Analysis of Exponential Distribution Under Adaptive Type-I Progressive Hybrid Censored Competing Risks Data

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Abstract

A competing risks model based on exponential distribution is considered under adaptive type-I progressive hybrid censoring scheme. We investigate the maximum likelihood estimation and Bayesian estimation for the distribution parameter. The Bayes estimate of the unknown parameter is obtained based on squared error and LINEX loss functions under the assumption of gamma prior. The asymptotic confidence intervals, the Bayes credible intervals and two parametric bootstrap confidence intervals are also proposed. To evaluate the performance of the estimators, a simulation study is carried out.

Keywords: Competing risks, Adaptive type-I progressive hybrid censoring, Exponential distribution, Maximum likelihood estimation, Bayesian estimation, Bootstrap confidence intervals.

1. Introduction

Lin and Huang (2012), introduced a new progressive hybrid censoring scheme called an adaptive type-I progressive hybrid censoring scheme (*AT-I PHCS*), and it can be described as follows: suppose n identical units are put on test with progressive censoring scheme $R_1, R_2, ..., R_m$, $1 \le m \le n$, and the experiment is terminated at a prefixed time T, where $T \in (0,\infty)$ and integers R_i 's are prefixed. At the time of the first failure $x_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure $x_{2:m:n}$, R_2 of the remaining units are randomly removed and so on. Let J denote the number of failures that occur before time T. If the *m-th* failure $x_{m:m:n}$ occurs before time T (i.e $x_{m:m:n} < T$), the process will not stop, but continue to observe failures without any further withdrawals until reach time T. Then, at time T all remaining units $R_J^* = n - J - \sum_{i=1}^{J} R_i$ are removed and the experiment is terminated. The progressive censoring scheme in this case will become $R_1, R_2, ..., R_m, R_{m+1}, ..., R_J$, where $R_m = R_{m+1} = ... = R_J = 0$. Otherwise, the process when $x_{m:m:n} > T$ will have a progressive censoring scheme as $R_1, R_2, ..., R_J$.

Progressive hybrid censoring scheme in the context of life testing studies has become quite popular. Kundu and Joarder. (2006) considered a type-I progressive hybrid censoring scheme (*T-I PHCS*), where the experiment is terminated at time $T^* = \min\{x_{m:m:n}, T\}$. They investigated the maximum likelihood and Bayesian estimation for the exponential distribution. Childs et al. (2008) proposed a type-II progressive hybrid censoring scheme, where the experiment is terminated at time $T^* = \max\{x_{m:m:n}, T\}$, and derived the exact distribution of the maximum likelihood estimator. Ng et al. (2009) introduced a new censoring scheme, called an adaptive type-II progressive hybrid censoring, where the number of failures *m* and the corresponding progressively scheme is given, but no units will be removed when the experimental time passes time *T*. Recently, Lin and Huang. (2012) proposed another adaptive progressive hybrid censoring scheme, *AT-I PHCS,* which assures the termination of the life testing experiment at a fixed time *T* and results a higher efficiency in estimations as compared with *T-I PHCS,* they studied point and interval estimation for the exponential distribution based on an *AT-I PHCS*. Lin et al. (2012) investigated the maximum likelihood and Bayesian estimation for a two-parameter Weibull distribution based on *AT-I PHCS*. They obtained the Bayes estimates of the unknown parameters by using the approximated form of Lindley (1980) and Tierney and Kadane (1986).

The main aim of this paper is analyzing the competing risk model when lifetimes have independent exponential distributions under an *AT-I PHCS*. We derive the maximum likelihood estimators (MLE) and Bayes estimators under squared error and LINEX loss functions using gamma priors. We also obtain the asymptotic confidence interval, credible interval and two bootstrap confidence intervals.

The rest of this paper is organized as follows: In section 2, we introduce the model and the notation used throughout this paper. In section 3, we discuss the maximum likelihood estimation. The Bayes estimators of the parameter under squared error and LINEX loss functions are derived in section 4. Different confidence intervals are presented in Section 5. In section 6, numerical illustration of the maximum likelihood and Bayes estimates and the corresponding confidence intervals are presented. Finally, in section 7, we obtain the ML and Bayes estimators when the causes of failure are unknown.

2. Model Description and Notation

In reliability analysis, the failure of items may be attributable to more than one cause at the same time. These "causes" are competing for the failure of the experimental unit. Consider a life time experiment with $n \in N$ identical units, where its lifetimes are described by independent and identically distributed (i.i.d) random variables $X_1, X_2, ..., X_n$. Without loss of generality; assume that there are only two causes of failure. We have $X_i = \min\{X_{1i}, X_{2i}\}$ for $i = 1, ..., n$, where $X_{ki}, k = 1,2$, denotes the latent failure time of the *i-th* unit under the *k-th* cause of failure. We assume that the latent failure times X_{1i} and X_{2i} are independent, and the pairs (X_{1i}, X_{2i}) are i.i.d. Assume that the failure times follows the exponential distribution with cumulative distribution function F_k and failure hazard function h_k have the form

$$F_k = 1 - e^{-\lambda_k x}, \quad h_k = \lambda_k, \quad x > 0, \lambda > 0, k = 1,2 \tag{1}$$

Under adaptive type-I progressive censoring scheme and in presence of competing risks data we have the following observation:

$$(X_{1:m:n}, c_1, R_1), ..., (X_{m-1:m:n}, c_m, R_{m-1}), (X_{m:m:n}, c_m, 0), ,..., (X_{J:m:n}, c_J, 0), (T, R_J^*)$$

where c_i is the indicator denoting the cause of failure, J denote the number of failures before time T and R_J^* is the number of remaining units left at the time point T with $R_m = R_{m+1}, ..., R_J = 0$. Let $c_i \in (1,2)$. Here, $c_i = k, k = 1,2$ means the unit i has failed due to cause *k*. Further, we define

$$I_1(c_i = 1) = \begin{cases} 1, & c_i = 1 \\ 0 & else \end{cases} \quad \text{and} \quad I_2(c_i = 2) = \begin{cases} 1, & c_i = 2 \\ 0 & else \end{cases}$$

Thus, the random variables $J_1 = \sum_{i=1}^{J} I_1(c_i = 1)$ and $J_2 = \sum_{i=1}^{J} I_2(c_i = 2)$ describe the number of failures due to the first and the second cause of failures, respectively. Both J_1 and J_2 follow binomial distributions with sample size *J*. using the independence of the latent failure times X_{1i}, X_{2i}, $i = 1, ..., n$, we can obtain the relative risk rate due to a particular cause (say, cause 1) as follows

$$P(X_{1i} \le X_{2i}) = \int_0^\infty \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 x} .dx = \frac{\lambda_1}{\lambda_1 + \lambda_2} \quad \text{similarly,} \quad P(X_{2i} \le X_{1i}) = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

For a given censoring scheme $R_1, R_2, ..., R_{m-1}, 0, ..., , 0, R_J^*$, the likelihood function of the observed data $(x_1, c_1), ..., (T, R_J^*)$ is given by

$$L(\lambda_1, \lambda_2) = c_J . \prod_{i=1}^{J} \left\{ \left[f_1(x_i).\bar{F}_2(x_i) \right]^{I(c_i=1)} \left[f_2(x_i).\bar{F}_1(x_i) \right]^{I(c_i=2)} . \left[\bar{F}_1(x_i).\bar{F}_2(x_i) \right]^{R_i} \right\}$$
$$\times \left[\bar{F}_1(T).\bar{F}_2(T) \right]^{R_J^*}$$

where $x_i = x_{i:m:n}$ for simplicity of notation, $c_J = \prod_{i=1}^{J} \gamma_i$ with $\gamma_i = m - i + 1 - \sum_{j=1}^{m} R_j$.

Applying the identity $f_k = h_k .\bar{F}_k$, we can write the likelihood function as follows

$$L(\lambda_1, \lambda_2) = c_J . \prod_{i=1}^{J} \left\{ \left[h_1(x_i) \right]^{I(c_i=1)} \left[h_2(x_i) \right]^{I(c_i=2)} . \left[\bar{F}_1(x_i).\bar{F}_2(x_i) \right]^{1+R_i} \right\} . \left[\bar{F}_1(T).\bar{F}_2(T) \right]^{R_J^*} \tag{2}$$

we denote $J = J_1 + J_2$, and $J > 0$.

3. Maximum Likelihood Estimation

From (1) and (2), the likelihood function ignoring the normalized constant can be written as follows

$$L(\lambda_1,\lambda_2)=\lambda_1^{J_1}.\lambda_2^{J_2}.\exp\left\{-\left(\lambda_1+\lambda_2\right)\left[\sum_{i=1}^{J}\left(1+R_i\right)x_i+R_J^{*}T\right]\right\} \tag{3}$$

and the log-likelihood function give $J\geq 1$

$$\ln L(\lambda_1,\lambda_2)=J_1\ln\lambda_1+J_2\ln\lambda_2-\left(\lambda_1+\lambda_2\right)\left[\sum_{i=1}^{J}\left(1+R_i\right)x_i+R_J^{*}T\right] \tag{4}$$

upon differentiating (4) with respect to λ_1 and λ_2 we get the likelihood equations as

$$\frac{\partial\ln L(\lambda_1,\lambda_2)}{\partial\lambda_1}=\frac{J_1}{\lambda_1}-\ell(x),$$

$$\frac{\partial\ln L(\lambda_1,\lambda_2)}{\partial\lambda_2}=\frac{J_2}{\lambda_2}-\ell(x) \tag{5}$$

where $\ell(x)=\sum_{i=1}^{J}\left(1+R_i\right)x_i+R_J^{*}T$ is the total time on test (TTT). Equating the first derivations (5) to zero, we get the MLE of λ_1 and λ_2 in the following form

$$\hat{\lambda}_k=\frac{J_k}{\ell(x)},\ k=1,2$$

From the log-likelihood function in (4), we have

$$\frac{\partial^2\ln L(\lambda_1,\lambda_2)}{\partial\lambda_1^2}=-\frac{J_1}{\lambda_1^2},$$

$$\frac{\partial^2\ln L(\lambda_1,\lambda_2)}{\partial\lambda_2^2}=-\frac{J_2}{\lambda_2^2},$$

and

$$\frac{\partial^2\ln L(\lambda_1,\lambda_2)}{\partial\lambda_2\partial\lambda_1}=0 \tag{6}$$

The Fisher information matrix $I(\lambda_1,\lambda_2)$ is then obtained by taking the expectation of minus equations (6), this expectation is difficult to obtained, so, under some regularity conditions, $(\hat{\lambda}_1,\hat{\lambda}_2)$ is approximately bivariately normal with mean (λ_1,λ_2) and covariance matrix $I^{-1}(\lambda_1,\lambda_2)$. Practically, we estimate $I^{-1}(\lambda_1,\lambda_2)$ by $I^{-1}(\hat{\lambda}_1,\hat{\lambda}_2)$, then

$$I^{-1}(\hat{\lambda}_1,\hat{\lambda}_2)=\left.\begin{bmatrix}-\dfrac{\partial^2\ln L(\lambda_1,\lambda_2)}{\partial\lambda_1^2} & -\dfrac{\partial^2\ln L(\lambda_1,\lambda_2)}{\partial\lambda_1\partial\lambda_2}\\ -\dfrac{\partial^2\ln L(\lambda_1,\lambda_2)}{\partial\lambda_2\partial\lambda_1} & -\dfrac{\partial^2\ln L(\lambda_1,\lambda_2)}{\partial\lambda_2^2}\end{bmatrix}^{-1}\right|_{(\lambda_1=\hat{\lambda}_1,\lambda_2=\hat{\lambda}_2)}=\begin{bmatrix}var(\hat{\lambda}_1) & 0\\ 0 & var(\hat{\lambda}_2)\end{bmatrix}$$

4. Bayesian Estimation

We consider the Bayesian estimation under the assumption that the random variables λ_k, $k=1,2$, has a gamma prior distribution with known shape and scale parameters a_k and b_k, with pdf given by

$$g(\lambda_k)=\frac{b_k^{a_k}}{\Gamma(a_k)}.\lambda_k^{a_k-1}.e^{-\lambda_k.b_k}, \quad \lambda_k>0,\ a_k,b_k>0 \tag{7}$$

combining (3) and (7), the joint posterior density of λ_1 and λ_2 given the data is

$$g(\lambda_1,\lambda_2\,|\,\underline{x})=\frac{1}{A}.\lambda_1^{a_1+J_1-1}.\lambda_2^{a_2+J_2-1}.\exp\left\{-\left[\lambda_1\left(b_1+\ell(x)\right)+\lambda_2\left(b_2+\ell(x)\right)\right]\right\}$$

where $A=\dfrac{\Gamma(a_1+J_1)\Gamma(a_2+J_2)}{\left(b_1+\ell(x)\right)^{a_1+J_1}.\left(b_2+\ell(x)\right)^{a_2+J_2}}$.

The marginal posterior of λ_1 and λ_2 are given as follows

$$g(\lambda_1\,|\,\underline{x})=\frac{\left(b_1+\ell(x)\right)^{a_1+J_1}}{\Gamma(a_1+J_1)}.\lambda_1^{a_1+J_1-1}.\exp\left\{-\lambda_1\left[b_1+\ell(x)\right]\right\}$$

and

$$g(\lambda_2\,|\,\underline{x})=\frac{\left(b_2+\ell(x)\right)^{a_2+J_2}}{\Gamma(a_2+J_2)}.\lambda_2^{a_2+J_2-1}.\exp\left\{-\lambda_2\left[b_2+\ell(x)\right]\right\}$$

It is important to state that, in Bayesian estimation, we consider two types of loss functions. The first is the squared error loss function. The second, the LINEX loss function introduced by Varian (1975). The LINEX loss function with parameters a and d is given by

$$\ell\left(\tilde{\theta},\theta\right)=a\left\{e^{d(\tilde{\theta}-\theta)}-d(\tilde{\theta}-\theta)-1\right\} \tag{8}$$

where a and d are constants. The sign and magnitude of d represent the direction and degree of symmetry, respectively. From (8) the Bayes estimator $\tilde{\theta}$ of θ is given by

$$\tilde{\theta}_{lin}=-\frac{1}{d}\ln E\left(e^{-d\,\theta}\right),\ d\neq 0 \tag{9}$$

for d closed to zero, the LINEX loss is approximately squared error loss.

Under squared error loss function, the Bayes estimator of λ_1 and λ_2 is the posterior mean which obtained as follows

$$\tilde{\lambda}_{Sq1}=\frac{a_1+J_1}{b_1+\ell(x)} \quad \text{and} \quad \tilde{\lambda}_{Sq2}=\frac{a_2+J_2}{b_2+\ell(x)}$$

For the non-informative priors $a_1 = b_1 = a_2 = b_2 = 0$, the Bayes estimators coincide with the corresponding MLEs. Also, the posterior risk associated with λ_k , $k = 1,2$ can be written as

$$R(\lambda_k) = E\left(\lambda_k^2\right) - \left[E\left(\lambda_k\right)\right]^2$$

where

$$E\left(\lambda_k^r\right) = \mu_k^r = \frac{1}{A}\frac{\Gamma(a_1 + J_1 + \delta r).\Gamma(a_2 + J_2 + \xi r)}{\left(b_1 + \ell(x)\right)^{a_1+J_1+\delta r}\left(b_2 + \ell(x)\right)^{a_2+J-J_1+\xi r}},\ r = 1,2$$

is the marginal posterior r th moments of λ_k , $k = 1,2$, and $\xi = 0, \delta = 1$ for $k = 1$ and $\xi = 1, \delta = 0$ for $k = 2$.

Under LINEX loss function (8), the Bayes estimator of λ_1 and λ_2 can be obtained as follows

$$\tilde{\lambda}_{lin1} = \frac{-(a_1 + J_1)}{d}\ln\left\{\frac{b_1 + \ell(x)}{d + b_1 + \ell(x)}\right\}$$

and

$$\tilde{\lambda}_{lin2} = \frac{-(a_2 + J_2)}{d}\ln\left\{\frac{b_2 + \ell(x)}{d + b_2 + \ell(x)}\right\},\ d \neq 0$$

One can use other asymmetric loss functions, such as, modified LINEX loss function proposed by Basu and Ibrahimi (1991), which appears to be suitable for the estimation of scale parameters and other quantities, also, entropy loss function suggested by Calabria and Pulcini (1994), that is alternative to the modified LINEX loss function.

5. Confidence Intervals

In this section, we propose four different confidence intervals. One is based on the asymptotic distribution of λ_1 and λ_2, the second is the credible intervals based on the posterior distribution, and two different bootstrap confidence intervals.

- **Asymptotic confidence interval (NA)**

The $100(1-\alpha)$ approximate confidence intervals for λ_1 and λ_2 can be obtained using the asymptotic normality of the MLEs as follows

$$\hat{\lambda}_1 \pm z_{\alpha/2}.\sqrt{\text{var}(\hat{\lambda}_1)} \text{ and } \hat{\lambda}_2 \pm z_{1-\alpha/2}.\sqrt{\text{var}(\hat{\lambda}_2)}$$

where $\text{var}(\hat{\lambda}_1) = \hat{\lambda}_1^2 / J_1$ and $\text{var}(\hat{\lambda}_2) = \hat{\lambda}_2^2 / J_2$ are the elements on the main diagonal of the covariance matrix $I^{-1}(\hat{\lambda}_1, \hat{\lambda}_2)$ and $z_{\alpha/2}$ is the upper $\frac{\alpha}{2}$ - *th* percentile point of a standard normal distribution.

- **Credible interval (BA)**

The credible interval of λ_1 and λ_2 can be obtained using the posterior distributions of λ_1 and λ_2. The posterior of $Z_1 = 2\lambda_1(b_1 + \ell(x))$ and $Z_2 = 2\lambda_2(b_2 + \ell(x))$ follows χ^2 distribution with $[2(a_1 + J_1)]$ and $[2(a_2 + J_2)]$ degrees of freedom respectively. Therefore, $100(1-\alpha)\%$ credible intervals for λ_1 and λ_2 are

$$\left\{\frac{\chi^2_{[2(a_1+J_1)],1-\frac{\alpha}{2}}}{2(b_1+\ell(x))}, \frac{\chi^2_{[2(a_1+J_1)],\frac{\alpha}{2}}}{2(b_1+\ell(x))}\right\} \text{ and } \left\{\frac{\chi^2_{[2(a_2+J_2)],1-\frac{\alpha}{2}}}{2(b_2+\ell(x))}, \frac{\chi^2_{[2(a_2+J_2)],\frac{\alpha}{2}}}{2(b_2+\ell(x))}\right\}$$

where $(a_1 + J_1) > 0$ and $(a_2 + J_2) > 0$. Note that if $[2(a_1 + J_1)]$ and $[2(a_2 + J_2)]$ are not integer values, then gamma distribution can be used to construct the credible intervals.

- **Bootstrap confidence intervals**

Here, we construct two parametric bootstrap confidence intervals for λ_1 and λ_2

A) Percentile bootstrap confidence interval (PB)

1- Compute the MLE of λ_k using the original adaptive type-I progressive hybrid censored sample with censoring scheme $R_1,...,R_{m-1},0,...,,0,R_J^*$ and prefixed *T*.

2- Generate a bootstrap sample using $\hat{\lambda}_k$, $R_1,...,R_{m-1},0,...,,0,R_J^*$ and *T* to obtain the bootstrap estimate of λ_k say $\hat{\lambda}_k^b$ using the bootstrap sample.

3- Repeat step (2) *B* times to have $\lambda_k^{b(1)},\lambda_k^{b(2)},...,\lambda_k^{b(B)}$.

4- Arrange $\lambda_k^{b(1)},\lambda_k^{b(2)},...,\lambda_k^{b(B)}$ in ascending order as $\lambda_k^{b[1]},\lambda_k^{b[2]},...,\lambda_k^{b[B]}$.

5- A two-sided $100(1-\alpha)\%$ percentile bootstrap confidence interval for the unknown parameter λ_k is given by

$$\left\{\hat{\lambda}_k^{b[B\alpha/2]}, \hat{\lambda}_k^{b[B(1-\alpha/2)]}\right\}$$

B) Bootstrap-*t* confidence interval (TB)

1-2) Same as the steps (1) in Boot-p

3) Compute the *t*-statistic $T = (\hat{\lambda}_k^b - \hat{\lambda}_k)/\sqrt{V(\hat{\lambda}_k^b)}$, where $V(\hat{\lambda}_k^b)$ is the asymptotic variances of $\hat{\lambda}_k^b$ and it can be obtained using the Fisher information matrix.

4. Repeat step 2-3 *B* times and obtain $T^{(1)},T^{(2)},...,T^{(B)}$.

5. Arrange $T^{(1)},T^{(2)},...,T^{(B)}$ in ascending order as $T^{[1]},T^{[2]},...,T^{[B]}$.

6. A two-sided $100(1-\alpha)\%$ bootstrap-*t* confidence interval for the unknown parameter λ_k is given by

$$\hat{\lambda}_k + T^{[B\alpha/2]}\sqrt{V(\hat{\lambda}_k)},\ \hat{\lambda}_k + T^{[B(1-\alpha/2)]}\sqrt{V(\hat{\lambda}_k)}$$

6. Numerical Results

The performance of the different results obtained in the previous sections can't be compared theoretically, to illustrate the behavior of the proposed methods as well as evaluate the statistical performances of these estimates a numerical illustration is conducted. We re-analyze a real data set analyzed by Hoel (1972), and reused by Kundu et al. (2004). Also, a simulations study is used to compare the performance of the different estimators, different confidence intervals using different parameter values and different schemes.

Example: In this section, we re-analyze one data set which was originally analyzed by Hoel (1972) and later by Kundu et al. (2004), Pareek et al. (2009) and Cramer and Schmiedt (2011). The data was obtained from a laboratory experiment in which male mice received a radiation dose of 300 roentgens at 35 days to 42 days (5-6 weeks) of age. The cause of death for each mouse was determined by reticulum cell sarcoma as cause 1 and other causes of death as cause 2, there were $n = 77$ observations remain in the analysis. The progressively type-II censored data was generated and first used by Kundu et al. (2004). Considering $T = 630$ and using the censoring scheme $m = 25$ and $R_1 = R_2 = ... = R_{24} = 2$, the adaptive progressive type-I censored sample from the original data is given by

(40, 2), (42, 2), (62, 2), (163, 2), (179, 2), (206, 2), (222, 2), (228, 2), (252, 2), (259,2), (318, 1), (385, 2), (407, 2), (420, 2), (462, 2), (517, 2), (517, 2), (524, 2), (525, 1),(558, 1),(536, 1), (605, 1), (612, 1), (620,2), (621, 1), (622, 2), (628,1).

The first component denotes the life time and the second component indicate the cause of failure. There were $J = 27$, $J_1 = 8$, $J_2 = 19$ and $R_{25} = R_{26} = 0$ and $R_{27}^* = 2$. From the above data, we obtain the following:

$\sum_{i=1}^{J}(1+R_i)x_i + R_J^* T = 29108$, which yields

Parameter	ML estimate	Bayes estimate under Squared error loss
λ_1	2.74838×10^{-4} (9.442×10^{-9})	3.092×10^{-4} (1.06212×10^{-8})
λ_2	6.52741×10^{-4} (2.242×10^{-8})	6.871×10^{-4} (2.3603×10^{-8})

where the variances and the Bayes risk reported within brackets. Also, the relative risk due to cause 1 is 0.296, and due to cause 2 is 0.704, The MLE's of the mean lifetimes due to cause 1 and cause 2 are given by

$$\hat{\lambda}_1^{-1} = 3638.5 \text{ and } \hat{\lambda}_2^{-1} = 1532.$$

Now we report the 95% asymptotic, credible intervals, Boot-P and Boot-t confidence intervals

Methods	λ_1	λ_2
NA	$(2.7482\times10^{-4}, 2.74857\times10^{-4})$	$(6.5269779\times10^{-4}, 6.5278524\times10^{-4})$
BA	$(1.414\times10^{-4}, 5.415\times10^{-4})$	$(4.197\times10^{-4}, 1.019\times10^{-3})$
PB	$(2.2932\times10^{-4}, 3.4131\times10^{-4})$	$(5.4465\times10^{-4}, 8.1061\times10^{-4})$
TB	$(2.2029\times10^{-4}, 7.0627\times10^{-4})$	$(5.2319\times10^{-4}, 7.7987\times10^{-4})$

The analysis of the previous real data set demonstrate the importance and usefulness of adaptive type-I progressive hybrid censoring scheme and inferential procedures based on them.

Simulation study: A simulation study is conducted to evaluate the behavior of the ML and Bayes estimates by considering different values of sample sizes $n = 30, 50$, different effective number of failures $m = 5, 10$ and $T = 0.4, 0.6$, and by choosing $(\lambda_1, \lambda_2) = (0.4, 0.6)$ and $(1, 0.8)$ in all the cases, and considered three different sampling schemes

Scheme 1: $R_1 = \cdots = R_{m-1} = 0$ and $R_m = n - m$,

Scheme 2: $R_1 = \cdots = R_{m-1} = 1$ *and* $R_m = n - 2m + 1$, and

Scheme 3: $R_1 = \cdots = R_{m-1} = R_m = (n - m)/m$,

For each case, the MLEs and Bayes estimates under squared error and LINEX (with $d = 0.1$) loss functions of λ_1 and λ_2 are computed based on 1000 simulations, with the assumption that the number of failures due to each cause of failures at least one, and the parameters distributed as a random variables with gamma prior distributions with parameter (1, 1.5) and (1, 1.5), respectively. The average values, average bias, root mean squared errors and average number of observed failures J_A for the ML and Bayes estimates of λ_1 and λ_2 are reported in tables 1 and 2 . The average 95% confidence length of asymptotic confidence intervals, the credible intervals with respect to the gamma prior distributions, Boot-p and Boot-t confidence intervals of λ_1 and λ_2 and the corresponding coverage probabilities are reported in tables 3 and 4. All of the computations were performed using MATLAB and MATHCAD program version 2007.

From table 1 and 2, we observed that in most cases the MLE of λ_1 and λ_2 has smaller biases than the Bayes estimates, and the ordering of performance of estimators in term of minimum root mean squared errors (from best to worst) for λ_1 and λ_2 is Bayes estimates under LINEX, squared error loss functions and MLE's. Comparing the three censoring scheme based on minimum root mean squared errors shows that the performance of estimation for scheme 1 is best followed by scheme 2 and then scheme 3. When T becomes larger, the root mean square errors decreases, this is not being very surprising, because when T increases some additional information is gathered. From table 3 and 4, in terms of coverage probabilities and average confidence lengths we observed that the Bayes credible intervals quite close to the nominal level than other three methods, Among these methods, PB has the shortest average lengths followed by BA, NA, and then TB, also, we observed that, in all cases when T increases the average length decreases.

7. Unknown Causes of Failure

In all procedures mentioned above, we assume that the cause of failure for all individuals to be known. We now consider the situation of unknown causes of failure, let $J_1 = \sum_{i=1}^{J} I_1(c_i = 1)$ and $J_2 = \sum_{i=1}^{J} I_2(c_i = 2)$ describe the number of failures due to the first and the second cause of failures, respectively, and $J_3 = \sum_{i=1}^{J} I_3(c_i = *)$ is the number of failures having failure times but corresponding causes of failure are unknown. Let us also denote $J^* = J_1 + J_2$, and therefore $J = J^* + J_3$. The likelihood function of the observed data ignoring the constant is

$$L(\lambda_1,\lambda_2) = \lambda_1^{J_1}\lambda_2^{J_2}\left(\lambda_1+\lambda_2\right)^{(J-J^*)}\exp\left\{-\left(\lambda_1+\lambda_2\right)\left[\sum_{i=1}^{J}\left(1+R_i\right)x_i + R_J^*T\right]\right\} \tag{10}$$

Taking the natural logarithm of (10), we obtain

$$\ln L(\lambda_1,\lambda_2) = J_1\ln\lambda_1 + J_2\ln\lambda_2 + (J-J^*)\ln\left(\lambda_1+\lambda_2\right) - \left(\lambda_1+\lambda_2\right)\left[\sum_{i=1}^{J}\left(1+R_i\right)x_{(i)} + R_J^*T\right] \tag{11}$$

Upon differentiating (11) with respect to λ_1 and λ_2 and equating the partial derivatives to zeros, we obtain the MLE's of λ_1 and λ_2 as

$$\hat{\lambda}_k = \frac{J.J_k}{J^*.\ell(x)}, k = 1,2$$

Consider the Bayesian estimation under the assumption of gamma prior distribution (7), the joint posterior density of λ_1 and λ_2 given the data is can be written as follows

$$g(\lambda_1,\lambda_2 \mid \underline{x}) = \frac{1}{A1}.\sum_{i=0}^{J-J^*}\binom{J-J^*}{i}.\lambda_1^{a_1+J_1+i-1}.\lambda_2^{a_2+J-J_1-i-1}.\exp\left\{-\left(\lambda_1+\lambda_2\right)\ell(x) - \left(b_1\lambda_1 + b_2\lambda_2\right)\right\} \tag{12}$$

where $A1 = \sum_{i=0}^{J-J^*} \binom{J-J^*}{i} \frac{\Gamma(a_1+J_1+i)\Gamma(a_2+J-J_1-i)}{\left(b_1+\ell(x)\right)^{a_1+J_1+i} . \left(b_2+\ell(x)\right)^{a_2+J-J_1-i}}$.

Under squared error loss function, the Bayes estimator of λ_1 and λ_2 is the posterior mean which obtained as follows

$$\tilde{\lambda}_{Sq1} = \frac{1}{A1} \sum_{i=0}^{J-J^*} \binom{J-J^*}{i} \frac{\Gamma(a_1+J_1+i+1).\Gamma(a_2+J-J_1-i)}{\left(b_1+\ell(x)\right)^{a_1+J_1+i+1} \left(b_2+\ell(x)\right)^{a_2+J-J_1-i}}$$

and

$$\tilde{\lambda}_{Sq2} = \frac{1}{A1} \sum_{i=0}^{J-J^*} \binom{J-J^*}{i} \frac{\Gamma(a_1+J_1+i).\Gamma(a_2+J-J_1-i+1)}{\left(b_1+\ell(x)\right)^{a_1+J_1+i} \left(b_2+\ell(x)\right)^{a_2+J-J_1-i+1}}$$

Under LINEX loss function (8), the Bayes estimator of λ_1 and λ_2 can be obtained as follows

$$\tilde{\lambda}_{lin1} = \frac{-1}{d} \ln\left\{ \frac{1}{A1} \sum_{i=0}^{J-J^*} \binom{J-J^*}{i} \frac{\Gamma(a_1+J_1+i).\Gamma(a_2+J-J_1-i)}{\left(d+b_1+\ell(x)\right)^{a_1+J_1+i} \left(b_2+\ell(x)\right)^{a_2+J-J_1-i}} \right\}$$

and

$$\tilde{\lambda}_{lin2} = \frac{-1}{d} \ln\left\{ \frac{1}{A1} \sum_{i=0}^{J-J^*} \binom{J-J^*}{i} \frac{\Gamma(a_1+J_1+i).\Gamma(a_2+J-J_1-i)}{\left(b_1+\ell(x)\right)^{a_1+J_1+i} \left(d+b_2+\ell(x)\right)^{a_2+J-J_1-i}} \right\}, \; d \neq 0$$

All the procedures discussed in sections (5) can be easily modified to the present situation.

Table (1)[c]: The average biases, root mean squared errors and average number of failures of the ML and Bayes estimates of $(\lambda_1, \lambda_2) = (0.4, 0.6)$ under different censoring scheme and different T 's.

Scheme	*(n ,m)*	*ML Estimates*		*Bayes Estimates*				J_A
		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\tilde{\lambda}_{Sq1}$	$\tilde{\lambda}_{Sq2}$	$\tilde{\lambda}_{lin1}$	$\tilde{\lambda}_{lin2}$	
				T = 0.4				
1	(30,5)	0.003	0.024	0.032	0.028	0.03	0.025	9.908
		0.209	0.267	0.185	0.232	0.184	0.23	
2		0.001	0.028	0.039	0.031	0.037	0.028	8.862
		0.216	0.281	0.189	0.24	0.188	0.239	
3		0.045	0.033	0.085	0.023	0.081	0.018	4.928
		0.32	0.391	0.252	0.287	0.248	0.283	
1	(50,5)	0.015	0.022	0.009	0.025	0.007	0.023	16.397
		0.179	0.214	0.164	0.197	0.164	0.196	
2		0.014	0.03	0.01	0.032	0.009	0.031	15.359
		0.177	0.226	0.161	0.206	0.161	0.205	
3		0.032	0.039	0.071	0.035	0.068	0.031	6.354
		0.272	0.328	0.229	0.262	0.226	0.259	
1	(30,10)	0.008	0.001	0.041	0.006	0.039	0.004	9.79
		0.206	0.249	0.184	0.216	0.183	0.215	
2		0.017	0.005	0.053	0.011	0.05	0.007	8.202
		0.238	0.277	0.207	0.233	0.206	0.232	
3		0.021	0.008	0.06	0.011	0.057	0.007	6.895
		0.253	0.309	0.216	0.248	0.213	0.245	
1	(50,10)	0.002	0.003	0.024	0.002	0.022	0.0005	16.331
		0.16	0.186	0.149	0.171	0.148	0.17	
2		0.004	0.002	0.028	0.003	0.027	0.001	14.092
		0.174	0.21	0.16	0.19	0.16	0.189	
3		0.018	0.011	0.049	0.013	0.047	0.009	8.667
		0.239	0.279	0.209	0.231	0.207	0.229	
				T = 0.6				
1	(30,5)	0.01	0.023	0.017	0.026	0.015	0.024	13.448
		0.189	0.237	0.171	0.214	0.17	0.213	
2		0.008	0.033	0.022	0.035	0.02	0.032	11.91
		0.206	0.253	0.184	0.225	0.183	0.223	
3		0.043	0.059	0.081	0.045	0.077	0.04	5.885
		0.305	0.396	0.248	0.302	0.244	0.298	
1	(50,5)	0.026	0.029	0.008	0.031	0.008	0.03	22.401
		0.165	0.197	0.154	0.186	0.153	0.185	
2		0.022	0.033	0.003	0.035	0.004	0.033	20.957
		0.173	0.216	0.16	0.202	0.159	0.201	
3		0.019	0.038	0.056	0.036	0.054	0.033	7.657
		0.258	0.32	0.221	0.262	0.219	0.259	
1	(30,10)	0.005	0.02	0.03	0.023	0.029	0.021	13.565
		0.176	0.214	0.161	0.193	0.161	0.192	
2		0.015	0.033	0.045	0.035	0.043	0.032	10.774
		0.197	0.251	0.179	0.218	0.178	0.217	
3		0.02	0.042	0.054	0.039	0.051	0.035	8.426
		0.229	0.292	0.2	0.239	0.198	0.237	
1	(50,10)	0.005	0.019	0.021	0.021	0.02	0.02	22.771
		0.136	0.167	0.13	0.157	0.129	0.157	
2		0.013	0.021	0.031	0.024	0.03	0.022	19.338
		0.156	0.182	0.148	0.169	0.148	0.168	
3		0.02	0.047	0.049	0.044	0.047	0.041	9.935
		0.216	0.282	0.193	0.238	0.192	0.236	

[c] The first and the second row in each cell represent the average biases and root mean squared errors of λ_1 and λ_2 respectively.

Table (2)[d]: The average biases, root mean squared errors and average number of failures of the ML and Bayes estimates of $(\lambda_1, \lambda_2) = (1, 0.8)$ under different censoring scheme and different T 's.

Scheme	*(n ,m)*	*ML Estimates* $\hat{\lambda}_1$	*ML Estimates* $\hat{\lambda}_2$	*Bayes Estimates* $\tilde{\lambda}_{Sq1}$	*Bayes Estimates* $\tilde{\lambda}_{Sq2}$	*Bayes Estimates* $\tilde{\lambda}_{lin1}$	*Bayes Estimates* $\tilde{\lambda}_{lin2}$	J_A
				T = 0.4				
1	(30,5)	0.019	0.044	0.037	0.015	0.042	0.011	15.497
		0.356	0.341	0.3	0.284	0.299	0.282	
2		0.044	0.034	0.023	0.003	0.029	0.002	13.699
		0.397	0.362	0.322	0.296	0.32	0.294	
3		0.093	0.11	0.061	0.01	0.071	0.001	6.396
		0.593	0.541	0.376	0.349	0.372	0.344	
1	(50,5)	0.0002	0.033	0.033	0.016	0.036	0.013	25.718
		0.283	0.281	0.256	0.251	0.255	0.25	
2		0.003	0.014	0.038	0.002	0.042	0.004	23.571
		0.283	0.268	0.255	0.24	0.254	0.239	
3		0.066	0.08	0.046	0.013	0.054	0.006	8.517
		0.491	0.473	0.348	0.338	0.345	0.334	
1	(30,10)	0.011	0.028	0.044	0.002	0.049	0.002	15.414
		0.346	0.304	0.294	0.255	0.293	0.254	
2		0.012	0.044	0.059	0.005	0.065	0.0003	11.814
		0.391	0.365	0.315	0.289	0.314	0.286	
3		0.025	0.067	0.075	0.007	0.082	0.0004	8.915
		0.48	0.45	0.349	0.32	0.347	0.316	
1	(50,10)	0.041	0.001	0.005	0.019	0.038	0.003	25.552
		0.241	0.215	0.266	0.24	0.241	0.216	
2		0.004	0.027	0.036	0.008	0.039	0.005	21.616
		0.285	0.263	0.253	0.231	0.252	0.23	
3		0.033	0.062	0.059	0.009	0.065	0.003	10.333
		0.458	0.415	0.343	0.307	0.341	0.304	
				T = 0.6				
1	(30,5)	0.016	0.022	0.029	0.001	0.033	0.002	19.786
		0.335	0.29	0.291	0.253	0.29	0.251	
2		0.014	0.044	0.037	0.017	0.042	0.013	17.331
		0.346	0.321	0.295	0.272	0.294	0.27	
3		0.086	0.069	0.049	0.004	0.058	0.012	7.412
		0.584	0.509	0.389	0.342	0.385	0.338	
1	(50,5)	0.005	0.021	0.022	0.009	0.024	0.007	33.072
		0.269	0.243	0.248	0.223	0.247	0.222	
2		0.007	0.04	0.022	0.024	0.025	0.022	30.638
		0.275	0.262	0.251	0.237	0.25	0.236	
3		0.05	0.06	0.043	0.01	0.05	0.004	10.159
		0.467	0.412	0.348	0.309	0.346	0.306	
1	(30,10)	0.015	0.01	0.03	0.01	0.034	0.013	19.705
		0.307	0.281	0.267	0.245	0.266	0.244	
2		0.046	0.016	0.02	0.011	0.025	0.016	14.66
		0.373	0.326	0.302	0.269	0.3	0.268	
3		0.064	0.03	0.038	0.018	0.045	0.024	9.979
		0.485	0.414	0.347	0.307	0.344	0.305	
1	(50,10)	0.007	0.003	0.02	0.008	0.022	0.01	32.847
		0.242	0.212	0.223	0.195	0.222	0.195	
2		0.015	0.006	0.018	0.008	0.021	0.01	27.391
		0.26	0.236	0.234	0.213	0.233	0.212	
3		0.066	0.028	0.025	0.012	0.031	0.017	11.467
		0.456	0.375	0.338	0.291	0.336	0.288	

[d] The first and the second row in each cell represent the average biases and root mean squared errors of λ_1 and λ_2 respectively.

Table (3)[e]: The average 95% confidence lengths and the coverage probabilities $(\lambda_1, \lambda_2) = (0.4, 0.6)$ for different methods and different censoring scheme and different T 's.

Scheme	*(n, m)*		*NA*	*BA*	*PB*	*TB*
			T = 0.4			
1		λ_1	0.761 (86.8)	0.736 (98.4)	0.472 (87.8)	1.207 (76.6)
		λ_2	0.968 (90.2)	0.898 (94.5)	0.489 (91.4)	1.112 (81.4)
2	(30,5)	λ_1	0.811 (84.8)	0.779 (98.1)	0.493 (91.2)	1.120 (75.8)
		λ_2	1.028 (91.8)	0.945 (96.2)	0.521 (93.5)	1.315 (79.7)
3		λ_1	1.154 (94.5)	1.034 (98.6)	0.653 (97.7)	1.401 (84.1)
		λ_2	1.392 (88.6)	1.183 (96.7)	0.801 (91.5)	1.751 (81.1)
1		λ_1	0.582 (83.9)	0.572 (87.9)	0.389 (86.3)	0.732 (79.8)
		λ_2	0.753 (92.1)	0.718 (94)	0.392 (93.3)	0.711 (88.9)
2	(50,5)	λ_1	0.605 (86.4)	0.593 (88.8)	0.401 (87.5)	0.871 (71.5)
		λ_2	0.786 (91.7)	0.747 (93.2)	0.404 (92.1)	0.849 (79.6)
3		λ_1	1.009 (91.2)	0.933 (98.1)	0.618 (94.4)	1.487 (85.8)
		λ_2	1.244 (89.9)	1.095 (95.1)	0.734 (93.2)	1.720 (76.1)
1		λ_1	0.774 (91.8)	0.745 (98)	0.470 (92.7)	1.328 (79.4)
		λ_2	0.948 (91.3)	0.881 (95.5)	0.486 (93.9)	1.121 (80.4)
2	(30,10)	λ_1	0.855 (87.6)	0.813 (97.9)	0.531 (91.3)	1.368 (77.4)
		λ_2	1.042 (91.7)	0.954 (96.1)	0.573 (94.3)	1.227 (78.8)
3		λ_1	0.937 (87.1)	0.878 (98.6)	0.736 (96.4)	1.173 (71)
		λ_2	1.14 (89.1)	1.022 (96.6)	0.855 (95.4)	1.205 (80)
1		λ_1	0.601 (89.9)	0.586 (94.4)	0.369 (91.9)	0.769 (78.6)
		λ_2	0.738 (93.8)	0.705 (96.3)	0.372 (94.6)	0.658 (83.3)
2	50,10	λ_1	0.647 (91.7)	0.629 (95.6)	0.411 (92.2)	0.910 (82.5)
		λ_2	0.793 (92.2)	0.753 (94.6)	0.418 (93.6)	0.774 (80.5)
3		λ_1	0.841 (89.6)	0.795 (97)	0.699 (95.90)	1.229 (83.3)
		λ_2	1.03 (91.7)	0.939 (97)	0.791 (96.7)	1.068 (83.8)
			T = 0.6			
1		λ_1	0.646 (85.8)	0.631 (89.3)	0.409 (87.8)	0.868 (75.3)
		λ_2	0.831 (90)	0.785 (93.3)	0.416 (90.4)	0.803 (83.9)
2	(30,5)	λ_1	0.691 (83.7)	0.672 (93.1)	0.409 (92)	1.074 (78.4)
		λ_2	0.894 (91.8)	0.837 (93.6)	0.420 (92.8)	0.978 (85.5)
3		λ_1	1.069 (89.9)	0.973 (97.8)	0.610 (90.8)	1.619 (71.5)
		λ_2	1.321 (87.9)	1.14 (95.2)	0.736 (94.7)	1.961 (78.1)
1		λ_1	0.49 (84.9)	0.485 (87.3)	0.328 (86.4)	0.543 (74.2)
		λ_2	0.649 (90.6)	0.627 (92.2)	0.331 (91.1)	0.585 (79)
2	(50,5)	λ_1	0.512 (84.8)	0.506 (86.3)	0.331 (85.2)	0.616 (70.4)
		λ_2	0.677 (88.9)	0.651 (90.9)	0.336 (90.2)	0.644 (76.5)
3		λ_1	0.906 (87.5)	0.853 (96.9)	0.544 (89.3)	1.606 (77.8)
		λ_2	1.134 (88.5)	1.019 (93.7)	0.626 (91.7)	1.749 (84.6)
1		λ_1	0.666 (91.8)	0.647 (95.9)	0.412 (93.7)	0.812 (87)
		λ_2	0.833 (94.2)	0.787 (97.1)	0.419 (94.5)	0.719 (85.5)
2	(30,10)	λ_1	0.763 (90.6)	0.732 (96.7)	0.475 (95.1)	0.999 (82)
		λ_2	0.953 (94.4)	0.883 (96.2)	0.491 (94.8)	0.861 (80.8)
3		λ_1	0.869 (89.6)	0.819 (97.9)	0.716 (95.5)	1.076 (81.2)
		λ_2	1.089 (93.3)	0.984 (96.1)	0.810 (96)	1.017 (84.5)
1		λ_1	0.52 (92.3)	0.51 (94.9)	0.336 (94)	0.522 (82.6)
		λ_2	0.647 (94.6)	0.624 (96.1)	0.340 (95.3)	0.496 (86.3)

Continue Table (3):

2	(50,10)	λ_1	0.571 (91.5)	0.557 (94)	0.343 (92.4)	0.665 (85.6)
		λ_2	0.705 (93.7)	0.676 (95.9)	0.349 (95.5)	0.606 (81.1)
3		λ_1	0.808 (90.5)	0.766 (96.5)	0.621 (93.5)	1.138 (80.2)
		λ_2	1.014 (93.2)	0.927 (95.7)	0.697 (94.2)	1.113 (80.1)

[e] The first and second rows represent the average 95% confidence lengths of asymptotic confidence intervals, the credible intervals with respect to the gamma prior distributions, Boot-p and Boot-t confidence intervals of λ_1 and λ_2 respectively, and the corresponding coverage probabilities are reported within brackets.

Table (4)[f]: The average 95% confidence lengths and the coverage probabilities of $(\lambda_1, \lambda_2) = (1, 0.8)$ for different methods and different censoring scheme and different T 's.

Scheme	*(n, m)*		*NA*	*BA*	*PB*	*TB*
			T = 0.4			
1	(30,5)	λ_1	1.346 (92.5)	1.197 (94.6)	0.697 (91.7)	1.317 (84.2)
		λ_2	1.221 (91.4)	1.097 (94.2)	0.693 (92.3)	1.49 (80.9)
2		λ_1	1.452 (92.6)	1.271 (94.5)	0.750 (92.8)	1.372 (75)
		λ_2	1.292 (91.4)	1.146 (94.5)	0.743 (93)	1.503 (82.3)
3		λ_1	2.209 (90.3)	1.667 (95.2)	1.284 (90.5)	3.04 (79)
		λ_2	2.009 (91.4)	1.547 (98.1)	1.286 (91.9)	3.081 (87.9)
1	(50,5)	λ_1	1.032 (92.7)	0.961 (94.5)	0.575 (94.3)	0.923 (87.7)
		λ_2	0.941 (92.9)	0.881 (94.7)	0.569 (93.6)	1.016 (86.4)
2		λ_1	1.07 (92)	0.991 (94.4)	0.594 (92.3)	0.951 (86.8)
		λ_2	0.966 (91.4)	0.901 (94.3)	0.590 (93.5)	1.299 (84.9)
3		λ_1	1.882 (92.3)	1.52 (95.3)	1.108 (94.8)	2.617 (84)
		λ_2	1.699 (91.8)	1.397 (97.3)	1.038 (94.4)	2.766 (81.2)
1	(30,10)	λ_1	1.338 (92.8)	1.191 (94.9)	0.718 (92.8)	1.178 (57.8)
		λ_2	1.207 (93.2)	1.086 (96.3)	0.715 (94.3)	1.346 (65.6)
2		λ_1	1.529 (91.8)	1.316 (95.5)	0.847 (95.1)	1.531 (62.5)
		λ_2	1.391 (93)	1.212 (96.3)	0.832 (93.8)	1.825 (67.3)
3		λ_1	1.784 (91.9)	1.457 (95)	1.294 (94.45)	1.915 (79.7)
		λ_2	1.632 (91.9)	1.353 (96)	1.226 (95.4)	2.186 (83.1)
1	50,10	λ_1	1.031 (92.4)	0.959 (94.2)	0.561 (92.7)	0.793 (82.7)
		λ_2	0.934 (93.6)	0.874 (96.1)	0.552 (94.9)	0.840 (81.2)
2		λ_1	1.13 (93.4)	1.037 (95.1)	0.636 (94.3)	0.994 (80.9)
		λ_2	1.023 (93.1)	0.946 (95.7)	0.627 (93.4)	1.076 (88.7)
3		λ_1	1.701 (92.5)	1.415 (95.2)	1.227 (92.8)	2.189 (82.2)
		λ_2	1.52 (91.2)	1.286 (94.2)	1.152 (93.8)	2.297 (82.2)
			T = 0.6			
1	(30,5)	λ_1	1.184 (92.2)	1.079 (93.2)	0.626 (93.1)	1.075 (88.4)
		λ_2	1.065 (92.4)	0.978 (95.2)	0.615 (93.9)	1.143 (85.8)
2		λ_1	1.269 (92.4)	1.141 (93.7)	0.651 (92.9)	1.206 (86.0)
		λ_2	1.156 (92.5)	1.048 (95.1)	0.638 (92.7)	1.330 (83.7)
3		λ_1	2.028 (89.9)	1.587 (94.3)	1.321 (90.4)	2.857 (77.2)
		λ_2	1.802 (90.2)	1.446 (97.7)	1.206 (95.8)	2.749 (70.7)
1	(50,5)	λ_1	0.912 (92.2)	0.862 (93.1)	0.509 (92.7)	0.783 (89.8)
		λ_2	0.825 (93.3)	0.784 (94.1)	0.502 (93.3)	0.828 (87.8)
2		λ_1	0.953 (92.6)	0.896 (94)	0.542 (92.8)	0.821 (84.3)

Continue Table (4):

		λ_2	0.872 (93.1)	0.823 (94.1)	0.533 (94)	0.888 (83.1)
3		λ_1	1.701 (91.6)	1.42 (94.4)	1.025 (93.4)	2.225 (81.8)
		λ_2	1.533 (91.7)	1.301 (96.6)	0.961 (94.7)	2.366 (88.4)
1		λ_1	1.186 (94)	1.08 (96.2)	0.624 (95.7)	1.111 (89.5)
		λ_2	1.056 (92.3)	0.971 (94.9)	0.611 (93.7)	1.142 (85.4)
2	(30,10)	λ_1	1.394 (93)	1.229 (94.7)	0.767 (94.5)	1.341 (84.5)
		λ_2	1.232 (91.8)	1.101 (95.4)	0.743 (92.4)	1.358 (81.8)
3		λ_1	1.728 (93.3)	1.429 (95.6)	1.323 (94.1)	1.973 (89.2)
		λ_2	1.515 (90.8)	1.28 (95.9)	1.233 (91.9)	1.939 (88.5)
1		λ_1	0.916 (93)	0.865 (94.3)	0.521 (93.5)	0.787 (84.6)
		λ_2	0.817 (93.8)	0.776 (94.8)	0.511 (94.3)	0.793 (81.6)
2	(5010)	λ_1	1.009 (94.5)	0.942 (95.5)	0.522 (95)	0.887 (80.7)
		λ_2	0.897 (93.5)	0.843 (94.8)	0.515 (94.1)	0.895 (89.5)
3		λ_1	1.618 (91.5)	1.368 (93.9)	1.245 (92.8)	2.031 (89)
		λ_2	1.418 (90.9)	1.222 (95.2)	1.148 (93.5)	1.980 (89.8)

[f] The first and second rows represent the average 95% confidence lengths of asymptotic confidence intervals, the credible intervals with respect to the gamma prior distributions, Boot-p and Boot-t confidence intervals of λ_1 and λ_2 respectively, and the corresponding coverage probabilities are reported within brackets.

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