Simulation MLE of Parameters of the Mixture Distribution in the Presence of Two Outliers

Einolah Deiri
Department of Statistics, Islamic Azad University
Qaemshahr Branch, Mazandaran, Iran
deiri53@gmail.com

Abstract

In the presence paper, we deal with the estimation of parameters of the Exponentiated Gamma (EG) distribution with presence of multiple(r=2) outliers. The maximum likelihood and moment of the estimators are derived. These estimators are compared empirically using Monte Carlo simulation when all the parameters are unknown. There bias and MSE are investigated with help of numerical technique.

Keywords: Exponentiated Gamma distribution, Maximum and Moment Estimators, Outliers, Newton-Raphson method.

1. Introduction

Dixit, Moore and Barnett (1996), assume that a set of random variables $(X_1, X_2, ..., X_n)$ represent the distance of an infected sampled plant from a plant from a plot of plants inoculated with a virus. Some of the observations are derived from the airborne dispersal of the spores and are distributed according to the exponential distribution. The other observations out of *n* random variables (say *k*) are present because aphids which are know to be carriers of barley yellow mosaic dwarf virus (BYMDV) have passed the virus into the plants when the aphids feed on the sap. Dixit and Nasiri (2001) considered estimation of parameters of the exponential distribution in the presence of outliers generated from uniform distribution. Deiri (2011,a,b,c) considered the estimation of the parameters of the exponentiated gamma distribution in the presence of outlier generated from exponentiated gamma distribution. In this paper, we obtain the maximum likelihood and moment estimators of the parameters of the exponentiated gamma distribution in the presence of multiple outliers generated from exponentiated gamma distribution.

Recently a new distribution, called Exponentiated Gamma (EG) distribution, has been introduced. This distribution was introduced by Gupta et al. (1998) which has a probability density function (p.d.f.) of the form

$$f(x;\alpha) = \alpha\, x e^{-x}(1-e^{-x}(1+x))^{\alpha-1} \quad ; \quad x>0\ ,\ \alpha>0, \tag{1}$$

and a cumulative distribution function (c.d.f.)

$$F(x;\alpha) = (1-e^{-x}(1+x))^{\alpha} \quad ; \quad x>0\ ,\ \alpha>0, \tag{2}$$

where α is the shape parameter. It is important to mention that when $\alpha=1$, the Exponentiated Gamma p.d.f. is that of gamma distribution with shape parameter $\alpha=2$ and scale parameter $\beta=1$, i.e., $G(2,1)$. For more details about this distribution, see Shawky and Bakoban (2008c & 2009). Also, characterization from EG distribution based on record values and Bayesian estimations on the EG distribution discussed by Shawky and Bakoban (2008b,a). The density functions of the Exponentiated Gamma distribution can take different shapes. Figure 1 shows the shape of $f(x;\alpha)$ for different values of α.

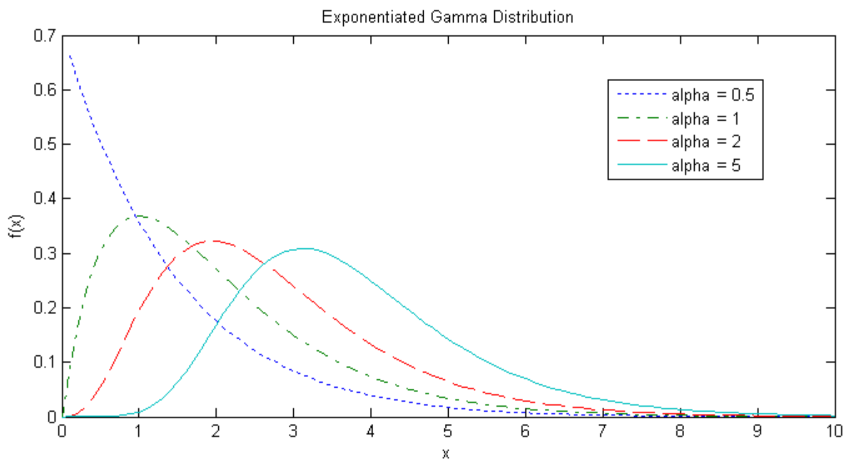

Figure 1: p.d.f. of EG(α) for different values of α.

We assume that the random variables $(X_1, X_2, ..., X_n)$ are such that two of them are distributed with p.d.f $g(x;\theta)$,

$$g(x;\theta) = \theta x e^{-x}(1-e^{-x}(1+x))^{\theta-1} \quad ; \quad x>0 \ , \ \theta>0 \tag{3}$$

and remaining $(n-2)$ random variables are distributed with p.d.f $f(x;\alpha)$ given in (1).

The paper is organized as follows: In section 2, we have obtained the joint distribution of $(X_1, X_2, ..., X_n)$ in the presence of two outliers. Section 3 and 4 discusses the methods of moment and maximum likelihood estimators. The different proposed methods have been compared using Monte Carlo simulations and the results have been reported in section 5.

2. Joint Distribution of $(X_1, X_2, ..., X_n)$ with Presence of Multiple Outliers

The joint distribution of $(X_1, X_2, ..., X_n)$ in the presence of two outliers can be expressed as

$$f(x_1, x_2, ..., x_n; \alpha, \theta) = \frac{1}{c} \prod_{i=1}^{n} f(x_i;\alpha) \cdot \sum_{\underline{A}} \prod_{r=1}^{2} \frac{g(x_{A_r};\theta)}{f(x_{A_r};\alpha)}$$

$$= \frac{1}{c} \prod_{i=1}^{n} \alpha x_i e^{-x_i}\left(1-e^{-x_i}(1+x_i)\right)^{\alpha-1} \cdot \sum_{\underline{A}} \prod_{r=1}^{2} \frac{\theta x_{A_r} e^{-x_{A_r}}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-1}}{\alpha x_{A_r} e^{-x_{A_r}}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\alpha-1}}$$

$$= \frac{\alpha^{n-k}\theta^k}{c} e^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left[x_i\left(1-e^{-x_i}(1+x_i)\right)^{\alpha-1}\right] \cdot \sum_{\underline{A}} \prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha} \tag{4}$$

where $c=(n(n-1))/2$ and $\sum_{\underline{A}} = \sum_{A_1=1}^{n-1} \sum_{A_2=A_1+1}^{n}$. See Dixit (1996), Dixit and Nasiri (2001), and

Deiri (2011a,b,c). From (4), the marginal distribution of X is

$$f(x;\alpha,\theta)=\frac{n-2}{n}\alpha\, x\, e^{-x}\left(1-e^{-x}(1+x)\right)^{\alpha-1}+\frac{2}{n}\theta\, x\, e^{-x}\left(1-e^{-x}(1+x)\right)^{\theta-1}\;;\;\; x>0. \quad (5)$$

3. Method of Moment

The moments can also be obtained in the form of a series which is finite or infinite depending on whether α and θ are integers or not. The raw moments of X may be determined from (5) be direct integration. For $r\in \mathrm{N}$, we find that

$$E(X^r)=\int_0^\infty x^r\left[\frac{n-2}{n}\alpha\, x\, e^{-x}\left(1-e^{-x}(1+x)\right)^{\alpha-1}+\frac{2}{n}\theta\, x\, e^{-x}\left(1-e^{-x}(1+x)\right)^{\theta-1}\right]dx$$

$$=\frac{n-2}{n}\int_0^\infty \alpha\, x^{r+1}e^{-x}\left(1-e^{-x}(1+x)\right)^{\alpha-1}dx+\frac{2}{n}\int_0^\infty \theta\, x^{r+1}e^{-x}\left(1-e^{-x}(1+x)\right)^{\theta-1}dx \quad (6)$$

Since $0<e^{-x}(1+x)<1$ for $x>0$ by using the binomial series expansion we have

$$(1-e^{-x}(1+x))^m=\sum_{i=0}^{\infty}\binom{m}{i}(-1)^i e^{-ix}(1+x)^i. \quad (7)$$

Hence,

$$E(X^r)=\frac{(n-2)\alpha}{n}\int_0^\infty x^{r+1}e^{-x}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i e^{-ix}(1+x)^i\,dx$$

$$+\frac{2\theta}{n}\int_0^\infty x^{r+1}e^{-x}\sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i e^{-ix}(1+x)^i\,dx$$

Also, whereas i is integer, by using the binomial series expansion we have

$$(1+x)^i=\sum_{j=0}^{i}\binom{i}{j}x^j$$

Hence,

$$E(X^r)=\frac{(n-2)\alpha}{n}\int_0^\infty x^{r+1}e^{-x}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\alpha-1}{i}\binom{i}{j}(-1)^i e^{-ix}x^j\,dx$$

$$+\frac{2\theta}{n}\int_0^\infty x^{r+1}e^{-x}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\theta-1}{i}\binom{i}{j}(-1)^i e^{-ix}x^j\,dx \quad (8)$$

Since the quantity inside the summation is absolutely integrable, interchanging the summation and integration we have

$$E(X^r)=\frac{(n-2)\alpha}{n}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\alpha-1}{i}\binom{i}{j}(-1)^i\int_0^\infty x^{r+j+1}e^{-(i+1)x}\,dx$$

$$+\frac{2\theta}{n}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\theta-1}{i}\binom{i}{j}(-1)^{i}\int_{0}^{\infty}x^{r+j+1}e^{-(i+1)x}\,dx$$

$$=\frac{(n-2)\alpha}{n}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\alpha-1}{i}\binom{i}{j}(-1)^{i}\frac{\Gamma(r+j+2)}{(i+1)^{(r+j+2)}}$$

$$+\frac{2\theta}{n}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\theta-1}{i}\binom{i}{j}(-1)^{i}\frac{\Gamma(r+j+2)}{(i+1)^{(r+j+2)}}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i}\frac{(r+j+1)!}{(i+1)^{(r+j+2)}}\left[\frac{(n-2)\alpha}{n}\binom{\alpha-1}{i}+\frac{2\theta}{n}\binom{\theta-1}{i}\right]. \tag{9}$$

For $\alpha=\theta=\beta$ in case of no outlier presence, from (9) we get

$$E(X^{r})=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\beta-1}{i}\binom{i}{j}(-1)^{i}\frac{\beta(r+j+1)!}{(i+1)^{(r+j+2)}}, \tag{10}$$

it is proposed by Gupta et al. (1998), also in case of two outlier presence, it is given by Deiri (2011, 2012). We observe that the infinite series is summable. For *r* = 1, 2 and 3, $E(X^{r})$ is given by

$$E(X)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i}\frac{(j+2)!}{(i+1)^{(j+3)}}\left[\frac{(n-2)\alpha}{n}\binom{\alpha-1}{i}+\frac{2\theta}{n}\binom{\theta-1}{i}\right] \tag{11}$$

$$E(X^{2})=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i}\frac{(j+3)!}{(i+1)^{(j+4)}}\left[\frac{(n-2)\alpha}{n}\binom{\alpha-1}{i}+\frac{2\theta}{n}\binom{\theta-1}{i}\right] \tag{12}$$

$$E(X^{3})=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i}\frac{(j+4)!}{(i+1)^{(j+5)}}\left[\frac{(n-2)\alpha}{n}\binom{\alpha-1}{i}+\frac{2\theta}{n}\binom{\theta-1}{i}\right]. \tag{13}$$

4. Method of Maximum Likelihood

One sees from (6) that moment estimates for the parameters of the EG distribution with presence of two outliers can not be obtained in closed forms and therefore that is little point in considering the method any further. Proceeding with the method of maximum likelihood, the log likelihood function from a sample of *n* observations, $(X_1, X_2, ..., X_n)$ is given by

$$L(\alpha,\theta)=\ln f(x_1,x_2,...,x_n;\alpha,\theta)$$

$$=(n-2)\ln\alpha+2\ln\theta-\ln c+\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}x_i$$

$$+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-x_i}(1+x_i)\right)+\ln\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha} \tag{14}$$

where $c=(n(n-1))/2$.

Taking the derivative with respect to α and θ and equating to *0*, we obtain the normal equations as

$$\frac{\partial L(\alpha,\theta)}{\partial\alpha}=\frac{n-2}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-x_i}(1+x_i)\right)-\frac{\frac{\partial}{\partial\alpha}\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}$$

$$=\frac{n-2}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-x_i}(1+x_i)\right)-\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}\overset{set}{=}0 \quad (15)$$

$$\frac{\partial L(\alpha,\theta)}{\partial\theta}=\frac{2}{\theta}+\frac{\frac{\partial}{\partial\theta}\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}$$

$$=\frac{2}{\theta}+\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}\overset{set}{=}0 \quad (16)$$

Here, we need to use either the scoring algorithm or the Newton-Raphson algorithm to solve the two non-linear equations simultaneously, so we will solve for $\hat{\alpha}$ and $\hat{\theta}$ iteratively, using the Newton-Raphson method, a tangent method for root finding. In our case we will estimate $\beta=(\alpha,\theta)$ iteratively:

$$\hat{\beta}_{i+1}=\hat{\beta}_i-G^{-1}g\ , \quad (17)$$

where g is the vector of normal equations for which we want

$$g=[g_1\quad g_2]\ , \quad (18)$$

with

$$g_1=\frac{n-2}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-x_i}(1+x_i)\right)-\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}\ , \quad (19)$$

$$g_2=\frac{2}{\theta}+\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}\ , \quad (20)$$

and G is the matrix of second derivatives

$$G=\begin{bmatrix}\dfrac{d\,g_1}{d\alpha} & \dfrac{d\,g_1}{d\theta}\\ \dfrac{dg_2}{d\alpha} & \dfrac{dg_2}{d\theta}\end{bmatrix} \tag{21}$$

where

$$\frac{\partial g_1}{\partial\alpha}=\frac{2-n}{\alpha^2}+\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\cdot\left(\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right)^2\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}$$
$$-\left(\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\cdot\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}\right)^2, \tag{22}$$

$$\frac{\partial g_1}{\partial\theta}=\frac{\partial g_2}{\partial\alpha}=-\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\cdot\left(\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right)^2\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}$$
$$+\left(\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\cdot\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}\right)^2, \tag{23}$$

$$\frac{\partial g_2}{\partial\theta}=\frac{-1}{\theta^2}+\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\cdot\left(\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right)^2\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}$$
$$-\left(\frac{\sum_{\underline{A}}\prod_{r=1}^{2}\left[\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}\cdot\ln\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)\right]}{\sum_{\underline{A}}\prod_{r=1}^{2}\left(1-e^{-x_{A_r}}(1+x_{A_r})\right)^{\theta-\alpha}}\right)^2. \tag{24}$$

The Newton-Raphson algorithm converges, as our estimates of α and θ change by less than a tolerated amount with each successive iteration, to $\hat{\alpha}$ and $\hat{\theta}$.

Note that for $\alpha=\theta=\beta$, in case of no outlier presence, $\hat{\beta}$ can be obtain as

$$\hat{\beta}=\frac{-n}{\sum_{i=1}^{n}\ln\left(1-e^{-x_i}(1+x_i)\right)}$$

it is given by Gupta et al. (1998), also in case of two outlier presence, it is obtained by Deiri (2011).

5. Numerical Experiments and Discussions

In this paper, we have addressed the problem of estimating parameters of Exponentiated Gamma distribution in presence of two outliers. In order to have some idea about Bias and Mean Square Error (MSE) of methods of moment and MLE, we perform sampling experiments using a MATLAB. The results are given in Tables 1 and 2, for $\alpha=0.5$ and $\theta=2$. We report the average estimates and the MSEs based on *1500* replications. It is observed that the maximum likelihood estimator work quit well.

Table 1: $\alpha=0.5$ *and* $\theta=2$

n	Bias $\hat{\alpha}_{MOM}$	MSE $\hat{\alpha}_{MOM}$	Bias $\hat{\alpha}_{MLE}$	MSE $\hat{\alpha}_{MLE}$
10	-0.943	0.862	0.890	0.662
15	-0.989	0.729	0.850	0.569
20	-0.882	0.571	0.781	0.461
25	-0.675	0.382	0.592	0.307
30	-0.740	0.214	0.734	0.171
35	-0.518	0.172	0.541	0.108
40	-0.382	0.097	0.307	0.079
45	-0.386	0.063	0.329	0.043
50	-0.158	0.055	0.137	0.037

Table 2: $\alpha=0.5$ *and* $\theta=2$

n	Bias $\hat{\theta}_{MOM}$	MSE $\hat{\theta}_{MOM}$	Bias $\hat{\theta}_{MLE}$	MSE $\hat{\theta}_{MLE}$
10	0.804	0.932	0.438	0.628
15	0.853	0.899	0.552	0.575
20	0.754	0.825	0.448	0.516
25	0.827	0.622	0.397	0.401
30	0.661	0.548	0.313	0.382
35	0.504	0.486	0.240	0.339
40	0.379	0.291	0.278	0.272
45	0.298	0.183	0.183	0.164
50	0.318	0.096	0.196	0.073

References

1. Deiri, E. (2011a), Estimation of Reliability for Exponential Case in the Presence of One Outlier. *Financial Mathematics and Applications, 1, 1-8.*

2. Deiri, E. (2011b), Estimation of Parameters of the Gamma Distribution in the Presence of Two Outliers. *Australian Journal of Basic and Applied Science*.

3. Deiri, E. (2011c), Estimations of the Exponentiated Gamma Distribution Parameters with Presence of Two Outliers. *International Journal of Academic Research*.

4. Deiri, E. (2012d), Classical Estimator of Parameters of the Gamma Case With Presence of Two Outliers. Communications in Statistics-Simulation and Computation, 41: 590-597.

   DOI: 10.1080/03610918.2011.598993.

5. Dixit, U. J., More, K, L. and Barnett, V. (1996) on the estimation of the power of the scale parameter of the exponential distribution in the presence of outliers generated from uniform distribution, *Metron, 54, 201-211.*

6. Dixit, U. J. & Nasiri, P. (2001) estimation of parameter of the exponential distribution in the presence of outliers generated from uniform distribution, *Metron, 49(3-4), 187-198.*

7. Gupta, R.C, Gupta, R.D. & Gupta, P. L. (1998) Modeling failure time data by Lehman alternatives, *Communications in Statistics-Theory and Methods, 27(4), 887-904*.

8. Shawky, A.I. & Bakoban, R. A. (2008a) Bayesian and Non-Bayesian Estimations on the exponentiated Gamma Distribution. *Applied Mathematical Sciences, 51(2) 2521-2530*.

9. Shawky, A.I. & Bakoban, R. A. (2008b) Characterizations from exponentiated gamma distribution based on record values. *Accepted for publication in JSTA, vol.7.*

10. Shawky, A.I. & Bakoban, R. A. (2008c) Certain characterizations of the exponentiated gamma distribution. *Accepted for publication in JATA.*

11. Shawky, A.I. & Bakoban, R. A. (2009) Order Statistics from Exponentiated Gamma distribution and Associated Inference. *International Journal Contemporary Mathematical Sciences, 2(4) 71-91*.

12. Zheng, G. (2002). On the fisher information matrix in type-II censored data from the exponentiated exponential family. *Biometrical journal, 44, 353-357*.