Bayesian and Non–Bayesian Estimation for Two Generalized Exponential Populations Under Joint Type II Censored Scheme

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Abstract

In this paper, Bayesian and non-Bayesian estimators have been obtained for two generalized exponential populations under joint type II censored scheme, which generalize results of Balakrishnan and Rasouli (2008) and Shafay et al. (2013). The maximum likelihood estimators (MLEs) of the parameters and Bayes estimators have been developed under squared error loss function as well as under LINEX loss function. Moreover, approximate confidence region are also discussed and compared with two Bootstrap confidence regions. Also the MLE and three confidence intervals for the stress–strength parameter $R = P(Y < X)$ are explored. A numerical illustration for these new results is given.

Keywords: Generalized exponential distribution; Joint type-II censoring; Maximum likelihood estimation; Confidence bounds; Bootstrap intervals; Coverage probabilities; Bayesian estimation; Squared-error loss; Linear-exponential loss; Stress–strength reliability.

1. Introduction

Recently a new distribution, named as generalized exponential (GE) distribution or exponentiated exponential distribution was introduced and studied quite extensively by Gupta and Kundu. 1999; 200la; 2001b; 2002. The two parameters of an exponentiated exponential distribution represent the shape and the scale parameter like a gamma distribution or a Weibull distribution. The density function varies significantly depending of the shape parameter. It is observed that it has lots of properties which are quite similar to those of a gamma distribution but it has an explicit expression of the distribution function or the survival function like a Weibull distribution.

The two-parameter GE distribution has the following distribution function;

$$F_i(x) = \left(1 - e^{-\theta_i x}\right)^{\alpha_i}, \quad i = 1,2,$$

and it has a density function

$$f_i(x) = \alpha_i \theta_i \left(1 - e^{-\theta_i x}\right)^{\alpha_i - 1} e^{-\theta_i x} \qquad \alpha_i, \theta_i > 0,\ x > 0, \qquad (1)$$

Here α is the shape parameter and θ is the scale parameter. When the shape parameter α equals 1 it coincides with the 1-parameter exponential distribution. Therefore, GE distribution is a generalization of an 1-parameter exponential distribution having a shape parameter α.

The joint type-II censoring may occur while conducting comparative life-tests of products from different lines of production, for example. To be more precise, suppose products are being manufactured by two different lines under the same conditions and that two independent samples of sizes *m* and *n* are selected from these two lines, respectively, and are placed simultaneously on a life-testing experiment. Then, due to cost and time considerations, the experimenter may choose to terminate the life-testing experiment as soon as a certain number of failures occur. The successive failure times and the corresponding product types will be recorded, and the life-testing experiment will get terminated as soon as a pre-specified number of failures are observed.

Suppose that $X_1,...,X_m$, the lifetimes of *m* specimens of product *A*, are i.i.d. random variables from distribution function $F(x)$ and density function $f(x)$, and $Y_1,...,Y_n$, the lifetimes of *n* specimens of product *B*, are i.i.d. random variables from distribution function *G(x)* and density function *g(x)*. Further, suppose $W_{(1)}<W_{(2)}<...<W_{(N)}$ denote the order statistics of the $N=m+n$ random variables $\{X_1,...,X_m;Y_1,...,Y_n\}$. Then, under the joint Type-II censoring scheme, the observable data consist of (Z,W), where $\mathrm{W}=(\mathrm{W}_{(1)},\mathrm{W}_{(2)},...,\mathrm{W}_{(r)})$, with r $(1\le r<N)$ being a pre-fixed integer, and $\mathrm{Z}=(\mathrm{Z}_1,...,\mathrm{Z}_r)$ with $z_i=1$ or *0* according as w_i is from an *X*- or *Y*-failure.

Letting $M_r=\sum_{i=1}^{r}Z_i$ denote the number of *X*-failures in W and $N_r=\sum_{i=1}^{r}(1-Z_i)=r-M_r$ (i.e., the number of *Y*-failures in W), the likelihood of *(Z, W)* is given by Balakrishnan and Rasouli (2008) as:

$$L=C\prod_{i=1}^{r}[\{f(w_i)\}^{Z_i}\{g(w_i)\}^{1-Z_i}]\{\bar{F}(w_r)\}^{m-m_r}\{\bar{G}(w_r)\}^{n-n_r}, \tag{2}$$

where $\bar{F}=1-F,\bar{G}=1-G$ are the survival functions of the two populations and $C=\dfrac{m!n!}{(m-m_r)!(n-n_r)!}$.

Balakrishnan and Rasouli (2008) developed likelihood inference for the parameters of two exponential populations under joint type-II censoring. They developed inferential methods based on maximum likelihood estimates (MLE) and compared their performance with those based on some other approaches such as Bootstrap. Shafay et al. (2013) derived the Bayesian inference for the unknown parameters of two exponential populations under joint type II censoring they developed with the use of squared-error, linear-exponential and general entropy loss functions. The problem of predicting the future failure times, both point and interval prediction, based on the observed joint type-II censored data is obtained; see also Rasouli and Balakrishnan (2010) for a generalization of their results to progressive type-II censoring. Finally Balakrishnan and Feng (2014).

generalized Balakrishnan and Rasouli (2008), Rasouli and Balakrishnan (2010) and Shafay et al. (2013) works by considered a jointly type II censored sample arising from h independent exponential populations.

Succeeding section deals with the computational procedure to obtain the MLEs of $\alpha_1, \alpha_2, \theta_1$ and θ_2 and their asymptotic variance-covariance matrix. Section 3 describes the various bootstrap confidence intervals. While section 4 is concerned with Bayes estimators under squared error loss function as well as under LINEX loss function for the parameters. Section 5 the maximum likelihood estimation and confidence intervals using asymptotic distribution and two parametric bootstrap resampling methods for the stress–strength parameter $R = P(Y < X)$ are explored. All estimators are not in nice closed forms, therefore, numerical examples are considered to illustrate the proposed estimators in section 6. Last section includes a brief conclusion.

Not that If $\alpha_1 = \alpha_2 = 1$ we obtain MLEs based on a jointly type-II censored sample from two exponential populations which introduced by Balakrishnan and Rasouli (2008) and Shafay et al. (2013).

2. Maximum Likelihood Estimators

Suppose that the two populations are GE distribution with distribution function;

$$F(x;\alpha_1,\theta_1) = \left(1-e^{-\theta_1 x}\right)^{\alpha_1} \quad \text{and} \quad G(x;\alpha_2,\theta_2) = \left(1-e^{-\theta_2 x}\right)^{\alpha_2},$$
$$\alpha_i, \theta_i > 0,\ x > 0, \quad i = 1,2,$$

where α_i are the shape parameters and θ_i are the scale parameters. In this case, the likelihood function in (2) becomes

$$L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z) = \frac{m!\,n!\,\alpha_1^{m_r}\,\theta_1^{m_r}\,\alpha_2^{n_r}\,\theta_2^{n_r}}{(m-m_r)!(n-n_r)!}\prod_{i=1}^{r}\left[\left(1-u_i\right)^{\alpha_1-1}u_i\right]^{z_i}\left[\left(1-q_i\right)^{\alpha_2-1}q_i\right]^{(1-z_i)}$$

$$\times\left[1-\left(1-v_1\right)^{\alpha_1}\right]^{(m-m_r)}\left[1-\left(1-v_2\right)^{\alpha_2}\right]^{(n-n_r)} \qquad (3)$$

where $u_i = e^{-\theta_1 w_i}$, $q_i = e^{-\theta_2 w_i}$, $v_1 = e^{-\theta_1 w_r}$, $v_2 = e^{-\theta_2 w_r}$, $\lambda_1 = (1-v_1)$ and $\lambda_2 = (1-v_2)$.

Therefore, to obtain the MLE's of α_i and θ_i we find the first derivatives of the natural logarithm of the likelihood function (3) with respect to α_i and θ_i and equating them to zero, we get the following four equations

$$\frac{\partial \ln L}{\partial \alpha_1} = \frac{m_r}{\hat{\alpha}_1} + \sum_{i=1}^{r} z_i \ln\left(1-\hat{u}_i\right) - \frac{(m-m_r)\hat{\lambda}_1^{\hat{\alpha}_1}\ln\hat{\lambda}_1}{1-\hat{\lambda}_1^{\hat{\alpha}_1}} = 0,$$

$$\frac{\partial \ln L}{\partial \alpha_2} = \frac{n_r}{\hat{\alpha}_2} + \sum_{i=1}^{r} (1-z_i) \ln\left(1-\hat{q}_i\right) - \frac{(n-n_r)\hat{\lambda}_2^{\hat{\alpha}_2}\ln\hat{\lambda}_2}{1-\hat{\lambda}_2^{\hat{\alpha}_2}} = 0,$$

$$\frac{\partial \ln L}{\partial \theta_1} = \frac{m_r}{\hat{\theta}_1} + (\hat{\alpha}_1 - 1)\sum_{i=1}^{r}\frac{z_i w_i \hat{u}_i}{1-\hat{u}_i} - \sum_{i=1}^{r} w_i z_i - \frac{\hat{\alpha}_1 (m-m_r) w_r \hat{\lambda}_1^{\hat{\alpha}_1 - 1}\hat{v}_1}{1-\hat{\lambda}_1^{\hat{\alpha}_1}} = 0,$$

and

$$\frac{\partial \ln L}{\partial \theta_2} = \frac{n_r}{\hat{\theta}_2} + (\hat{\alpha}_2 - 1)\sum_{i=1}^{r} \frac{(1-z_i)w_i \hat{q}_i}{1-\hat{q}_i} - \sum_{i=1}^{r} w_i (1-z_i) - \frac{\hat{\alpha}_2 (n-n_r) w_r \hat{\lambda}_2^{\hat{\alpha}_2 - 1} \hat{v}_2}{1-\hat{\lambda}_2^{\hat{\alpha}_2}} = 0 \qquad (4)$$

where $\hat{u}_i = e^{-\hat{\theta}_1 w_i}$, $\hat{q}_i = e^{-\hat{\theta}_2 w_i}$, $\hat{v}_1 = e^{-\hat{\theta}_1 w_r}$, $\hat{v}_2 = e^{-\hat{\theta}_2 w_r}$ $\hat{\lambda}_1 = (1-\hat{v}_1)$ and $\hat{\lambda}_2 = (1-\hat{v}_2)$. Then the maximum likelihood estimates of the parameters $\alpha_1, \alpha_2, \theta_1$ and θ_2 can be obtained by solving system of equations (4). No explicit form for these estimates, we use a numerical technique using Mathcad2007 Package to obtain $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta}_1$ and $\hat{\theta}_2$.

The approximate asymptotic variance-covariance matrix for $\alpha_1, \alpha_2, \theta_1$ and θ_2 can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivatives of logarithms of the likelihood functions. Cohen (1965) concluded that the approximate variance covariance matrix may be obtained by replacing expected values by their MLEs. Now the Fisher information matrix associated with $\alpha_1, \alpha_2, \theta_1$ and θ_2 is defined as:

$$I(\alpha_1, \alpha_2, \theta_1, \theta_2) = E\begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha_1^2} & 0 & -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \theta_1} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \alpha_2^2} & 0 & -\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \theta_2} \\ -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \theta_1} & 0 & -\frac{\partial^2 \ln L}{\partial \theta_1^2} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \theta_2} & 0 & -\frac{\partial^2 \ln L}{\partial \theta_2^2} \end{bmatrix},$$

where

$$-\frac{\partial^2 \ln L}{\partial \alpha_1^2} = \frac{m_r}{\alpha_1^2} + \frac{(m-m_r)\lambda_1^{\alpha_1} (\ln \lambda_1)^2}{\left[1-\lambda_1^{\alpha_1}\right]^2},$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_2^2} = \frac{n_r}{\alpha_2^2} + \frac{(n-n_r)\lambda_2^{\alpha_2} (\ln \lambda_2)^2}{\left[1-\lambda_2^{\alpha_2}\right]^2},$$

$$-\frac{\partial^2 \ln L}{\partial \theta_1^2} = \frac{m_r}{\theta_1^2} + (\alpha_1 - 1)\sum_{i=1}^{r} \frac{w_i^2 u_i z_i}{(1-u_i)^2} +$$

$$\alpha_1 (m-m_r)(w_r). \frac{\left[\left(1-\lambda_1^{\alpha_1}\right) v_1 w_r \lambda_1^{\alpha_1 - 1}\right]\left[\left[(\alpha_1 - 1) v_1 \lambda_1^{-1}\right] - 1\right] + \left[\alpha_1 w_r \lambda_1^{2(\alpha_1 - 1)} v_1^2\right]}{\left[1-\lambda_1^{\alpha_1}\right]^2},$$

$$-\frac{\partial^2 \ln L}{\partial \theta_2^2} = \frac{n_r}{\theta_2^2} + (\alpha_2 - 1)\sum_{i=1}^{r} \frac{w_i^2 q_i (1 - z_i)}{(1 - q_i)^2} +$$

$$\alpha_2 (n - n_r)(w_r).\frac{\left[\left(1-\lambda_2^{\alpha_2}\right)v_2 w_r \lambda_2^{\alpha_2 - 1}\right]\left[\left[(\alpha_2 - 1) v_2 \lambda_2^{-1}\right] - 1\right] + \left[\alpha_2 w_r \lambda_2^{2(\alpha_2 - 1)} v_2^2\right]}{\left[1-\lambda_2^{\alpha_2}\right]^2},$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \theta_1} = -\sum_{i=1}^{r} \frac{w_i u_i z_i}{(1-u_i)} + \alpha_1 (m - m_r)(w_r) v_1.\frac{\left(1-\lambda_1^{\alpha_1}\right)\left(\alpha_1 \lambda_1^{\alpha_1 - 1} \ln \lambda_1 + \lambda_1^{\alpha_1 - 1}\right) + \left(\alpha_1 \lambda_1^{2\alpha_1 - 1} \ln \lambda_1\right)}{\left[1-\lambda_1^{\alpha_1}\right]^2},$$

and

$$-\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \theta_2} = -\sum_{i=1}^{r} \frac{w_i q_i (1-z_i)}{(1-q_i)} + \alpha_2 (n - n_r)(w_r) v_2.\frac{\left(1-\lambda_2^{\alpha_2}\right)\left(\lambda_2^{\alpha_2 - 1} \ln \lambda_2\right) + \left(\alpha_2 \lambda_2^{2\alpha_2 - 1} \ln \lambda_2\right)}{\left[1-\lambda_2^{\alpha_2}\right]^2} \qquad (5)$$

Using the asymptotic normality of the MLEs, we can express the approximate $100(1-\alpha)\%$ confidence intervals for $\alpha_1, \alpha_2, \theta_1$ and θ_2.

Suppose that $\hat{\delta}$ is the MLE of the parameter vector $\delta = (\alpha_1, \alpha_2, \theta_1, \theta_2)$. Denote the Fisher information matrix corresponding to δ by I_δ and $\phi = \lim_{n\to\infty} n I_\delta^{-1}$. Then, $\hat{\delta}$ is asymptotically normal distributed (see Serfling (1980)), i.e., $\sqrt{n}(\hat{\delta} - \delta) \sim N(0, \phi)$. In particular, let $\left(\hat{S}_{\hat{\alpha}_i}\right)^2 = \hat{\phi}_{(i,i)}/n, i = 1,2,$ are the (i,i) elements in the matrix $\hat{\phi} = n \hat{I}_\delta^{-1}$ and $\hat{I}_\delta$ is the estimator of I_δ. Therefore, asymptotic normality confidence intervals of δ_i, $i = 1,2,$ with confidence level $100(1-\alpha)\%$ are given by

$$\hat{\alpha}_i \pm z_{(1-\alpha/2)} \hat{S}_{\hat{\alpha}_i},\ i = 1,2, \text{ and } \hat{\theta}_i \pm z_{(1-\alpha/2)} \hat{S}_{\hat{\theta}_i},\ i = 1,2.$$

where $z_{(1-\alpha)/2}$ denotes the upper $(1-\alpha)/2$ percentage point of the standard normal distribution.

3. Bootstrap Confidence Intervals

In this section, we present several bootstrap methods to construct confidence intervals for $\alpha_1, \alpha_2, \theta_1$ and θ_2, viz., Studentized-t interval (Boot-t) and Percentile interval (Boot-p) (see Efron (1982) and Efron and Tibshirani (1994) for details).

a) Bootstrap Percentile Interval Procedure (Boot-p)

The bootstrap percentile method defines the lower and upper bounds of the confidence intervals just using the $100\alpha/2th$ and $100(1-\alpha/2)th$ quantiles of the empirical bootstrap distribution of $\hat{\alpha}_i^*$ and $\hat{\theta}_i^*, i = 1,2$ respectively. In particular:

(1) Compute the MLE $(\hat{\alpha}_i, \hat{\theta}_i)$ of (α_i, θ_i) based on joint type II censored sample (w, z).

(2) Use $(\hat{\alpha}_i, \hat{\theta}_i)$ to generate a bootstrap joint type II censored sample (w^*, z^*) and compute the bootstrap estimate of (α_i, θ_i), say $(\hat{\alpha}_i^*, \hat{\theta}_i^*)$, based on this bootstrap sample.

(3) Repeat step 2 B times to have $\hat{\alpha}_i^{*(1)}, \hat{\alpha}_i^{*(2)}, \ldots, \hat{\alpha}_i^{*(B)}$ and $\hat{\theta}_i^{*(1)}, \hat{\theta}_i^{*(2)}, \ldots, \hat{\theta}_i^{*(B)}$.

(4) Arrange $\hat{\alpha}_i^{*(1)}, \hat{\alpha}_i^{*(2)}, \ldots, \hat{\alpha}_i^{*(B)}$ and $\hat{\theta}_i^{*(1)}, \hat{\theta}_i^{*(2)}, \ldots, \hat{\theta}_i^{*(B)}$ in ascending order and obtain $\hat{\alpha}_i^{*[1]}, \hat{\alpha}_i^{*[2]}, \ldots, \hat{\alpha}_i^{*[B]}$ and $\hat{\theta}_i^{*[1]}, \hat{\theta}_i^{*[2]}, \ldots, \hat{\theta}_i^{*[B]}$.

(5) A two-sided $100(1-\alpha)\%$ percentile bootstrap confidence interval for (α_i, θ_i), say $[\hat{\alpha}_{iL}^*, \hat{\alpha}_{iU}^*]$ and $[\hat{\theta}_{iL}^*, \hat{\theta}_{iU}^*]$ is given by

$$(\hat{\alpha}_{iL}^*, \hat{\alpha}_{iU}^*) = \left(\hat{\alpha}_i^{*([B\alpha/2])}, \hat{\alpha}_i^{*([B(1-\alpha/2])}\right) \text{ and } (\hat{\theta}_{iL}^*, \hat{\theta}_{iU}^*) = \left(\hat{\theta}_i^{*([B\alpha/2])}, \hat{\theta}_i^{*([B(1-\alpha/2])}\right).$$

b) Studentized-t Interval Procedure (Boot-t)

The Boot-t confidence intervals estimators are computed according to the following steps:

(1–2) Same as the steps 1–2 in (a).

(3) Compute the t –statistic $T_{\hat{\alpha}_i^*} = (\hat{\alpha}_i^* - \hat{\alpha}_i) \big/ \hat{S}_{\hat{\alpha}_i^*}$ and $T_{\hat{\theta}_i^*} = (\hat{\theta}_i^* - \hat{\theta}_i) \big/ \hat{S}_{\hat{\theta}_i^*}$ where $\hat{S}_{\hat{\alpha}_i^*}$ and $\hat{S}_{\hat{\theta}_i^*}$ are the bootstrap versions.

(4) Repeat steps 2–3 B times and obtain $T_{\hat{\alpha}_i^*}^{(1)}, T_{\hat{\alpha}_i^*}^{(2)}, \ldots, T_{\hat{\alpha}_i^*}^{(B)}$ and $T_{\hat{\theta}_i^*}^{(1)}, T_{\hat{\theta}_i^*}^{(2)}, \ldots, T_{\hat{\theta}_i^*}^{(B)}$.

(5) Arrange $T_{\hat{\alpha}_i^*}^{(1)}, T_{\hat{\alpha}_i^*}^{(2)}, \ldots, T_{\hat{\alpha}_i^*}^{(B)}$ and $T_{\hat{\theta}_i^*}^{(1)}, T_{\hat{\theta}_i^*}^{(2)}, \ldots, T_{\hat{\theta}_i^*}^{(B)}$ in ascending order and obtain $T_{\hat{\alpha}_i^*}^{[1]}, T_{\hat{\alpha}_i^*}^{[2]}, \ldots, T_{\hat{\alpha}_i^*}^{[B]}$ and $T_{\hat{\theta}_i^*}^{[1]}, T_{\hat{\theta}_i^*}^{[2]}, \ldots, T_{\hat{\theta}_i^*}^{[B]}$.

(6) A two-sided $100(1-\alpha)\%$ bootstrap-t confidence interval for (α_i, θ_i) say $[\hat{\alpha}_{i,tL}^*, \hat{\alpha}_{i,tL}^*]$ and $[\hat{\theta}_{i,tL}^*, \hat{\theta}_{i,tU}^*]$, is given by

$$\left(\hat{\alpha}_i + T_{\hat{\alpha}_i^*}^{([B\alpha/2])} \hat{S}_{\hat{\alpha}_i}, \; \hat{\alpha}_i + T_{\hat{\alpha}_i^*}^{([B(1-\alpha/2)])} \hat{S}_{\hat{\alpha}_i}\right), \; i = 1, 2,$$

and

$$\left(\hat{\theta}_i + T_{\hat{\theta}_i^*}^{([B\alpha/2])} \hat{S}_{\hat{\theta}_i}, \hat{\theta}_i + T_{\hat{\theta}_i^*}^{([B(1-\alpha/2)])} \hat{S}_{\hat{\theta}_i}\right), \; i = 1, 2,$$

In section 6, we will have a simulation study in order to evaluate the performance of the three confidence intervals.

4. Bayes Estimators

In this section, Bayesian method is used to obtain the estimators for the unknown parameters $\alpha_i,\ i=1,2$ and $\theta_i,\ i=1,2$ using symmetric squared error loss function and asymmetric LINEX loss functions.

We consider that $\alpha_1,\alpha_2,\theta_1$ and θ_2 have the following independent gamma prior distributions;

$$\pi(\alpha_k)\propto \frac{b_k^{a_k}}{\Gamma(a_k)}\alpha_k^{a_k-1}e^{-b_k\alpha_k},\quad a_k,b_k,\alpha_k>0,$$

and

$$\pi(\theta_k)\propto \frac{b_k^{a_k}}{\Gamma(a_k)}\theta_k^{a_k-1}e^{-b_k\theta_k},\quad \theta_k>0,\ k=1,2 \tag{6}$$

Here all the hyper parameters a_k and b_k are assumed to be known and non-negative.

Combining (6) with equation (3) and using Bayes theorem, the joint posterior density function of $\alpha_1,\alpha_2,\theta_1$ and θ_2 can be written as:

$$l(data\setminus\alpha_1,\alpha_2,\theta_1,\theta_2)=\frac{1}{\psi}L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z)\,\pi(\alpha_k)\,\pi(\theta_k)$$

where $$\psi=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z)\,\pi(\alpha_k)\,\pi(\theta_k)\,d\alpha_k d\theta_k$$

Therefore, the Bayes estimator of any function of $\alpha_1,\alpha_2,\theta_1$ and θ_2, say $\delta(\alpha_1,\alpha_2,\theta_1,\theta_2)$ under the squared error loss function is

$$\hat{\delta}_B=E_{\alpha_1,\alpha_2,\theta_1,\theta_2\setminus data}\left(\delta(\alpha_1,\alpha_2,\theta_1,\theta_2)\right)$$

$$=\frac{1}{\psi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \delta(\alpha_1,\alpha_2,\theta_1,\theta_2)\,L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z)\,\pi(\alpha_k)\,\pi(\theta_k)\,d\alpha_k d\theta_k \tag{7}$$

Under a LINEX loss function the Bayes estimate of a function $\delta(\alpha_1,\alpha_2,\theta_1,\theta_2)$ is given by

$$\hat{\delta}_L=-\frac{1}{c}\ln E(e^{-c\delta}),\ c\neq 0, \tag{8}$$

where

$$E(e^{-c\delta})=\frac{1}{\psi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-c\delta}\,L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z)\,\pi(\alpha_k)\,\pi(\theta_k)\,d\alpha_k d\theta_k$$

Equations (4), (5), (7) and (8) are hard to obtain. An iterative procedure is applied to solve these equations numerically.

5. Estimation of the stress–strength reliability $R = P(Y < X)$

In this section, we consider the problem of estimating reliability in the stress strength model when the strength of a unit or a system, *X*, has cumulative distribution function $F_1(x)$ and the stress subject to it, *Y*, has cumulative distribution function $F_2(y)$. The main purpose of this section is the focus on the inference on $R = P(Y < X)$, where, *X* and *Y* are independent generalized exponential random variables under joint type II censoring scheme. The maximum likelihood estimation and confidence intervals using asymptotic distribution and two parametric bootstrap resampling methods for parameter *R* are explored.

If *X* is the strength of a system which is subjected to a stress *Y*, then *R* is a measure of system performance, the system fails if at any time the applied stress is greater than its strength, The estimation of *R* is very common in the statistical literature. For example, Tong (1974, 1975 and 1977), Constantine and Karson (1986), Ahmad *et al.* (1997), Surles and Padgett (2001), Kundu and Gupta (2005 and 2006), and Raqab and Kundu (2005) have discussed this problem when *X* and *Y* are two independent exponential, gamma, Burr type X, and generalized exponential random variables. However, the censoring sampling has not been taken into account in these works. Finally, Chien-Tai and Shun (2013) consider the problem of estimation for $R = P(Y < X)$, where *X* and *Y* are two independent but not identically general location-scale distributed random variables under the joint progressively type-II right censoring scheme.

5.1 Maximum Likelihood Estimation of Reliability *R*

Let *X*, *Y* have generalized exponential distribution with pdfs $f_1(x)$ and $f_2(y)$ and cumulative distributions functions $F_1(x)$ and $F_2(y)$ in equation (1). Here, we consider the problem of estimating reliability $R = P(Y < X)$ based on a joint type-II censored sample mentioned in section 1. The reliability function is defined as

$$R = P(Y < X) = \int_0^\infty F_2(y) f_1(x)\, dx \tag{9}$$

By using the MLEs $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta}_1$ and $\hat{\theta}_2$ from likelihood equations (4), can be obtained the MLE of *R* (invariance property) as

$$\hat{R} = \int_0^\infty \hat{\theta}_1 \hat{\alpha}_1 e^{-\hat{\theta}_1 x} \left[1 - e^{-\hat{\theta}_2 x}\right]^{\hat{\alpha}_2} \left[1 - e^{-\hat{\theta}_1 x}\right]^{\hat{\alpha}_1 - 1} dx\ , \tag{10}$$

which can be solved by using an iterative numerical method.

5.2 Interval Estimation for *R*

In this sub section, we propose different methods of constructing confidence intervals for *R*. The first method is based on the asymptotic distribution of $\hat{R}$. Other methods are based on two parametric bootstrap methods: the percentile bootstrap (Boot- *p*) and bootstrap- *t* methods (Boot- *t*).

a) Asymptotic Normality Procedure

By best asymptotically normal property of the MLE, the asymptotic variance of $\hat{R}$ is found to be

$$Var(\hat{R}) = Q' I_{\beta}^{-1} Q \tag{11}$$

where $Q' = \left(\frac{\partial R}{\partial \alpha_1}, \frac{\partial R}{\partial \alpha_2}, \frac{\partial R}{\partial \theta_1}, \frac{\partial R}{\partial \theta_2} \right)$, Q' denotes the transpose of Q and I_{β} Fisher information matrix presented in equation (5). If we replace the variance by its estimate, we can easily obtain an approximate $100(1-\alpha)$ confidence interval for *R* as

$$\hat{R} \pm z_{(1-\alpha/2)} \sqrt{Var(\hat{R})}$$

b) Parametric Boot-*p* confidence interval

To obtain the Boot- p confidence intervals for *R*, we use the following algorithms

1. Based on joint type-II censored sample (w, z), compute the MLEs $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta}_1$ and $\hat{\theta}_2$ and then the MLE $\hat{R}$ of *R*.
2. Generate random samples from two independent generalized exponential with parameters $(\hat{\alpha}_1, \hat{\theta}_1)$ and $(\hat{\alpha}_2, \hat{\theta}_2)$ of sizes *m* and *n*, respectively. Then, generate a bootstrap joint type-II censored sample (w^*, z^*).
3. Compute the MLE $\hat{R}$ based on (w^*, z^*).
4. Repeat Steps 2–3 B times and obtain $\hat{R}^{*(1)}, \hat{R}^{*(2)}, \ldots, \hat{R}^{*(B)}$.
5. Arrange $\hat{R}^{*(1)}, \hat{R}^{*(2)}, \ldots, \hat{R}^{*(B)}$ in ascending order to obtain the bootstrap sample $(\hat{R}^{*[1]}, \hat{R}^{*[2]}, \ldots, \hat{R}^{*[B]})$.

Then, a two-sided $100(1-\alpha)$ Boot- p confidence interval for *R* is given by

$$\left(\hat{R}^{*[B\alpha/2]}, \hat{R}^{*[B(1-\alpha/2)]} \right)$$

c) Parametric Boot- *t* confidence interval

1. Repeat the same Steps 1 to 3 above.
2. Compute $Var(\hat{R}^*)$ from equation (11), and compute the t –statistic

$$T^* = \frac{(\hat{R}^* - \hat{R})}{\sqrt{Var(\hat{R}^*)}}$$

3. Arrange $T^{*(1)}, T^{*(2)}, \ldots, T^{*(B)}$ in ascending order to obtain the bootstrap sample $(T^{*[1]}, T^{*[2]}, \ldots, T^{*[B]})$.

Then, a two-sided $100(1-\alpha)$ Boot- t confidence interval for *R* is given by

$$\left(\hat{R} + T^{*[B\alpha/2]} \sqrt{Var(\hat{R})}, \hat{R} + T^{*[B(1-\alpha/2)]} \sqrt{Var(\hat{R})} \right)$$

6. Numerical Illustration

It clear that, there are no explicit solutions for obtaining new estimators in both non-Bayesian and Bayesian approaches. Therefore artificial data, numerical solution and computer facilities are needed. The main object of this section is to illustrate numerically most of the new theoretical result obtained in the previous two sections.

6.1 Illustrative Example

We consider two samples of size $m = n = 10$ each from Nelson's data (1982), (groups 3 and 5 in Table 4.1, [27, p.462]) which correspond to breakdown in minutes of an insulating fluid subjected to high voltage stress. These failure times, denoted here as groups *X* and *Y*, are presented in table 1.

Table 2 presents the jointly type-II censored data that have been obtained from the two samples in table 1 with $r = 15$. We then computed the MLE and Bayesian estimates of $\alpha_1, \alpha_2, \theta_1$ and θ_2 (with the choice of ($a_i, i = 1,2,3,4$ and $b_i, i = 1,2,3,4$) as hyperparameters) based on the data in table 2, and these results are presented in table 3.

Table 4 presents the 95% approximate, Boot-p and Boot-t confidence intervals for $\alpha_1, \alpha_2, \theta_1$ and θ_2 corresponding to case $r = 15$. From these results, we observe that Boot-*p* and Boot-*p* confidence intervals are satisfactory compared to the approximate confidence.

Table 1: The failure time data for groups *X* and *Y*

Group	Data									
X	1.99,	0.64,	2.15,	1.08,	2.57,	0.93,	4.75,	0.82,	2.06,	0.49
Y	8.11,	3.17,	5.55,	0.80,	0.20,	1.13,	6.63,	1.08,	2.44,	0.78

Table 2: The jointly type-II censored data, with *r* = 15, from groups *X* and *Y* in table 1

W	.20	.49	.64	.78	.80	.82	.93	1.08	1.08	1.13	1.99	2.06	2.15	2.44	2.57
Z	0	1	1	0	0	1	1	1	0	0	1	1	1	0	1

Table 3: The MLE, Bayesian estimates, root mean squared errors ($\sqrt{MSE}$) and estimated risks (ER) for $\alpha_1, \alpha_2, \theta_1$ and θ_2

	MLE	$\sqrt{MSE}$	SE	ER	LINEX			
					c =3	ER	c = -0.5	ER
$\hat{\alpha}_1$	3.578	2.119	3.43	0.0047	3.507	0.000044	3.46	0.00157
$\hat{\alpha}_2$	1.378	0.767	1.44	0.0033	1.513	0.0002	1.422	0.0062
$\hat{\theta}_1$	1.202	0.432	1.21	0.0002	1.26	0.0036	1.191	0.0001
$\hat{\theta}_2$	0.477	0.281	0.53	0.00069	0.483	0.0003	0.603	0.0105

Table 4: The 95% approximate, Bootstrap-*p* and Bootstrap-*t* confidence intervals for $\alpha_1, \alpha_2, \theta_1$ and θ_2

r = 15				
	CI for α_1	CI for α_2	CI for θ_1	CI for θ_2
Approximate	(-0.571, 7.728)	(-0.107, 2.863)	(0.361, 2.043)	(-0.072, 1.025)
Boot-p	(2.077, 8.305)	(0.776, 3.058)	(0.508, 3.468)	(0.213, 0.905)
Boot-t	(-0.581, 5.46)	(-0.216, 1.952)	(1.047, 2.375)	(0.275, 0.827)

6.2 Monte Carlo Simulation

A simulation study was conducted in order to evaluate the performance of MLEs and also all the confidence intervals discussed in the preceding sections. We considered different sample sizes for the two populations as m = 15, 20, 40, 50 and n = 15, 20, 40, 50, and different choices for r = 12, 16, 18, 24, 32, 40, 48, 60, 64, 80. We also chose the parameters $(\alpha_1, \alpha_2, \theta_1, \theta_2)$ to be $(1, 0.6, 0.15, 0.2)$. For these cases, we computed the MLEs, root mean squared errors $\sqrt{MSE}$ and the 95% confidence intervals for $(\alpha_1, \alpha_2, \theta_1, \theta_2)$ and R (with $(R = P(Y < X) = 0.683)$) using approximate, Boot-*t* and Boot-*p* methods (with *N*-Boot as 1000) and the corresponding coverage probabilities. We computed the Bayesian estimates of $(\alpha_1, \alpha_2, \theta_1, \theta_2)$ under the SE and LINEX loss functions. We repeated this process 1000 times and computed the average values of all the estimates and the estimated risk (ER) for each estimate. The average value of the MLEs $(\alpha_1, \alpha_2, \theta_1, \theta_2)$ and ($\sqrt{MSE}$) summarized in tables 5.

From table 6 we observe that the Bayesian estimates under the SE and LINEX loss functions and their ERs. In tables 7–10, the coverage probabilities and the average widths of 95% CIs $(\alpha_1, \alpha_2, \theta_1, \theta_2)$ for all the methods are presented for some small, moderate and large values of *m* and *n*.

From table 11 we observe The average value of the MLE R, ($\sqrt{MSE}$), and average widths of the 95% confidence intervals based on approximate, Boot- p , and Boot- t methods with corresponding coverage probabilities for small, moderate and large values of m , n and r with $(R = P(Y < X) = 0.683)$.

Table 5: The average value of the MLEs $(\alpha_1, \alpha_2, \theta_1, \theta_2)$ and ($\sqrt{MSE}$) for small, moderate and large values of m , n and r

$\alpha_1 = 1, \alpha_2 = 0.6, \theta_1 = 0.15$ and $\theta_2 = 0.20$									
(m, n)	r	$\hat{\alpha}_1$	$\sqrt{MSE}$	$\hat{\alpha}_2$	$\sqrt{MSE}$	$\hat{\theta}_1$	$\sqrt{MSE}$	$\hat{\theta}_2$	$\sqrt{MSE}$
	12	1.084	0.332	0.644	0.187	0.165	0.042	0.213	0.035
(15,15)	18	1.081	0.322	0.645	0.187	0.160	0.036	0.209	0.035
	24	1.078	0.315	0.645	0.185	0.160	0.034	0.209	0.035
	16	1.061	0.273	0.630	0.157	0.161	0.035	0.208	0.030
(20,20)	24	1.053	0.249	0.628	0.155	0.159	0.029	0.208	0.030
	32	1.049	0.241	0.628	0.154	0.159	0.028	0.208	0.029
	32	1.039	0.186	0.618	0.099	0.156	0.024	0.203	0.021
(40,40)	48	1.033	0.175	0.618	0.098	0.155	0.021	0.203	0.021
	64	1.032	0.173	0.617	0.098	0.155	0.020	0.203	0.020
	40	1.024	0.16	0.611	0.091	0.154	0.02	0.204	0.019
(50,50)	60	1.019	0.153	0.611	0.089	0.154	0.019	0.204	0.019
	80	1.019	0.151	0.611	0.089	0.153	0.018	0.204	0.019

Table 6: Bayesian estimates and (ER) of $(\alpha_1,\alpha_2,\theta_1,\theta_2)$ for different choices of $m,n,r,$ $a_1=1,a_2=1,a_3=1,a_4=1,b_1=1.5,b_2=1.75,b_3=2.5,b_4=3$, and c = -0.5 with $\alpha_1=1,\alpha_2=0.6,\theta_1=0.15$ and $\theta_2=0.20$

(m,n)	r	SE				LINEX (c = -0.5)			
		$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$
	12	1.073	0.644	0.165	0.21	1.064	0.644	0.165	0.193
	ER	0.106	0.035	0.0018	0.0012	0.0124	0.0042	0.00022	0.0001
(15,15)	18	1.07	0.645	0.16	0.207	1.061	0.645	0.16	0.189
	ER	0.1	0.035	0.0013	0.0012	0.0117	0.0041	0.00016	0.0002
	24	1.067	0.645	0.16	0.207	1.058	0.645	0.16	0.19
	ER	0.096	0.034	0.0012	0.0012	0.0112	0.0041	0.00014	0.0002
	16	1.076	0.636	0.162	0.204	1.067	0.636	0.162	0.186
	ER	0.973	0.026	0.0012	0.0009	0.0231	0.0032	0.000147	0.0001
(20,20)	24	1.04	0.635	0.159	0.204	1.03	0.635	0.159	0.186
	ER	0.067	0.026	0.001	0.0009	0.008	0.0031	0.000118	0.0001
	32	1.037	0.635	0.159	0.204	1.028	0.635	0.159	0.186
	ER	0.064	0.026	0.0008	0.0009	0.0077	0.0031	0.000101	0.0001
	24	1.029	0.626	0.158	0.225	0.996	0.620	0.154	0.209
	ER	0.0026	0.0006	0.0006	0.00062	0.000002	0.00041	0.00009	0.00001
(40,40)	48	1.029	0.626	0.158	0.225	0.996	0.620	0.154	0.209
	ER	0.0026	0.0006	0.0006	0.00062	0.000002	0.00041	0.00009	0.00001
	64	1.029	0.626	0.158	0.225	0.996	0.620	0.154	0.209
	ER	0.0026	0.0006	0.0006	0.00062	0.000002	0.00041	0.00009	0.00001
	40	0.99	0.61	0.152	0.205	1.071	0.641	0.154	0.209
	ER	0.0001	0.0001	0.000006	0.00002	0.000623	0.00021	0.0000025	0.000009
(50,50)	60	0.99	0.61	0.152	0.205	1.071	0.641	0.154	0.209
	ER	0.0001	0.0001	0.000006	0.00002	0.000623	0.00021	0.0000025	0.000009
	80	0.99	0.61	0.152	0.205	1.071	0.641	0.154	0.209
	ER	0.0001	0.0001	0.000006	0.00002	0.000623	0.00021	0.0000025	0.000009

Table 7: Simulated coverage probabilities (CP) and the average widths of the 95% confidence intervals of α_1 for some small, moderate and large values of m,n and r with $\alpha_1=1$

(m,n)	r	Approximate		Boot-p		Boot-t	
		CP(%)	Length	CP(%)	Length	CP(%)	Length
	12	99.994	2.589	71.4	0.826	73.05	16.621
(15,15)	18	99.884	1.926	91.11	1.076	76.86	0.917
	24	99.508	1.643	89.22	1.221	71.65	0.882
	16	61.872	1.533	91.19	0.66	97.14	1.287
(20,20)	24	99.869	1.641	92.57	0.746	90.47	0.747
	32	99.472	1.388	91.92	0.845	86.72	0.724
	32	99.997	1.431	93.23	0.491	93.39	0.542
(40,40)	48	99.94	1.118	95.57	0.536	95.56	0.625
	64	99.692	0.953	94.04	0.675	93.24	0.708
	40	99.994	1.266	92.22	0.536	86.25	0.48
(50,50)	60	99.884	0.987	93.17	0.432	95.5	0.477
	80	99.486	0.841	93.81	0.499	94.72	0.553

Table 8: Simulated CP and the average widths of the 95% confidence intervals of α_2 for some small, moderate and large values of m,n and r with $\alpha_2 = 0.6$

(m,n)	r	Approximate		Boot-p		Boot-t	
		CP(%)	Length	CP(%)	Length	CP(%)	Length
	12	99.717	1.055	91.33	0.743	98.91	0.492
(15,15)	18	99.197	0.93	90.84	0.826	74.18	0.551
	24	98.572	0.856	90.5	0.829	75.6	0.568
	16	98.957	0.713	92.46	0.443	97.99	0.742
(20,20)	24	99.307	0.75	92.77	0.506	96.05	0.731
	32	98.767	0.694	92.14	0.769	75.91	0.533
	32	99.636	0.597	94.2	0.311	90.67	0.288
(40,40)	48	99.035	0.528	93.67	0.39	89.37	0.346
	64	98.404	0.489	92.82	0.465	86.36	0.401
	40	99.639	0.524	94.21	0.331	85.16	0.254
(50,50)	60	99.092	0.463	93.51	0.369	82.25	0.283
	80	98.47	0.431	93.36	0.454	82.2	0.316

Table 9: Simulated CP and the average widths of the 95% confidence intervals of θ_1 for some small, moderate and large values of m,n and r with $\theta_1 = 0.15$

(m,n)	r	Approximate		Boot-p		Boot-t	
		CP(%)	Length	CP(%)	Length	CP(%)	Length
	12	100	0.675	94.92	0.087	46.53	0.049
(15,15)	18	100	0.4	95.52	0.121	92.13	0.106
	24	99.999	0.276	95.27	0.146	96.26	0.153
	16	100	0.434	94.04	0.074	89.14	0.068
(20,20)	24	100	0.344	95.74	0.096	88.13	0.078
	32	99.998	0.241	94.54	0.134	91.21	0.122
	32	100	0.383	95.84	0.066	95.21	0.064
(40,40)	48	100	0.235	95.21	0.08	93.15	0.074
	64	99.999	0.165	95.05	0.103	94.71	0.101
	40	100	0.342	95.73	0.064	94.27	0.06
(50,50)	60	100	0.211	94.64	0.064	94.28	0.063
	80	99.997	0.148	95.02	0.088	93.32	0.084

Table 10: Simulated CP and the average widths of the 95% confidence intervals of θ_2 for some small, moderate and large values of m,n and r with $\theta_2 = 0.20$

(m,n)	r	Approximate		Boot-p		Boot-t	
		CP(%)	Length	CP(%)	Length	CP(%)	Length
(15,15)	12	100	0.621	96.29	0.159	70.93	0.046
	18	100	0.451	95.34	0.157	93.59	0.149
	24	100	0.365	94.31	0.216	81.05	0.152
(20,20)	16	100	0.474	95.61	0.113	85.94	0.08
	24	100	0.39	95.28	0.137	87.51	0.112
	32	100	0.317	94.77	0.209	92.7	0.194
(40,40)	32	100	0.37	95.11	0.074	94.32	0.072
	48	100	0.271	94.64	0.09	93.76	0.088
	64	100	0.219	94.95	0.16	95.54	0.169
(50,50)	40	100	0.332	94.79	0.066	94.38	0.065
	60	100	0.243	95.21	0.079	95.6	0.08
	80	100	0.197	94.62	0.125	90.47	0.108

Table 11: The average value of the MLE R, ($\sqrt{MSE}$), and average widths of the 95% confidence intervals based on approximate, Boot- *p*, and Boot- *t* methods with corresponding coverage probabilities for small, moderate and large values of m, n and r with $(R = P(Y < X) = 0.683)$

(m,n)	r	$\hat{R}$	$\sqrt{MSE}$	Approximate		Boot-p		Boot-t	
				CP(%)	Length	CP(%)	Length	CP(%)	Length
(15,15)	12	0.669	0.0988	99.6	0.529	82.7	0.412	87.9	0.397
	18	0.673	0.092	99.2	0.513	79.8	0.402	88.1	0.387
	24	0.675	0.0924	99.1	0.447	84.5	0.287	91.2	0.268
(20,20)	16	0.669	0.0867	99.8	0.569	92.7	0.438	95.9	0.490
	24	0.671	0.0837	98.8	0.418	98.4	0.481	100	0.714
	32	0.672	0.0818	97.9	0.368	100	0.635	100	0.832
(40,40)	32	0.679	0.0622	99.5	0.373	95.6	0.256	97	0.269
	48	0.681	0.0589	98.6	0.284	97.4	0.289	94.6	0.259
	64	0.6825	0.0577	96.9	0.246	99.6	0.384	99.8	0.406
(50,50)	40	0.681	0.0548	99.8	0.33	92.8	0.209	90.9	0.194
	60	0.681	0.0522	98.5	0.244	93.7	0.209	92	0.190
	80	0.680	0.0524	96.2	0.217	92.6	0.191	91.5	0.183

7. Conclusions

In this paper, the MLEs and Bayesian estimation based on the SE and LINEX loss functions for the unknown parameters of two Generalized exponential distributions has been discussed based on a joint type- II censored sample. We obtained the MLEs of the parameters and found corresponding Fisher information matrix. Also, we studied three approximate methods, Asymptotic Normality, Bootstrap-t and Parametric Bootstrap percentile procedures for constructing intervals for the parameters. The MLEs and the Bayesian estimates have then been compared through a Monte Carlo simulation study and a numerical example has also been presented to illustrate all the inferential results established here. The computational results show that the Bayesian estimation based on the SE, and LINEX loss functions is more precise than the MLEs estimation. Also, the ERs and MSE of all the estimates decrease with increasing r even when the sample sizes m and n are small. We assessed the performance of the mentioned three confidence intervals. According to the simulation study, when the sample sizes of two populations, n and m, and the total number of failures r, are large, the estimators' biases are small and the confidence intervals have desirable coverage probabilities. Also, we observed that the approximate better than the two bootstrap methods often perform as well as each other. Finally, the estimation of the stress strength parameter $R = P(Y < X)$ has been considered.

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