The New Weibull-Pareto Distribution

Suleman Nasiru
Department of Statistics, Faculty of Mathematical Sciences
University for Development Studies, P. O. Box 24, Navrongo
Upper East Region, Ghana, West Africa
sulemanstat@gmail.com

Albert Luguterah
Department of Statistics, Faculty of Mathematical Sciences
University for Development Studies, P. O. Box 24, Navrongo
Upper East Region, Ghana, West Africa
adlugu@yahoo.com

Abstract

A new distribution, the New Weibull-Pareto, is defined and studied. Various properties of the distributions are obtained and the method of maximum likelihood used to estimate the parameters of the distribution. The usefulness of the distribution has also been demonstrated by applying it to real life data.

Keywords: New Weibull-Pareto, Moments, Survival, Hazard, Maximum likelihood estimates.

1. Introduction

Many lifetime data used for statistical analysis follows a particular statistical distribution. Knowledge of the appropriate distribution that any phenomenon follows, greatly improves the sensitivity, power and efficiency of the statistical tests associated with it. Several distributions exist for modeling these lifetime data however, some of these lifetime data do not follow these existing distributions or are inappropriately described by them. This therefore creates room for developing new distributions which could better describe some of these phenomena and therefore provide greater flexibility in the modeling of lifetime data. As a result, umpteen of distributions have been developed and studied by researchers.

Gupta *et al*. (1998) developed the exponentiated exponential distribution, Mudholkar *et al*. (1995) proposed the exponentiated-Weibull distribution, Akinsete *et al*. (2008) developed the beta-Pareto distribution, Alzaatreh *et al*. (2012) developed the gamma-Pareto distribution and Alzaatreh *et al*. (2013) developed the Weibull-Pareto distribution. Also, Merovci and Puka (2014) developed the transmuted Pareto distribution while Kareema and Boshi (2013), developed the Exponential Pareto distribution. In this article, we present another form of the Weibull-Pareto distribution called the New Weibull-Pareto Distribution (NWPD).

2. New Weibull-Pareto Distribution (NWPD)

Let X be a random variable from a Pareto distribution with its cumulative distribution function (cdf) for $x \geq \theta$ given by

$$F_1(x;\ \theta, k) = 1 - \left(\frac{\theta}{x}\right)^k \tag{1}$$

where $\theta > 0$ is a scale parameter and $k > 0$ is the shape parameter. The probability density function (pdf) corresponding to (1) is

$$f_1(x; \theta, k) = \frac{k\theta^k}{x^{k+1}}$$

The NWPD has a cdf of the form

$$G(x) = \int_0^{\frac{1}{R(x)}} f_2(x)\, dx \tag{2}$$

where $R(x)$ is the survival function of the Pareto distribution and is given by $R(x) = 1 - F_1(x;\ \theta, k)$ while $f_2(x)$ is the pdf of a Weibull distribution and is given by

$$f_2(x) = \alpha\lambda(\lambda x)^{\alpha-1} e^{-(\lambda x)^\alpha} \tag{3}$$

where $x > 0,\ \alpha > 0,$ and $\lambda > 0$.

Using (2) and (3), and given that $R(x) = \left(\frac{\theta}{x}\right)^k$, the cdf of the NWPD is given by

$$G(x) = \int_0^{\frac{1}{\left(\frac{\theta}{x}\right)^k}} \alpha\lambda(\lambda x)^{\alpha-1} e^{-(\lambda x)^\alpha}\, dx$$

$$G(x) = 1 - e^{-\lambda^\alpha \left(\frac{x}{\theta}\right)^{\alpha k}}$$

If we let $\delta = \lambda^\alpha$ and $\beta = \alpha k$, then the cdf of the NWPD can be written as

$$G(x) = 1 - e^{-\delta\left(\frac{x}{\theta}\right)^\beta} \tag{4}$$

The pdf is obtained by finding the first derivate of (4) with respect to x. Thus the pdf is

$$g(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} \tag{5}$$

where $0 < x < \infty,\ \beta > 0,\ \theta > 0$ and $\delta > 0$.

Lemma 2.1. The limit of the pdf of the NWPD, $g(x)$, as $x \to \infty$ is 0 and as $x \to 0$ is 0. This can be proved by taking the limit of (5) as $x \to \infty$ and as $x \to 0$.

Proof.

$$\lim_{x\to\infty} g(x) = \frac{\beta\delta}{\theta} \lim_{x\to\infty} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta}$$

$$= \frac{\beta\delta}{\theta} \times \infty \times 0 = 0$$

and

$$\lim_{x\to 0} g(x) = \frac{\beta\delta}{\theta}\lim_{x\to 0}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}$$
$$= \frac{\beta\delta}{\theta} \times 0 \times 1 = 0$$

Lemma 2.2. The limit of the cdf of the NWPD, $G(x)$, as $x \to \infty$ is 1 and as $x \to 0$ is 0. This can be proved by taking the limit of (4) as $x \to \infty$ and as $x \to 0$.

Proof.

$$\lim_{x\to\infty} G(x) = 1 - \lim_{x\to\infty} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}$$
$$= 1 - 0$$
$$= 1$$

and

$$\lim_{x\to 0} G(x) = 1 - \lim_{x\to 0} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}$$
$$= 1 - 1$$
$$= 0$$

Figure 1 is the plot of the pdf of the NWPD for the different values of the parameters of the distribution.

3. Survival and hazard functions

The NWPD can be a useful characterization of the survival time of a given system because of it analytical structure. The survival function is given by $S(x) = 1 - G(x)$. Thus using (4),

$$S(x) = e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \tag{6}$$

Figure 2 is the plot of the survival function of the NWPD for different values of the parameters of the distribution.

Another characteristics of interest of a random variable is the hazard function defined by

$$h(x) = \frac{g(x)}{S(x)}$$

Thus using (5) and (6), another characteristics of interest of a random variable is the hazard function defined by

$$h(x) = \frac{g(x)}{S(x)}$$

Thus using (5) and (6),

$$h(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \tag{7}$$

From the hazard function the following can be observed:

i. If $\beta = 1$, the failure rate is constant and given by:

$$h(x) = \frac{\delta}{\theta}$$

This makes the NWPD suitable for modeling systems or components with constant failure rate.

ii. If $\beta > 1$, the hazard is an increasing function of x, which makes the NWPD suitable for modeling components that wears faster with time.

iii. If $\beta < 1$, the hazard is a decreasing function of x, which makes the NWPD suitable for modeling components that wears slower with time.

Figure 3 is the plot of the hazard function of the NWPD for different values of the parameters of the distribution.

4. Moments and generating functions

4.1 Moments

If X is a random variable distributed as a NWPD, then the r^{th} non-central moment is given by:

$$E(X^r) = \theta^r \delta^{\frac{-r}{\beta}} \Gamma\left(\frac{\beta+r}{\beta}\right) \quad (8)$$

Proof.

$$E(X^r) = \int_0^\infty x^r \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx$$

Let $y = \delta\left(\frac{x}{\theta}\right)^\beta$, $dy = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} dx$ and $x = \theta\left(\frac{y}{\delta}\right)^{\frac{1}{\beta}}$

$$E(X^r) = \int_0^\infty \left(\theta\left(\frac{y}{\delta}\right)^{\frac{1}{\beta}}\right)^r e^{-y} dy$$

$$= \theta^r \gamma^{\frac{-r}{\beta}} \int_0^\infty y^{\frac{r}{\beta}} e^{-y} dy$$

$$= \theta^r \delta^{\frac{-r}{\beta}} \Gamma\left(\frac{\beta+r}{\beta}\right)$$

If $r = 1$, $E(X) = \theta\delta^{\frac{-1}{\beta}} \Gamma\left(\frac{\beta+1}{\beta}\right)$

If $r = 2$, $E(X^2) = \theta^2\delta^{\frac{-2}{\beta}} \Gamma\left(\frac{\beta+2}{\beta}\right)$

Therefore the variance is given by

$$Var(X) = \theta^2\delta^{\frac{-2}{\beta}} \Gamma\left(\frac{\beta+2}{\beta}\right) - \left\{\theta\delta^{\frac{-1}{\beta}} \Gamma\left(\frac{\beta+1}{\beta}\right)\right\}^2$$

4.2 Incomplete moments

If X is a random variable distributed as a NWPD with parameters β, θ and γ, the r^{th} incomplete moment of X is given by:

$$M_r(z) = \theta^r \delta^{\frac{-r}{\beta}} \gamma\left(\frac{\beta + r}{\beta}, \delta\left(\frac{z}{\theta}\right)^\beta\right) \tag{9}$$

Proof.

$$M_r(z) = \int_0^z x^r \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx$$

Let $y = \delta\left(\frac{x}{\theta}\right)^\beta, dy = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} dx$ and $x = \theta\left(\frac{y}{\delta}\right)^{\frac{1}{\beta}}$. If $x = 0, y = 0$ and if $x = z, y = \delta\left(\frac{z}{\theta}\right)^\beta$

This implies

$$M_r(z) = \theta^r \delta^{\frac{-r}{\beta}} \int_0^{\delta\left(\frac{z}{\theta}\right)^\beta} y^{\frac{r}{\beta}} e^{-y}\, dy$$

The integral $\int_0^{\delta\left(\frac{z}{\theta}\right)^\beta} y^{\frac{r}{\beta}} e^{-y}\, dy$ is the lower incomplete gamma and is given by $\gamma\left(\frac{\beta+r}{\beta}, \delta\left(\frac{z}{\theta}\right)^\beta\right)$

$$= \theta^r \gamma^{\frac{-r}{\beta}} \gamma\left(\frac{\beta + r}{\beta}, \delta\left(\frac{z}{\theta}\right)^\beta\right)$$

4.3 Moment generating function

Let X have a NWPD. The moment generating function (mgf) of X denoted by $M_x(t)$ is given by

$$M_x(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \theta^i \delta^{\frac{-i}{\beta}} \Gamma\left(\frac{\beta + i}{\beta}\right) \tag{10}$$

Proof.

By definition

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx$$

Using Taylor series

$$M_x(t) = \int_0^\infty \left(1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots + \frac{t^n x^n}{n!} + \cdots\right) g(x) dx$$

$$= \sum_{i=0}^{\infty} \frac{t^i E\left(X^i\right)}{i!}$$

$$M_x(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \theta^i \delta^{\frac{-i}{\beta}} \Gamma\left(\frac{\beta + i}{\beta}\right)$$

5. Quantile function and simulation

Let $Q(q), 0 < q < 1$ denote the quantile function for the NWPD. Then $Q(q)$ is given by

$$Q(q) = \theta\left\{\frac{1}{\delta}\ln\left(\frac{1}{1-q}\right)\right\}^{\frac{1}{\beta}} \tag{11}$$

In particular, the distribution of the median is:

$$Q(0.5) = \theta\left\{\frac{1}{\delta}\ln(2)\right\}^{\frac{1}{\beta}}$$

Let U be a uniform variate on the unit interval $(0,1)$. Thus by means of the inverse transformation method, we consider the random variable X given by:

$$X = \theta\left\{\frac{1}{\delta}\ln\left(\frac{1}{1-U}\right)\right\}^{\frac{1}{\beta}}$$

This follows the NWPD.

6. Skewness and Kurtosis

In this study, the quantile based measures of skewness and kurtosis was employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness based on quartiles is given by

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and the Moors' kurtosis is on octiles and is given by

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}$$

where $Q(.)$ represents the quantile function.

7. Mode and Mean deviations

The mode of the NWPD is obtained by finding the first derivate of $\ln g(x)$ with respect to x and equating it to zero. That is $\frac{d\ln g(x)}{dx} = 0$. Therefore the mode at $x = x_0$ is given by

$$x_0 = \theta\left\{\frac{(\beta-1)}{\beta\delta}\right\}^{\frac{1}{\beta}}$$

The mean deviation about the mean and the median are useful measures of variation for a population. Let $\mu = E(X)$ and m be the mean and median of the NWPD respectively. The mean deviation about the mean is

$$D(\mu) = E(|X-\mu|) = \int_0^\infty |x-\mu| g(x)dx = 2\mu G(\mu) - 2\int_0^\mu x g(x)dx$$

$$D(\mu) = 2\mu G(\mu) - 2\theta\delta^{\frac{-1}{\beta}}\gamma\left(\frac{\beta+1}{\beta}, \delta\left(\frac{\mu}{\theta}\right)^{\beta}\right)$$

The mean deviation from the median is

$$D(m) = E(|X - m|) = \int_0^\infty |x - m| g(x) dx = \mu - 2 \int_0^m x g(x) dx$$

$$D(m) = \mu - 2\theta\delta^{\frac{-1}{\beta}} \gamma\left(\frac{\beta + 1}{\beta}, \delta\left(\frac{m}{\theta}\right)^\beta\right)$$

8. Order statistics

Often, sample values such as the smallest, largest, or middle observation from a random sample provide important information. For example, the highest flood water or minimum temperature recorded during past years might be useful when planning for future emergencies. Let $X_{(1)}$ denote the smallest of $\{X_1, X_2, \dots, X_n\}$, $X_{(2)}$ denote the second smallest of $\{X_1, X_2, \dots, X_n\}$, and similarly $X_{(r)}$ denote the r^{th} smallest of $\{X_1, X_2, \dots, X_n\}$. Then the random variables $X_{(1)}$, $X_{(2)}, \dots, X_{(n)}$, called the order statistics of the sample $X_1, X_2, \dots, X_n$, has probability density function of the r^{th} order statistic, $X_{(r)}$, as:

$$g_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!} g_X(x) [G_X(x)]^{r-1} [1 - G_X(x)]^{n-r}$$

for $r = 1, 2, 3, \dots, n$.

The pdf of the r^{th} order statistic of the NWPD is

$$g_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!} \frac{\beta\delta}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} \times \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^{r-1} \times \left[e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^{n-r} \qquad (12)$$

The pdf of the largest order statistic $X_{(n)}$ is therefore

$$g_{X_{(n)}}(x) = \frac{n\beta\delta}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} \times \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^{n-1}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$g_{X_{(1)}}(x) = \frac{n\beta\delta}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} \times \left[e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^{n-1}$$

9. Parameter Estimation and inference

In this section, the maximum likelihood estimation and inference of the parameters of the NWPD have been discussed. Let $X \sim \text{NWPD}(\delta, \beta, \theta)$ and let $(\delta, \beta, \theta)^T$ be the vector of the model parameters. The log-likelihood function for $(\delta, \beta, \theta)^T$ is given by

$$L = \prod_{i=1}^{n} \frac{\beta\delta}{\theta} \left(\frac{x_i}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x_i}{\theta}\right)^\beta} \qquad (13)$$

The log-likelihood function LL, is

$$LL = n\ln\beta + n\ln\delta - n\ln\theta - \delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} + (\beta - 1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\theta}\right) \qquad (14)$$

The score functions are given by:

$$\begin{cases} \dfrac{\partial LL}{\partial \delta} = \dfrac{n}{\delta} - \displaystyle\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} \\ \dfrac{\partial LL}{\partial \beta} = \dfrac{n}{\beta} + \displaystyle\sum_{i=1}^{n}\left[1 - \delta\left(\frac{x_i}{\theta}\right)^{\beta}\right]\ln\left(\frac{x_i}{\theta}\right) \\ \dfrac{\partial LL}{\partial \theta} = \dfrac{-n}{\theta} + \dfrac{\beta\delta}{\theta^{(\beta+1)}}\displaystyle\sum_{i=1}^{n} x_i^{\beta} + \dfrac{n(1-\beta)}{\theta} \end{cases} \qquad (15)$$

The maximum likelihood estimators $\left(\hat{\delta}, \hat{\beta}, \hat{\theta}\right)^T$ are obtained by equating (15) to zero and solving for the non-linear system of equations iteratively. In order to compute the standard errors and asymptotic confidence interval, the large sample approximation in which the maximum likelihood estimator of a parameter can be treated as being approximately multivariate normal was used. Thus as $n \to \infty$, the asymptotic distribution of the maximum likelihood estimators $\left(\hat{\delta}, \hat{\beta}, \hat{\theta}\right)^T$ is given by

$$\begin{pmatrix} \hat{\delta} \\ \hat{\beta} \\ \hat{\theta} \end{pmatrix} \sim MN\left[\begin{pmatrix} \delta \\ \beta \\ \theta \end{pmatrix}, \begin{pmatrix} I_{\delta\delta} & I_{\delta\beta} & I_{\delta\theta} \\ & I_{\beta\beta} & I_{\beta\theta} \\ & & I_{\theta\theta} \end{pmatrix}^{-1}\right]$$

where $\begin{pmatrix} I_{\delta\delta} & I_{\delta\beta} & I_{\delta\theta} \\ & I_{\beta\beta} & I_{\beta\theta} \\ & & I_{\theta\theta} \end{pmatrix}^{-1}$ is the approximate variance-covariance matrix. The diagonal elements are the variances of the parameters and the off-diagonal elements are the covariance between the parameters. The elements of the variance-covariance matrix are obtained from

$$I_{ij}(\varphi) = -E\left(\frac{\partial^2 LL}{\partial\varphi_i \partial\varphi_j}\right)$$

The second derivative of the log-likelihood function with respect to the parameters is given by

$$\begin{cases} \dfrac{\partial^2 LL}{\partial \delta^2} = \dfrac{-n}{\delta^2} \\ \dfrac{\partial^2 LL}{\partial \beta^2} = \dfrac{-n}{\beta^2} - \delta \sum_{i=1}^{n} \left(\dfrac{x_i}{\theta}\right)^{\beta} \left(\ln\left(\dfrac{x_i}{\theta}\right)\right)^2 \\ \dfrac{\partial^2 LL}{\partial \theta^2} = \dfrac{n}{\theta^2} - \dfrac{\delta\beta(\beta+1)}{\theta^{\beta+2}} \sum_{i=1}^{n} x_i^{\beta} + \dfrac{n(\beta-1)}{\theta^2} \\ \dfrac{\partial^2 LL}{\partial \delta \partial \beta} = -\sum_{i=1}^{n} \left(\dfrac{x_i}{\theta}\right)^{\beta} \ln\left(\dfrac{x_i}{\theta}\right) \\ \dfrac{\partial^2 LL}{\partial \delta \partial \theta} = \dfrac{\beta}{\theta^{\beta+1}} \sum_{i=1}^{n} x_i^{\beta} \\ \dfrac{\partial^2 LL}{\partial \beta \partial \theta} = \dfrac{\delta\beta}{\theta^{\beta+1}} \sum_{i=1}^{n} \left[x_i^{\beta} \ln\left(\dfrac{x_i}{\theta}\right) + \dfrac{1}{\theta}\left(\dfrac{x_i}{\theta}\right)^{\beta} \right] - \dfrac{n}{\theta} \end{cases} \tag{16}$$

An approximate $100(1-\alpha)\%$ two side confidence intervals for δ, β and θ are respectively:

$$\hat{\delta} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{\delta\delta}^{-1}}, \qquad \hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{\beta\beta}^{-1}} \qquad \text{and} \qquad \hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{\theta\theta}^{-1}}.$$

10. Application

In this section we demonstrated that the NWPD is useful in modeling real life situation. The newly proposed distribution was used to model the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consists of 72 exceedances for the year 1958-1984, rounded to one decimal place as shown in Table 1. Recently, Merovci and Puka (2014), and Bourguignon *et al*. (2013) analysed this data using the Transmuted Pareto (TP) distribution and Kumaraswamy Pareto (Kw-P) distribution respectively, demonstrating the superiority of their distributions over the weibull and Pareto distributions. We therefore fitted the NWPD to this data and compared our results to theirs.

Table 2 displays the Maximum Likelihood Estimates (MLEs) of the model parameters. Since for the Pareto distribution $x \geq \theta$, the MLE of θ is the first-order statistic. From Table 2, it was obvious that the NWPD provides a better fit compared to the other candidate models since it has the lowest value of $-2LL$, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Akiake Information Criterion Corrected (AICC).

11. Conclusion

This article defined a New Weibull-Pareto Distribution (NWPD) and studied various properties of the distribution. The moments, deviations from the mean and median, mode, survival function, hazard function and the maximum likelihood estimates of the parameters, have been investigated. The application of the new distribution has also been demonstrated with real life data. The results, compared with other known distributions, revealed that the NWPD provides a better fit for modeling real life data.

References

1. Akinsete, A., Famoye, F., and Lee, C. (2008). The Beta-Pareto distribution. *Statistics*, 42(6): 547-563.

2. Alzaatreh, A., Famoye, F., and Lee, C. (2013). Weibull-Pareto distribution and its applications. *Communication in Statistics-Theory and Methods*, 42: 1673-1691.

3. Alzaatreh, A., Famoye, F., and Lee, C. (2012). Gamma-Pareto distribution and its applications. *Journal of Modern Applied Statistical Methods*, 11(1): 78-94.

4. Bourguignon, M., Silva, R. B., Zea, L. M., and Cordeiro, G. M. (2013). The Kumaraswamy-Pareto distribution. *Journal of Statistical Theory and Applications*, 12(2): 129-144.

5. Gupta, R. C., Gupta, R. D., and Gupta, P. L. (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics-Theory and Methods*, 27: 887-904.

6. Kareema, A. K. and Boshi, M. A., (2013). Exponential Pareto Distribution. *Mathematical Theory and Modeling*, 3(5): 135-146.

7. Merovci, F., and Puka, L., (2014). Transmuted Pareto Distribution. *Probstat Forum*, 7: 1-11.

8. Mudhokar, G. S., Srivastava, D. K., and Freimer, M. (1995). The expnentiated Weibull family: a reanalysis of the bus-motor-failure data. *Technometrics*, 37(4): 436-445.

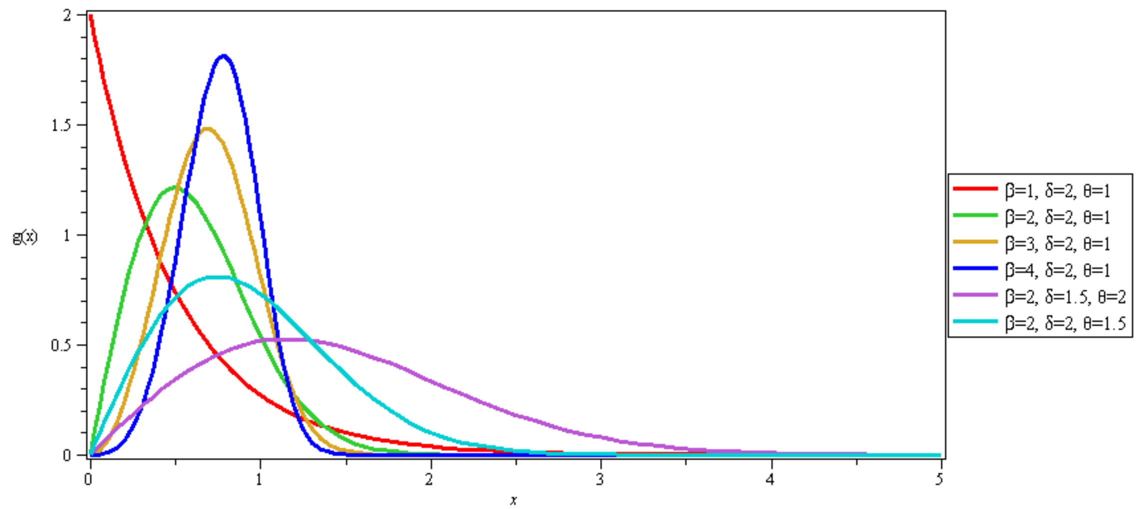

Figure 1: Plot of the pdf of the NWPD

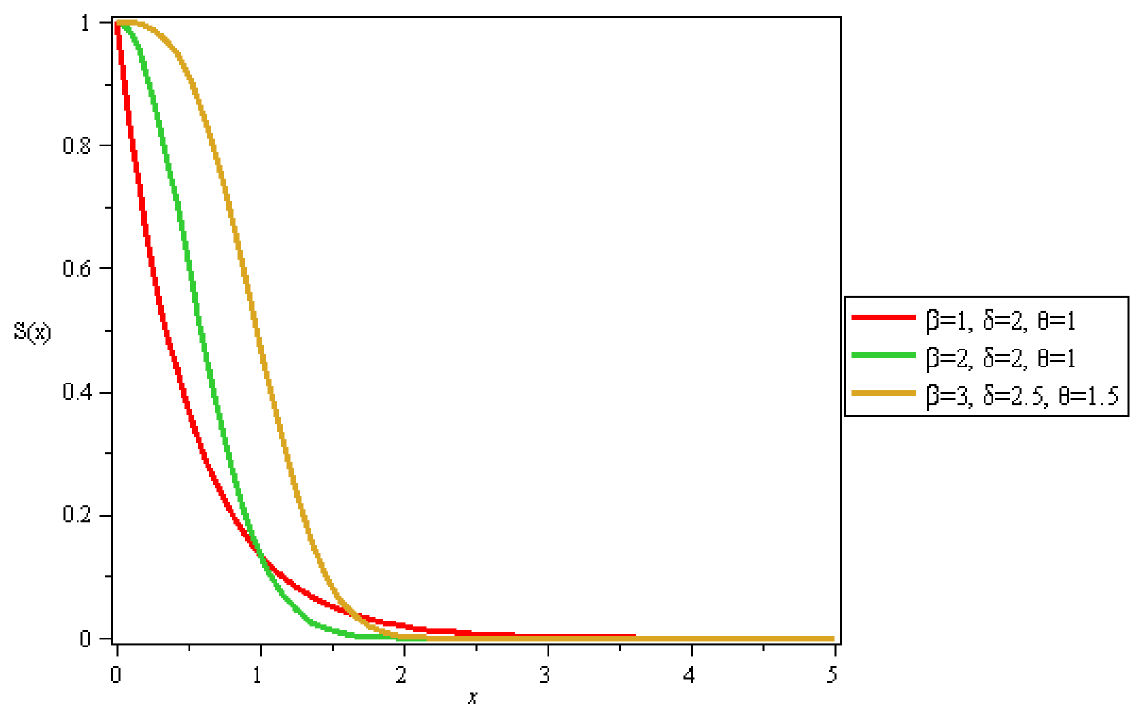

Figure 2: Survival function plot of the NWPD

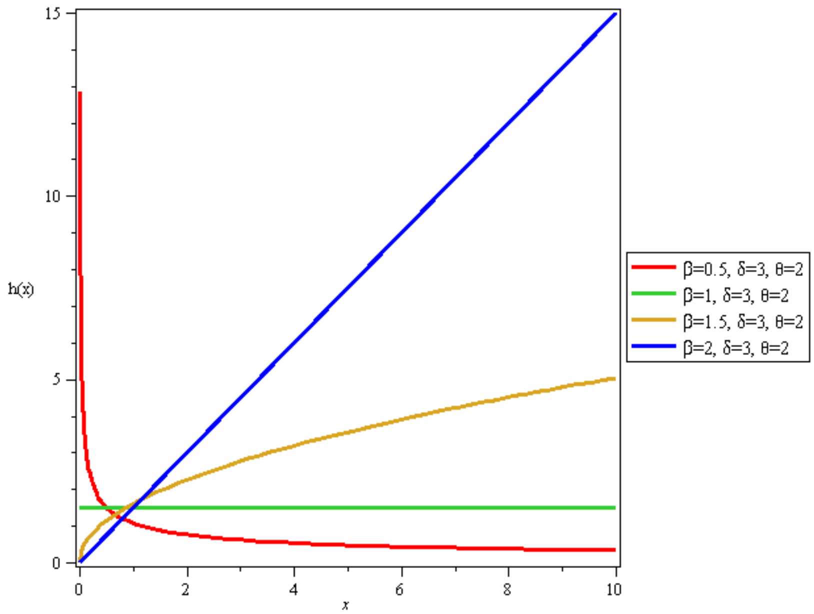

Figure 3: Hazard plot of the NWPD

Table 1: Exceedances of Wheaton River flood data

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7
1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1
0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9
0.9	1.7	7.0	20.1	0.4	2.8	14.1	9.9
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3
1.9	10.4	13.0	10.7	12.0	30.0	9.3	3.6
2.5	27.6	14.4	36.4	1.7	2.7	37.6	64.0
1.7	9.7	0.1	27.5	1.1	2.5	0.6	27.0

Table 2: Maximum likelihood estimates and statistics for model selection

Model	Parameter estimates	Standard error	-2LL	AIC	AICC	BIC
NWPD	$\beta = 0.1999$	0.0236	158.3258	162.3258	162.6787	166.8791
	$\delta = 11.7450$	2.8664				
	$\theta = 0.1$	-				
TP	$a = 0.349$	0.031	572.401	578.402	578.755	580.955
	$\lambda = -0.952$	0.047				
	$x_0 = \min(x) = 0.1$	-				
Kw-P	$a = 2.8553$	0.3371	542.4	548.4	549.0	555.3
	$b = 85.8468$	60.4213				
	$k = 0.0528$	0.0185				
	$\beta = 0.1$	-				