The Multivariate Order Statistics for Exponential and Weibull Distributions

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Abstract

In this paper we have derived the distribution of multivariate order statistics for multivariate exponential & multivariate weibull distribution. The moment expression for multivariate order statistics has also been derived.

Keywords: Multivariate exponential distribution, Multivariate Weibull distribution, Order Statistics, Moments.

1. Introduction:

Let $X_1, X_2, ..., X_p$ be a set of p random variables with density function $f_i(x_i)$ and distribution function $F_i(x_i)$. Gumbel (1958) provided following method of obtaining joint density of the vector $\mathbf{x}$ as:

$$f(\mathbf{x}) = \prod_{i=1}^{p} f_i(x_i)\left[1+\theta\prod_{i=1}^{p}\left\{2F_i(x_i)-1\right\}\right] \tag{1.1}$$

Suppose Y is another random variable with density function $f(y)$ and distribution function $F(y)$ then the joint density of Y and $\mathbf{x}$ may be obtained by using:

$$f(y,\mathbf{x}) = f(y)\times\prod_{i=1}^{p} f_i(x_i)\left[1+\theta\left\{2F(y)-1\right\}\times\prod_{i=1}^{p}\left\{2F_i(x_i)-1\right\}\right] \tag{1.2}$$

which can be used to generate multivariate distributions based on any available marginal. Several authors have used (1.2) to propose multivariate distributions.

Order Statistics has been extensively studied by many authors. Specifically, the distribution of *r*th order statistics as given in David and Nagaraja (2003) is

$$f_{(r:n)}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r};$$

where $F(x)$ is distribution function of the parent distribution.

The univariate order statistics for special distributions has been extensively studied by several researchers in context of their application in extreme events. The order statistics has also been used to study the concomitants for bivariate distribution as discussed by Veena and Thomas (2008).

The multivariate order statistics has been area of researcher in recent years. Belzunce et al (2003) has presented the idea of multivariate order statistics but not with much applicability. Barakat (2009) provided some results for multivariate order statistics for dependent random variables. Arnold (2009) has developed the density of multivariate order statistics of a random vector $\mathbf{X}$ by using the idea of concomitants. The density of multivariate order statistics is obtained by using:

$$f_{(r:n)}(\mathbf{x}) = \frac{n!}{(r-1)!(n-r)!} f(\mathbf{x})\int_{-\infty}^{\infty}\left[F(y)\right]^{r-1}\left[1-F(y)\right]^{n-r} f(y|\mathbf{x})dy \qquad (1.3)$$

where $f(y|\mathbf{x})$ is the conditional distribution of Y given $\mathbf{x}$. Hafeez et al (2011) discussed the bivariate order statistics for exponential distribution by using (1.3).

In this paper we have studied the multivariate order statistics for exponential and Weibull distribution. These distributions have been discussed in the following sections.

We first define a multivariate exponential distribution by using (1.2). Suppose that every component of vector $\mathbf{x}$ has common Exponential Distribution with density and distribution function given as:

$$f_i(x_i) = \alpha_i e^{-\alpha_i x_i} \qquad \textit{and} \qquad F_i(x_i) = 1 - e^{-\alpha_i x_i};$$

Further, suppose that the random variable Y also has Exponential distribution with density and distribution function as:

$$f(y) = \beta e^{-\beta y} \quad \textit{and} \quad F(y) = 1 - e^{-\beta y}.$$

The joint density of Y and $\mathbf{x}$ is obtained by using (5.2.2) as:

$$f(y,\mathbf{x}) = \beta k e^{-\beta y}\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right] \qquad (1.4)$$

where $k = \prod_{i=1}^{p}\alpha_i$.

The joint marginal distribution of $\mathbf{x}$ is obtained from (5.2.4) as:

$$f(\mathbf{x}) = \int_0^\infty \beta k e^{-\beta y} \exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right]dy$$

$$= k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\int_0^\infty \beta e^{-\beta y}\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right]dy$$

$$= k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)$$

The conditional distribution of Y given $\mathbf{x}$ is:

$$f(y|\mathbf{x}) = \frac{f(y,\mathbf{x})}{f(\mathbf{x})}$$

$$f(y|\mathbf{x}) = \frac{\beta k e^{-\beta y}\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right]}{k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)}$$

$$f(y|\mathbf{x}) = \beta e^{-\beta y}\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right] \tag{1.5}$$

The distribution of rth multivariate order statistics is derived in the following section.

2. Rth Multivariate Order Statistics for Exponential Distribution

In this section the rth multivariate order statistics for exponential distribution has been derived. The density of rth multivariate order statistics is obtained by using (5.2.5) in (5.2.2) as:

$$f_{r:n}(\underline{x}) = \frac{n!}{(r-1)!(n-r)!} f(x)\int_{-\infty}^{\infty} f(y/x)[F(y)]^{r-1}[1-F(y)]^{n-r}dy \tag{2.1}$$

$$f_{r:n}(\underline{x}) = \frac{n!}{(r-1)!(n-r)!} k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\int_0^\infty \left(1-e^{-\beta y}\right)^{r-1}\left(e^{-\beta y}\right)^{n-r}\beta e^{-\beta y}[1+\theta\left(1-2e^{-\beta y}\right)$$

$$\prod_{i=1}^{p}\left(1-2e^{-\alpha_i x_i}\right)]dy$$

$$f_{r:n}(\underline{x}) = \frac{n!}{(r-1)!(n-r)!} k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\frac{(n-r)!(r-1)!}{n!} + \theta\prod_{i=1}^{p}\left(1-2e^{-\alpha_i x_i}\right)$$

$$\left[\int_0^\infty \beta e^{-\beta y}\left(1-e^{-\beta y}\right)^{r-1}\left(e^{-\beta y}\right)^{n-r}dy - 2\int_0^\infty \beta e^{-2\beta y}\left(1-e^{-\beta y}\right)^{r-1}\left(e^{-\beta y}\right)^{n-r}dy\right]$$

$$f_{r:n}\left(\underline{x}\right)=\frac{n!}{\left(r-1\right)!\left(n-r\right)!}k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\frac{\left(n-r\right)!\left(r-1\right)!}{n!}\left[1+\theta\prod_{i=1}^{p}\left(1-2e^{-\alpha_i x_i}\right)\left(\frac{2r}{n+1}-1\right)\right]$$

$$f_{r:n}\left(\underline{x}\right)=k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\left[1+\theta\prod_{i=1}^{p}\left(1-2e^{-\alpha_i x_i}\right)\left(\frac{2r-n-1}{n+1}\right)\right] \tag{2.2}$$

$where\, k=\prod_{i=1}^{p}\alpha_i$.

The moments of rth multivariate order statistics are derived in the following section.

3. Moments for Multivariate Order Statistics for Exponential Distribution

In this section the Moments for rth multivariate order statistics has been derived. The kth joint moment of a set of random variables is given as:

$$E\left(X_{1r:n}^{k},X_{2r:n}^{k},......X_{pr:n}^{k}\right)=\int_0^\infty\int_0^\infty.....\int_0^\infty x_1^k x_2^k......x_p^k f_{r;n}\left(x_1,x_2,....x_p\right)dx_1dx_2....dx_p \tag{3.1}$$

Using (5.3.2) in (5.4.1), the moments are given as:

$$E\left(X_{1r:n}^{k},X_{2r:n}^{k},......X_{pr:n}^{k}\right)=\int_0^\infty\int_0^\infty.....\int_0^\infty x_1^k x_2^k......x_p^k k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)$$
$$\left[1+\theta\prod_{i=1}^{p}\left(1-2e^{-\alpha_i x_i}\right)\left(\frac{2r-n-1}{n+1}\right)\right]dx_1dx_2....dx_p$$

$$E\left(X_{1r:n}^{k},X_{2r:n}^{k},......X_{pr:n}^{k}\right)=\int_0^\infty x_1^k\alpha_1 e^{-\alpha_1 x_1}dx_1\int_0^\infty x_2^k\alpha_2 e^{-\alpha_2 x_2}dx_2.....\int_0^\infty x_p^k\alpha_p e^{-\alpha_p x_p}dx_p$$
$$+\theta\left(\frac{2r-n-1}{n+1}\right)\int_0^\infty x_1^k\alpha_1 e^{-\alpha_1 x_1}\left(1-2e^{-\alpha_1 x_1}\right)dx_1\int_0^\infty x_2^k\alpha_2 e^{-\alpha_2 x_2}\left(1-2e^{-\alpha_2 x_2}\right)dx_2$$
$$\times........\times\int_0^\infty x_p^k\alpha_p e^{-\alpha_p x_{2p}}\left(1-2e^{-\alpha_p x_p}\right)dx_p$$

Or

$$E\left(X_{1r:n}^{k},X_{2r:n}^{k},......X_{pr:n}^{k}\right)=\prod_{t=1}^{p}I_t+\theta\left(\frac{2r-n-1}{n+1}\right)\prod_{t=1}^{p}M_t \tag{3.2}$$

$$where\quad I_t=\int_0^\infty x_t^k\alpha_t e^{-\alpha_t x_t}dx_t;\qquad t=1,2,\ldots,p$$

$$M_t=\int_0^\infty x_t^k\alpha_t e^{-\alpha_t x_t}\left(1-2e^{-\alpha_t x_t}\right)dx_t;\qquad t=1,2,\ldots,p$$

Consider: $I_t = \int_0^\infty x_t^k \alpha_t e^{-\alpha_t x_t} dx_t$

$$Let\ w = \alpha_t x_t \Rightarrow dw = \alpha_t dx_t\,;\, x_t = \frac{w}{\alpha_t}$$

$$I_t = \frac{1}{\alpha_t^k}\int_0^\infty w^k e^{-w} dw = \frac{1}{\alpha_t^k}\Gamma(k+1)$$

Now consider M_t as:

$$M_t = \int_0^\infty x_t^k \alpha_t e^{-\alpha_t x_t}\left(1-2e^{-\alpha_t x_t}\right)dx_t$$

$$M_1 = \frac{1}{\alpha_t^k}\int_0^\infty w_t^k e^{-w_t}\left(1-2e^{-w_t}\right)dw;\ with\ w_t = \alpha_t x_t$$

$$M_t = \frac{\Gamma(k+1)}{\alpha_t^k}\left(\frac{2^k-1}{2^k}\right)$$

Using values of $I_1, I_2, ...I_p$ & $M_1, M_2, ...M_p$ in (5.4.2) we have:

$$E\left(X_{1r:n}^{k}, X_{2r:n}^{k}, ...X_{pr:n}^{k}\right) = \frac{1}{\alpha_1^k}\left\lceil k+1\right.\frac{1}{\alpha_2^k}\left\lceil k+1\right. ....\frac{1}{\alpha_p^k}\left\lceil k+1\right. + \theta\left(\frac{2r-n-1}{n+1}\right)$$
$$\frac{\left\lceil k+1\right.}{\alpha_1^k}\frac{\left(2^k-1\right)}{\left(2^k\right)}.\frac{\left\lceil k+1\right.}{\alpha_2^k}\frac{\left(2^k-1\right)}{\left(2^k\right)}....\frac{\left\lceil k+1\right.}{\alpha_p^k}\frac{\left(2^k-1\right)}{\left(2^k\right)}$$

$$E\left(X_{1r:n}^{k}, X_{2r:n}^{k}, ...X_{pr:n}^{k}\right) = \prod_{i=1}^{p}\frac{\left(\left\lceil k+1\right.\right)}{\alpha_i^k} + \prod_{i=1}^{p}\frac{\left(\left\lceil k+1\right.\right)}{\alpha_i^k}\frac{\left(2^k-1\right)}{\left(2^k\right)}\theta\left(\frac{2r-n-1}{n+1}\right)$$

$$E\left(X_{1r:n}^{k}, X_{2r:n}^{k}, ...X_{pr:n}^{k}\right) = \frac{\left(\left\lceil k+1\right.\right)^p}{\prod_{i=1}^{p}\alpha_i^k} + \frac{\left(\left\lceil k+1\right.\right)^p}{\prod_{i=1}^{p}\alpha_i^k}\left\{\theta\left(\frac{2r-n-1}{n+1}\right)\left(\frac{2^k-1}{2^k}\right)\right\} \qquad (3.3)$$

The moments of specific order can be obtained from (3.3).

4. Multivariate Order Statistics for Weibull Distribution

In this section the multivariate order statistics has been discussed. For this suppose that every component of vector **x** has common Weibull Distribution with density and distribution function given as:

$$f_i(x_i) = \alpha_i x^{\alpha_i - 1}\exp\left(-x_i^{\alpha_i}\right) \quad and \quad F_i(x_i) = 1-\exp\left(-x_1^{\alpha_1}\right);$$

Further, suppose that the random variable Y also has weibull distribution with density and distribution function as:

$$f\left(y\right)=\beta y^{\beta-1}\exp\left(-y^{\beta}\right) \qquad and \qquad F\left(y\right)=1-\exp\left(-y^{\beta}\right).$$

The joint density of Y and $\mathbf{x}$ is obtained by using (1.1) as:

$$f\left(y,\mathbf{x}\right)=\beta k e^{-\beta y}\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right] \qquad (4.1)$$

where $k=\prod_{i=1}^{p}\alpha_i$.

The joint marginal distribution of $\mathbf{x}$ is obtained from (4) as:

$$f\left(\mathbf{x}\right)=\int_0^{\infty}\beta k e^{-\beta y}\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right]dy$$

$$f\left(\mathbf{x}\right)=k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\int_0^{\infty}\beta e^{-\beta y}\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right]dy$$

$$f\left(\mathbf{x}\right)=k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)$$

The conditional distribution of Y given $\mathbf{x}$ is:

$$f\left(y|\mathbf{x}\right)=\frac{f\left(y,\mathbf{x}\right)}{f\left(\mathbf{x}\right)}$$

$$f\left(y|\mathbf{x}\right)=\frac{\beta k e^{-\beta y}\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right]}{k\exp\left(-\sum_{i=1}^{p}\alpha_i x_i\right)}$$

$$f\left(y|\mathbf{x}\right)=\beta e^{-\beta y}\left[1+\theta\left(1-2e^{-\beta y}\right)\prod_{i=1}^{p}\left\{1-2e^{-\alpha_i x_i}\right\}\right] \qquad (4.2)$$

5. Rth Multivariate Order Statistics for Trivariate Weibull Distribution

In this section the rth multivariate order statistics for multivariate weibull distribution has been derived. The density of rth multivariate order statistics is obtained by using (4.2) in (1.2) as:

$$f\left(y/x\right)=\beta y^{\beta-1}e^{-y^{\beta}}\left[1+\theta\left(1-2e^{-y^{\beta}}\right)\prod_{i=1}^{p}\left(1-2e^{-x_i^{\alpha_i}}\right)\right]$$

$$f_{r:n}\left(\underline{x}\right)=\frac{n!}{(r-1)!(n-r)!}\,f\left(x\right)\int_{-\infty}^{\infty} f\left(y/x\right)[F\left(y\right)]^{r-1}[1-F\left(y\right)]^{n-r}\,dy$$

$$f_{r:n}\left(\underline{x}\right)=\frac{n!}{(r-1)!(n-r)!}k\prod_{i=1}^{p}x_i^{\alpha_i-1}\exp\left(-\sum_{i=1}^{p}x_i^{\alpha_i}\right)$$

$$\int_0^{\infty}\beta y^{\beta-1}e^{-y^{\beta}}\left(1-e^{-y^{\beta}}\right)^{r-1}\left(e^{-y^{\beta}}\right)^{n-r}\left(1+\theta\left(1-2e^{-y^{\beta}}\right)\right)\prod_{i=1}^{p}\left(1-2e^{-x_i^{\alpha_i}}\right)dy \qquad (5.1)$$

$$f_{r:n}\left(\underline{x}\right)=\frac{n!}{(r-1)!(n-r)!}k\prod_{i=1}^{p}x_i^{\alpha_i-1}\exp\left(-\sum_{i=1}^{p}x_i^{\alpha_i}\right)[\frac{(n-r)!(r-1)!}{n!}+\theta\prod_{i=1}^{p}\left(1-2e^{-x_i^{\alpha_i}}\right)$$

$$\{\int_0^{\infty}\beta y^{\beta-1}e^{-y^{\beta}}\left(1-e^{-y^{\beta}}\right)^{r-1}\left(e^{-y^{\beta}}\right)^{n-r}dy-2\int_0^{\infty}\beta y^{\beta-1}e^{-2y^{\beta}}\left(1-e^{-y^{\beta}}\right)^{r-1}\left(e^{-y^{\beta}}\right)^{n-r}dy]$$

$$f_{r:n}\left(\underline{x}\right)=k\prod_{i=1}^{p}x_i^{\alpha_i-1}\exp\left(-\sum_{i=1}^{p}x_i^{\alpha_i}\right)\left[1+\theta\prod_{i=1}^{p}\left(1-2e^{-x_i^{\alpha_i}}\right)\left(\frac{2r-n-1}{n+1}\right)\right] \qquad (5.2)$$

$$where\, k=\prod_{i=1}^{p}\alpha_i$$

6. Moments for Multivariate Order Statistics for Trivariate Weibull Distribution

In this section the Moments for r[th] multivariate order statistics has been derived;

$$E\left(X_{1r:n}^{\;k},X_{2r:n}^{\;k},......X_{pr:n}^{\;k}\right)=\int_0^{\infty}\int_0^{\infty}.....\int_0^{\infty}x_1^kx_2^k......x_p^k\,f_{r;n}\left(x_1,x_2,....x_p\right)dx_1dx_2....dx_p \qquad (6.1)$$

$$E\left(X_{1r:n}^{\;k},X_{2r:n}^{\;k},......X_{pr:n}^{\;k}\right)=\int_0^{\infty}\int_0^{\infty}.....\int_0^{\infty}x_1^kx_2^k......x_p^k k\prod_{i=1}^{p}x_i^{\alpha_i-1}\exp\left(-\sum_{i=1}^{p}x_i^{\alpha_i}\right)$$

$$\left[1+\theta\prod_{i=1}^{p}\left(1-2e^{-x_i^{\alpha_i}}\right)\left(\frac{2r-n-1}{n+1}\right)\right]dx_1dx_2....dx_p$$

$$E\left(X_{1r:n}^{\;k},X_{2r:n}^{\;k},......X_{pr:n}^{\;k}\right)=\prod_{t=1}^{p}I_t+\theta\left(\frac{2r-n-1}{n+1}\right)\times\prod_{t=1}^{p}M_t \qquad (6.2)$$

where $I_t=\int_0^{\infty}x_t^k\alpha_tx_t^{\alpha_t-1}e^{-x_t^{\alpha_t}}dx_t$

and $M_t=\int_0^{\infty}x_t^k\alpha_tx_t^{\alpha_1-1}e^{-x_t^{\alpha_t}}\left(1-2e^{-x_t^{\alpha_t}}\right)dx_t$

Consider I_t as:

$$I_t=\int_0^{\infty}x_t^k\alpha_tx_t^{\alpha_t-1}e^{-x_t^{\alpha_t}}dx_1$$

$$I_t=\frac{1}{\alpha_t^k}\int_0^{\infty}w_t^{k/\alpha_t}e^{-w_t}dw_t=\Gamma\left(k/\alpha_1+1\right)$$

Now consider M_t as:

$$M_t = \int_0^\infty x_t^k \alpha_t x_t^{\alpha_1 - 1} e^{-x_t^{\alpha_t}} \left(1 - 2e^{-x_t^{\alpha_t}}\right) dx_t$$

$$= \int_0^\infty w_t^{k/\alpha_1} e^{-w_t} \left(1 - 2e^{-w_t}\right) dw_t = \Gamma\left(k/\alpha_t + 1\right)\left(\frac{2^{k/\alpha_t} - 1}{2^{k/\alpha_t}}\right)$$

Using values of $I_1, I_2, ...I_p$ & $M_1, M_2, ...M_p$ in (5.7.2) we have:

$$E\left(X_{1r:n}^{\,k}, X_{2r:n}^{\,k}, ...X_{pr:n}^{\,k}\right) = \Gamma\left(\frac{k}{\alpha_1}+1\right)\Gamma\left(\frac{k}{\alpha_2}+1\right), ..., \Gamma\left(\frac{k}{\alpha_p}+1\right) + \theta\left(\frac{2r-n-1}{n+1}\right)$$

$$\Gamma\left(\frac{k}{\alpha_1}+1\right)\left(\frac{2^{k/\alpha_1}-1}{2^{k/\alpha_1}}\right).\Gamma\left(\frac{k}{\alpha_2}+1\right)\left(\frac{2^{k/\alpha_2}-1}{2^{k/\alpha_2}}\right)....\Gamma\left(\frac{k}{\alpha_p}+1\right)\left(\frac{2^{k/\alpha_p}-1}{2^{k/\alpha_p}}\right)$$

$$E\left(X_{1r:n}^{\,k}, X_{2r:n}^{\,k}, ...X_{pr:n}^{\,k}\right) = \prod_{i=1}^{p}\Gamma\left(\frac{k}{\alpha_i}+1\right) + \prod_{i=1}^{p}\Gamma\left(\frac{k}{\alpha_i}+1\right)\left(\frac{2^{k/\alpha_i}-1}{2^{k/\alpha_i}}\right)\theta\left(\frac{2r-n-1}{n+1}\right)$$

$$E\left(X_{1r:n}^{\,k}, X_{2r:n}^{\,k}, ...X_{pr:n}^{\,k}\right) = \prod_{i=1}^{p}\Gamma\left(\frac{k}{\alpha_i}+1\right) + \theta\left(\frac{2r-n-1}{n+1}\right)\prod_{i=1}^{p}\Gamma\left(\frac{k}{\alpha_i}+1\right)\left(\frac{2^{k/\alpha_i}-1}{2^{k/\alpha_i}}\right) \quad (6.3)$$

The moments of specific order can be obtained from (6.3).

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