HYPOTHESIS TESTING OF PARAMETERS FOR ORDINARY LINEAR CIRCULAR REGRESSION

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Abstract

This paper presents the hypothesis testing of parameters for ordinary linear circular regression model assuming the circular random error distributed as von Misses distribution. The main interests are in testing of the intercept and slope parameter of the regression line. As an illustration, this hypothesis testing will be used in analyzing the wind and wave direction data recorded by two different techniques which are HF radar system and anchored wave buoy.

Key words*:* circular random variable, continuous linear variables, regression model, hypothesis testing, von Mises distribution.

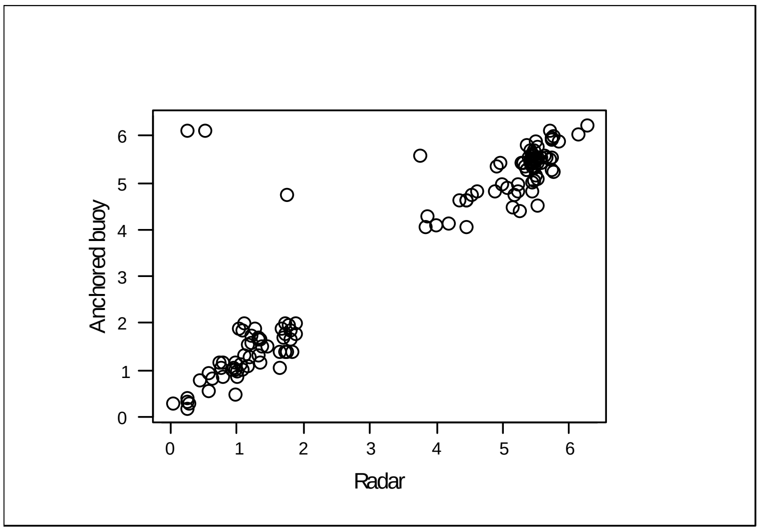
1. Introduction

A circular random variable is a variable which takes values on the circumference of a circle, i.e. the angle is in the range (0, 2π) radians or (0^0, 360^0). This random variable must be analysed by techniques differing from those appropriate for the usual Euclidean type variables because the circumference is a bounded closed space, for which the concept of origin is arbitrary or undefined. A continuous linear variable is a random variable with realisations on the straight line which may be analysed by usual techniques.

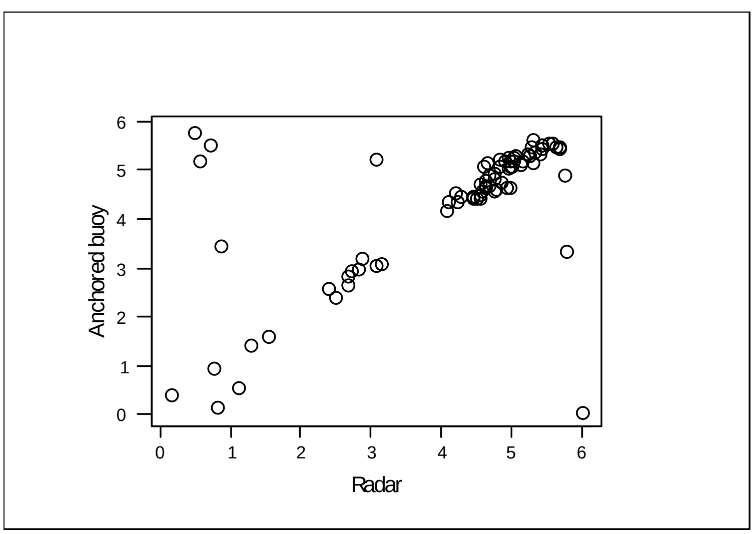
Ordinary linear circular regression model is applied when we wish to determine the relationship between a single circular explanatory variable *X* and a circular response variable *Y*. As an example is in the relationship of the measurements of wind and wave directions measured by an anchored buoy (*Y*) and radar (*X*) as shown in figures below, Sova (1995).

Figure below gives a simple (or even simplistic) scatter plot for each data set. In both cases the observations (direction, in radians) has been measured simultaneously by an anchored buoy (*Y*) and by radar (*X*). Although measurements are in radians, we will often use the more familiar degrees in informal discussion of the data. In this case we arbitrarily chose variate *Y* as the anchored buoy and variate *X* as radar.

We would anticipate an ideal model $y = x$ as appropriate for the data. The scatter plots of one direction against the other (as measured in radians anticlockwise from North) give a cluster of points along the *X*=*Y* diagonal and then a few in the top left and bottom right corners (359^o against 1^o, etc.). If we think of these data as arising from an ordinary linear regression model, we would regard those points at the top left and bottom right as outliers, but in relation to an ordinary linear circular regression model this is not so, because the measurements are on the circle or circumference, not a straight line. Since 1^o is only 2^o from 359^o, the point (1^o, 359^o) on the simple scatter plot should not really be far from the ideal model $y = x$. In this respect, the simple scatter plot is misleading but it illustrate that an ordinary linear regression model which ignore the circularity of the data is equally misleading. Perhaps such scatter plots should be drawn on a torus which maintains the "wrapping" of the measurements scales. This shows the chief problem with ordinary linear regression when applied to circular variables and below we will propose an ***ordinary linear circular regression model*** which is more suited to this form of data, Hussin (2004).



(a) Wind direction data.



(b) Wave direction data.

Fig. 1: Scatter plot for wind and wave direction data (in radians).

2. The Model and Parameters Estimation

Suppose for any circular observations $(x_1, y_1),...(x_n, y_n)$ of circular variables *X* and *Y*, we propose a model of

$$y_i = \alpha + \beta x_i + \varepsilon_i \pmod{2\pi} \tag{1}$$

where ε_i is a circular random error having a von Mises distribution with mean circular 0 and concentration parameter κ. An application when both *X* and *Y* are circular, for example in modelling a relationship for calibration between two instruments for wind direction, requires $\beta \approx 1$.

Parameters α and β may be estimate by maximum likelihood estimation. Based on the von Mises density function, the log likelihood function for model (1) is given by

$$\log L(\alpha, \beta, \kappa; x_1, ..., x_n, y_1, ..., y_n) = \\ -n\log(2\pi) - n\log I_0(\kappa) + \kappa \sum \cos(y_i - \alpha - \beta x_i).$$

We differentiate log *L* with respect to α , β and κ to get $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\kappa}$ which are given by

$$\hat{\alpha} = \begin{cases} \tan^{-1}\left(\frac{S}{C}\right), & S > 0, C > 0 \\ \tan^{-1}\left(\frac{S}{C}\right) + \pi, & C < 0 \\ \tan^{-1}\left(\frac{S}{C}\right) + 2\pi, & S < 0, C > 0 \end{cases} . \tag{2}$$

$$\hat{\beta}_1 \approx \hat{\beta}_0 + \frac{\sum x_i \sin(y_i - \hat{\alpha} - \hat{\beta}_0 x_i)}{\sum x_i^2 \cos(y_i - \hat{\alpha} - \hat{\beta}_0 x_i)} \tag{3}$$

This expression for $\hat{\beta}$ may be solved iteratively given some suitable “initial guesses” at the estimate. We can then update α and β and proceed iteratively. This iteration procedure will continue until the convergence criterion satisfied.

Further the estimate of parameter concentration is given by

$$\hat{\kappa} = A^{-1}\left(\frac{1}{n}\sum \cos(y_i - \hat{\alpha} - \hat{\beta} x_i)\right). \tag{4}$$

We can now proceed to obtain useful approximations using the results of Dobson (1978) who gives various simple approximations for the function *A* (ratio of the modified Bessel functions for the first kind of order one, and the first kind of order zero) for the von Mises concentration parameter, κ. For example, for $0.65 < w < 1$, the approximation is

$$A^{-1}(w) \approx \frac{9 - 8w + 3w^2}{8(1 - w)}.$$

Thus by using maximum likelihood estimation, we have shown that $\hat{\alpha}, \hat{\beta}$ and $\hat{\kappa}$ can be estimated iteratively. Since both the *x* and *y* are measurements of the same quantity, unity would be logical initial estimate of β, and so a possible initial estimate for iteration is $\beta_0 = 1.0$ in (2) and (3).

3. The Hypothesis Testing of Parameters

This section is concerned with testing hypotheses on α and β, i.e testing the hypothesis $H_0: \beta = 0 \,(\text{mod} 2\pi)$ vs $H_1: \beta \neq 0 \,(\text{mod} 2\pi)$ for the ordinary linear circular regression of the form

$$y_i = \alpha + \beta x_i + \varepsilon_i \quad (\text{mod}\, 2\pi), \quad i = 1, \ldots, n,$$

using observations $(x_1, y_1), \ldots, (x_n, y_n)$ and assuming that ε_i are $VM(0, \kappa)$. It follows directly that the observations y_i are $VM(\alpha + \beta x_i, \kappa)$. In particular, our interest is in testing the hypotheses that $H_0: \beta = 1$ against $H_1: \beta \neq 1$ and $H_0: \alpha = 0 (\text{mod} 2\pi)$ against $H_1: \alpha \neq 0 (\text{mod}\, 2\pi)$. In a practical application $\alpha = 0$ and $\beta = 1.0$ implies an exact relationship between the two variables.

For large κ, Mardia (1972), has shown that $2\kappa S_0$ has a χ^2 distribution with *n* degrees of freedom where

$$S_0 = \left\{ n - \sum \cos(y_i - \alpha - \beta x_i) \right\}.$$

Further, if $S_{0,0}$ and $S_{0,1}$ are the values of S_0 under H_0 and H_1 respectively, i.e. $S_{0,1} = S_0(\hat{\alpha}, \hat{\beta})$ and $S_{0,0} = S_0(\alpha^*, 1)$, say then for large κ,

$$2\kappa S_{0,1} \sim \chi^2_{n-2},$$

which gives $2\kappa(S_{0,1} - S_{0,0}) \sim \chi_1^2$. Further $2\kappa S_{0,1}$ and $2\kappa(S_{0,1} - S_{0,0})$ may be assumed to be independently distributed. Consequently, for testing H_0 against H_1, the following F-statistic may be used

$$F_{1,\, n\text{-}2} = \frac{(n-2)\left(S_{0,1} - S_{0,0}\right)}{S_{0,1}}.$$

In this section we will use the above result to test the hypotheses $H_0{:}\beta = 1$ against $H_1{:}\beta \neq 1$ and $H_0{:}\alpha = 0 (\mathrm{mod}\, 2\pi)$ against $H_1{:}\alpha \neq 0$ $(\mathrm{mod}\, 2\pi)$, in order to make a comparison between the fitted model and the ideal model of $y = x$. Suppose that $(\alpha^*, 1)$ and $(\hat{\alpha}, \hat{\beta})$ are the maximum likelihood estimates of the parameters under $H_0{:}\beta = 1$ and $H_1{:}\beta \neq 1$ respectively. Under $H_0{:}\beta = 1$, the log likelihood function is given by

$$\log L^* = -n\log(2\pi) - n\log I_0(\kappa) + \kappa\sum\cos(y_i - \alpha - x_i).$$

Differentiating $\log L^*$ with respect to α gives,

$$\frac{\partial \log L^*}{\partial \alpha} = \kappa\sum\sin(y_i - \alpha - x_i).$$

Setting this equal to zero and simplifying we get

$$\tan\alpha^* = \frac{\sum\sin(y_i - x_i)}{\sum\cos(y_i - x_i)} = \frac{S^*}{C^*}, \quad \text{say.}$$

Hence

$$\alpha^* = \begin{cases} \tan^{-1}\left(\dfrac{S^*}{C^*}\right), & S^* > 0, C^* > 0 \\ \tan^{-1}\left(\dfrac{S^*}{C^*}\right) + \pi, & C^* < 0 \\ \tan^{-1}\left(\dfrac{S^*}{C^*}\right) + 2\pi, & S^* < 0, C^* > 0 \end{cases}$$

Therefore

$$S_{0,0} = \left(n - \sum\cos(y_i - \alpha^* - x_i)\right) \quad \text{and} \quad S_{0,1} = \left(n - \sum\cos(y_i - \hat{\alpha} - \hat{\beta}x_i)\right)$$

and the test statistic is given by

$$F_{1,\,n\text{-}2} = \frac{(n-2)\left(S_{0,1} - S_{0,0}\right)}{S_{0,1}} .$$

The same technique can be used to test the hypothesis that $H_0 : \alpha = 0 \,(\text{mod} 2\pi)$ against $H_1 : \alpha \neq 0 (\text{mod}\, 2\pi)$. Suppose that $(0, \beta^*)$ and $(\hat{\alpha}, \hat{\beta})$ are the maximum likelihood estimates of the parameters under $H_0 : \alpha = 0 \,(\text{mod} 2\pi)$ and $H_1 : \alpha \neq 0 (\text{mod}\, 2\pi)$ respectively. Under $H_0 : \alpha = 0 \,(\text{mod} 2\pi)$, the log likelihood function is given by

$$\log L^* = -n \log(2\pi) - n \log I_0(\kappa) + \kappa \sum \cos(y_i - \beta x_i) .$$

Differentiating log L^* with respect to β gives,

$$\frac{\partial \log L^*}{\partial \beta} = \kappa \sum x_i \cos(y_i - \beta x_i) .$$

This may be solved iteratively for β^* and it can be shown that

$$\beta^* = \beta_0^* + \frac{\sum x_i \sin(y_i - \beta_0^* x_i)}{\sum x_i^2 \cos(y_i - \beta_0^* x_i)} ,$$

where β_0^* is an initial estimate of β^*. Hence

$$S_{0,0} = \left\{ n - \sum \cos(y_i - \beta^* x_i) \right\}, \text{ and } S_{0,1} = \left\{ n - \sum \cos(y_i - \hat{\alpha} - \hat{\beta} x_i) \right\} .$$

We will use the same F-statistic as above to test the hypothesis for intercept α.

4. Application and Conclusion

In this section we will apply the results of the hypothesis testing and will use the wind and wave direction data as our examples. Suppose variable *X* is the observation (directions) measured by radar and variable *Y* is the observation (directions) measured by anchored buoy. The scatter plots of the data set in Figure 1, indicate that there is generally a linear relationship between the directions taken by radar and anchored buoy with $\beta \approx 1$. We assume that each y_i, observation on *Y*, and x_i, observation on *X*, can be described by the model

$$y = \alpha + \beta x + \varepsilon \;\; (\text{mod}\, 2\pi) ,$$

where ε represents departures from the straight line implicit in the model, also called circular random errors, and are distributed as a von Mises distribution with circular mean 0, and concentration parameter κ, i.e. $\varepsilon \sim VM(0,\kappa)$.

Estimates for the wind direction data are given in Table 1, together with their approximate standard errors.

Parameter	Estimate	St. error
α	0.153	6.479×10^{-2}
β	0.976	1.596×10^{-2}
κ	7.337	0.882

Table 1: Parameter estimates for wind direction data.

Testing the null hypothesis that α is equal to $0 \bmod (2\pi)$, gave an F-statistic equal to 5.275 which we compare with F(1,127). The null hypothesis is rejected at the 5% significance level indicating a non-zero α is required. However, testing the null hypothesis that $\hat{\beta}$ equals 1, gives F=2.217, and we draw the conclusion that the null hypothesis can not be rejected at the 5% significance level.

Table 2 give the estimates of wave direction data together with their approximate standard errors.

Parameter	Estimate	St. error
α	6.008	0.165
β	1.074	3.732×10^{-2}
κ	4.508	0.680

Table 2: Parameter estimates for wave direction data.

Testing the null hypothesis that α is equal to $0 \bmod (2\pi)$ gives F = 1.815, and the null hypothesis that β is equal to 1, gives F = 2.707. We compare these F values with F(1, 76), and we draw the conclusion that both null hypotheses can not be rejected at 5% significance level.

References

1. Dobson, A. J. *Applied Statistics*, 27, (1978), pp. 345-347.
2. Mardia, K. V. *Statistics of Directional Data*. Academic Press, London, (1972).
3. Hussin, A.G., Fieller, N. R. J. and Eleanor, E. C. *Journal of Applied Science & Technology*, 8, Nos. 1 & 2 (2004), pp. 1 – 6.
4. Sova, M. S., *The sampling variability and the validation of high frequency radar measurements of the sea surface*. Ph.D. Thesis, School of Mathematics and Statistics, University of Sheffield, (1995).