Designing a Composite Service Organization
(Through Mathematical Modeling)

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Abstract

Suppose we have a class of similar service organizations each of which is characterized by the same numerically measurable input/output characteristics. Even if the amount of any input does not differ in them, one or more organizations can be expected to outperform the others in one or more production aspects. Our interest lies in comparing the output efficiency levels of all service organizations. For it we use mathematical modeling, mainly linear programming to design a composite organization with new input measures which relative to a specific organization should have a higher level of efficiency with regard to all output measures. The other purpose of this paper is to evaluate the output characteristics of this proposed service organization. The paper also touches some other highly important planning features of this organization.

1. Introduction

Suppose we have a class of similar service organizations each of which is characterized by the same numerically measurable input and output traits. As for the input resource, the members of this class may differ quantitatively in this regard but the economic, technical or other motivational factors can disproportionately stimulate, encourage and influence their respective performance. Where the class consists of two or more operating members, the one that is not producing enough is normally regarded as inefficient. The remark of this kind is made in the context of relative performance with reference to resource consumed.

The organizations use the similar material to pursue their similar objectives but the variable effort, motivational influences and awareness of competition cause difficulties in precisely measuring their output. It is so because the collective output of an organization is dependent on more than one conceptually non-comparable variables. And of course we cannot justify for any reason to set aside the input resource which again is a separate function of other variables. Canonical analysis for output studies does not offer here much help for two reasons – one that motivational factors are liable to induce non-linear relationships between input and output measures, and the other is that the number of organizations may be too small for statistical consideration.

Our interest in this paper lies in comparing the performance levels of organizations by means of mathematical modeling. For this purpose we aim to design a hypothetical service organization with desirable input and output traits - a composite service organization. An application of linear programming was casually and briefly mentioned by Sweeney, Anderson and Williams (2003) to evolve such an organization aiming to be more efficient relative to some specific organization. The organization under reference applies to each member a certain percentage to his input resource and forming a linearly weighted combination of quantities regarding each specific input based on all members expects the same proportionate response from the members in their all output measures. It should be at least as productive as some selected organization in the class.

We study here critically this aspect of creating a composite and suggests various criteria to monitor the relative performance of organizations in the class. Some theorems on specific LP models are developed and their applications are given.

To illustrate the importance of the above concept in educational institutions we have the input resource comprising the faculty, its cadres and competence, staff, students' quality, class rooms, labs, discipline, facilities etc. The number of graduates, seminars held, participation in research and other activities are generally the expected consequences of their services. For hospitals, we may consider physicians, their specialization, nurses, equipment, supply expense, beds etc as input measures; and that patient days, patients treated, nurses / interns trained, facilities culminate in the form of productivity aspects.

2. Notations, assumptions and the Concept

Suppose the class C has n service organizations $S_1, S_2, \ldots, S_n$. The $p > 1$ input and $q > 1$ output characteristics pertaining to each organization are denoted by $I_1, I_2, \ldots, I_p$ and $O_1, O_2, \ldots, O_q$ respectively.

Presuming the availability of information regarding the indicated characteristics, let Iij be the quantity of the jth input for the ith organization. Similarly Oij stands for the amount of the jth output relating to the ith organization.

We assign the weights $w_{1k}, w_{2k}, \ldots, w_{nk}$ to $S_1, S_2, \ldots, S_n$ to design a composite with the essential condition that for its formulation in comparison with some member S_k of the class C it outperforms S_k. Each weight is a nonnegative number and their sum $w_{1k} + w_{2k} + \ldots + w_{nk} = 1$.

3. Model Generation for a Composite Service Organization

We proceed in this section to explore various mathematical models to obtain a hypothetical composite organization of the kind described above.

There may be any number of criteria probably as many criteria as the number of characteristics, or even more; or one may generate a composite with bias towards the minimum input and the other towards the maximum output. But the important point is to understand the purpose for which it is required. Therefore, mainly the management has to decide about the choice of criteria.

Let C_k be the composite organization associated with the specific S_k, which we evolve under the condition that with regard to both input and output it is, if not better, as good as S_k. So, the input I_i for C_k is,

$$w_{1k} I_{1i} + w_{2k} I_{2i} + \ldots\ldots + w_{nk} I_{ni}, \qquad i = 1, 2, \ldots, p.$$

a weighted linear combination. Similarly, for it we form jth linear output

$$w_{1k} O_{1j} + w_{2k} O_{2j} + \ldots\ldots + w_{nk} O_{nj} \qquad j = 1, 2, \ldots, q.$$

3.1 Model I

If C_k is to be at least as efficient as S_k then its each input resource must not exceed the similar input of S_k and its output should not be less than that of S_k. We have therefore now a model describing these criteria.

$$w_{1k} + w_{2k} + \ldots\ldots + w_{nk} = 1$$
$$w_{1k} I_{1i} + w_{2k} I_{2i} + \ldots\ldots + w_{nk} I_{ni} \le I_{ki}$$
$$w_{1k} O_{1j} + w_{2k} O_{2j} + \ldots\ldots + w_{nk} O_{nj} \ge O_{kj}$$

for each i, j thus entailing p + q + 1 linear constraints.

Model Solutions

It is not difficult to see that this model may have various solutions, meaning different sets of weights and therefore different composites. So if $w_{\mathbf{k}} = (w_{1k}, w_{2k}, \ldots, w_{nk})$, the solutions $\{w_k\}$ generate a family of composites $\{C_k\}$ bearing the quality of desired performance.

There must exist at least one solution for which except $w_{kk.}$ all weights are zero. Here this particular weight = 1, which obviously means that C_k is in fact S_k itself.

The mathematical system of linear inequalities may not produce a unique solution. So if we find more than one composite organization outperforming S_k, the question is how to pick up the best performer. We cannot avoid the limitation of identifying the most efficient composite. We propose the following:

i) Include some more reasonable constraints. We may assign the values zero to some slack and surplus variables to find solutions. But in doing so it is imperative to assess the repercussions on the objectives aimed.
ii) If the management shows a collective concern for both input and output the above model may be modified by incorporating a pertinent expectation

in terms of an efficiency factor relating to composite's input or output. For example, a linear programming model or goal programming may be used.

iii) If an optimal composite is identified, find how far it is from the particular organization under comparison. In this case the weights determined give an idea about the composite's dependence on other members of the class. The larger dependence means the lower efficiency of the particular organization.

iv) If an organization is to be ranked for its performance it is necessary to find an optimal composite for each member of the class.

3.2 Linear Programming Models

As already indicated we may adopt a rigorous principle to formulate an imaginary organization of our interest. One way of doing it is to minimize its input and yet achieve output at least higher than that of the particular member. Sweeney, Anderson and Williams [1] use this approach in a limited context. We attempt to develop this idea.

For a better understanding of C it is vital to address both issues of resource employed and productivity, and it is in this perspective that we propose the following criteria. The models arisen are different because their requirements are not uniform.

Model II

If the amount of each input of C_k is not to exceed that of S_k we assign a collective efficiency factor E_k for all its input measures. Even if C_k and S_k display the same output level a more efficient C_k should need desirably an economic input ($E_k\, I_{ki}$) where $E_k \le 1$. The smaller the number E_k is the more superior the latter is over the former. The equality $E_k = 1$ is introduced to cover the possibility of `at least one` solution. To accommodate this element of preference we set up the following linear programming model:

Minimize E_k

st

$$w_{1k} + w_{2k} + \ldots + w_{nk} = 1$$
$$w_{1k} I_{1i} + w_{2k} I_{2i} + \ldots + w_{nk} I_{ni} \le E_k I_{ki} \quad \text{for each } i$$
$$w_{1k} O_{1j} + w_{2k} O_{2j} + \ldots + w_{nk} O_{nj} \ge O_{kj} \quad \text{for each } j$$
$$E_k \le 1$$

All the known and unknown constants are non-negative. Obviously this model does not produce an empty feasible region.

The chance is that $E_k = 1$, which implies that a more economical composite does not exist. The point to note is:

This model gauges the efficiency level of the composite relative to one particular organization. Even if this level is very small, care is needed to support the composite's superiority over other organizations. A comprehensive analysis based on surplus/ slack quantities, dual prices, and ranges of feasibility, may be helpful to examine this aspect.

Theorem 1: For the above model the efficiency

a) $E_k < 1$ if and only if $W_{kk} < 1$
b) $E_k = 1$ if and only if $W_{kk} = 1$

Proof:

Case a) Let $E_k < 1$. Then each ith input of the composite is

$$w_{1k} I_{1i} + w_{2k} I_{2i} + \ldots\ldots\ldots\ldots + w_{nk} I_{ni} \le E_k I_{ki} < I_{ki}$$

So if $W_{kk} < 1$ does not hold then this weight must be equal to 1. In this case the left side comes to I_{ik}, which causes a contradiction.
Conversely, let $W_{kk} < 1$.

We have $w_{1k} I_{1i} + w_{2k} I_{2i} + \ldots\ldots\ldots\ldots + w_{nk} I_{ni} \le E_k I_{ki}$.
If $E_k < 1$ is not true, suppose that $E_k =1$. In this case by the definition of efficiency, the left side of the above expression is I_{ki} for all i, which is possible only if $w_{kk} = 1$.

Case b) If $E_k =1$ then by the argument given above we must have $W_{kk} = 1$. Conversely, for $W_{kk} = 1$ the system simplifies to $I_{ki} \le E_k I_{ki}$, again a contradiction as the factor E_k cannot be more than 1.

Remarks on Theorem:

1. The efficiency factor E_k close 1 suggests that the corresponding composite is nearly as as efficient S_k. When its value = 1, S_k is efficient because for its formulation the composite does not depend on other organizations.
2. If each composite has the efficiency factor = 1, all organizations are equally performing.
3. If the model is not followed as such and is modified the theorem may cease to be applicable.

Model III

Suppose now that the focus converges on output and we assign an efficiency factor F_k where each output of C_k must not be less than that of S_k. Here the jth output of the composite is $F_k O_{kj}$, where F_k should be at least 1. Since a larger efficiency factor means the better performance of C_k, the linear programming model for this situation is as follows:

Maximize F_k
st

$$w_{1k} + w_{2k} + \ldots\ldots\ldots\ldots + w_{nk} = 1$$
$$w_{1k} I_{1i} + w_{2k} I_{2i} + \ldots\ldots\ldots\ldots + w_{nk} I_{ni} \le I_{ki} \quad \text{for each } i$$
$$w_{1k} O_{1j} + w_{2k} O_{2j} + \ldots\ldots\ldots\ldots + w_{nk} O_{nj} \ge F_k O_{kj} \quad \text{for each } j$$
$$F_k \ge 1$$

All the known and unknown constants are non-negative. Here as well $F_k = 1$ is possible. Again here,

- A composite relative to a particular organization demands at the most the same amount of input as that of the particular organization but it may not turn up better than other organizations with regard to their individual productivity. Like as indicated above a further analysis is needed to rank them.

Theorem 2: For the above model the efficiency

a) $F_k > 1$ if and only if $W_{kk} < 1$
b) $F_k = 1$ if and only if $W_{kk} = 1$

The proof of this theorem is left to the reader.

Model IV

Let us now be a little more ambitious and consider a situation where for input we minimize the efficiency factor as in Model II but we expect the composite output to achieve at least a particular highest output. So if it is the first output the Model II needs the following modification.

Minimize E_k
st

$$w_{1k} + w_{2k} + \ldots\ldots\ldots\ldots + w_{nk} = 1$$
$$w_{1k} I_{1i} + w_{2k} I_{2i} + \ldots\ldots\ldots\ldots + w_{nk} I_{ni} \le E_k I_{ki} \quad \text{for each } i$$
$$w_{1k} O_{11} + w_{2k} O_{21} + \ldots\ldots\ldots\ldots + w_{nk} O_{n1} \ge \max\{O_{11},\ldots,O_{n1}\}$$
$$w_{1k} O_{1j} + w_{2k} O_{2j} + \ldots\ldots\ldots\ldots + w_{nk} O_{nj} \ge O_{kj} \quad \text{for each } j > 1$$
$$E_k \le 1$$

We can have a large class of models. The above models may be modified for certain purposes. So, we may state:

The above models describe particular situations and depending on one's purpose a model may be set up. Ambitious aims may fail to provide feasible solutions as well as usefulness.

4. Applications

We consider the problem from Sweeney, Anderson and Williams (2003) and apply the above methods. The problem is based on a class of four similar hospitals which we denote here by HG, HU, HC, HS. The input measures are full-time equivalent non-physicians, supply expense ($1000s), bed-days (1000s) available, and the output measures are medicare patient days (1000s), non-medicare patient days (1000s), nurses and interns trained.

The objective is to monitor their relative performance and identify the hospitals not functioning efficiently in view of the resources used by the class. The authors provide the following data.

Input

Hospital	FTE non physicians	Supply expense	Bed days available
HG	285.20	123.80	106.72
HU	162.30	128.70	64.21
HC	275.70	348.50	104.10
HS	210.40	154.10	104.04

Output

Hospital	Medi patient days	non-medi patient days	Nurses trained	Interns trained
HG	48.14	43.10	253	41
HU	34.62	27.11	148	27
HC	36.72	45.98	175	23
HS	33.16	56.46	160	84

We apply the unknown weights WG, WU, WC and WS for HG, HU, HC and HS to design the composite hospital relative to HC. We set up a number of models:

Model I

$$WG + WU + WC + WS = 1$$
$$48.14\ WG + 34.62\ WU + 36.72\ WC + 33.16\ WS \geq 36.72$$
$$43.1\ WG + 27.11\ WU + 45.98\ WC + 56.46\ WS \geq 45.98$$
$$253\ WG + 148\ WU + 175\ WC + 160\ WS \geq 175$$
$$41\ WG + 27\ WU + 23\ WC + 84\ WS \geq 23$$
$$285.2\ WG + 162.3\ WU + 275.7\ WC + 210.4\ WS \leq 275.7$$
$$123.8\ WG + 128.7\ WU + 348.5\ WC + 154.1\ WS \leq 348.5$$
$$106.72\ WG + 64.21\ WU + 104.1\ WC + 104.04\ WS \leq 104.1$$
$$WG, WU, WC, WS \geq 0$$

This system may have a large number of solutions. One obvious solution is:

$$WG = 0, \quad WU = 0, \quad WC = 1, \quad WS = 0$$

meaning that HC itself. If another solution does not exist it means HC is not one of inefficient hospitals. On the contrary if there exists other solution it shows the relative inefficiency of HC. The larger the weights are the greater the efficiency of the composite implying other hospitals are better. Consider the solution:

WG = 0.686, WU = 0..045, WC = 0 , WS = 0.269

which shows composite's considerable dependence on HW and HS demanding an input of about 259, 132, 104 for FTE non physicians, supply expense, bed days respectively. Yet another solution is:

WG = 0.126, WU = 0.212, WC = 0, WS = 0.527

Relative to HC under this solution, its associated composite is slightly better with respect to output but it may be attractive for its lower input (about 235, 140, 94 for FTE non physicians, supply expense, bed days). So that the composite identified here is desirable but the question is:

'Is it the best available composite?',

a limitation of Model I as already indicated. However, the weights determined for the two solutions suggest that HC is a poor performer (if not the worst). If it is not so the composite would not ignore HC altogether.

Model II

We consider the following models to evolve a hypothetical composite hospital based on input measures at least as efficient as:

HG – Model IIG
HU – Model IIU
HC – Model IIC
HS – Model IIS

The above model is set up for each situation and its solution is discussed in the light of Theorem I. Model II based on HC for comparison, and renamed as Model IIC, the objective here is to minimize E subject to the Model I constraints but limiting its input constraints above by 275.7E, 348.5E and 104.1E.

The solution is obtained by LINDO software. The superiority of the composite is clearly manifested. Relative to HC, the weights applied by C_k are 21.2%, 26%, 0%, 52.7% for HG, HU, HC, HS respectively (total about 100%). For its input this composite requires 90.5% of HC's input.

Let us exactly compare the current input of HC and the composite using information on LINDO surplus / slack variables. The associated dual prices are non-significant, so we ignore it for interpretation of solution. The composite for HC has the characteristics:

Efficiency: 0.905
Weight: WG = 0.212, WU = 0.260, WC = 0, WS = 0.527

Input	Composite	HC	Difference
FTE non physicians	213.69	275.7	- 59.03
Supply expense	140.97	348.5	-207.53
Bed days available	94.21	104.1	- 9.89
Output			
Days medicare	36.72	36.72	0
Patients treated	45.98	45.98	0
Nurses trained	175.0	174.42	+0.58
Interns trained	23.00	60.02	+37.02

The solutions of other three models indicate that none of HG, HU and HS has a better composite.

Remark (resource oriented)

We find that the ***oriented composite*** is clearly better than HC, and for its development it depends on other hospitals. On the contrary, better composites do not exist for other hospitals. HC uses some input parameters that are abnormally high (in particular 'supply expenses'). Invariably, even by applying modified models it is found that the consequent composite hospital applies no weight to HC.

Model III

We now change our focus to output for a composite that has an input not more than that of HC. We maximize F subject to its constraints as in Model III., using F as a multiplier for each output measure. We set up models for all four situations and obtain their solutions. A higher value of F is desirable for a new formulation.

It was discovered that except Model IIIC no other model generates a useful solution. The particulars of this hospital are:
Efficiency: 1.083
Weight: WG = 0.438, WU = 0.028, WC = 0, WS = 0.534

Input	Composite	Difference relative to HC
FTE non physicians	241.8	- 33.9
Supply expense	140.1	-208.4
Bed days available	104.1	0

Output

Days medicare	39.77	+ 3.05
Patients treated	49.80	+ 3.82
Nurses trained	200.42	+25.42
Interns trained	63.58	+40.58

Remark (results oriented)

From the above information we conclude that if the objective is to design a ***results oriented*** composite even then HC turns to be a poor performer with supply expense as an abnormal input.

A further analysis of HC is possible through Model IV to investigate the contribution of its supply expenses in adversely affecting its efficiency.

Conclusion

We conclude from the above remarks that HC is the least efficient among competing four hospitals. The abnormally high supply expenses contribute prominently to its relative inefficiency. The management may investigate the underlying causes of this problem. Other hospitals perform equally well in view of input consideration.

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Reference

1. Sweeney D. J., Anderson D. R. and Williams T. A., (2003), *An Introduction to Management Science*, 10th edition. Thomson South-Western.