Diagnostic Test for GARCH Models Based on Absolute Residual Autocorrelations

Farhat Iqbal
Department of Statistics, University of Balochistan
Quetta-Pakistan
farhatiqb@gmail.com

Abstract

In this paper the asymptotic distribution of the absolute residual autocorrelations from generalized autoregressive conditional heteroscedastic (GARCH) models is derived. The correct asymptotic standard errors for the absolute residual autocorrelations are also obtained and based on these results, a diagnostic test for checking the adequacy of GARCH-type models are developed. Our results do not depend on the existence of higher moments and is therefore robust under heavy-tailed distributions.

Keywords: GARCH, Diagnostic test, Residual autocorrelations.

1. Introduction

To capture the conditional variance structure of many financial time series the autoregressive conditional heteroscedastic (ARCH) model was introduced by Engle (1982). Later Bollerslev (1986) proposed the generalized ARCH (GARCH) model. Since then many extensions of GARCH models have been developed and among them the asymmetric model also known as GJR model is very popular among practitioners (see Glosten et al., 1993).

There is a wide range of literature on modelling of these conditional heteroscedastic time series models but model checking and diagnostics have not been given due attention. Testing the adequacy of conditional heteroscedastic models is undoubtedly important for several economic and statistical reasons. Diagnostic is one of the important stages of model building and any misspecification in the model (in mean and variance) results in inconsistency and also loss of efficiency in the estimated parameters of the model. Residual autocorrelations are used to identify possible departure from the assumption that the white noise disturbances in the specified model are uncorrelated (see Box and Jenkins, 1970).

To check the model adequacy the distribution of residual autocorrelations might be useful. The asymptotic distribution of autocorrelations was first used by Box and Pierce (1970). They formulate a portmanteau statistic for model checking and showed that this statistic follows a chi-square distribution. Ljung and Box (1978) modified the Box-Pierce test and showed that their modified statistic is much closer to that of a chi-square. A new portmanteau statistic using the squared residual autocorrelations was given by McLeod and Li (1983). Li and Mak (1994) showed that the Box-Pierce type statistic does not follow an asymptotic chi-square distribution and a portmanteau test was developed for checking the adequacy of ARCH/GARCH models.

Wong and Li (1995) presented a portmanteau test using rank of squared residuals and showed through simulations that their test using ranks is a more robust alternative to the McLeod-Li (1983) statistic. Monte Carlo results was reported by Tse and Zuo (1997) for the finite sample performance of some commonly used diagnostics and found that the Li-Mak (1994) test based performs favorably among the other versions of statistics. A simulation study of Chen (2002) showed that Ljung-Box (1978) and McLeod-Li (1983) tests are not robust to heavy-tailed data. Kwan et al. (2005) carried out a comparative study of the finite-sample performance of some frequently used portmanteau tests. Based on their Monte Carlo results they reported that for skewed data the empirical size of these tests are severely undersized and that the better alternative with mode power is the non-parametric test.

In this paper we examine goodness-of-fit test based on the absolute residual autocorrelations in the class of conditional heteroscedastic time series models. We are motivated by Li and Mak (1994) who developed a diagnostic test for time series model with conditional variance. They used the squared residual autocorrelations for which a finite fourth moment is needed. This excludes many heavy-tailed distributions, hence we derive asymptotic distribution of absolute residual autocorrelations from GARCH-type models. Our result is robust under heavy-tailed since it does not depend on higher order moments. Based on this result we construct a portmanteau statistic, for checking the model adequacy, which is asymptotically distributed as a chi-square,

The rest of the paper is organized as follows: In Section 2, GARCH-type models and estimation method are introduced. In Section 3, the asymptotic distribution of absolute residual autocorrelations are derived and a useful diagnostic statistic is developed. Finally, Section 4 concludes the results.

2. GARCH-TYPE Model

For the simple GJR (1, 1) model, the following representation of the return series $\{X_t; t \in \mathbb{Z}\}$ is assumed. Observer $\{X_t; 1 \leq t \leq T\}$ such that

$$\begin{aligned} X_t &= h_t^{\frac{1}{2}}\epsilon_t, \\ h_t &= \omega_0 + \alpha_0 X_{t-1}^2 + \beta_0 h_{t-1} + \gamma_0 D_{t-1} X_{t-1}^2, \end{aligned} \tag{2.1}$$

where $D_t = 1$ if $X_t < 0$ and 0 otherwise, with $\{\epsilon_t\}$ a sequence of independent and identically distributed (i.i.d.) unobservable real-valued random variables and $\boldsymbol{\theta_0} = [\omega_0, \alpha_0, \gamma_0, \beta_0]'$, the unknown parameter vector in the parameter space

$$\boldsymbol{\Theta} = \left\{\boldsymbol{\theta} = [\omega, \alpha, \gamma, \beta]'; \omega > 0, \alpha, \gamma, \beta \geq 0 \ , \left(\alpha + \beta + \tfrac{1}{2}\gamma\right) < 1\right\}.$$

Under these parameter constratints, model (1) is strictly stationary and hence covariance stationary under finite second moment. The GJR (1, 1) model reduces to the GARCH (1, 1) model when $\gamma_0 = 0$ and to the ARCH (1) model when both $\beta_0 = \gamma_0 = 0$.

By recursive substituions, we get

$$h_t = \frac{\omega_0}{1-\beta_0} + \alpha_0 \sum_{j=1}^{\infty} \beta_0^{j-1} X_{t-j}^2 + \gamma_0 \sum_{j=1}^{\infty} D_{t-j}\, \beta_0^{j-1} X_{t-j.}^2$$

For $\boldsymbol{\theta} \in \Theta$, define the variance function

$$v_t(\boldsymbol{\theta}) = \frac{\omega}{1-\beta} + \alpha \sum_{j=1}^{\infty} \beta^{j-1} X_{t-j}^2 + \gamma \sum_{j=1}^{\infty} D_{t-j}\, \beta^{j-1} X_{t-j,}^2$$

and note that $v_t(\boldsymbol{\theta}_0) = h_t$.

In (2.1), if f denotes the error density, then the conditional density of $\{X_t\}$ given past will be $v_t^{-\frac{1}{2}}(\boldsymbol{\theta}_0) f\left\{v_t^{-\frac{1}{2}}(\boldsymbol{\theta}_0) X_t\right\}, 1 \le t \le T$. Now a random quantity as a minimizer of the negative log-likelihood function can be defined as

$$L_T(\boldsymbol{\theta}) = \frac{1}{T} \sum_{t=1}^{T} l_t(\boldsymbol{\theta}), \qquad \boldsymbol{\theta} \in \Theta,$$

where

$$l_t(\boldsymbol{\theta}) = \left[\left(\frac{1}{2}\right) \log v_t(\boldsymbol{\theta}) - \log f\{X_t / v_t^{1/2}(\boldsymbol{\theta})\}\right].$$

Then the derivative of the log-likelihood is

$$\frac{\partial L_T(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})} = \sum_{t=1}^{T} \left(\frac{1}{2}\right) \left[1 - \frac{X_t^2}{v_t(\boldsymbol{\theta})}\right] \left\{\frac{\dot{v}_t(\boldsymbol{\theta})}{v_t(\boldsymbol{\theta})}\right\}.$$

Throughout for a function $g,\ \dot{g}\ and\ \ddot{g}$ will denote the first and second derivatives, respectively, whenever they exist.

We can then define $\boldsymbol{\theta}_T$ in the model (2.1) as a solution of the equation

$$\mathbf{M}_T(\boldsymbol{\theta}) = \sum_{t=1}^{T} m_t(\boldsymbol{\theta}) = \mathbf{0},$$

where

$$m_t(\boldsymbol{\theta}) = \frac{\partial l_T(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})} = \left(\frac{1}{2}\right) \left[1 - \frac{X_t^2}{v_t(\boldsymbol{\theta})}\right] \left\{\frac{\dot{v}_t(\boldsymbol{\theta})}{v_t(\boldsymbol{\theta})}\right\}. \qquad (2.2)$$

Since $\{X_t; t \le 0\}$ are unobservable, $\{v_t(\boldsymbol{\theta})\}$'s are non observable and hecne $\boldsymbol{\theta}_T$'s are noncomputable. Define an observable approximation $\{\hat{v}_t(\boldsymbol{\theta}); t \ge 1\}$ to the variance functions $\{v_t(\boldsymbol{\theta}); t \ge 1\}$ as

$$\hat{v}_t(\boldsymbol{\theta}) = \frac{\omega}{1-\beta} + \left\{\alpha \sum_{j=1}^{t-1} \beta^{j-1} X_{t-j}^2 + \gamma \sum_{j=1}^{t-1} D_{t-j}\, \beta^{j-1} X_{t-j,}^2\right\} I(t \ge 2)$$

Then $\widehat{\boldsymbol{\theta}}_T$ is defined as a solution of

$$\widehat{\mathbf{M}}_T(\boldsymbol{\theta}) = \sum_{t=1}^{T} \widehat{m}_t(\boldsymbol{\theta}) = \mathbf{0},$$

where

$$\widehat{m}_t(\boldsymbol{\theta}) = \left(\frac{1}{2}\right)\left[1 - \frac{X_t^2}{\hat{v}_t(\boldsymbol{\theta})}\right]\left\{\frac{\dot{\hat{v}}_t(\boldsymbol{\theta})}{\hat{v}_t(\boldsymbol{\theta})}\right\}.$$

Under the usual regualrity conditions (see Hall and Heyde, 1980), it can be shown that for maximum likelihood estimators,

$$T^{\frac{1}{2}}\left(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right) \xrightarrow{Dist} \mathbf{N}\,[\mathbf{0}, 2\mathbf{G}^{-1}],$$

where

$$\mathbf{G} = \mathbf{G}(\boldsymbol{\theta}_0) := \mathrm{E}\left\{\frac{\dot{v}_1(\boldsymbol{\theta}_0)\dot{v}_1'(\boldsymbol{\theta}_0)}{v_1^2(\boldsymbol{\theta}_0)}\right\},$$

and $\xrightarrow{Dist}$ denotes convergence in distribution.

It can also be shown that

$$\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0 = (T\mathbf{B})^{-1}\sum_{t=1}^{T}\left(\frac{1}{2}\right)\left(1 - \frac{X_t^2}{v_t(\boldsymbol{\theta})}\right)\frac{\dot{v}_t(\boldsymbol{\theta}_0)}{v_t(\boldsymbol{\theta}_0)} + o_p\left(T^{-\frac{1}{2}}\right) \qquad (2.3)$$

with $\mathbf{B} = -\mathrm{E}\left(\mathrm{T}^{-1}\,\ddot{\mathbf{M}}_{\mathrm{T}}\,(\boldsymbol{\theta}_0)\right)$ and

$$\ddot{\mathbf{M}}_T(\boldsymbol{\theta}_0) = \left(\frac{1}{2}\right)\sum_{t=1}^{T}\left[\frac{X_t^2}{v_t(\boldsymbol{\theta}_0)}\left\{\frac{\dot{v}_t(\boldsymbol{\theta}_0)\dot{v}_t'(\boldsymbol{\theta}_0)}{v_t^2(\boldsymbol{\theta}_0)}\right\}\right] + \left(\frac{1}{2}\right)\sum_{t=1}^{T}\left[\left\{1 - \frac{X_t^2}{v_t(\boldsymbol{\theta}_0)}\right\}\{v_t(\boldsymbol{\theta}_0)\ddot{v}_t(\boldsymbol{\theta}_0) - \dot{v}_t(\boldsymbol{\theta}_0)\dot{v}_t'(\boldsymbol{\theta}_0)\}/v_t^2(\boldsymbol{\theta}_0)\right]$$

It can be shown that

$$\frac{\ddot{\mathbf{M}}_T(\boldsymbol{\theta}_0)}{2T} \to \mathbf{V},$$

where $\mathbf{V} = \left(\frac{1}{4}\right)\mathbf{G}$. Therefore, we have $\mathbf{B}^{-1} = -2\mathbf{G}^{-1}$.

3. Asymptotic Distribution of the Absolute Residual Autocorrelations

The asymptotic distribution of the absolute residual autocorrelations is derived in this section. Based on results, a useful portmanteau test is developed that can be used to check the adequacy of GARCH-type models.

The estimated residuals are defined as

$$\hat{\epsilon}_t = \frac{X_t}{\{\hat{v}_t(\boldsymbol{\theta}_T)\}^{\frac{1}{2}}}, \qquad 1 \le t \le T.$$

Following Li and Mak (1994), we defined the lag-k standardized residual autocorrelation as

$$\tilde{\rho}_k = \frac{\sum_{t=k+1}^{T}\left(|X_t|/\hat{v}_t^{\frac{1}{2}} - \bar{\epsilon}\right)\left(|X_{t-k}|/\hat{v}_{t-k}^{\frac{1}{2}} - \bar{\epsilon}\right)}{\sum_{t=k+1}^{T}\left(|X_t|/\hat{v}_t^{\frac{1}{2}} - \bar{\epsilon}\right)^2}$$

for $k = 1, \dots, M,$ where $\bar{\epsilon}_t = \frac{1}{T}\sum_{t=1}^{T}|X_t|/\hat{v}_t^{\frac{1}{2}}$, and for convenience we write $\hat{v}^{\frac{1}{2}} = \hat{v}^{\frac{1}{2}}(\boldsymbol{\theta}_T)$.

If the model is correct, then by the ergodic theorem

$$\bar{\epsilon}_t = \frac{1}{T}\sum_{t=1}^{T}|X_t|/\hat{v}_t^{\frac{1}{2}} \xrightarrow{a.s.} \mathrm{E}\left(|\epsilon_t|\right) \qquad \text{as} \qquad T \to \infty,$$

and $\mathrm{E}\left(|\epsilon_t|\right) = \mu_\epsilon$. So $\tilde{\rho}_k$ can be replaced by

$$\hat{\rho}_k = \frac{\sum_{t=k+1}^{T}\left(|X_t|/\hat{v}_t^{\frac{1}{2}} - \mu_\epsilon\right)\left(|X_{t-k}|/\hat{v}_{t-k}^{\frac{1}{2}} - \mu_\epsilon\right)}{\sum_{t=k+1}^{T}\left(|X_t|/\hat{v}_t^{\frac{1}{2}} - \mu_\epsilon\right)^2} \text{ for } k = 1, \cdots, M.$$

Now condsider the asymptotic distribution of the absolute residual autocorrelation $\tilde{\boldsymbol{\rho}} = (\tilde{\rho}_1, \tilde{\rho}_2, \dots, \tilde{\rho}_M)'$ for some integer $M > 0$. If the model is correct,

$$\frac{1}{T}\sum_{t=k+1}^{T}\left(\frac{|X_t|}{\hat{v}_t^{\frac{1}{2}}} - \mu_\epsilon\right)^2 \xrightarrow{a.s.} \mathrm{E}\left(|\epsilon_t| - \mu_\epsilon\right)^2$$

and $\mathrm{var}\left(|\epsilon_t|\right) = \sigma_\epsilon^2$.

Hence we only need to consider the asymptotic distribution of

$$\hat{C}_k^a = \frac{1}{T}\sum_{t=k+1}^{T}\left(\frac{|X_t|}{\hat{v}_t^{\frac{1}{2}}} - \mu_\epsilon\right)\left(\frac{|X_{t-k}|}{\hat{v}_{t-k}^{\frac{1}{2}}} - \mu_\epsilon\right).$$

Let $\hat{\mathbf{C}}^a = (\hat{C}_1^a, \dots, \hat{C}_M^a)'$ and $\mathbf{C}^a = (C_1^a, \dots, C_M^a)'$, for some integer $M > 0$. Similarly, $\hat{\boldsymbol{\rho}}$ and $\boldsymbol{\rho}$ can be defined. By Taylor's expansion of $\hat{\mathbf{C}}^a$ about $\boldsymbol{\theta}_0$ and evaluated at $\hat{\boldsymbol{\theta}}_T$, we have

$$\hat{\mathbf{C}}^a \approx \mathbf{C}^a + \frac{\partial \mathbf{C}^a}{\partial \boldsymbol{\theta}}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right), \tag{3.1}$$

where $\frac{\partial \mathbf{C}^{\mathrm{a}}}{\partial \boldsymbol{\theta}} = (\frac{\partial C_1^a}{\partial \boldsymbol{\theta}}, \dots, \frac{\partial C_M^a}{\partial \boldsymbol{\theta}})'$, and for $k = 1, \dots, M$,

$$\frac{\partial C_k^a}{\partial \boldsymbol{\theta}} = -\frac{1}{2T} \sum_{t=k+1}^{T} \frac{|X_t|}{v_t^{\frac{1}{2}}} \left(\frac{|X_{t-k}|}{\hat{v}_{t-k}^{\frac{1}{2}}} - \mu_\epsilon \right) \frac{\dot{v}_t}{v_t}$$
$$- \frac{1}{2T} \sum_{t=k+1}^{T} \frac{|X_{t-k}|}{v_{t-k}^{\frac{1}{2}}} \left(\frac{|X_t|}{\hat{v}_t^{\frac{1}{2}}} - \mu_\epsilon \right) \frac{\dot{v}_{t-k}}{v_{t-k}}.$$

By the ergodic theorem, we obtain

$$\frac{\partial C_k^a}{\partial \boldsymbol{\theta}} \xrightarrow{a.s.} -Y_k^a, \qquad \text{as} \quad T \to \infty,$$

where $$Y_k^a = \mu_\epsilon \mathrm{E}\left[\left(\frac{|X_{t-k}|}{\hat{v}_{t-k}^{\frac{1}{2}}} - \mu_\epsilon \right) \frac{\dot{v}_t}{v_t} \right].$$

Then $\hat{\mathbf{C}}$ in (3.1) can be approximated by

$$\hat{\mathbf{C}} \approx \mathbf{C}^{\mathrm{a}} - \mathbf{Y}^{\mathrm{a}} \left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0 \right), \tag{3.2}$$

where $\mathbf{Y}^a = (Y_1^a, \dots, Y_M^a)'$.

The proof of the following lemma may be shown by simple calculations.

Lemma 1:

For any constant vector $\mathbf{Z} = (Z_1, \dots, Z_M)'$

$$\sqrt{T}\, \mathbf{Z}'\mathbf{C}^a = \frac{1}{\sqrt{T}} \sum_{t=M+1}^{T} \mathbf{U}_t^a + O_p(1), \tag{3.3}$$

where

$$\mathbf{U}_t^a = \sum_{k=1}^{M} Z_k \left(\frac{|X_t|}{v_t^{\frac{1}{2}}} - \mu_\epsilon \right) \left(\frac{|X_{t-k}|}{v_{t-k}^{\frac{1}{2}}} - \mu_\epsilon \right) \tag{3.4}$$

and

$$\mathrm{E}(\mathbf{U}_t^{\mathrm{a}})^2 = \sigma_\epsilon^4 \mathbf{Z}'\mathbf{Z} < \infty. \tag{3.5}$$

Lemma 2:

$$\mathrm{E}\left\{ \left(\frac{\partial l_t}{\partial \boldsymbol{\theta}} \right) U_t^a \right\} = \frac{d\, \mathbf{Y}^{a\prime}\mathbf{Z}}{\mu_\epsilon}$$

where $\mathbf{U}_t^a$ *is defined in (3.4) and* $d = \left(\mu_\epsilon - \mathrm{E}\left(\frac{|X_t^3|}{v_t^{\frac{3}{2}}} \right) \right)$.

Proof:

By (2.1), (2.2) and (3.4), we have

$$\frac{\partial l_t}{\partial \boldsymbol{\theta}} \mathbf{U}_t^a = \left[\left(\frac{1}{2}\right)\left(1 - \frac{X_t^2}{v_t}\right)\frac{\dot{v}_t}{v_t}\right] \times \left[\sum_{k=1}^{M} Z_k \left(\frac{|X_t|}{v_t^{\frac{1}{2}}} - \mu_\epsilon\right)\left(\frac{|X_{t-k}|}{v_{t-k}^{\frac{1}{2}}} - \mu_\epsilon\right)\right].$$

Hence it follows that

$$\mathrm{E}\left\{\left(\frac{\partial l_t}{\partial \boldsymbol{\theta}}\right)\mathbf{U}_t^a\right\} = \mathrm{E}\left\{\mathrm{E}\left\{\left(\frac{\partial l_t}{\partial \boldsymbol{\theta}}\right)\mathbf{U}_t^a | F_{t-1}\right\}\right\}$$

$$= \frac{1}{2\mu_\epsilon}\left[\mu_\epsilon - \mathrm{E}\left(\frac{|X_t^3|}{v_t^{\frac{3}{2}}}\right)\right]\sum_{k=1}^{M} Z_k \,\mathrm{E}\left[\mu_\epsilon\left(\frac{|X_{t-k}|}{v_{t-k}^{\frac{1}{2}}} - \mu_\epsilon\right)\frac{\dot{v}_t}{v_t}\right]$$

$$= \frac{1}{\mu_\epsilon}\left[\mu_\epsilon - \mathrm{E}\left(\frac{|X_t^3|}{v_t^{\frac{3}{2}}}\right)\right]\sum_{k=1}^{M} Z_k Y_k^a$$

$$= \frac{d\,\mathbf{Y}^{a\prime}\mathbf{Z}}{\mu_\epsilon}.$$

Lemma 3:

The joint distribution of $\sqrt{T}\mathbf{C}^a$ *and* $\sqrt{T}(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)$ *is asymptotically normal with mean zero and covariance*

$$\begin{pmatrix} \sigma_\epsilon^4 \mathbf{I}_M & \dfrac{d\,\mathbf{Y}^{a}\mathbf{B}^{-1}}{\mu_\epsilon} \\ \dfrac{d\,\mathbf{B}^{-1}\mathbf{Y}^{a\prime}}{\mu_\epsilon} & 2\,\mathbf{G}^{-1} \end{pmatrix}.$$

Proof:

Let $\tilde{\mathbf{Z}} = (\mathbf{Z}', \mathbf{V}')'$ be any constant vector and $\tilde{\mathbf{Z}}'\tilde{\mathbf{Z}} \neq \mathbf{0}$, where the dimension of $\mathbf{V}$ is same as that of $\boldsymbol{\theta}$.

Now by (2.3) and (3.3), we have

$$\sqrt{T}\tilde{\mathbf{Z}}'\left(\mathbf{C}^{a\prime}, \widehat{\boldsymbol{\theta}}'_T - \boldsymbol{\theta}'\right)' = \frac{1}{\sqrt{T}}\left(\sum_{t=M+1}^{T} \mathbf{U}_t^a + \sum_{t=1}^{T} \mathbf{V}'\mathbf{B}^{-1}\frac{\partial l_t}{\partial \boldsymbol{\theta}}\right) + O_p(1)$$

$$= \frac{1}{\sqrt{T}}\sum_{t=M+1}^{T}\left(\mathbf{U}_t^a + \mathbf{V}'\mathbf{B}^{-1}\frac{\partial l_t}{\partial \boldsymbol{\theta}}\right) + O_p(1) \qquad (3.6)$$

It can be shown that $\left(1/\sqrt{T}\right)\sum_{t=M+1}^{T}(\mathbf{U}_t^a + \mathbf{V}'\mathbf{B}^{-1}(\partial l_t)/\partial\boldsymbol{\theta})$ is a martingale and by (3.5), Lemma 1 and Lemma 2,

$$\mathrm{E}\left(\mathbf{U}_t^a + \mathbf{V}'\mathbf{B}^{-1}\frac{\partial l_t}{\partial\boldsymbol{\theta}}\right)^2 = \sigma_\epsilon^4\mathbf{Z}'\mathbf{I}_M\mathbf{Z} + \frac{d\mathbf{Z}'\mathbf{Y}^a\mathbf{B}^{-1}\mathbf{V}}{\mu_\epsilon} + \frac{d\mathbf{V}'\mathbf{B}^{-1}\mathbf{Y}^{a\prime}\mathbf{Z}}{\mu_\epsilon} + 2\mathbf{V}'\mathbf{G}^{-1}\mathbf{V}$$

$$= \tilde{\mathbf{Z}}'\begin{pmatrix} \sigma_\epsilon^4\mathbf{I}_M & \dfrac{d\,\mathbf{Y}^{\mathrm{a}}\mathbf{B}^{-1}}{\mu_\epsilon} \\ \dfrac{d\,\mathbf{B}^{-1}\mathbf{Y}^{a\prime}}{\mu_\epsilon} & 2\,\mathbf{G}^{-1} \end{pmatrix}\tilde{\mathbf{Z}}' < \infty.$$

Hence the proof completes by using Bilingsley's (1961) martingale central limit theorem and (3.6).

Theorem:

$$\sqrt{T}\hat{\mathbf{C}}^{\mathrm{a}} \xrightarrow{Dist} \mathbf{N}[\mathbf{0}, \sigma_\epsilon^4\mathbf{W}^a] \qquad \text{as} \quad T \to \infty,$$

$$\sqrt{T}\tilde{\boldsymbol{\rho}} \xrightarrow{Dist} \mathbf{N}[\mathbf{0}, \mathbf{W}^a] \qquad \text{as} \quad T \to \infty,$$

where

$$\mathbf{W}^{\mathrm{a}} = \mathbf{I}_M + \frac{4\mathbf{Y}^a\mathbf{G}^{-1}\mathbf{Y}^{a\prime}}{\sigma_\epsilon^4}\left\{\frac{d}{\mu_\epsilon} + \frac{1}{2}\right\}$$

Proof:

This follows from (3.2) and Lemma 3.

Hence for the absolute residual autocorrelations the correct asymptotic standard errors are obtained. In general, the matrix $\mathbf{W}^a$ is not an idempotent matrix even asymptotically, therefore $T\hat{\boldsymbol{\rho}}'\hat{\boldsymbol{\rho}}$ is not asymptotically distributed as a chi-squared. However, if the model is specified correctly, the portmanteau statistic

$$Q^a(M) = T\hat{\boldsymbol{\rho}}'[\mathbf{W}^a]^{-1}\hat{\boldsymbol{\rho}}$$

will be asymptotically distributed as a χ^2 with M degrees of freedom. Rejecting this statistic will imply that there is a temporal dependence in variances of the series under investigation. The adequacy of GARCH-type models can checked using this new portmanteau statistic. It is worth mentioning here that only the existence of a second-order-moment is required in this case.

If the distribution of $\{\epsilon_t\}$ is known, the exact values of σ_ϵ^4, d and μ_ϵ can be obtained. For example when ϵ_t follows the standard normal distribution and the squared residual autocorrelations are considered, we have $\sigma_\epsilon^4 = 4$, $\mu_\epsilon = 1$, and $d = -2$. Hence, after some calculation we get the asymptotic covariance matrix of squared residual autocorrelations as in Li and Mak (1994). Furthermore, if ν_t is constant over time, the asymptotic standard error of the squared residual autocorrelations is exactly $1/\sqrt{T}$ and we get McLeod and Li (1983) result.

Since the distribution of $\{\epsilon_t\}$ is not known in practice, the estimates of d and $\mathbf{Y}^a$ can be obtained as

$$\hat{d} = \left(\hat{\mu}_\epsilon - \frac{1}{T}\sum_{t=1}^{T}\left(\frac{|X_t^3|}{\hat{v}_t^{\frac{3}{2}}}\right)\right)$$

and $\hat{\mathbf{Y}}^a = (\hat{Y}_1^a, \cdots, \hat{Y}_M^a)'$, where

$$\hat{Y}_k^a = \hat{\mu}_\epsilon \frac{1}{T}\sum_{t=k+1}^{T}\left[\left(\frac{|X_{t-k}|}{\hat{v}_{t-k}^{\frac{1}{2}}} - \hat{\mu}_\epsilon\right)\frac{\dot{\hat{v}}_t}{\hat{v}_t}\right].$$

An estimate of $\mathbf{G}^{-1}$ can be obtained from $\left[\hat{\mathbf{G}}\right]^{-1}$, where

$$\hat{\mathbf{G}} = \frac{1}{T}\sum_{t=1}^{T}\left\{\frac{\dot{\hat{v}}_t\, \dot{\hat{v}}_t'}{\hat{v}_t^2}\right\}.$$

Also, σ_ϵ^4 can be replaced by $\left(\hat{C}_0^{\mathrm{a}}\right)^2$. Using these sample estimates, we can define an estimate of $\mathbf{W}^a$as

$$\hat{\mathbf{W}}^{\mathrm{a}} = \mathbf{I}_M + \frac{4\hat{\mathbf{Y}}^a\hat{\mathbf{G}}^{-1}\hat{\mathbf{Y}}^{a\prime}}{\hat{\sigma}_\epsilon^4}\left\{\frac{\hat{d}}{\hat{\mu}_\epsilon} + \frac{1}{2}\right\}.$$

4. Conclusion

This paper deals with the diagnostic checking of ARCH/GARCH models. The asymptotic covariance matrix of the absolute residual autocorrelations is derived. The correct asymptotic standard errors are obtained and these give more accurate standard errors than $1/\sqrt{T}$ for the residual autocorrelations. Based on these results, a portmanteau test is developed for diagnostic checking of GARCH-type models. Our results are valid under weaker moment condition.

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