Characterizations of Transmuted Complementary Weibull Geometric Distribution

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Abstract

We present certain characterizations of a recently introduced distribution (Afify et al., 2014), called Transmuted Complementary Weibull Geometric distribution based on: (i) hazard function ; (ii) a simple relation between two truncated moments. We like to mention that the characterization (ii) which is expressed in terms of the ratio of truncated moments is stable in the sense of weak convergence. It does not require a closed form for the cumulative distribution function and serves as a bridge between a first order differential equation and probability.

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. In this short note, we present here, characterizations of Transmuted Complementary Weibull Geometric Distribution (TCWGD for short). These characterizations are based on: (i) hazard function; (ii) a simple relationship between two truncated moments.

Afify et al. (2014) introduced a new generalization of the complementary Weibull geometric distribution whose probablity density function (pdf) and corresponding cumulative distribution function (cdf) are given, respectively, by

$$f(x) = f(x;\alpha,\beta,\gamma,\delta) = \frac{\alpha\beta\gamma^{\beta}x^{\beta-1}\left(\alpha(1-\delta)+(1-\alpha+\delta+\alpha\delta)e^{-(\gamma x)^{\beta}}\right)}{\left(\alpha+(1-\alpha)e^{-(\gamma x)^{\beta}}\right)^{3}}, \tag{1.1}$$

and

$$F(x) = F(x;\alpha,\beta,\gamma,\delta) = \frac{\alpha^{2}+\left((\alpha\delta+\alpha-2\alpha^{2})-(\alpha\delta+\alpha-\alpha^{2})e^{-(\gamma x)^{\beta}}\right)e^{-(\gamma x)^{\beta}}}{\left(\alpha+(1-\alpha)e^{-(\gamma x)^{\beta}}\right)^{2}}, \tag{1.2}$$

where $x>0$, α,β,γ all positive and $|\delta|\le 1$ are parameters.

They established different properties of their new distribution. For the detailed treatment of TCWGD, we refer the interested reader to Afify et al. (2014). The present author believes that, their paper would have been more complete if the authors had considered some characterizations of TCWGD. In what follows, we present certain characterizations of TCWGD in terms of hazard function and truncated moments.

2. Characterization Results

2.1. Characterizations based on hazard function

It is obvious that the hazard function of twice differentiable distribution function satisfies the first order differential equation

$$\frac{h_F'(x)}{h_F(x)} - h_F(x) = p(x),$$

where $p(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x), \tag{2.1.1}$$

for many univariate continuous distributions $(2.1.1)$ seems to be the only differential equation in terms of the hazard function. The goal of the characterization based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form $(2.1.1)$. For some general families of distributions like TCWGD, this may not be possible. Here we present a characterization of TCWGD for the special case of $\delta = 1$.

Proposition 2.1.1. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf (1.1) for $\delta = 1$, if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - (\beta - 1)x^{-1} h_F(x) = \frac{2\alpha(\alpha - 1)\beta^2\gamma^{2\beta}x^{2(\beta-1)}e^{-(\gamma x)^\beta}}{\left\{\alpha + (1-\alpha)e^{-(\gamma x)^\beta}\right\}^2}, \quad x > 0. \tag{2.1.2}$$

Proof: If X has pdf (1.1) with $\delta = 1$, then clearly $(2.1.2)$ holds. Now, if $(2.1.2)$ holds, then

$$x^{-(\beta-1)}h_F'(x) - (\beta - 1)x^{-\beta} h_F(x) = \frac{2\alpha(\alpha - 1)\beta^2\gamma^{2\beta}x^{(\beta-1)}e^{-(\gamma x)^\beta}}{\left\{\alpha + (1-\alpha)e^{-(\gamma x)^\beta}\right\}^2},$$

or

$$\frac{d}{dx}\left\{x^{-(\beta-1)}h_F(x)\right\} = \frac{d}{dx}\left\{2\alpha\beta\gamma^\beta\left\{\alpha + (1-\alpha)e^{-(\gamma x)^\beta}\right\}^{-1}\right\},$$

or

$$\frac{f(x)}{1-F(x)} = h_F(x) = 2\alpha\beta\gamma^\beta x^{\beta-1}\left\{\alpha + (1-\alpha)e^{-(\gamma x)^\beta}\right\}^{-1}$$
$$= \frac{2\alpha\beta\gamma^\beta x^{\beta-1}e^{(\gamma x)^\beta}}{\alpha e^{(\gamma x)^\beta} + (1-\alpha)}.$$

Integrating both sides of the above equation from 0 to x, we arrive at

$$-\log\left(1-F(x)\right)=\log\left\{\alpha e^{(\gamma x)^{\beta}}+(1-\alpha)\right\}^{2}$$

from which we have

$$1-F(x)=\left\{\alpha e^{(\gamma x)^{\beta}}+(1-\alpha)\right\}^{-2}$$

$$=\frac{e^{-2(\gamma x)^{\beta}}}{\left\{\alpha+(1-\alpha)e^{-(\gamma x)^{\beta}}\right\}^{2}},$$

or

$$F(x)=\frac{\alpha^{2}+\left\{2\alpha(1-\alpha)+\alpha(\alpha-2)e^{-(\gamma x)^{\beta}}\right\}e^{-(\gamma x)^{\beta}}}{\left\{\alpha+(1-\alpha)e^{-(\gamma x)^{\beta}}\right\}^{2}},$$

which is (1.2) for $\delta=1$.

2.2. Characterizations based on two truncated moments

In this subsection we present characterizations of TCWGD in terms of a simple relationship between two truncated moments. Our characterization results presented here will employ an interesting result due to Glänzel (1987) (Theorem 2.2.1 below). The advantage of the characterizations given here is that, cdf F need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

Theorem 2.2.1. Let $(\Omega,,\mathbf{P})$ be a given probability space and let $H=[a,b]$ be an interval for some $a<b$ $(a=-\infty, b=\infty$ mightaswellbeallowed). Let $X:\Omega\rightarrow H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}[q_1(X)|X\geq x]=\mathbf{E}[q_2(X)|X\geq x]\,\eta(x),\qquad x\in H\,,$$

is defined with some real function η. Assume that $q_1,q_2\in C^1(H)$, $\eta\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $q_2\eta=q_1$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x)=\int_a^x C\left|\frac{\eta'(u)}{\eta(u)q_2(u)-q_1(u)}\right|\exp\left(-s(u)\right)du,$$

where the function s is a solution of the differential equation $s'=\frac{\eta' q_2}{\eta q_2-q_1}$ and C is a constant, chosen to make $\int_H dF=1$.

Remarks 2.2.2. (a) In Theorem 2.2.1, the interval H need not be closed since the condition is only on the interior of H. (b) Clearly, Theorem 2.2.1 can be stated in terms of two functions q_1 and η by taking $q_2(x)\equiv 1$, which will reduce the condition given in Theorem 2.2.1 to $E[q_1(X)|X\geq x]=\eta(x)$. However, adding an extra function will give a lot more flexibility, as far as its application is concerned.

Proposition 2.2.3. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_2(x)=\left\{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}\right\}^4\left\{\alpha(1-\delta)+(1-\alpha+\alpha\delta+\delta)e^{-(\gamma x)^\beta}\right\}^{-1}$ and $q_1(x)=q_2(x)\left\{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}\right\}$ for $x>0$. The pdf of X is (1.1) if and only if the function η defined in Theorem 2.2.1 has the form

$$\eta(x)=\frac{2}{3}\left\{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}\right\},\quad x>0.$$

Proof. Let X have pdf (1), then

$$\big(1-F(x)\big)\mathbf{E}[q_2(X)|X\geq x]=\frac{\alpha}{2(1-\alpha)}\left\{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}\right\}^2,\quad x>0,$$

and

$$\big(1-F(x)\big)\mathbf{E}[q_1(X)|X\geq x]=\frac{\alpha}{3(1-\alpha)}\left\{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}\right\}^3,\quad x>0,$$

and finally

$$\eta(x)q_2(x)-q_1(x)=-\frac{1}{3}q_2(2)\left\{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}\right\}^2\neq 0\quad for\quad x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)q_2(x)}{\eta(x)q_2(x)-q_1(x)}=\frac{2(1-\alpha)\beta\gamma^\beta x^{\beta-1}e^{-(\gamma x)^\beta}}{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}}$$

and hence

$$s(x)=-2\log\left\{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}\right\},\quad x>0.$$

Now, in view of Theorem 2.2.1, X has pdf (1.1) .

Corollary 2.2.4. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_2(x)$ be as in Proposition 2.2.3. The pdf of X is (1.1) if and only if there exist functions q_1 and η defined in Theorem 2.2.1 satisfying the differential equation

$$\frac{\eta'(x)q_2(x)}{\eta(x)q_2(x)-q_1(x)}=\frac{2(1-\alpha)\beta\gamma^\beta x^{\beta-1}e^{-(\gamma x)^\beta}}{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}},\quad x>0.$$

Remarks 2.2.5.

(a) The general solution of the differential equation in Corollary 2.2.4 is

$$\eta(x)=\left\{\alpha+(1-\alpha)e^{-(\gamma x)^\beta}\right\}^{-2}\times\left[-\int 2(1-\alpha)\beta\gamma^\beta x^{\beta-1}e^{-(\gamma x)^\beta}[q_2(x)]^{-1}q_1(x)dx+D\right],$$

for $x>0$, where D is a constant. One set of appropriate functions is given in Proposition 2.2.3 with $D=0$.

(b) Clearly there are other triplets of functions (q_1,q_2,η) satisfying the conditions of Theorem 2.2.1. We presented one such triplet in Proposition 2.2.3.

Reference

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2. Glänzel, W. A. (1987). Characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, $75 - 84$.