, we have

$$\int_0^\infty x^{\alpha-1}e^{-p x^2}\,\gamma(\nu,c\,x)dx=\frac{c^\nu\,(p)^{-(\alpha+\nu)/2}}{2\nu}\,\Gamma\left(\frac{\alpha+\nu}{2}\right)\ _2F_2\left(\frac{\nu}{2},\frac{\alpha+\nu}{2};\frac{1}{2},\frac{\nu+2}{2};\frac{c^2}{4p}\right)$$
$$-\frac{c^{\nu+1}\,(p)^{-(\alpha+\nu+1)/2}}{2(\nu+1)}\,\Gamma\left(\frac{\alpha+\nu+1}{2}\right)\ _2F_2\left(\frac{\nu+1}{2},\frac{\alpha+\nu+1}{2};\frac{3}{2},\frac{\nu+3}{2};\frac{c^2}{4p}\right)$$

LEMMA 2 (Prudnikov et al. (1986), Volume 2, [18], Equations (2.10.3.10), Page 151).

For $\mathrm{Re}(c)>0$, $\mathrm{Re}(p)>0$, and $\mathrm{Re}(\nu)>-2$, we have

$$\int_0^\infty x\,e^{-p x^2}\,\gamma(\nu,c\,x)dx=\frac{c^\nu\,\Gamma(\nu)}{(2\,p)^{\left(\frac{\nu+2}{2}\right)}}\,e^{\left(\frac{c^2}{8p}\right)}D_{-\nu}\left(\frac{c}{\sqrt{2\,p}}\right)$$

LEMMA 3 (Gradshteyn and Ryzhik (2000), [5], Equation (3.462.1), Page 337).

For $\mathrm{Re}(p)>0$, and $\mathrm{Re}(\alpha)>0$, we have

$$\int_0^\infty x^{\alpha-1}e^{-p x^2-c\,x}\,dx=(2\,p)^{-\left(\frac{\alpha}{2}\right)}\,\Gamma(\alpha)\,e^{\left(\frac{c^2}{8p}\right)}D_{-\alpha}\left(\frac{c}{\sqrt{2\,p}}\right).$$

LEMMA 4 (Gradshteyn and Ryzhik (2000), [5], Equation (7.612.2), Page 858).

For $0 < \mathrm{Re}(b) < \mathrm{Re}(b)$, and $\mathrm{Re}(c) < \mathrm{Re}(b) + 1$, we have

$$\int_0^\infty x^{b-1}\,\Psi(a, c; x)\,dx = \frac{\Gamma(b)\,\Gamma(a-b)\,\Gamma(b-c+1)}{\Gamma(a)\,\Gamma(a-c+1)} .$$

Distribution of the Ratio $\left|\frac{X}{Y}\right|$

Let X and Y be gamma and Rayleigh random variables respectively, distributed independently of each other and defined as follows.

Gamma Distribution: A continuous random variable X is said to have a gamma distribution if its pdf $f_X(x)$ and cdf $F_X(x) = P(X \le x)$ are, respectively, given by

$$f_X(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\beta x}, \; x > 0,\; \alpha > 0,\; \beta > 0 \tag{1}$$

and

$$F_X(x) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}, \tag{2}$$

where $\gamma(\alpha, \beta x)$ denotes incomplete gamma function.

Rayleigh Distribution: A continuous random variable Y is said to have a Rayleigh distribution if its pdf $f_Y(y)$ and cdf $F_Y(y) = P(Y \le y)$ are, respectively, given by

$$f_Y(y) = \left(\frac{y}{\sigma^2}\right) e^{-y^2/2\sigma^2}, \quad y > 0,\; \sigma > 0, \tag{3}$$

and

$$F_Y(y) = 1 - e^{-y^2/2\sigma^2} \tag{4}$$

In what follows, we consider the derivation of the distribution of the ratio $\left|\frac{X}{Y}\right|$, when X and Y are gamma and Rayleigh random variables respectively, distributed independently of each other and defined as above. An explicit expression for the cdf of $\left|\frac{X}{Y}\right|$ in terms of parabolic-cylinder function $D_\nu(z)$ has been derived in the following subsection and provided in Theorem 2.1. In Theorem 2.2, another explicit expression for the cdf of $\left|\frac{X}{Y}\right|$ in terms of the generalized hypergeometric function ${}_2F_2$ has been given.

2. Derivation of CDF of the ratio Z

THEOREM 2.1

Suppose X is a gamma random variable with pdf $f_X(x)$ as given in (1) and cdf $F_X(x) = P(X \le x)$ given by (2). Also, suppose Y is a Rayleigh random variable with pdf $f_Y(y)$ given by (3). Then the cdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$F(z) = (\sigma\beta z)^{\alpha} e^{(\sigma^2\beta^2 z^2)/4} D_{-\alpha}(\sigma\beta z) \qquad (5)$$

where $D_{-\alpha}(\sigma\beta z)$ denotes parabolic-cylinder function.

PROOF

Using the expressions (2) for the cdf of gamma random variable X and expression (3) for the pdf of Rayleigh random variable Y, the cdf $F(z) = \Pr\left(\left|\frac{X}{Y}\right| \le z\right)$ can be expressed as

$$F(z) = \Pr(|X| \le z|Y|) = \int_0^{\infty} F_X(zy)\, f_Y(y)\, dy$$

$$= \left[\frac{1}{\sigma^2\,\Gamma(\alpha)}\right] \int_0^{\infty} y\, e^{-y^2/2\sigma^2}\, \gamma(\alpha, \beta z y)\, dy, \qquad (6)$$

where $y > 0,\ z > 0,\ \alpha > 0,\ \beta > 0,\ \sigma > 0$. The proof of Theorem 2.1 easily follows by using Lemma 2 in the integral (6) above.

THEOREM 2.2

Suppose X is a gamma random variable with pdf $f_X(x)$ as given in (1) and cdf $F_X(x) = P(X \le x)$ given by (2). Also, suppose Y is a Rayleigh random variable with pdf $f_Y(y)$ given by (3). Then the cdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$F(z) = \frac{\left(\sqrt{2}\,\sigma\beta z\right)^{\alpha}}{2\,\Gamma(\alpha)} \left\{ \Gamma\left(\frac{\alpha}{2}\right) {}_2F_2\left(\frac{\alpha}{2}, \frac{\alpha+2}{2}; \frac{1}{2}, \frac{\alpha+2}{2}; \frac{\sigma^2\beta^2 z^2}{2}\right) \right.$$
$$\left. - \Gamma\left(\frac{\alpha+1}{2}\right)\left(\sqrt{2}\,\sigma\beta z\right) {}_2F_2\left(\frac{\alpha+1}{2}, \frac{\alpha+3}{2}; \frac{3}{2}, \frac{\alpha+3}{2}; \frac{\sigma^2\beta^2 z^2}{2}\right) \right\} \qquad (7)$$

where ${}_2F_2(.)$ denotes the generalized hypergeometric function of order $(2, 2)$.

PROOF

Using the expressions (2) for cdf of gamma random variable X and expression (3) for pdf of Rayleigh random variable Y, the cdf $F(z) = \Pr\left(\left|\frac{X}{Y}\right| \le z\right)$ can be expressed as

$$F(z) = \Pr(|X| \le z|Y|) = \int_0^\infty F_X(z\,y)\; f_Y(y)\; dy$$

$$= \left[\frac{1}{\sigma^2\,\Gamma(\alpha)}\right] \int_0^\infty y\, e^{-y^2/2\sigma^2}\; \gamma(\alpha,\, \beta\, z\, y)\; dy\,, \qquad (8)$$

where $y > 0,\ z > 0,\ \alpha > 0,\ \beta > 0,\ \sigma > 0$. The proof of Theorem 2.2 easily follows by using Lemma 1 in the integral (8) above.

Plots of CDF of the ratio Z

The possible shapes of the cdfs of the ratio in (5) or (7) for $\alpha = 1, \sigma = 1$, and different values of β = 0.2, 0.5, 1, 2, and for $\alpha = 2, \sigma = 5$, and different values of β = 0.2, 0.5, 1, 2, are provided respectively, in Figures 1 and 2 below. The effects of the parameters can easily be seen from these graphs. Similar plots can be drawn for others values of the parameters.

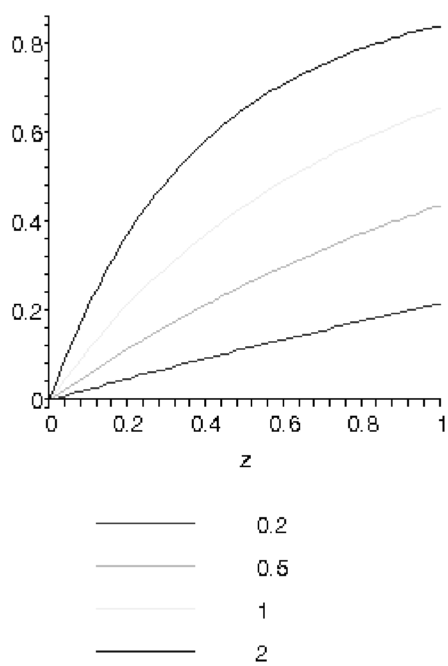

Figure 1: Three_Parameters_Gamma_Rayleigh_Ratio_CDFS for $\alpha = 1, \sigma = 1$, and different values of β = 0.2, 0.5, 1, 2

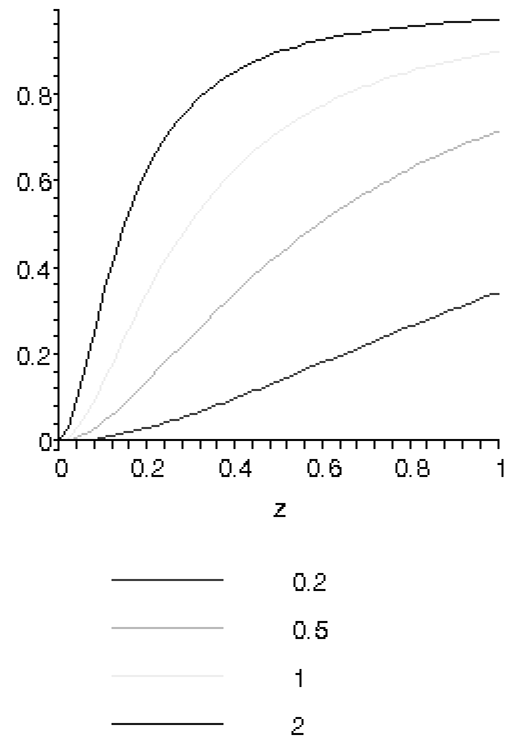

Figure 2: Three_Parameters_Gamma_Rayleigh_Ratio_CDFS for $\alpha = 2, \sigma = 5$, and different values of β = 0.2, 0.5, 1, 2

PDF of the Ratio $Z = \left|\frac{X}{Y}\right|$

3. Derivation of PDF of the ratio Z

This section derives the pdf of the ratio $Z = \left|\frac{X}{Y}\right|$, when X and Y are gamma and Rayleigh random variables distributed according to (1) and (3), respectively, and independently of each other. An explicit expression for the pdf of the ratio $Z = \left|\frac{X}{Y}\right|$ in terms of parabolic-cylinder function $D_\nu(z)$ has been derived in Theorem 3.1. Another explicit expression for the cdf of $\left|\frac{X}{Y}\right|$ in terms of the confluent hypergeometric function $\Psi(.)$ of Tricomi has been given in Corollary 3.1. To describe the possible shapes of the associated pdfs, the respective plots are provided in Figures 3, 4, and 5.

THEOREM 3.1

Suppose X is a gamma random variable with pdf $f_X(x)$ given by (1) and Y is a Rayleigh random variable with pdf $f_Y(y)$ given by (3). Then the pdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$f_Z(z) = \alpha(\alpha+1)(\sigma\beta z)^{\alpha} z^{-1} e^{(\sigma^2\beta^2 z^2)/4} D_{-(\alpha+2)}(\sigma\beta z), \quad z>0,\ \alpha>0, \beta>0, \sigma>0 \qquad (9)$$

PROOF

The pdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$f_Z(z) = \int_0^{\infty} y\, f_X(zy)\, f_Y(y)\, dy$$

$$= \left[\frac{\beta^{\alpha} z^{\alpha-1}}{\sigma^2\,\Gamma(\alpha)}\right] \int_0^{\infty} y^{(\alpha+2)-1} e^{-\left(\beta z\, y + \frac{1}{2\sigma^2} y^2\right)}\, dy, \qquad (10)$$

where $y>0,\ z>0,\ \alpha>0, \beta>0, \sigma>0$. The proof of Theorem 3.1 easily follows by using Lemma 3 in the integral (10) above.

COROLLARY 3.1

Using the definition of parabolic-cylinder function in terms of confluent hypergeometric function of Tricomi $\Psi(.)$, as given above, it is easy to see that the pdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$f_Z(z) = \frac{\alpha(\alpha+1)(\sigma\beta)^{\alpha}}{2^{(\alpha+2)/2}} z^{\alpha-1} \psi\left(\frac{\alpha+2}{2}, \frac{1}{2}; \frac{\sigma^2\beta^2 z^2}{2}\right), z>0,\ \alpha>0, \beta>0, \sigma>0 \qquad (11)$$

REMARK: Using the above expression (11) for the pdf of the ratio and Lemma 4, one can easily see that $\int_0^{\infty} f_Z(z)\, dz = 1$.

4. Plots of PDF of the ratio Z

The possible shapes of the pdfs of the ratio in (9) or (11) for different values of α, σ, and β are provided, respectively, in Figures 3, 4, and 5 below. These graphs evident that the distribution of *z* is right skewed. The effects of the parameters can easily be seen from these graphs. Similar plots can be drawn for others values of α, σ, and β.

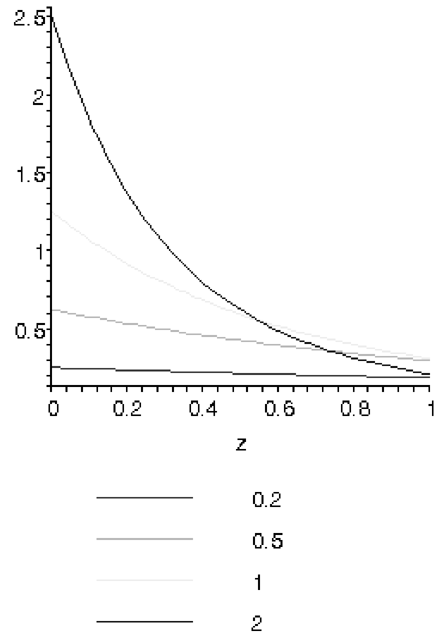

Figure 2: Three_Parameters_ Gamma_Rayleigh_Ratio_PDFS for $\alpha = 1, \sigma = 1$, and different values of β

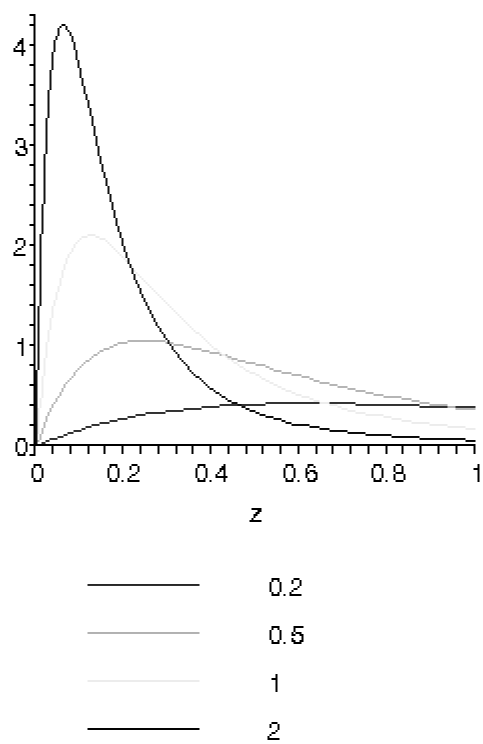

Figure 3: Three_Parameters_Gamma_Rayleigh_Ratio_PDFS for $\alpha = 2, \sigma = 5$, and different values of β

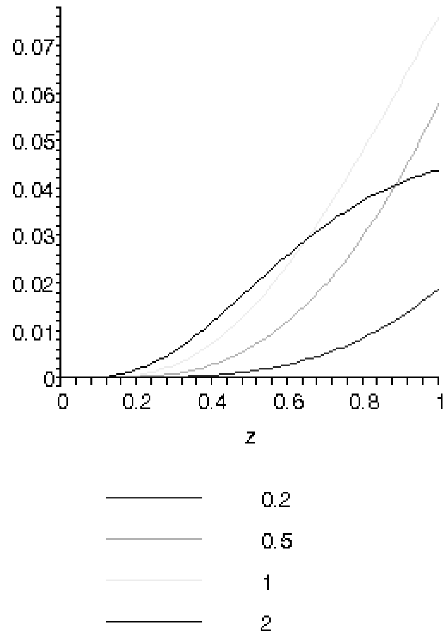

Figure 4: Three_Parameters_ Gamma_Rayleigh_Ratio_PDFS for $\alpha = 5, \sigma = 1$, and different values of β

The kth Moment of the ratio $Z = \left|\frac{X}{Y}\right|$

In this section, the expression for the kth moment of RV $Z = \left|\frac{X}{Y}\right|$ in terms of gamma or beta function has been derived.

THEOREM 4.1

If Z is a random variable with pdf given by (11), then its kth moment can be expressed as

$$E\left(Z^k\right) = \left(\frac{\sqrt{2}}{\sigma\,\beta}\right)^k \frac{\Gamma\left(\frac{\alpha + k}{2}\right)\Gamma\left(\frac{\alpha + k + 1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha + 1}{2}\right)}, \ \forall\, k \geq 1,\ k \text{ is an integer.} \qquad (12)$$

PROOF

We have

$$E\left(Z^k\right) = \left[\frac{\alpha\,(\alpha + 1)\left(\sigma\,\beta\right)^{\alpha}}{2^{(\alpha+2)/2}}\right]\int_0^{\infty} z^{k+\alpha-1}\ \psi\left(\frac{\alpha + 2}{2}, \frac{1}{2}; \frac{\sigma^2\,\beta^2\,z^2}{2}\right) dz \qquad (13)$$

Substituting $\frac{\sigma^2 \beta^2 z^2}{2} = t$, and using Lemma 4 in the integral (13) above, the result of Theorem 4.1 easily follows. It is evident from Theorem 4.1 that the moment exists $\forall\, k \geq 1$, where k is an integer.

COROLLARY 4.1

Using the definition of beta function, the kth moment given by (12) can be easily expressed in terms of beta function as follows:

$$E\left(Z^k\right) = \left(\frac{\sqrt{2}}{\sigma \beta}\right)^k \frac{B\left(\frac{\alpha + k}{2}, \frac{\alpha + k + 1}{2}\right) \Gamma\left(\alpha + k + \frac{1}{2}\right)}{B\left(\frac{\alpha}{2}, \frac{\alpha + 1}{2}\right) \Gamma\left(\alpha + \frac{1}{2}\right)}, \ \forall\, k \geq 1, \ k \text{ is an integer.}$$

COROLLARY 4.2

It is easy to see from (12) that the first few moments are given by

$$\mu'_1 = Mean = E(Z) = \frac{\alpha}{\sqrt{2}\, \beta\, \sigma},$$

$$\mu'_2 = E\left(Z^2\right) = \frac{\alpha\, (\alpha + 1)}{2\, \beta^2\, \sigma^2},$$

$$\mu'_3 = E\left(Z^3\right) = \frac{\alpha\, (\alpha + 1)\, (\alpha + 2)}{2^{3/2}\, \beta^3\, \sigma^3}, \text{ etc.}$$

Also, one can easily obtain the variance given by

$$Var(Z) = \frac{\alpha}{2 \beta^2 \sigma^2}.$$

5. Concluding Remarks

This paper has derived the exact probability distribution of the ratio of two independent random variables X and Y, where X has a gamma and Y has a Rayleigh distribution respectively. The expressions for the cdf, pdf and moments of the ratio of two variables are given as function of some special functions. The plots for the cdf and pdf have been provided. We hope the findings of the paper will be useful for the practitioners in various fields.

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