

Concomitants of Order Statistics from Bivariate Inverse Rayleigh Distribution

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Abstract

The probability density function (pdf) of the r th, $1 \leq r \leq n$ and joint pdf of the r th and s th, $1 \leq r < s \leq n$ concomitants of order statistics are derived for Bivariate Inverse Rayleigh Distribution and their moments, product moments are obtained. Its percentiles are also obtained.

Keywords: Order Statistics, Concomitants of Order Statistics, Single and Product moments, Percentiles.

1. Introduction

Let (X_i, Y_i) , $i=1,2,3, \dots, n$ be n pairs of independent random variables from some bivariate population with distribution function $F(x,y)$. If we arrange the X -variates in ascending order as $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ then the y -variates paired (not necessarily in ascending order) with these ordered statistics are called the Concomitants of Order Statistics (David 1973), Induced Order Statistics (Bhattacharya 1984) and are denoted by $Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]}$. The probability density function (pdf) of $Y_{[r:n]}$ is given by (David 1981)

$$g_{[r:n]}(y) = \int_x f(y/x) f_{r:n}(x) dx \tag{1.1}$$

where $f_{r:n}(x)$ is the pdf of $X_{r:n}$ which is given as:

$$f_{r:n}(x) = C_{r:n} [F(x)]^{n-1} [1-F(x)]^{n-r} f(x) \tag{1.2}$$

with $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$

The joint pdf of the $Y_{[r:n]}$ and $Y_{[s:n]}$ $1 \leq r < s \leq n$ is

$$g_{[r,s:n]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f(y_1/x_1) f(y_2/x_2) f_{r,s:n}(x_1/x_2) dx_1 dx_2 \tag{1.3}$$

Where $f_{r,s:n}(x_1, x_2)$ is the joint pdf of $(X_{r:n}, X_{s:n})$ is given as:

$$f_{r,s:n}(x_1, x_2) = C_{r,s:n} [F(x_1)]^{r-1} [F(x_2) - F(x_1)]^{s-r-1} [1-F(x_2)]^{n-s} f(x_1) f(x_2) \tag{1.4}$$

with $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$

For properties of concomitants of order statistics refer to David and Nagaraga (1998), Anguman and Khan (1998), Ahasanullah and Nevzoror (2001). More refer to Ayesha and Ahmad (2001) and Aleem and Ahmad (2003).

Here we consider Bivariate Inverse Rayleigh distribution (Mukherjee and Saran 1984) with distribution function (df)

$$F = F_1 F_2 (1 + \alpha \bar{F}_1 \bar{F}_2) \tag{1.5}$$

with $\bar{F}_1 = 1 - F_1(x)$

and the pdf

$$f(x, y) = f_1 f_2 [1 + \alpha(1 - 2f_1)(1 - 2f_2)] \tag{1.6}$$

with $f_1 = F_1'$, $-1 \leq \alpha \leq 1$

and $f(x, \theta) = \frac{2\theta}{x^3} \exp\left[-\frac{\theta}{x^2}\right]$ $x, \theta > 0$ (1.7)

$$F(x) = \exp\left[-\frac{\theta}{x^2}\right] \tag{1.8}$$

Expressions for single i.e rth and joint pdf of rth and sth concomitants of order statistics from bivariate inverse rayleigh distribution are obtained.

2. Probability density function of $Y_{[r:n]}$

For the Bivariate Inverse Rayleigh distribution with distribution function (1.5), the pdf of the rth concomitant of the order statistics, in view of (1.1) and (1.2) is:

$$g_{[r:n]}(y) = 4\theta_1\theta_2 C_{r:n} \frac{1}{y^3} e^{-\frac{\theta_2}{y^2}} \int_0^\infty \frac{1}{x^3} e^{-\frac{\theta_1}{x^2}} [1 - e^{-\frac{\theta_1}{x^2}}]^{n-r} [1 + \alpha(1 - 2e^{-\frac{\theta_1}{x^2}})(1 - 2e^{-\frac{\theta_2}{y^2}})] dx$$

where $y, \theta_1, \theta_2 \geq 0$

after integrating and simplifying, we get

$$g_{[r:n]}(y) = \frac{2\theta_2}{y^3} e^{-\frac{\theta_2}{y^2}} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \frac{1}{(r+i)} \left[1 + \frac{\alpha}{(r+i+1)} (1 - 2e^{-\frac{\theta_2}{y^2}})\right] \tag{2.1}$$

where $y, \theta_2 \geq 0$, $-1 \leq \alpha \leq 1$

we have checked that $\int g_{[r:n]}(y) dy = 1$

3. Single Moments

The Kth moment of $y_{[r:n]}$, using the basic definition $\mu_{[r:n]}^{(k)} = E[y_{[r:n]}^{(k)}]$ of (2.1) is obtained as:

$$\mu_{[r:n]}^{(k)} = \sqrt{1 - \frac{k}{2}} \theta_2^{k/2} C_{r:n} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \frac{1}{(r+i)} \left\{1 + \frac{\alpha}{(r+i+1)} \left(1 - \frac{2^{k/2}}{\theta_2}\right)\right\} \tag{3.1}$$

(3.1) can be used to obtain moments and inverse moments (when $k = -k$). It is observed that the higher moments of order 2 or more for the distribution of the concomitant $y_{[r:n]}$ of rth order statistics does not exist but the inverse moments of any order exist.

4. Percentiles of $y_{[r:n]}$

The pth percentile of the concomitant of rth order statistics of Bivariate Inverse Rayleigh distribution is given as:

$$\xi_p = \left[\frac{2\theta_2 B}{Ln \left\{ A \pm \sqrt{A^2 - 4BP} \right\}^2} \right]^{\frac{1}{2}} \tag{4.1}$$

where

$$A = C_{r:n} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \frac{1}{(r+i)} \left[1 + \frac{\alpha}{(r+i+1)} \right]$$

and

$$B = C_{r:n} \alpha \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \frac{1}{(r+i)(r+i+1)}$$

when $P=0.5$ in (4.1), we get the median of the distribution.

5. Joint Distribution of two Concomitants $y_{[r:n]}$ and $y_{[s:n]}$

For the Bivariate Inverse Rayleigh distribution using (1.7) and (1.8) in (1.3), then after lengthily integration and simplification, we get

$$g_{[r,s:n]}(y_1, y_2) = 8C_{r,s:n} \frac{\theta_2^2}{y_1^3 y_2^3} e^{-\theta_2 \left[\frac{1}{y_1^2} + \frac{1}{y_2^2} \right]} \sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s} (-1)^{i+j} \binom{s-r-1}{i} \binom{n-s}{j} \left[\frac{H(y_1, y_2)}{(s+j)(r+i)} - \frac{P(y_1, y_2)}{(s+j+1)(r+i+1)} - \frac{2\alpha H(y_2) + 2\alpha^2 H(y_1)H(y_2)}{(s-r-i+j+2)} + \frac{2\alpha^2 H(y_1)H(y_2)}{(s-r-i+j+3)} \right] \tag{5.1}$$

where

$$H(y_1) = 1 - 2e^{-\frac{\theta_2}{y_1^2}}$$

$$H(y_2) = 1 - 2e^{-\frac{\theta_2}{y_2^2}}$$

$$H(y_1, y_2) = 1 + \alpha H(y_1) + \alpha H(y_2) + \alpha^2 H(y_1)H(y_2)$$

$$P(y_1, y_2) = 2\alpha H(y_1) + 2\alpha^2 H(y_1)H(y_2)$$

and $y_1, y_2, \theta_2 > 0, -1 \leq \alpha \leq 1$

6. Product moments of two Concomitants $y_{[r:n]}$ and $y_{[s:n]}$

The product moments of two concomitants $y_{[r:n]}$ and $y_{[s:n]}$ are given by

$$E[y_{[r:n]}^m \cdot y_{[s:n]}^n] = \int_0^\infty \int_0^\infty y_1^m y_2^n g_{[r,s:n]}(y_1, y_2) dy_1 dy_2$$

putting the values of $g_{[r,s;n]}(y_1, y_2)$ from (5.1), integrating and after simplification, we get:

$$E[y_{[r;n]}^m \cdot y_{[s;n]}^n] = 2C_{r,s;n} \theta_2^{\frac{m+n}{2}} \left(1 - \frac{m}{2}\right) \left(1 - \frac{n}{2}\right) \sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s} (-1)^{i+j} \binom{s-r-1}{i} \binom{n-s}{j} \left[\frac{A_1}{(r+i)(s+j)} - \frac{A_2}{(r+i+1)(s+j+1)} - \frac{A_3}{(s-r-i+j+2)} + \frac{A_4}{(s-r-i+j+3)} \right]$$

where $m, n < 2$ and

$$A_1 = 1 + 2\alpha + \alpha^2 - (1 + \alpha)\alpha \left[2^{m/2} + 2^{n/2} \right] + \alpha^2 2^{m+n/2}$$

$$A_2 = 2\alpha(1 + \alpha) - 2\alpha(1 + \alpha)2^{1+m/2} - \alpha^2 2^{1+n/2} \left[1 - 2^{m/2} \right]$$

$$A_3 = 2\alpha(1 + \alpha) - 2\alpha \left[2^{n/2} - \alpha 2^{m/2} \right] - 2\alpha^2 \left[2^{n/2} - 2^{m/2} \right]$$

$$A_4 = 2\alpha^2 \left[1 - 2^{m/2} - 2^{n/2} + 2^{\frac{m+n}{2}} \right]$$

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