Odds Generalized Exponential-Pareto Distribution: Properties and Application

Sudhansu S. Maiti
Department of Statistics, Visva-Bharati University, Santiniketan, India
dssm1@rediffmail.com

Sukanta Pramanik
Siliguri College, North Bengal University, India
skantapramanik@gmai.com

Abstract

A new distribution, called Odds Generalized Exponential-Pareto distribution (OGEPD) is proposed for modeling lifetime data. A comprehensive account of the mathematical properties of the new distribution including estimation and simulation issues is presented. A data set has been analyzed to illustrate its applicability.

Keywords: Exponential distribution; Maximum likelihood estimation; Odds function; T-X family of distributions.

1. Introduction

Statistical distributions are very useful in describing the real world phenomena. Though a modest number of distributions have been developed, there are always scope for developing distributions, studying their properties which are either more flexible or for fitting real world scenarios. There are always urge among the researchers for developing new and more flexible distributions. As a result, many new distributions have come up and studied.

There are several ways of adding one or more parameters to a distribution function. Such an addition of parameters makes the resulting distribution richer and more flexible for modeling data. Proportional hazard model (PHM), Proportional reversed hazard model (PRHM), Proportional odds model (POM), Power transformed model (PTM) are few such models originated from this idea to add a shape parameter. In these models, a few pioneering works are by Box and Cox (1964), Cox (1972), Mudholkar and Srivastava (1993), Shaked and Shantikumar (1994), Marshall and Olkin (1997), Gupta and Kundu (1999), Gupta and Gupta (2007) among others.

Many distributions have been developed in recent years that involves the logit of the beta distribution. Under this generalized class of beta distribution scheme, the cumulative distribution function (cdf) for this class of distributions for the random variable X is generated by applying the inverse of the cdf of X to a beta distributed random variable to obtain,

$$F(x) = \frac{1}{B(\alpha,\beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1}\,dt; \alpha, \beta > 0,$$

Where G(x) is the cdf of any other distribution. This class has not only generalized the beta distribution but also added parameter(s) to it. Among this class of distributions are, the beta-Normal Eugene et el. (2002); beta-Gumbel Nadarajah and Kotz (2004); beta-Exponential Nadarajah and Kotz (2006); beta-Weibull Famoye et al. (2005); beta-Rayleigh Akinsete and Lowe (2009); beta-Laplace Kozubowski and Nadarajah (2008); and beta-Pareto Akinsete et al. (2008), among a few others. Many useful statistical properties arising from these distributions and their applications to real life data have been discussed in the literature.

In the generalized class of beta distribution, since the beta random variable lies between 0 and 1, and the distribution function also lies between 0 and 1, to find out cdf of generalized distribution, the upper limit is replaced by cdf of the generalized distribution. Alzaatreh et al. (2013) has proposed a new generalized family of distributions, called T-X family, and the cumulative distribution function (cdf) is defined as

$$F(x;\lambda,\theta)=\int_a^{W(F_\theta(x))} f_\lambda(t)dt, \quad (1.1)$$

where, the random variable $T\in[a,b]$, for $-\infty<a,b<\infty$ and $W(F_\theta(x))$ be a function of the cdf $F_\theta(x)$ so that $W(F_\theta(x))$ satisfies the following conditions:

(i) $W(F_\theta(x))\in[a,b]$,

(ii) $W(F_\theta(x))$ is differentiable and monotonically non-decreasing,

(iii) $W(F_\theta(x))\to a$ as $x\to-\infty$ and $W(F_\theta(x))\to b$ as $x\to\infty$.

We have defined a generalized class of any distribution having positive support. Taking $W(F_\theta(x))=\dfrac{F_\theta(x)}{1-F_\theta(x)}$, the odds function, the cdf of the proposed generalized class of distribution is given by

$$F(x;\lambda,\theta)=\int_0^{\frac{F_\theta(x)}{1-F_\theta(x)}} f_\lambda(t)dt. \tag{1.2}$$

The support of the resulting distribution will be that of $F_\theta(.)$. Here, $\dfrac{F_\theta(x)}{1-F_\theta(x)}=\dfrac{F_\theta(x)}{\overline{F}_\theta(x)}=\infty$ as $x\to\infty$ (assuming $\dfrac{1}{0}=\infty$). The resulting distribution is not only generalized but also added with some parameter(s) to the base distribution. We call this class of distributions as Odds Generalized family of distributions (OGFD).

Throughout this paper we use the following notations. We write upper incomplete gamma function and lower incomplete gamma function as $\Gamma(p,x)=\int_x^\infty w^{p-1}e^{-w}dw$ and $\gamma(p,x)=\int_0^x w^{p-1}e^{-w}dw$, for $x\ge 0, p>0$ respectively. The j-th derivative with respect to

p is denoted by $\Gamma^{(j)}(p,x)=\int_x^{\infty}(\ln w)^j w^{p-1}e^{-w}dw$ and $\gamma^{(j)}(p,x)=\int_0^{x}(\ln w)^j w^{p-1}e^{-w}dw$, for $x \geq 0, p > 0$ respectively.

In the present paper, we choose particular choice of $F_{\lambda}(x)=1-e^{-\lambda x}$ i.e. the Exponential distribution and $F_{\theta;a}(x)=1-\left(\frac{a}{x}\right)^{\theta}$ i.e. the Pareto distribution in (1.2). Hence, we call this distribution as Odds Generalized Exponential-Pareto distribution (OGEPD).

The paper is organized as follows. The new distribution is developed in section 2. A comprehensive account of mathematical properties including structural and reliability of the new distribution is provided in section 3. Maximum likelihood method of estimation of parameters of the distribution is discussed in section 4. A real life data set has been analyzed and compared with other fitted distributions with respect to Akaike Information Criterion (AIC) in section 5. Section 6 concludes.

2. The Probability Density Function of the OGEPD

The c.d.f. of the OGEPD is given by the form as

$$F_{e.p}(x)=\int_0^{\frac{F(x)}{1-F(x)}} f_{\lambda}(x)dx$$

where $F(x)=1-\left(\frac{a}{x}\right)^{\theta}$ and $f_{\lambda}(x)=\lambda e^{-\lambda x}$, so that

$$F_{e.p}(X;\lambda,\theta,a)=\int_0^{\left(\frac{x}{a}\right)^{\theta}-1}\lambda e^{-\lambda x}dx$$

$$=1-e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)} \tag{2.3}$$

Also the p.d.f. of the OGEPD is given by

$$f_{e.p}(x;\lambda,\theta,a)=\frac{\lambda\theta e^{\lambda}}{a^{\theta}}x^{\theta-1}e^{-\lambda\left(\frac{x}{a}\right)^{\theta}} \tag{2.4}$$

with range$(0,\infty)$. Figures 1- 3 show the pdfs for different λ, θ and a.

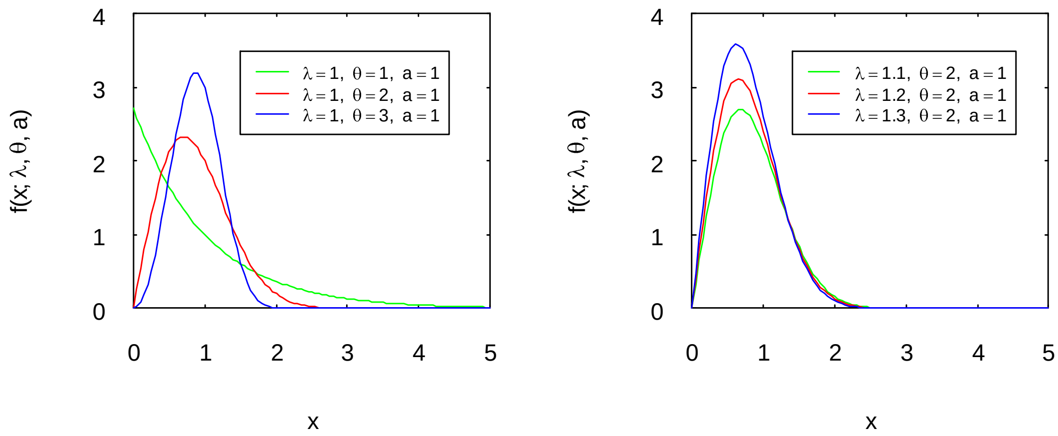

Figure 1: The probability density function of the OGEPD with $\theta > 1$

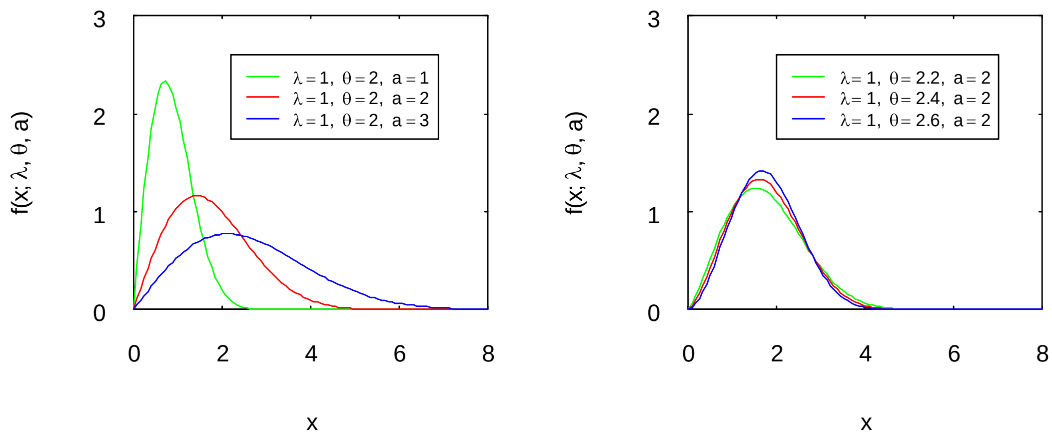

Figure 2: The probability density function of the OGEPD with $\theta > 1$

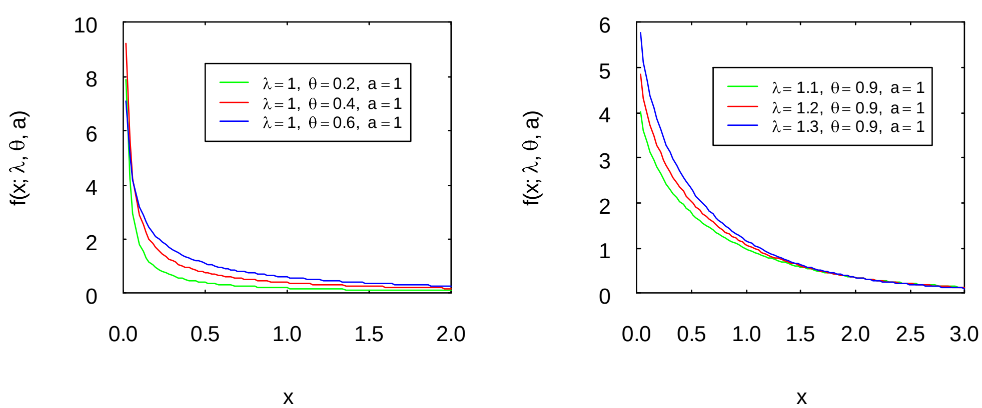

Figure 3: The probability density function of the OGEPD with $\theta < 1$

3. Statistical and Reliability Properties

3.1 Limit of the Probability Distribution Function

Since the c.d.f. of this distribution is $F_{e.p}(X;\lambda,\theta,a) = 1 - e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}$ so

$$\lim_{x\to a} F_{e.p}(X;\lambda,\theta,a) = \lim_{x\to a}(1-e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}) = 0$$

i.e. $F_{e.p}(a) = 0$

Now $\lim_{x\to\infty} F_{e.p}(X;\lambda,\theta,a) = \lim_{x\to\infty}(1-e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}) = 1$

i.e. $F_{e.p}(\infty) = 1$

3.2 Descriptive Statistics of the OGEPD

The mean of this OGEPD is as follows:

$$\mu_1' = E(X) = \frac{\lambda\theta e^{\lambda}}{a^{\theta}}\int_a^{\infty} x^{\theta} e^{-\lambda\left(\frac{x}{a}\right)^{\theta}} dx$$

Put $u = x^{\theta}$, we get

$$E(X) = \frac{\lambda e^{\lambda}}{a^{\theta}}\int_{a^{\theta}}^{\infty} u^{\frac{1}{\theta}} e^{-\frac{\lambda}{a^{\theta}}u} du$$

$$= \frac{\lambda e^{\lambda}}{a^{\theta}} \frac{\Gamma(\frac{1}{\theta}+1,\lambda)}{(\frac{\lambda}{a^{\theta}})^{\frac{1}{\theta}+1}}$$

$$= \frac{e^{\lambda} a}{\lambda^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,\lambda)$$

So the mean of the OGEPD is $\frac{e^{\lambda} a}{\lambda^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,\lambda)$.

The median of the OGEPD is given by

$$0.5 = \int_a^M f_{e.p}(x;\lambda,\theta,a)dx$$

$$= \int_a^M \frac{\lambda\theta e^{\lambda}}{a^{\theta}} x^{\theta-1} e^{-\frac{\lambda}{a^{\theta}}x^{\theta}}$$

Put $u = x^{\theta} \Rightarrow du = \theta x^{\theta-1}dx,$ with $x = \theta \Rightarrow u = a^{\theta}$ and $x = M \Rightarrow u = M^{\theta}$. So

$$0.5 = \frac{\lambda e^{\lambda}}{a^{\theta}}\int_{a^{\theta}}^{M^{\theta}} e^{-\frac{\lambda}{a^{\theta}}u} du = 1 - e^{-\lambda\left(\left(\frac{M}{a}\right)^{\theta}-1\right)}$$

gives $M = a(1+\frac{\ln 2}{\lambda})^{\frac{1}{\theta}}$.

Hence the median of the OGEPD is $a(1+\frac{\ln 2}{\lambda})^{\frac{1}{\theta}}$.

The mode of the OGEPD is given as:

mode= arg max(f(x))

So that $\ln f_{e.p}(x) = \ln\lambda + \ln\theta - \theta\ln a + (\theta-1)\ln x + \lambda - \lambda\left(\frac{x}{a}\right)^{\theta}$

Now $\frac{d}{dx}\ln f_{e.p}(x) = \frac{\theta-1}{x} - \frac{\lambda\theta}{a^{\theta}}x^{\theta-1} = 0$

i.e. $x = a\left(\frac{\theta-1}{\lambda\theta}\right)^{\frac{1}{\theta}}$,

So the mode of the OGEPD is $a\left(\frac{\theta-1}{\lambda\theta}\right)^{\frac{1}{\theta}}$.

The r^th order raw moment of the OGEPD is as follows:

$$E(X^r) = \frac{\lambda\theta e^{\lambda}}{a^{\theta}}\int_a^{\infty} x^{r+\theta-1} e^{-\lambda(\frac{x}{a})^{\theta}}dx$$

Put $u = x^{\theta}$, we get

$$E(X^r) = \frac{\lambda e^{\lambda}}{a^{\theta}}\int_{a^{\theta}}^{\infty} u^{\frac{r}{\theta}} e^{-\frac{\lambda}{a^{\theta}}u}du$$

$$= \frac{\lambda e^{\lambda}}{a^{\theta}}\frac{\Gamma(\frac{r}{\theta}+1,\lambda)}{(\frac{\lambda}{a^{\theta}})^{\frac{r}{\theta}+1}}$$

$$= \frac{e^{\lambda}a^{r}}{\lambda^{\frac{r}{\theta}}}\Gamma\left(\frac{r}{\theta}+1,\lambda\right). \tag{3.5}$$

Now **variance** of the OGEPD is

$$\mu_2 = \mu_2^{'} - \mu_1^{'2} = \frac{a^2}{\lambda^{\frac{2}{\theta}}}\left[e^{\lambda}\Gamma(\frac{2}{\theta}+1,\lambda) - e^{2\lambda}\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^2\right]$$

Now the **3rd** and **4th central moments** of the OGEPD are

$$\mu_3 = \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}$$

$$= \frac{a^3}{\lambda^{\frac{3}{\theta}}}\left[e^{\lambda}\Gamma(\frac{3}{\theta}+1,\lambda) - 3e^{2\lambda}\Gamma(\frac{2}{\theta}+1,\lambda)\Gamma(\frac{1}{\theta}+1,\lambda) + 2e^{3\lambda}\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^3 \right]$$

$$\mu_4 = \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3\mu_1^{'4}$$

$$= \frac{a^4}{\lambda^{\frac{4}{\theta}}}\left[e^{\lambda}\Gamma(\frac{4}{\theta}+1,\lambda) - 4e^{2\lambda}\Gamma(\frac{3}{\theta}+1,\lambda)\Gamma(\frac{1}{\theta}+1,\lambda) + 6e^{3\lambda}\Gamma(\frac{2}{\theta}+1,\lambda)\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^2 - 3e^{4\lambda}\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^4 \right]$$

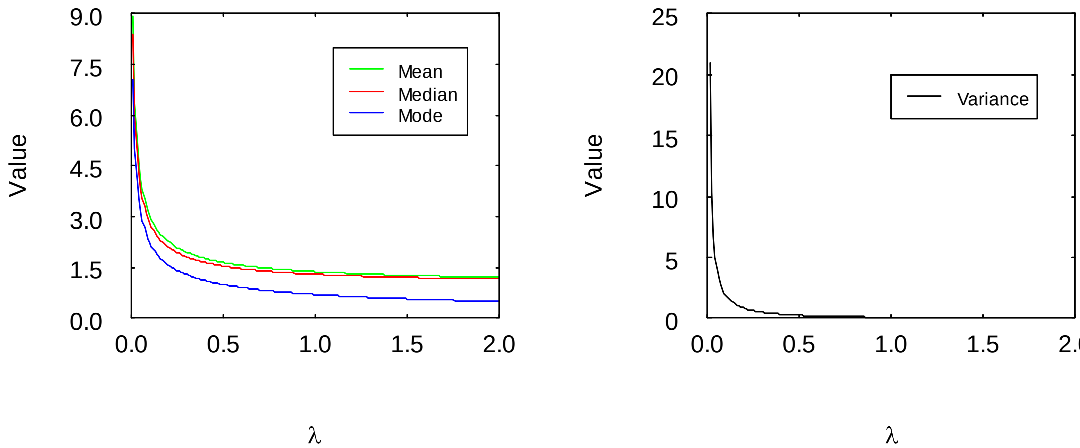

Figure 4: Mean, median, mode and variance of the OGEPD

The **Skewness (** g_1 **)** of the OGEPD is given by

$$g_1 = \frac{\mu_3^2}{\mu_2^{3}} = \frac{\left[e^{\lambda}\Gamma(\frac{3}{\theta}+1,\lambda) - 3e^{2\lambda}\Gamma(\frac{2}{\theta}+1,\lambda)\Gamma(\frac{1}{\theta}+1,\lambda) + 2e^{3\lambda}\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^3 \right]^2}{\left[e^{\lambda}\Gamma(\frac{2}{\theta}+1,\lambda) - e^{2\lambda}\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^2 \right]^3}$$

The **Kurtosis (** g_2 **)** of the OGEPD is given by

$$g_2 = \frac{\mu_4}{\mu_2^{2}} = \frac{e^{\lambda}\Gamma(\frac{4}{\theta}+1,\lambda) - 4e^{2\lambda}\Gamma(\frac{3}{\theta}+1,\lambda)\Gamma(\frac{1}{\theta}+1,\lambda) + 6e^{3\lambda}\Gamma(\frac{2}{\theta}+1,\lambda)\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^2 - 3e^{4\lambda}\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^4}{\left[e^{\lambda}\Gamma(\frac{2}{\theta}+1,\lambda) - e^{2\lambda}\left(\Gamma(\frac{1}{\theta}+1,\lambda)\right)^2 \right]^2}$$

Moment Generating Function(MGF):

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots\ldots + \frac{t^rX^r}{r!} + \ldots..\right] \\
&= \sum_{r=0}^{\infty} \frac{t^r}{r!} a_r^{'} \\
&= \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{e^{\lambda} a^r}{\lambda^{\frac{r}{\theta}}} \Gamma\left(\frac{r}{\theta} + 1, \lambda\right) \qquad (3.6)
\end{aligned}
$$

Characteristic Function(CF):

$$
\begin{aligned}
\Psi_X(t) &= E(e^{itX}) \\
&= E\left[1 + itX + \frac{i^2t^2X^2}{2!} + \frac{i^3t^3X^3}{3!} + \ldots\ldots + \frac{i^rt^rX^r}{r!} + \ldots..\right] \\
&= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} a_r^{'} \\
&= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{e^{\lambda} a^r}{\lambda^{\frac{r}{\theta}}} \Gamma\left(\frac{r}{\theta} + 1, \lambda\right) \qquad (3.7)
\end{aligned}
$$

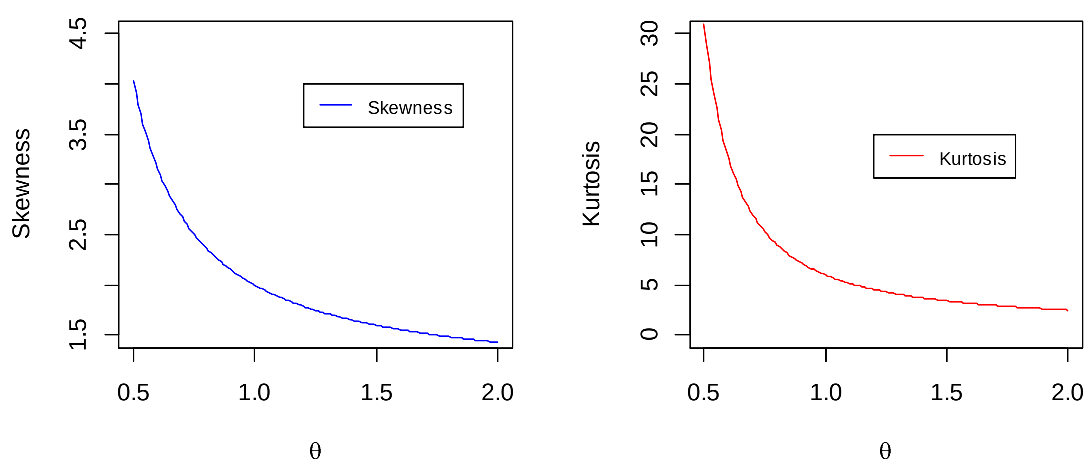

Figure 5: Skewness and Kurtosis of the OGEPD with different values of θ

Cumulant Generating Function(CGF):

$$K_X(t) = \ln_e(M_X(t))$$

$$= \ln_e\left[\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{e^{\lambda}a^r}{\lambda^{\frac{r}{\theta}}}\Gamma\left(\frac{r}{\theta}+1,\lambda\right)\right] \tag{3.8}$$

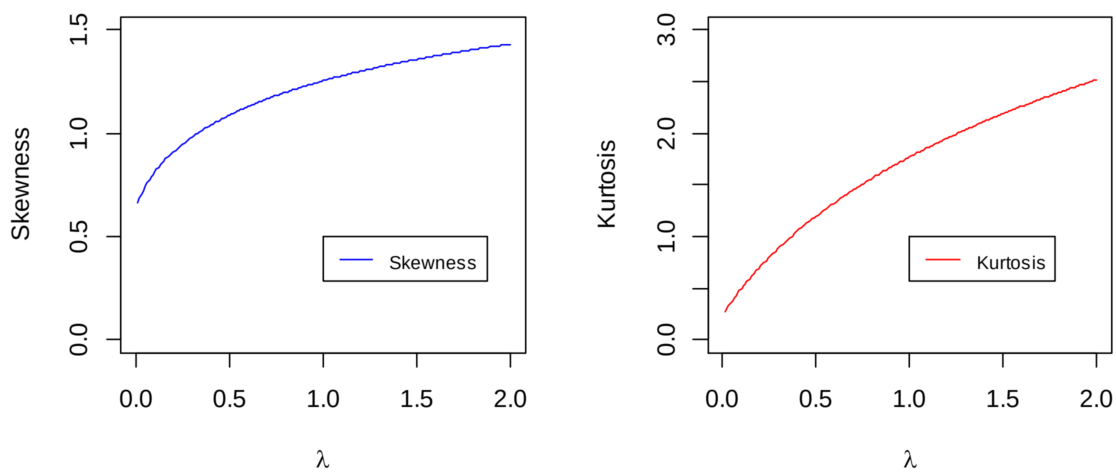

Figure 6: Skewness and Kurtosis of the OGEPD with different values of λ

Mean Deviation:

The mean deviation about the mean and the mean deviation about the median is defined by

$$MD_\mu = \int_a^\infty |x-\mu| f(x)dx$$

and

$$MD_M = \int_a^\infty |x-M| f(x)dx$$

respectively, where $\mu = E(X)$ and $M = Median(X)$ denotes the mean and median rspectively. The measures MD_μ and MD_M can be calculated using the relationships

$$\begin{aligned} MD_\mu &= \int_a^\infty |x-\mu| f(x)dx \\ &= \int_a^\mu (\mu-x) f(x)dx + \int_\mu^\infty (x-\mu) f(x)dx \\ &= \mu F(\mu) - \int_a^\mu xf(x)dx - \mu\{1-F(\mu)\} + \int_\mu^\infty xf(x)dx \\ &= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty xf(x)dx \end{aligned}$$

and

$$\begin{aligned} MD_M &= \int_a^\infty |x-M| f(x)dx \\ &= \int_a^M (M-x) f(x)dx + \int_M^\infty (x-M) f(x)dx \\ &= MF(M) - \int_a^M xf(x)dx - M\{1-F(M)\} + \int_M^\infty xf(x)dx \\ &= 2MF(M) - M - \int_a^M xf(x)dx + \int_M^\infty xf(x)dx \\ &= -\mu + 2\int_M^\infty xf(x)dx \end{aligned}$$

Now

$$\int_\mu^\infty xf(x)dx = \frac{\lambda\theta e^\lambda}{a^\theta}\int_\mu^\infty x^\theta e^{-\lambda\left(\frac{x}{a}\right)^\theta} dx$$

Put $u = x^\theta$, we get

$$\begin{aligned} \int_\mu^\infty xf(x)dx &= \frac{\lambda e^\lambda}{a^\theta}\int_{\mu^\theta}^\infty u^{\frac{1}{\theta}} e^{-\frac{\lambda}{a^\theta}u} du \\ &= \frac{\lambda e^\lambda}{a^\theta}\frac{\Gamma(\frac{1}{\theta}+1, \frac{\lambda\mu^\theta}{a^\theta})}{(\frac{\lambda}{a^\theta})^{\frac{1}{\theta}+1}} \\ &= \frac{e^\lambda a}{\lambda^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1, \frac{\lambda\mu^\theta}{a^\theta}) \end{aligned}$$

Thus

$$\begin{aligned} MD_\mu &= 2\mu\left[1-e^{-\lambda((\frac{\mu}{a})^\theta-1)}\right] - 2\mu + 2\frac{e^\lambda a}{\lambda^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1, \frac{\lambda\mu^\theta}{a^\theta}) \\ &= 2\left[\frac{e^\lambda a}{\lambda^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1, \frac{\lambda\mu^\theta}{a^\theta}) - \mu e^{-\lambda\left(\left(\frac{\mu}{a}\right)^\theta-1\right)}\right] \end{aligned} \tag{3.9}$$

and

$$MD_M = -\mu + 2\frac{e^\lambda a}{\lambda^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1, \frac{\lambda M^\theta}{a^\theta}) \tag{3.10}$$

Conditional Moments:

The residual life and the reversed residual life play an important role in reliability theory and other branches of statistics. Here, the r-th order raw moment of the residual life is given by

$$\mu_r(t) = E[(X-t)^r \mid X > t] = \frac{1}{\overline{F}(t)}\int_t^{\infty}(x-t)^r f(x)dx$$

$$= \frac{\lambda\theta}{a^{\theta}e^{-\frac{\lambda}{a^{\theta}}t^{\theta}}}\int_t^{\infty}(x-t)^r x^{\theta-1}e^{-\frac{\lambda}{a^{\theta}}x^{\theta}}dx$$

Put $u = x^{\theta} \Rightarrow du = \theta x^{\theta-1}dx,$ with $x = t \Rightarrow u = t^{\theta}$ and $x = \infty \Rightarrow u = \infty$

So,

$$\mu_r(t) = \frac{\lambda}{a^{\theta}e^{-\frac{\lambda}{a^{\theta}}t^{\theta}}}\int_{t^{\theta}}^{\infty}(u^{\frac{1}{\theta}}-t)^r e^{-\frac{\lambda}{a^{\theta}}u}du$$

$$= \frac{\lambda}{a^{\theta}e^{-\frac{\lambda}{a^{\theta}}t^{\theta}}}\int_{t^{\theta}}^{\infty}\left[\sum_{j=0}^{r}(-1)^j\binom{r}{j}u^{\frac{j}{\theta}}t^{r-j}\right]e^{-\frac{\lambda}{a^{\theta}}u}du$$

$$= \frac{\lambda}{a^{\theta}e^{-\frac{\lambda}{a^{\theta}}t^{\theta}}}\sum_{j=0}^{r}(-1)^j\binom{r}{j}t^{r-j}\int_{t^{\theta}}^{\infty}u^{\frac{j}{\theta}}e^{-\frac{\lambda}{a^{\theta}}u}du$$

$$= \frac{1}{e^{-\frac{\lambda}{a^{\theta}}t^{\theta}}}\sum_{j=0}^{r}(-1)^j\binom{r}{j}t^{r-j}\frac{a^j}{\lambda^{\frac{j}{\theta}}}\Gamma\left(\frac{j}{\theta}+1,\frac{\lambda t^{\theta}}{a^{\theta}}\right)$$

The r^{th} order raw moment of the reversed residual life is given by

$$m_r(t) = E[(t-X)^r \mid X < t] = \frac{1}{F(t)}\int_a^{t}(t-x)^r f(x)dx$$

Now,

$$\int_a^{t}(t-x)^r f(x)dx = \frac{\lambda\theta e^{\lambda}}{a^{\theta}}\int_a^{t}(t-x)^r x^{\theta-1}e^{-\frac{\lambda}{a^{\theta}}x^{\theta}}dx$$

Put $u = x^{\theta} \Rightarrow du = \theta x^{\theta-1}dx,$ with $x = a \Rightarrow u = a^{\theta}$ and $x = t \Rightarrow u = t^{\theta}$

$$\int_a^{t}(t-x)^r f(x)dx = \frac{\lambda e^{\lambda}}{a^{\theta}}\int_{a^{\theta}}^{t^{\theta}}(t-u^{\frac{1}{\theta}})^r e^{-\frac{\lambda}{a^{\theta}}u}du$$

$$= \frac{\lambda e^{\lambda}}{a^{\theta}}\int_{a^{\theta}}^{t^{\theta}}\left[\sum_{j=0}^{r}(-1)^j\binom{r}{j}u^{\frac{j}{\theta}}t^{r-j}\right]e^{-\frac{\lambda}{a^{\theta}}u}du$$

$$= \frac{\lambda e^{\lambda}}{a^{\theta}}\sum_{j=0}^{r}(-1)^j\binom{r}{j}t^{r-j}\int_{a^{\theta}}^{t^{\theta}}u^{\frac{j}{\theta}}e^{-\frac{\lambda}{a^{\theta}}u}du$$

Thus,

$$m_r(t) = \frac{\lambda e^{\lambda}}{a^{\theta}[1-e^{-\lambda((\frac{t}{a})^{\theta}-1)}]}\sum_{j=0}^{r}(-1)^j \binom{r}{j} t^{r-j}\int_{a^{\theta}}^{t^{\theta}} u^{\frac{j}{\theta}} e^{-\frac{\lambda}{a^{\theta}}u}\,du$$

$$= \frac{e^{\lambda}}{1-e^{-\lambda((\frac{t}{a})^{\theta}-1)}}\sum_{j=0}^{r}(-1)^j \binom{r}{j} t^{r-j}\frac{a^j}{\lambda^{\frac{j}{\theta}}}\left[\Gamma(\frac{j}{\theta}+1,\lambda)-\Gamma(\frac{j}{\theta}+1,\frac{\lambda t^{\theta}}{a^{\theta}})\right]$$

L- Moments:

Define $X_{k:n}$ be the k^{th} smallest moment in a sample of size n. The L-moments of X are defined by

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left[X_{r-k:r}\right], \quad r = 1,2,..$$

Now for OGEPD with parameter λ, θ and a, we have

$$E\left[X_{j:r}\right] = \frac{r!}{(j-1)!(r-j)!}\int_a^{\infty} x[F(x)]^{j-1}[1-F(x)]^{r-j}\,dF(x)$$

$$= \frac{r!}{(j-1)!(r-j)!}\frac{\lambda\theta e^{\lambda}}{a^{\theta}}\int_a^{\infty} x^{\theta}[1-e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}]^{j-1}[e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}]^{r-j}e^{-\lambda\left(\frac{x}{a}\right)^{\theta}}\,dx$$

So the first four L- Moments are,

$$\lambda_1 = E\left[X_{1:1}\right] = \frac{ae^{\lambda}}{\lambda^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,\lambda)$$

$$\lambda_2 = \frac{1}{2}E\left[X_{2:2}-X_{1:2}\right] = \frac{ae^{\lambda}}{\lambda^{\frac{1}{\theta}}}\left[\Gamma(\frac{1}{\theta}+1,\lambda)-\frac{e^{\lambda}}{2^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,2\lambda)\right]$$

$$\lambda_3 = \frac{1}{3}E\left[X_{3:3}-2X_{2:3}+X_{1:3}\right]$$

$$= \frac{ae^{\lambda}}{\lambda^{\frac{1}{\theta}}}\left[\Gamma(\frac{1}{\theta}+1,\lambda)-3\frac{e^{\lambda}}{2^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,2\lambda)+2\frac{e^{2\lambda}}{3^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,3\lambda)\right]$$

$$\lambda_4 = \frac{1}{4}E\left[X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4}\right]$$

$$= \frac{ae^{\lambda}}{\lambda^{\frac{1}{\theta}}}\left[\Gamma(\frac{1}{\theta}+1,\lambda)-6\frac{e^{\lambda}}{2^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,2\lambda)+10\frac{e^{2\lambda}}{3^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,3\lambda)-5\frac{e^{3\lambda}}{4^{\frac{1}{\theta}}}\Gamma(\frac{1}{\theta}+1,4\lambda)\right]$$

Quantile function:

Let X denote a random variable with the probability density function 4. The quantile function, say $Q(p)$, defined by $F(Q(p)) = p$ is the root of the equation

$$1-e^{-\frac{\lambda}{a^{\theta}}(Q(p)^{\theta}-a^{\theta})} = p$$

So,

$$Q(p) = a(1-\frac{\ln(1-p)}{\lambda})^{\frac{1}{\theta}} \tag{3.11}$$

3.3 Bonferroni curve, Lorenz curve and Ginis index

The Bonferroni and Lorenz curves are defined by

$$B(p) = \frac{1}{p\mu}\int_a^q xf(x)dx \tag{3.12}$$

and

$$L(p) = \frac{1}{\mu}\int_a^q xf(x)dx \tag{3.13}$$

respectively, or equivalently by

$$B(p) = \frac{1}{p\mu}\int_a^p F^{-1}(x)dx \tag{3.14}$$

and

$$L(p) = \frac{1}{\mu}\int_a^p F^{-1}(x)dx \tag{3.15}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. The Bonferroni and Gini indices are defined by

$$B = 1-\int_0^1 B(p)dp \tag{3.16}$$

and

$$G = 1-2\int_0^1 L(p)dp \tag{3.17}$$

By using Eq. 3.11, we calculate Eq. 3.14 and 3.15 as

$$\int_a^p F^{-1}(x)dx = a\int_a^p (1-\frac{\ln(1-x)}{\lambda})^{\frac{1}{\theta}} dx$$

Now, put $u = (1-\frac{\ln(1-x)}{\lambda})$, we get

$$\int_a^p F^{-1}(x)dx = \frac{ae^{\lambda}}{\lambda^{\frac{1}{\theta}}}\left[\gamma(\frac{1}{\theta}+1, \lambda-\ln(1-p)) - \gamma(\frac{1}{\theta}+1, \lambda-\ln(1-a))\right]$$

So,

$$B(p) = \frac{ae^{\lambda}}{p\mu\lambda^{\frac{1}{\theta}}}\left[\gamma(\frac{1}{\theta}+1, \lambda-\ln(1-p)) - \gamma(\frac{1}{\theta}+1, \lambda-\ln(1-a))\right] \tag{3.18}$$

and

$$L(p)=\frac{ae^{\lambda}}{\mu\lambda^{\frac{1}{\theta}}}\left[\gamma(\frac{1}{\theta}+1,\lambda-\ln(1-p))-\gamma(\frac{1}{\theta}+1,\lambda-\ln(1-a))\right] \tag{3.19}$$

Integrating Eqs. 3.18 and 3.19 with respect to p, we can calculate the Bonferroni and Gini indices given by Eqs. 3.16 and 3.17, respectively, as

$$B=1-\frac{ae^{\lambda}}{p\mu\lambda^{\frac{1}{\theta}}}\left[\int_0^1\gamma(\frac{1}{\theta}+1,\lambda-\ln(1-p))dp-\gamma(\frac{1}{\theta}+1,\lambda-\ln(1-a))\right] \tag{3.20}$$

and

$$G=1-\frac{2ae^{\lambda}}{\mu\lambda^{\frac{1}{\theta}}}\left[\int_0^1\gamma(\frac{1}{\theta}+1,\lambda-\ln(1-p))dp-\gamma(\frac{1}{\theta}+1,\lambda-\ln(1-a))\right] \tag{3.21}$$

3.4 Order Statistics

Suppose $X_1, X_2, X_3, ..., X_n$ is a random sample from Eq.2.4. Let $X_{(1)}, X_{(2)}, X_{(3)}, ..., X_{(n)}$, denote the corresponding order statistics. It is well known that the probability density function and the cumulative distribution function of the k^{th} order statistic, say $Y = X_{(k)}$, are given by

$$\begin{aligned} f_Y(y) &= \frac{n!}{(k-1)!(n-k)!}F^{k-1}(y)[1-F(y)]^{n-k} f(y) \\ &= \frac{n!}{(k-1)!(n-k)!}\left[1-e^{-\lambda\left(\left(\frac{y}{a}\right)^{\theta}-1\right)}\right]^{k-1}\left[e^{-\lambda\left(\left(\frac{y}{a}\right)^{\theta}-1\right)}\right]^{n-k}\frac{\lambda\theta}{a^{\theta}}y^{\theta-1}e^{-\lambda\left(\frac{y}{a}\right)^{\theta}} \\ &= \frac{n!}{(k-1)!(n-k)!}\frac{\lambda\theta e^{\lambda}}{a^{\theta}}y^{\theta-1}\left[1-e^{-\lambda\left(\left(\frac{y}{a}\right)^{\theta}-1\right)}\right]^{k-1}\left[e^{-\lambda\left(\left(\frac{y}{a}\right)^{\theta}-1\right)}\right]^{n-k} e^{-\lambda\left(\frac{y}{a}\right)^{\theta}} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} F_Y(y) &= \sum_{j=k}^{n}\binom{n}{j}F^{j}(y)[1-F(y)]^{n-j} \\ &= \sum_{j=k}^{n}\binom{n}{j}\left[1-e^{-\lambda\left(\left(\frac{y}{a}\right)^{\theta}-1\right)}\right]^{j} e^{-\lambda(n-j)\left(\left(\frac{y}{a}\right)^{\theta}-1\right)} \end{aligned} \tag{3.23}$$

3.5 Entropies

An entropy of a random variable X is a measure of variation of the uncertainty. A popular entropy measure is Renyi entropy (Renyi 1961). If X has the probability density function f(x), then Renyi entropy is defined by

$$H_R(\beta)=\frac{1}{1-\beta}\ln\left\{\int_a^{\infty}f^{\beta}(x)dx\right\} \tag{3.24}$$

where $\beta > 0$ and $\beta \neq 1$. Suppose X has the probability density function Eq.2.4. Then, one can calculate

$$\int_a^\infty f^\beta(x)dx = \frac{(\lambda\theta)^\beta e^{\lambda\beta}}{a^{\beta\theta}} \int_a^\infty x^{\beta(\theta-1)} e^{-\frac{\lambda\beta}{a^\theta}x^\theta} dx$$

Put $u = x^\theta \Rightarrow du = \theta x^{\theta-1}dx,$ with $x = a \Rightarrow u = a^\theta$ and $x = \infty \Rightarrow u = \infty$

So

$$\int_a^\infty f^\beta(x)dx = \frac{(\lambda\theta)^\beta e^{\lambda\beta}}{\theta a^{\beta\theta}} \int_{a^\theta}^\infty u^{\beta-\frac{\beta}{\theta}+\frac{1}{\theta}-1} e^{-\frac{\lambda\beta}{a^\theta}u} du$$

$$= \frac{(\lambda\theta)^\beta e^{\lambda\beta}}{\theta a^{\beta\theta}} \frac{\Gamma\left(\beta - \frac{\beta}{\theta} + \frac{1}{\theta}, \lambda\beta\right)}{\left(\frac{\lambda\beta}{a^\theta}\right)^{\beta-\frac{\beta}{\theta}+\frac{1}{\theta}}}$$

$$= \frac{\lambda^{\frac{\beta-1}{\theta}} \theta^{\beta-1} a^{1-\beta} e^{\lambda\beta}}{\beta^{(\beta-\frac{\beta}{\theta}+\frac{1}{\theta})}} \Gamma\left(\beta - \frac{\beta}{\theta} + \frac{1}{\theta}, \lambda\beta\right)$$

So **Renyi** entropy is

$$H_R(\beta) = \frac{1}{1-\beta} \ln\left\{ \frac{\lambda^{\frac{\beta-1}{\theta}} \theta^{\beta-1} a^{1-\beta} e^{\lambda\beta}}{\beta^{(\beta-\frac{\beta}{\theta}+\frac{1}{\theta})}} \Gamma\left(\beta - \frac{\beta}{\theta} + \frac{1}{\theta}, \lambda\beta\right) \right\}$$

$$= -\frac{\ln\lambda}{\theta} - \ln\theta + \frac{\lambda\beta}{1-\beta} + \ln a - \frac{(\beta - \frac{\beta}{\theta} + \frac{1}{\theta})}{1-\beta} \ln\beta + \frac{1}{1-\beta} \ln\Gamma\left(\beta - \frac{\beta}{\theta} + \frac{1}{\theta}, \lambda\beta\right) \quad (3.25)$$

Shannon measure of entropy is defined as

$$H(f) = E[-\ln f(x)] = -\int_a^\infty f(x) \ln f(x)dx$$

$$= -\ln\lambda\theta - \lambda + \theta\ln a - (\theta-1)\int_a^\infty \ln x f(x)dx + \frac{\lambda}{a^\theta}\int_a^\infty x^\theta f(x)dx$$

Now $\quad (\theta-1)\int_a^\infty \ln x f(x)dx = \frac{\lambda\theta(\theta-1)e^\lambda}{a^\theta} \int_a^\infty \ln x x^{\theta-1} e^{-\frac{\lambda}{a^\theta}x^\theta} dx$

Put $u = \frac{\lambda x^\theta}{a^\theta},\ (\theta-1)\int_a^\infty \ln x f(x)dx = (\theta-1)e^\lambda \int_\lambda^\infty \left[\frac{\ln u}{\theta} + \ln a - \frac{\ln\lambda}{\theta}\right].e^{-u}du$

$$= \frac{(\theta-1)e^\lambda}{\theta}\Gamma^{(1)}(1,\lambda) + (\theta-1)\ln a - \frac{(\theta-1)}{\theta}\ln\lambda \ = \frac{(\theta-1)}{\theta}\left[\ln\frac{a^\theta}{\lambda} + e^\lambda\Gamma^{(1)}(1,\lambda)\right]$$

Also $\quad \dfrac{\lambda}{a^{\theta}}\int_a^{\infty} x^{\theta} f(x)dx = e^{\lambda}\Gamma(2,\lambda)$

So $\quad H(f) = -\ln\lambda\theta - \lambda + \theta\ln a - \dfrac{(\theta-1)}{\theta}\left[\ln\dfrac{a^{\theta}}{\lambda} + e^{\lambda}\Gamma^{(1)}(1,\lambda)\right] + e^{\lambda}\Gamma(2,\lambda)$

$$H(f) = -\lambda - \ln\theta - \frac{\ln\lambda}{\theta} + \ln a - \frac{(\theta-1)e^{\lambda}}{\theta}\Gamma^{(1)}(1,\lambda) + e^{\lambda}\Gamma(2,\lambda) \tag{3.26}$$

3.6 Reliability and related properties

The Reliability function of the OGEPD is given by

$$R(x) = 1 - F(x) = e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta} - 1\right)} \tag{3.27}$$

and the Hazard rate of the OGEPD is given by

$$\begin{aligned} r(t) &= \frac{f(t)}{1-F(t)} \\ &= \frac{\frac{\lambda\theta}{a^{\theta}} t^{\theta-1} e^{-\lambda\left(\left(\frac{t}{a}\right)^{\theta}-1\right)}}{e^{-\lambda\left(\left(\frac{t}{a}\right)^{\theta}-1\right)}} \\ &= \frac{\lambda\theta}{a^{\theta}} t^{\theta-1} \end{aligned} \tag{3.28}$$

Now $f(x) = \dfrac{\lambda\theta}{a^{\theta}} x^{\theta-1} e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}$

i.e. $\ln f(x) = \ln\lambda + \ln\theta - \theta\ln a + (\theta-1)\ln x + \lambda - \lambda\left(\dfrac{x}{a}\right)^{\theta}$

So, $\dfrac{d}{dx}\ln f(x) = \dfrac{\theta-1}{x} - \dfrac{\lambda\theta x^{\theta-1}}{a^{\theta}}$

$$\frac{d^2}{dx^2}\ln f(x) = -\frac{\theta-1}{x^2} - \frac{\lambda\theta(\theta-1)x^{\theta-2}}{a^{\theta}} = -(\theta-1)\left(\frac{1}{x^2} + \frac{\lambda\theta x^{\theta-2}}{a^{\theta}}\right)$$

For $\lambda > 0$, $\theta > 1$, $a > 0$ and $x > 0$, $\dfrac{d^2}{dx^2}\ln f(x) < 0$.

So, the distribution is log-concave. Therefore, the distribution posses Increasing failure rate (IFR) and Decreasing Mean Residual Life (DMRL) property.

For $\lambda > 0$, $0 < \theta < 1$, $a > 0$ and $x > 0$, $\dfrac{d^2}{dx^2}\ln f(x) > 0$.

So, the distribution is log-convex. Therefore, the distribution posses Decreasing failure rate (DFR) and Increasing Mean Residual Life (IMRL) property.

Mean Residual Life (MRL) function is defined as

$$e_x(t) = \frac{1}{e^{-\frac{\lambda}{a^\theta}t^\theta}} \sum_{j=0}^{1} (-1)^j \binom{1}{j} t^{1-j} \frac{a^j}{\lambda^{\frac{j}{\theta}}} \Gamma\left(\frac{j}{\theta}+1, \frac{\lambda t^\theta}{a^\theta}\right)$$

$$= \frac{1}{e^{-\frac{\lambda}{a^\theta}t^\theta}} \left[t\Gamma\left(1, \frac{\lambda t^\theta}{a^\theta}\right) - \frac{a}{\lambda^{\frac{1}{\theta}}} \Gamma\left(\frac{1}{\theta}+1, \frac{\lambda t^\theta}{a^\theta}\right) \right]. \tag{3.29}$$

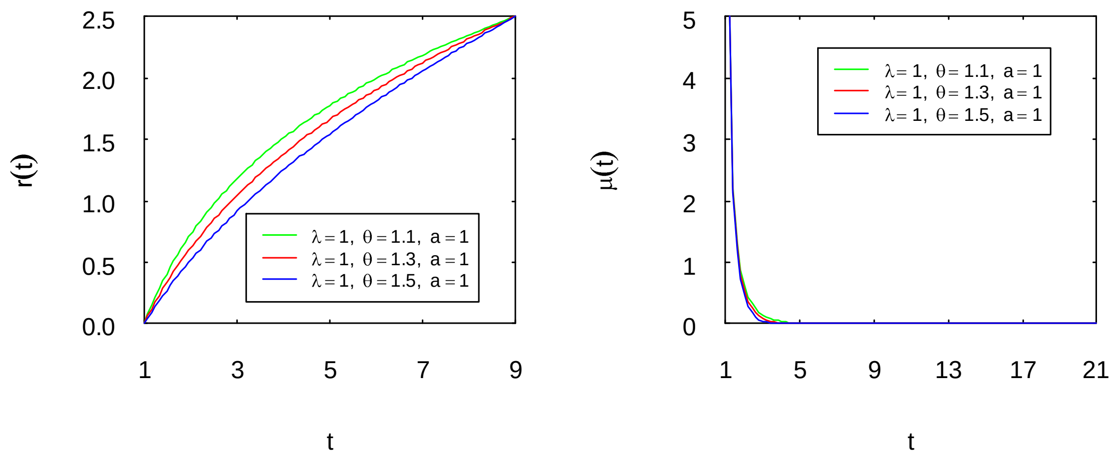

Figure 7: Hazard rate and Reversed Hazard Rate of the OGEPD with $\theta > 1$

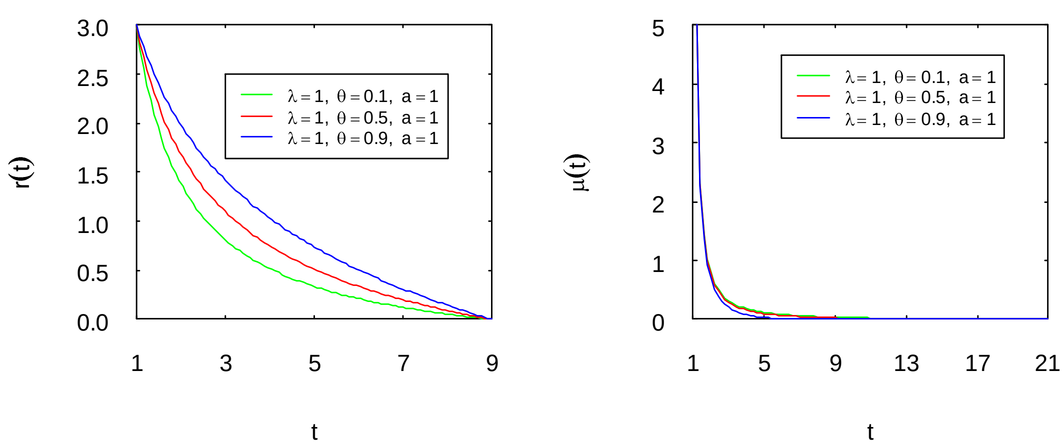

Figure 8: Hazard rate and Reversed Hazard Rate of the OGEPD with $\theta < 1$

Reversed Hazard rate:

$$\mu(x) = \frac{f(x)}{F(x)}$$

$$= \frac{\frac{\lambda\theta}{a^{\theta}} x^{\theta-1} e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}}{1-e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}}$$

$$= \frac{\lambda\theta x^{\theta-1} e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}}{a^{\theta}\left(1-e^{-\lambda\left(\left(\frac{x}{a}\right)^{\theta}-1\right)}\right)} \tag{3.30}$$

Expected Inactivity Time (EIT) or Mean Reversed Residual Life (MRRL) function is defined as

$$\bar{e}_x(t) = E(t - X \mid X < t)$$

$$= \frac{e^{\lambda}}{1-e^{-\lambda\left(\left(\frac{t}{a}\right)^{\theta}-1\right)}} \sum_{j=0}^{1} (-1)^j \binom{1}{j} t^{1-j} \frac{a^j}{\lambda^{\frac{j}{\theta}}} \left[\Gamma\left(\frac{j}{\theta}+1, \lambda\right) - \Gamma\left(\frac{j}{\theta}+1, \frac{\lambda t^{\theta}}{a^{\theta}}\right)\right]$$

$$= \frac{e^{\lambda}}{1-e^{-\lambda\left(\left(\frac{t}{a}\right)^{\theta}-1\right)}} \left[t\left\{\Gamma(1,\lambda) - \Gamma\left(1, \frac{\lambda t^{\theta}}{a^{\theta}}\right)\right\} - \frac{a}{\lambda^{\frac{1}{\theta}}} \left\{\Gamma\left(\frac{1}{\theta}+1, \lambda\right) - \Gamma\left(\frac{1}{\theta}+1, \frac{\lambda t^{\theta}}{a^{\theta}}\right)\right\} \right]. \tag{3.31}$$

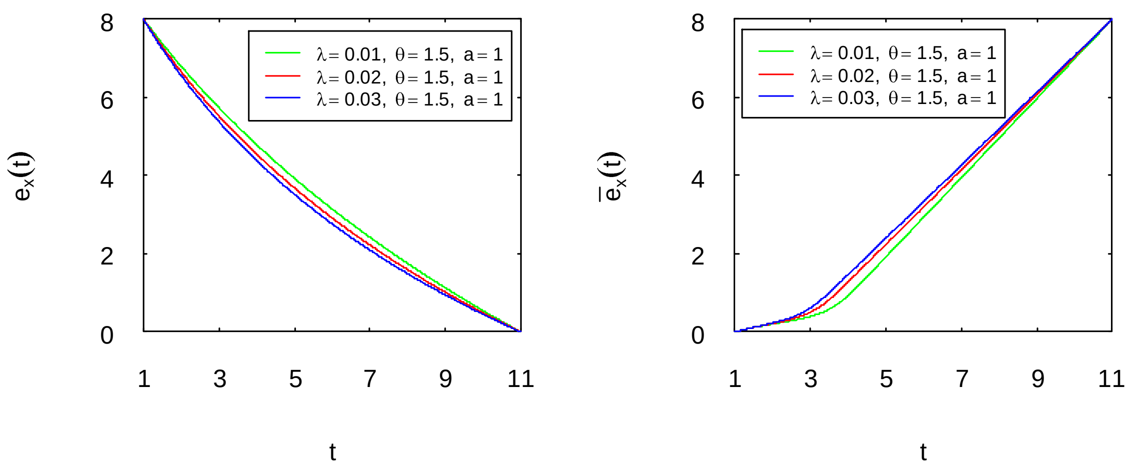

Figure 9: Mean Residual Life (MRL) and Expected Inactivity Time (EIT) of the OGEPD with $\theta > 1$

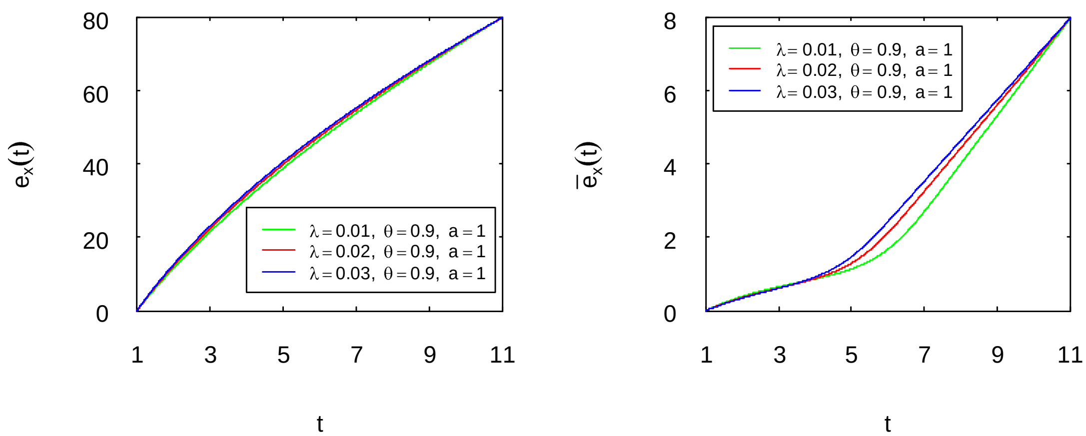

Figure 10: Mean Residual Life (MRL) and Expected Inactivity Time (EIT) of the OGEPD with $\theta < 1$

3.7 Stress-Strength Reliability

The Stress-Strength model describes the life of a component which has a random strength X that is subjected to a random stress Y. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X > Y$. So, Stress-Strength Reliability is $R = P(Y < X)$.

Let $X : OGEPD(\lambda_1, \theta_1, a_1)$ and $Y : OGEPD(\lambda_2, \theta_2, a_2)$ be independent random variables. Then Stress-Strength Reliability

$$R = P(Y < X)$$

$$= \int_{a_1}^{\infty} G_y(x) f(x) dx$$

$$= \int_{a_1}^{\infty} \left[1 - e^{-\lambda_2 \left(\left(\frac{x}{a_2} \right)^{\theta_2} - 1 \right)} \right] \frac{\lambda_1 \theta_1 e^{\lambda_1}}{a_1^{\theta_1}} x^{\theta_1 - 1} e^{-\lambda_1 \left(\frac{x}{a_1} \right)^{\theta_1}} dx$$

$$= 1 - \frac{\lambda_1 \theta_1 e^{\lambda_1 + \lambda_2}}{a_1^{\theta_1}} \int_{a_1}^{\infty} x^{\theta_1 - 1} e^{-\lambda_1 \left(\frac{x}{a_1} \right)^{\theta_1} - \lambda_2 \left(\frac{x}{a_2} \right)^{\theta_2}} dx$$

If $\theta_1 = \theta_2 = \theta$, then

$$R = 1 - \frac{\lambda_1 \theta e^{\lambda_1 + \lambda_2}}{a_1^{\theta}} \int_{a_1}^{\infty} x^{\theta - 1} e^{-\left(\frac{\lambda_1}{a_1^{\theta}} + \frac{\lambda_2}{a_2^{\theta}} \right) x^{\theta}} dx$$

$$= 1 - \frac{\lambda_1 e^{\lambda_1 + \lambda_2}}{\lambda_1 + \lambda_2 \left(\frac{a_1}{a_2} \right)^{\theta}} e^{-\left(\lambda_1 + \lambda_2 \left(\frac{a_1}{a_2} \right)^{\theta} \right)}$$

Also if $a_1 = a_2$, then

$$R = 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}$$
$$= \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

4. Maximum Likelihood Method of Estimation of the Parameters

Using the method of Maximum Likelihood Estimation (MLE), we estimate the parameters of the OGEPD.

Since

$$f_{e.p}(x;\lambda,\theta,a) = \frac{\lambda\theta e^{\lambda}}{a^{\theta}} x^{\theta-1} e^{-\lambda\left(\frac{x}{a}\right)^{\theta}}$$

The Likelihood function is given by

$$L(x;\lambda,\theta,a) = \prod_{i=1}^{n} f(x_i)$$
$$= \prod_{i=1}^{n} \frac{\lambda\theta e^{\lambda}}{a^{\theta}} x_i^{\theta-1} e^{-\frac{\lambda}{a^{\theta}} x_i^{\theta}}$$
$$= \frac{\lambda^n \theta^n e^{n\lambda}}{a^{n\theta}} \prod_{i=1}^{n} x_i^{\theta-1} e^{-\frac{\lambda}{a^{\theta}} \sum_{i=1}^{n} x_i^{\theta}} \qquad (4.32)$$

Then the logarithm of likelihood is

$$\ln L(x;\lambda,\theta,a) = n\ln\lambda + n\ln\theta + n\lambda - n\theta\ln a + (\theta-1)\sum_{i=1}^{n}\ln x_i - \frac{\lambda}{a^{\theta}}\sum_{i=1}^{n} x_i^{\theta}$$

Here the likelihood function will be maximized at $\hat{a} = x_{(1)}$, the smallest order statistic in the given sample of size n. The MLEs of λ, θ are the roots of

$$\frac{\partial \ln L(x;\lambda,\theta,\hat{a})}{\partial\lambda} = 0 \text{ and } \frac{\partial \ln L(x;\lambda,\theta,\hat{a})}{\partial\theta} = 0.$$

Now

$$\frac{\partial \ln L(x;\lambda,\theta,\hat{a})}{\partial\lambda} = \frac{n}{\lambda} + n - \sum_{i=1}^{n}\left(\frac{x_i}{\hat{a}}\right)^{\theta} = 0$$

$$i.e.\hat{\lambda} = \frac{n}{\sum_{i=1}^{n}\left(\frac{x_i}{\hat{a}}\right)^{\theta} - n} \qquad (4.33)$$

and

$$\frac{\partial \ln L(x;\lambda,\theta,\hat{a})}{\partial\theta} = \frac{n}{\theta} - n\ln p + \sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n}\left(\frac{x_i}{\hat{a}}\right)^{\theta}\ln\left(\frac{x_i}{\hat{a}}\right) = 0$$

Now put the values of λ in the above equation we get,

$$\frac{n}{\theta} - n\ln p + \sum_{i=1}^{n} \ln x_i - \frac{n\sum_{i=1}^{n}\left(\frac{x_i}{\hat{a}}\right)^{\theta} \ln\left(\frac{x_i}{\hat{a}}\right)}{\sum_{i=1}^{n}\left(\frac{x_i}{\hat{a}}\right)^{\theta} - n} = 0 \tag{4.34}$$

Estimation of the parameters λ and θ are to be done by solving the two equations using numerical method.

5. Data Analysis

In this section, we fit the odds generalized exponential-pareto model to a real data set obtained from Linhart and Zucchini (1986). The data set consists of failure times of the air conditioning system of an airplane and which is given as 1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62, 71, 71, 87, 90, 95, 120, 120, 225, 246 and 261. Histogram shows that the data set is positively skewed. Lee et al. (2007) fitted this data to the Beta Weibull distribution (BW). We have fitted this data set with the Odds Generalized Exponential-Pareto distribution. The estimated values of the parameters are $\hat{\lambda} = 0.03661283$, $\hat{\theta} = 0.8305798$, $\hat{a} = 1$, log-likelihood $= -150.8575$ and AIC $= 307.715$. Histogram and fitted exponential pareto curve to data have been shown in Figure 11.

Table 1: Summarized results of fitting different distributions to data set of Linhart and Zucchini (1986)

Distribution	Estimate of the parameter	Log-likelihood	AIC
BW	$\hat{a} = 3.087, \hat{b} = 0.132, \hat{c} = 0.667, \hat{\lambda} = 1.798$	-151.076	310.152
OGEPD	$\hat{\lambda} = 0.03661283, \hat{\theta} = 0.8305798, \hat{a} = 1$	-150.8575	307.715

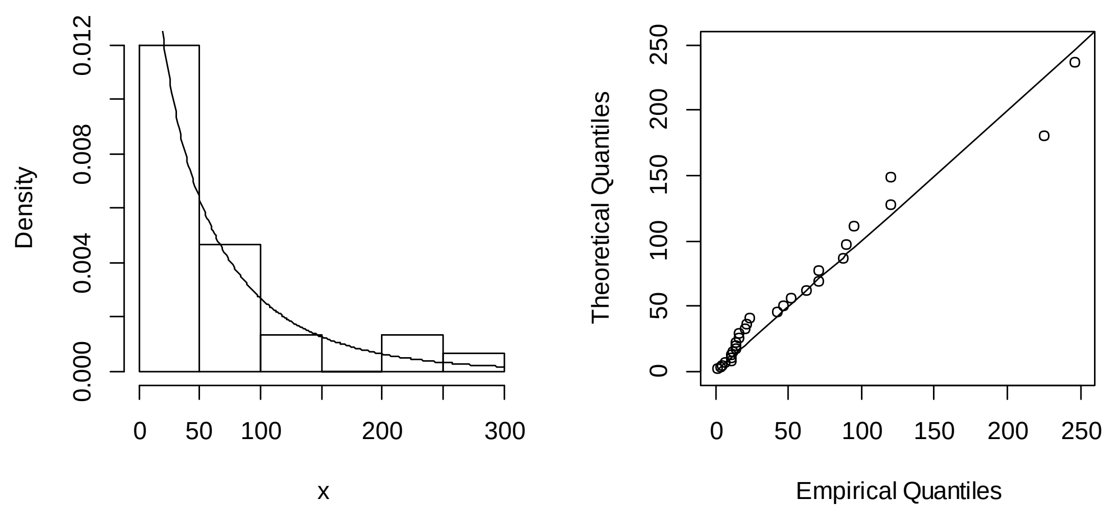

Figure 11: Plots of the fitted pdf and estimated quantiles versus observed quantiles of the OGEPD

6. Concluding Remark

In this article, we have studied a new probability distribution called Odds Generalized Exponential-Pareto Distribution. This is a particular case of T-X family of distributions proposed by Alzaatreh et al. (2013). The structural and reliability properties of this distribution have been studied and inference on parameters have also been mentioned. The advantage is that the distribution has only three parameters that are to be estimated. The appropriateness of fitting the Odds Generalized Exponential-Pareto distribution has been established by analyzing a real life data set.

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