The Marginal Distributions of a Crossing Time and Renewal Numbers Related with Two Poisson Processes are as Ph-Distributions

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Abstract

In this paper we consider, how to find the marginal distributions of crossing time and renewal numbers related with two poisson processes by using probability arguments. The obtained results show that the one-dimension marginal distributions are N+1 order PH-distributions.

Keywords: Marginal distributions, Poisson Process, PH-distributions.

1. Introduction

Assumption: Let N be a constant $\{X_i\}$ and $\{Y_i\}$ be two sequences of random variables. Suppose that $\{X_i\}, i=1,2,3...;$ are independently and identically distributed (i.i.d.) F(t) with mean λ^{-1} and $\{Y_j\},\ j=1,2,3...;$ are i.i.d. G(t) with mean μ^{-1}.

$N_1(t)=\sup\{n|S_n \le t\}$ is the counting process associated with $\{X_i\}$

where $S_0 = 0$ and $S_n = \sum_{i=1}^{n} Xi$

$N_2(t)=\sup\{n|T_n \le t\}$ is the counting process associated with $\{Y_j\}$

where $T_0 = 0$ and $T_n = \sum_{j=1}^{n} Y_j$

Assume that X_i, Y_j are mutually independent.

In the case of F(t) and G(t) are exponentially distributed with parameter λ and μ respectively, consider the following problem

Let $\xi_n = \inf\{n|T_n \le t\}$, its taking value $j = 1, 2, 3, \ldots;$

$$T_{\xi_N} = \sum_{j=1}^{\xi_N} Y_j, \quad \text{its taking value } t \ge 0$$

and

$$\eta_N = N_1(T_{\xi_N}). \quad \text{its taking value } i = N,\ N+1, \ldots;$$

we are interesting in finding the marginal distributions of T_{ξ_N} , ξ_n , η_N and prove that the obtained results are (N+1) order PH-distributions

2. Probability Arguments

In this section, first of all one can find out the marginal distribution of random variables T_{ξ_N} , ξ_n, and η_N then their probability generating function, mean and variance respectively .

Since

$$\xi_n = \inf\{n | T_n \geq S_N\}, \qquad \text{hence}$$

$$P\{\xi_n = j\} = P\{T_{j-1} \leq S_N < T_j\}$$

$$= \int_0^\infty P\{T_{j-1} \leq t < T_j\} \; f_{S_N}(t)dt$$

$$= \int_0^\infty P\{N_2(t) = j-1\} \; f_{S_N}(t)dt$$

$$= \int_0^\infty \exp(-\mu t) \frac{(\mu t)^{j-1}}{(j-1)!} \; f_{S_N}(t)dt$$

$$= \int_0^\infty \lambda \exp(-(\mu+\lambda) t) \frac{(\mu t)^{j-1}}{(j-1)!} \; \frac{(\lambda t)^{N-1}}{(N-1)!} \; dt$$

$$= \binom{N+j-2}{j-1} \left(\frac{\lambda}{\mu+\lambda}\right)^N \left(\frac{\mu}{\mu+\lambda}\right)^{j-1} \int_0^\infty \frac{((\lambda+\mu)\, t)^{N+j-2}}{(N+j-2)!} (\lambda+\mu) \exp(-(\lambda+\mu)\, t)\, dt$$

The integral function is an Erlang density, it yields

$$P\{\xi_N = j-1\} = \binom{N+j-2}{j-1} \left(\frac{\lambda}{\mu+\lambda}\right)^N \left(\frac{\mu}{\mu+\lambda}\right)^{j-1}, \; j = 1, 2, L \; ; \qquad (2.1)$$

Obviously, this is a negative binomial distribution. Thus the probability generating function is given by

$$G_{\xi_N}(u) = \sum_{j=1}^{\infty} \binom{N+j-2}{j-1} \left(\frac{\lambda}{\mu+\lambda}\right)^N \left(\frac{\mu}{\mu+\lambda}\right)^{j-1} u^j$$

$$= u \left(\frac{\lambda}{\mu+\lambda}\right)^N \sum_{k=0}^{\infty} \binom{N+k-1}{k} \left(\frac{\mu u}{\mu+\lambda}\right)^k, k = j-1$$

$$= u \left(\frac{\lambda}{\mu+\lambda}\right)^N \sum_{k=0}^{\infty} (-1)^k \binom{-N}{k} \left(\frac{\mu u}{\mu+\lambda}\right)^k$$

$$= u \left(\frac{\lambda}{\mu+\lambda}\right)^N \left(1 - \frac{\mu u}{\lambda+\mu}\right)^{-N}$$

$$= u \left(\frac{\lambda}{\lambda+\mu-\mu u}\right)^N \qquad (2,2)$$

In which the distribution of $\binom{-N}{k}$ and Newton's binomial formula see Ref [1, 9] the mean and variance of the distribution

$$E[\xi_N] = \tilde{G}\xi_N(1) = \frac{N\mu + \lambda}{\lambda} \qquad (2,3)$$

$$var[\xi_N] = \tilde{G}\xi_N(1) - \left(G'_{\xi_N}(1)\right)^2 = \frac{N\mu^2 + \lambda^2}{\lambda^2} \qquad (2,4)$$

Let Φ_T (t) represent the distribution function of T, since

$$T_{\xi_N} = \sum_{j=1}^{\xi_N} Y_j, \text{ Thus}$$

$$\sum_{j=1}^{\infty} p\{T_j \le t, \ T_{j-1} \le S_N < T_j\}$$

$$= \sum_{j=1}^{\infty}\left[\int_0^t \phi_{S_N}(u)\, d\Phi_{T_j}(u) - \int_0^t G^* \phi_{S_N}(u)\, d\Phi_{T_j}(u)\right]$$

$$= \int_0^t \tilde{G}^* \phi_{S_N}(u)\, d \sum_{j=1}^{\infty} \Phi_{T_j}(u)$$

$$= \int_0^t \tilde{G} * \phi_{S_N}(u)\, d\ M_2(u)$$

$$= \int_0^t \mu \exp(-\mu(t-u))\, \phi_{S_N}(u)\, du$$

$$= \int_0^t \phi_{S_N}(u) dG(t-u)$$

$$= P\{x_1 + x_2 + L + x_N + Y \le t\}$$

This shows that $T_{\xi_N} \stackrel{d}{=} x_1 + x_2 + L + x_N + Y = S_N + Y$ in distribution

Therefore the L-transform of random variable T_{ξ_N} is as follows

$$\varphi^*{}_{T_{\xi_N}}(s) = \frac{\mu}{s+\mu}\left(\frac{\lambda}{s+\lambda}\right)^N \qquad (2.5)$$

Since $\varphi^*{}_{T_{\xi_N}}$ (s) is a rational function of s, it may be written following form

See Ref. [2,10]

$$\varphi^*{}_{T_{\xi_N}}(s) = \frac{B_{11}}{(s+\mu)} + \frac{B_{12}}{(s+\lambda)^N} + L + \frac{B_{2N}}{(s+\lambda)}$$

where

$$B_{11} = \mu\left(\frac{\lambda}{\lambda - \mu}\right)^N$$

$$B_{2j} = \mu\lambda^N \sum_{j=1}^{N} (-1)^{j-1} \frac{1}{(\mu-\lambda)^j}, \qquad j = 1, 2, L, N$$

Taking the inverse of the L-transform one can get the density function T_{ξ_N}

$$\varphi^{*}T_{\xi_N}(t)=\left(\frac{\lambda}{\lambda-\mu}\right)^{N}\mu\exp(-\mu t)+\mu\lambda^{N}\sum_{j=1}^{N}(-1)^{j-1}\frac{1}{(\mu-\lambda)^{j}},\frac{t^{N-j}}{(N-j)!}\exp(-\lambda t)$$

$$=\left(\frac{\lambda}{\lambda-\mu}\right)^{N}\left[\mu\exp(-\mu t)-\mu\sum_{k=0}^{N-1}\frac{((\lambda-\mu)t)^{k}}{k!}\exp(-\lambda t)\right] \qquad (2.6)$$

Note that $\int_0^t \frac{\lambda(\lambda x)^k}{k!}\exp(-\lambda x)dx = \sum_{r=k+1}^{\infty}\frac{(\lambda t)^r}{r!}\exp(-\lambda t)$, the distribution of T_{ξ_N} is given by

$$\Phi_{T_{\xi_N}}(t)=\left(\frac{\lambda}{\lambda-\mu}\right)^{N}\left(1-\exp(-\mu t)-\frac{\mu}{\lambda}\sum_{k=0}^{N-1}\sum_{r=k+1}^{\infty}\frac{((\lambda-\mu)t)^{k}}{k!}\frac{(\lambda t)^{r}}{r!}\exp(-\lambda t)\right) \qquad (2.7)$$

The mean and variance of the distribution are

$$E\left[T_{\xi_N}\right]=E\left[s_N+Y\right]=\frac{N\mu+\lambda}{\lambda\mu} \qquad (2.8)$$

$$\operatorname{var}(T_{\xi_N})=\operatorname{var}\left[s_N+Y\right]=\frac{N\mu^2+\lambda^2}{\lambda^2\mu^2} \qquad (2.9)$$

using $T_{\xi_N}\overset{d}{=}x_1+x_2+\cdots+x_N+Y=s_N+Y$, one can easily get the distribution of $\eta_N=N_1(T_{\xi_N})$

$$P\{\eta_N=i\}=P\{N_1(T_{\xi_N})=N+k\}$$
$$=P\{x_1+x_2+\cdots+x_{N+k}\le Y<x_1+x_2+\cdots+x_{N+k+1}\}$$
$$=P\{x_1+x_2+\cdots+x_k\le Y<x_1+x_2+\cdots+x_{k+1}\}$$
$$=P\{N_1(Y)=k\}$$

where $k=0,1,2,\cdots;$

Thus

$$=P\{\eta_N=i\}\int_0^{\infty}\frac{(\lambda t)^k}{k!}\exp(-(\lambda+\mu)t)dt$$

$$=\left(\frac{\lambda}{\mu+\lambda}\right)^{k}\left(\frac{\mu}{\mu+\lambda}\right)=\int_0^{\infty}\frac{(\mu+\lambda)[(\lambda+\mu)t]}{k!}\exp[(\lambda+\mu)t]dt$$

$$=\left(\frac{\lambda}{\mu+\lambda}\right)^{i-1}\left(\frac{\mu}{\mu+\lambda}\right), \qquad (2.10)$$

It is a geometric distribution. Therefore the probability generating function of the distribution is

$$\begin{aligned} G_{\eta_N}(Z) &= \sum_{i=N}^{\infty} P\{\eta_N = i\} Z^i \\ &= \sum_{k=0}^{\infty} Z^{N+k} \int_0^{\infty} \frac{(\lambda t)^k}{k!} \exp(-\lambda t)^k \, d\Phi_y(t) \\ &= Z^N \varphi_y^* [\lambda(1-Z)] \\ &= \frac{\mu Z^N}{\mu + \lambda(1-Z)} \end{aligned} \tag{2.11}$$

Thus

$$E[\eta_N] = G'_{\eta_N}(1) = \frac{N\mu + \lambda}{\mu} \tag{2.12}$$

$$\operatorname{var}[\eta_N] = G''_{\eta_N}(1) - \left(G'_{\eta_N}(1)\right)^2 = \frac{N\mu^2 + \lambda^2}{\mu^2} \tag{2.13}$$

3. One Dimension Marginal Distributions as Ph-Distributions.

A particular instance of the method of supplementary variables is known as the method of phases and involves ideas of remarkable simplicity which were first proposed by A. K. Erlang. He observed that gamma distributions, whose shape parameter is a positive integer, may be considered as the probability distributions of sums of independent, negative exponential random variables. In this manner a number of highly useful results for renewal processes of Erlang type can be derived from those of the much simpler poisson process.

The basic idea of Erlang, which ultimately rests on the memoryless property of the negative exponential distribution, has been applied and extended by many authors.

Most useful elementary applications are discussed in the monograph by D. R. Cox[4], no attempt will be made to survey the uses of the method of phases in the existing literature, but we must draw attention to the paper by D. R. Cox[3] which introduces interesting

Notion of complex-valued probabilities in an attempt to find phase representations for all probability distributions on the positive real line which have rational *Laplace-stieltjes* transforms. Many open questions raised there have essentially remained un answered up to this time, and other related to the numerical use and fitting of such distributions are deserving of much further investigation.

The analytical and computational simplifications resulting from the method of phases are clear. They permeate the discussions of a large number of paper in the theory of queues and have recently been exploited in the construction of algorithms for certain single server queues [8]. One may anticipate that the method of phases will permit the algorithmic solution of a growing number of system of queues, as well as an exact numerical investigation of the Less-tractable priority queues. Instances of such results may be found in Ref. [6, 7].

Theorem 3.1 T_{ξ_N} is a (N+1) order continuous PH-distribution with represent (α,T) where $\alpha =$ $\alpha = (1, 0, 0, L, 0)$, α_{N+2}=0

$$T = \begin{bmatrix} -\lambda & \lambda & & & & \\ & -\lambda & \lambda & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \\ & & & & & -\mu \end{bmatrix}, T^0 = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \mu \end{bmatrix}$$

Proof The definition and property of PH-distribution see ref [5]. For continuous PH-distribution its L-transform is

$$\varphi^*_{T_{\xi_N}}(s) = \alpha_{N+2} + \alpha [sI - T]^{-1} T^0$$

Let $A = sI - T$, since $A^{-1} = \frac{1}{|A|} \mathrm{adj}A$, $\alpha = (1, 0, 0, L, 0)$ and $T^0 = (0, L, \mu)'$ it implies

$$\varphi^*_{T_{\xi_N}}(s) = \frac{1}{|A|} A_{(N=1)\times 1} \mu$$

$$= \frac{1}{(s+\lambda)^N (s+\mu)} (-1)^{N+1+1} (-\lambda)^N \mu$$

$$= \frac{\mu}{s+\mu} \left(\frac{\lambda}{s+\lambda} \right)^N \tag{3.1}$$

Obviously, this is the same to eq. (2.5)

Theorem 3.2 η_N is a N+1 order discrete PH-distribution with representation (α, P) where $\alpha = (1, 0, 0, L, 0)$, and $\alpha_{N+2} = 0$

$$P = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & \frac{\lambda}{\lambda+\mu} \\ & & & & & \frac{\lambda}{\lambda+\mu} \end{bmatrix}, P^0 = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \frac{\lambda}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} \end{bmatrix}$$

Proof For discrete PH-distribution, its probability generating function is $G_{\eta_N}(Z) = \alpha_{N+2} + Z\alpha[I - ZP]^{-1}P^0$ on a similar plan, we get

$$G_{\eta_N}(Z) = Z\frac{A_{N,1} + A_{N+1,1}}{|A|}\frac{\mu}{\lambda+\mu}$$

$$= Z\frac{(-1)^{N+1}(1-Z)^{N-1}\left(1-\frac{\lambda Z}{\lambda+\mu}\right) + (-1)^{N+1+1}(-1)^N\frac{\lambda Z^N}{\lambda+\mu}}{\left(1-\frac{\lambda Z}{\lambda+\mu}\right)}\frac{\mu}{\lambda+\mu}$$

$$= \frac{\mu Z^N}{\lambda+\mu(1-Z)} \qquad (3.2)$$

Same to eq. (2.11)

Theorem 3.3 η_N is a N+1 order discrete PH-distribution with representation (α, P) where $\alpha = (1, 0, 0, \cdots, 0)$, and $\alpha_{N+2} = 0$

$$P = \begin{bmatrix} 0 & \frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & \left(\frac{\lambda}{\lambda+\mu}\right) & \cdots & \cdots & \cdots & \frac{\mu}{\lambda+\mu} & \left(\frac{\lambda}{\lambda+\mu}\right)^{N-1} \\ 0 & \frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & \left(\frac{\lambda}{\lambda+\mu}\right) & \cdots & \cdots & \cdots & \frac{\mu}{\lambda+\mu} & \left(\frac{\lambda}{\lambda+\mu}\right)^{N-1} \\ 0 & 0 & & \frac{\lambda}{\lambda+\mu} & \cdots & \cdots & \cdots & \frac{\mu}{\lambda+\mu} & \left(\frac{\lambda}{\lambda+\mu}\right)^{N-2} \\ & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots & & \cdots & \frac{\mu}{\lambda+\mu} \end{bmatrix}, P^0 = \begin{bmatrix} \left(\frac{\lambda}{\lambda+\mu}\right)^N \\ \left(\frac{\lambda}{\lambda+\mu}\right)^N \\ \left(\frac{\lambda}{\lambda+\mu}\right)^{N-1} \\ \cdots \\ \cdot \\ \left(\frac{\lambda}{\lambda+\mu}\right) \end{bmatrix}$$

Proof The conclusion may be similarly verified as theorem 3.2. Because negative binomial distribution is the N-fold convolution of geometric distribution it is also may be obtained from the convolution property of PH-distribution.

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