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On Complex Random Variables

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Abstract

In this paper, it is shown that a complex multivariate random variable $Z = (Z_1, Z_2, \cdots, Z_p)'$ is a complex multivariate normal random variable of dimensionality p if and only if all nondegenerate complex linear combinations of Z have a complex univariate normal distribution. The characteristic function of Z has been derived, and simpler forms of some theorems have been given using this characterization theorem without assuming that the variance-covariance matrix of the vector Z is Hermitian positive definite. Marginal distributions of Z have been given. In addition, a complex multivariate t-distribution has been defined and the density derived. A characterization of the complex multivariate t-distribution is given. A few possible uses of this distribution have been suggested.

1. Introduction

Wooding (1956) had initially considered the multivariate analogue of the complex Gaussian distribution. Later Goodman (1963) discussed similar aspects of the complex multivariate normal distribution, and also considered the analogue of the Wishart distribution and of multiple and partial correlations. Brillinger (1968), Goodman and Dubman (1968), and Young (1971) gave some practical uses of the complex analogue. Other authors, for example, Capon and Goodman (1970), Giri (1965), Kabe (1966a, b), Khatri (1964, 1965), Saxena (1965), Srivastava (1963), Tan (1969), and Young (1971), have discussed the complex analogues of the classical real variable multivariate analysis. For some recent references on complex random variables, see Van den Bos (1995, 1998), Dryden and Mardia (1998), Olhede (2006) and Eriksson et al. (2009) and references therein.

A complex random variable Z is a measurable function defined on a probability measure space taking values in the field of complex numbers. A complex random variable Z, defined in this unique way, is represented by the equation $Z = X + iY$ where (X, Y) is a bivariate real random variable.

The mean or expected value of complex random variables Z, defined as $E(Z) = E\{X + iY\} = E(X) + iE(Y)$, is said to be exist if both real expectations $E(X)$ and $E(Y)$ exist. As in real random variables case, $E(X)$ are exists iff $E(|X|)$ exists, $E(Z)$ implies that $E(|X|)$ and $E(|Y|)$ exist. Further, in complex random variables the inequality $|E(Z)| \le E(|Z|)$ holds as in the case of real random variables. Similarly, the variance of Z is defined as $V(Z) = E[(Z - E(Z))(Z - E(Z)^*]$ where $Z^* = X - iY$ denotes the complex conjugate transpose of Z. It can be shown that $V(Z) = V(X) + V(Y)$ and that it exists if both $V(X)$ and $V(Y)$ exist. Also if Z_1 and Z_2 are two complex random variables, then the covariance between Z_1 and Z_2 is defined as $\text{cov}(Z_1, Z_2) = E[(Z_1 - E(Z_1))(Z_2 - E(Z_2)^*]$ and exists if $V(Z_1)$ and $V(Z_2)$ both exist. For details see Andersen et al. (1995), Biglieri (2005) and Gubner (2006).

If $Z_0 = X_0 + iY_0$ is a fixed complex number, then the probability that $Z \le Z_0$, implies that

$$F(z_0) = P(Z \le Z_0) = P(X \le X_0, Y \le Y_0) = F(x_0, y_0)$$

Where the functions $F(z_0)$ and $F(x_0, y_0)$ are the cumulative distribution function (cdf) of the complex random variable Z and the joint cdf of the bivariate random vector (X_0, Y_0) respectively.

If it is assumed that $t = t_1 + it_2$ is a fixed complex number, the characteristic function of Z is defined as

$$\phi_{(t)}(Z) = E(e^{i(t_1X + t_2Y)}) = E[e^{iRP(t^HZ)}]$$

where $RP(\cdot)$ denotes the real part of $(\cdot)$ and t^H denotes the complex conjugate of t.

Similarly, $Z = (Z_1, Z_2, \cdots, Z_p)'$ is a dimensional complex random variable if the vectors of its real and imaginary parts are a 2p-dimensional real random variable taking values in E^{2p}, the 2p-dimensional Euclidean space. The characteristic function of Z with respect to the complex vector $T = (T_1, T_2, \cdots, T_p)'$ is defined as

$$\phi_T(Z) = E(e^{iRP(T^H)}), \quad \text{where } T^H \text{ is a complex conjugate transpose of } T.$$

2. Multivariate Complex Normal Distribution

A p-variate complex normal random variable (denoted as CN_p) $Z = (Z_1, Z_2, \cdots, Z_p)'$ is a multiple complex random variables such that the vector of real and imaginary parts,

$n = (X_1, Y_1, X_2, Y_2, ..., X_p, Y_p)'$, is a 2p-variate normal random variable (denoted as N_{2p}). Let

$$\delta' = E(n) = (\mu_1, \nu_1, \mu_2, \nu_2, \cdots, \mu_p, \nu_p)',$$

and

$$\sum_n = E[(n-\delta)(n-\delta)'],$$

be the variance-covariance matrix of n, where the 2×2 sub matrices of $\sum_n$ have the following special form:

$$\begin{bmatrix} \operatorname{cov}(X_j, X_k) & \operatorname{cov}(X_j, Y_k) \\ \operatorname{cov}(Y_j, X_k) & \operatorname{cov}(Y_j, Y_k) \end{bmatrix} = \begin{cases} \dfrac{\sigma_k^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } j = k, \\ \dfrac{\sigma_j \sigma_k}{2}\begin{bmatrix} \alpha_{jk} & -\beta_{jk} \\ \beta_{jk} & \alpha_{jk} \end{bmatrix} & \text{if } j \neq k. \end{cases} \tag{2.1}$$

Then the density function of Z, following Goodman (1963), is given as:

$$P(Z) = (\pi^p |\sum_Z|)^{-1} \exp\{-(Z-\mu)^H \sum_Z^{-1} (Z-\mu)\}, \tag{2.2}$$

where μ is the mean vector of Z defined as

$$\mu = E(Z) = (\mu_1 + i\nu_1, \mu_2 + i\nu_2, \cdots, \mu_p + i\nu_p)',$$

and $\sum_Z$ is the variance-covariance matrix of Z defined as

$$\sum_Z = [E(Z-\mu)(Z-\mu)^H].$$

The elements of $\sum_Z$ are given as

$$\sigma_{jk} = \begin{cases} \sigma_k^2 & \text{if } j = k, \\ (\alpha_{jk} + i\beta_{jk})\sigma_j \sigma_k & \text{if } j \neq k, \end{cases} \tag{2.3}$$

where $j, k = 1, 2, \cdots, p$. The density function of (2.2) exists only if $\sum_Z$ is a Hermitian positive definite matrix. Goodman (1963) derived (2.2) for the special case when $\mu = \underline{0}$; however, the generalizations are straightforward.

Goodman (1963) defined Z to be CN_p in those cases in which its density function $P(Z)$ is given by (2.2), which exists only if $\sum_Z$ is Hermitian positive definite. In this section, the characteristic function of Z, which always exists, is derived and this characteristic function is used to define a random vector to be CN_p. To derive the characteristic function of Z, the following results proved by Goodman (1963) are needed and stated without proof.

If it is assumed that

$$r_{jk} = \begin{bmatrix} \alpha_{jk} & -\beta_{jk} \\ \beta_{jk} & \alpha_{jk} \end{bmatrix} \text{and}$$

$c_{jk} = \alpha_{jk} + i\beta_{jk}$, $j, k = 1, 2, \cdots, p$, and R is the matrix of r_{jk}'s and C is the matrix of c_{jk}'s, then

Lemma 2.1 Matrix C is isomorphic to matrix R (written as $C \cong R$).

Lemma 2.2 R is symmetric if and only if (iff) C is Hermitian.

Lemma 2.3 The $\det(R) = |\det(C)|^2$, where $\det(R)$ denotes the determinant of the matrix R.

Lemma 2.4 If R is symmetric, the quadratic form $\eta' R \eta = Z^H C Z$.

Theorem 2.1 The characteristic function of a complex multivariate normal random variable Z is given by,

$$\phi_T(Z) = \exp[iRP(T^H \mu) - \frac{1}{4} T^H \Sigma_Z T], \tag{2.4}$$

where $RP(\cdot)$ denotes the real part of $(\cdot)$, and T is a vector of complex numbers.

Proof Let

$$T = (T_1, T_2, \cdots, T_p)' = (U_1 + iV_1, U_2 + iV_2, \cdots, U_p + iV_p)';$$
$$\mu = [E(Z)]' = (\mu_1 + i\nu_1, \mu_2 + i\nu_2, \cdots, \mu_p + i\nu_p); \text{and}$$
$$\omega' = (U_1, V_1, U_2, V_2, \cdots, U_p, V_p).$$

Because η is a 2p-variate normal, the characteristic function of Z is given by

$$\phi_T(Z) = \phi_{(U,V)}(\eta) = E[\exp\{i\sum_{j=1}^{p}(U_j X_j + V_j Y_j)\}]$$
$$= \exp\{i\sum_{j=1}^{p}(U_j \mu_j + V_j \nu_j) - \frac{1}{2}\omega' \Sigma_\eta \omega\}.$$

Because Σ_η is symmetric, Σ_Z is Hermitian by Lemma 2.2. Also, $\Sigma_Z \cong 2\Sigma_\eta$, therefore $\omega' \Sigma_\eta \omega = \frac{1}{2} T^H \Sigma_Z T$ by Lemma 2.4. Furthermore, $\sum_{j=1}^{p}(U_j \mu_j + V_j \nu_j) = RP(T^H \mu)$, and the characteristic function of Z is thus given by (2.4).

Definition 2.1 A complex random vector Z is CN_p its characteristic function is given by (2.4).

Theorem 2.2 (Characterization Theorem). A random vector Z is CN_p iff all nondegenerate complex linear combinations of Z are CN_1.

Proof Let Z be p-variate complex normal with mean vector μ and Hermitian covariance matrix Σ_Z (denoted as $Z \sim CN_p(\mu, \Sigma_Z)$) the characteristic function of Z is given by (2.4). Let ξ be a vector of complex components. Then the characteristic function $\xi^H Z$ is given as

$$\phi_T(\xi^H Z) = \phi_{T\xi}(Z) = \exp[iRP(T^H(\xi^H \mu) - \frac{1}{4} T^*(\xi^H \Sigma_Z \xi)],$$

which is a characteristic function of $CN_1(\xi^H \mu, \xi^H \Sigma_Z \xi)$. Hence, for any non-zero complex vector ξ, $\xi^H Z$ is a complex univariate normal with mean $\xi^H \mu$ and variance $\xi^H \Sigma_Z \underline{\xi}$.

Next, suppose that $Z=(Z_1,Z_2,\cdots,Z_p)'$ is a p-variate complex random vector such that for every nondegenerate complex vector ξ, $\xi^H Z$ is CN_1. Then it can be proven that $Z\sim CN_p$ with some mean vector μ and Hermitian covariance matrix $\sum_Z$.

If the components of the complex vector ξ, are $0+i0$ except at the kth place, where they are $1+i0$, $\xi^H Z=Z_k\sim CN_1$ with a mean μ_k and variance $\sigma_{kk}=\sigma_k^2$ that exist by assumption, and the $\text{cov}(Z_j,Z_k)=\sigma_{jk}$ also exists. Let $\mu=(\mu_1,\mu_2,\cdots,\mu_p)'$ and $\sum_Z$ be the matrix of σ_{jk}'s. Thus, the characteristic function of $\xi^H Z$, is distributed as $CN_1(\xi^H\mu,\xi^H\sum_Z\xi)$ is given by

$$\phi_t(\xi^H Z)=\exp\left[iRP(t^*(\xi^H\mu)-\frac{1}{4}t^*(\xi^H\sum_Z\xi)t\right],$$

which letting $t=1+i0,$ reduces to

$$\phi_{\xi}(Z)=\exp\left[iRP(\xi^H\mu)-\frac{1}{4}(\xi^H\sum_Z\xi)\right]$$

and is a characteristic function of $CN_p(\mu,\sum_Z)$. Thus, $\underline{Z}\sim CN_p(\mu,\sum_Z).$

Theorem 2.3 The components $Z_1,Z_2,\cdots,Z_p$ of Z are mutually independent iff $\sum_Z$ is a diagonal matrix. Moreover, each component is complex univariate normal.

Proof If $Z_1,Z_2,\cdots,Z_p$ are mutually independent, then

$$\text{cov}(Z_j,Z_k)=\begin{cases}\sigma_k^2 & \text{if j = k,}\\ 0 & \text{otherwise,}\end{cases}$$

and $\sum_Z$ is a diagonal matrix consisting of real numbers. Further, the characteristic function of Z factors implies that the components are mutually independent, and $Z_k\sim CN_1(\mu_k,\sigma_k^2),\ k=1,2,\cdots,p.$

It may be noted that the other results can be similarly derived using this characteristic function.

3. Complex Multivariate t-Distribution

A p-variate complex t random variable $t=(t_1,t_2,\cdots,t_p)'$ is a multiple complex t-random variable such that the vectors of its real and imaginary parts, $n=(t_{1R},t_{1I},t_{2R},t_{2I},\cdots,t_{pR},t_{PI})'$, is a 2p-variate real t-distribution. The p-variate complex t-distribution arises from the joint distribution of p variates $t_i=Z_i/s$, $i=1,2,\cdots,p$ where the $Z_i's$ have a nonsingular complex multivariate normal distribution with means 0 and complex covariance matrix $\sum=\sigma^2R,\ R=(\rho_{ij})$ with $\rho_{ij}=1,$ and the matrix R is known in certain cases, but σ^2 is unknown. It is assumed that $\sum$ is Hermitian positive definite (hpd) and hence R is hpd. Also, it is assumed that $\sum$ is of the form described in (2.3).

s^2 is an estimate of σ^2 based on $2n$ degrees of freedom and is independently distributed of $Z=(Z_1,Z_2,\cdots,Z_p)'$ Also, $2ns^2/\sigma^2$ has a chi-square distribution with $2n$ degrees of freedom. Under these assumptions we have the following theorem:

Theorem 3.1 The density of $t=(t_1,t_2,\cdots,t_p)'$ is given by,

$$p(\underline{t})=\frac{\Gamma(n+p)}{(n\pi)^p|R|\Gamma(n)}\left(1+\frac{t^HR^{-1}t}{n}\right)^{-(n+p)}$$, where $|R|$ denotes the determinant of R.

Proof Using (2.2), the joint density of $Z=(Z_1,Z_2,\cdots,Z_p)'$, and s is

$$P(Z,s)=\frac{1}{\pi^p\sigma^{2p}|R|}\exp\left\{-\frac{1}{\sigma^2}Z^HR^{-1}Z\right\}\frac{1}{2^n\Gamma(n)}\left(\frac{2ns^2}{\sigma^2}\right)^{n-1}\frac{2n2s}{\sigma^2}\exp\left\{-\frac{1}{\sigma^2}ns^2\right\}dz_1dz_2\cdots dz_kds,$$

$$=\frac{1}{\pi^p|R|\Gamma(n)}\frac{2n^n}{\sigma^{2(n+p)}}s^{2n-1}\exp\left\{-\frac{1}{\sigma^2}(Z^HR^{-1}Z+ns^2)\right\}\ dz_1dz_2\cdots dz_kds,$$

where $dZ_k=dZ_{Rk}dZ_{Ik,}\quad k=1,2,\cdots,p.$

When the transformation $Z=ts$ is made, and it is noted that the Jacobian of the transformation, $J(Z,t),$ is $s^2p,$ the joint density of $t=(t_1,t_2,\cdots,t_p)'$, becomes

$$P(t,s)\ =\frac{1}{\pi^p|R|\Gamma(n)}\frac{2n^n}{\sigma^{2(n+p)}}s^{2(n+p)-1}\exp\left\{-\frac{s^2}{\sigma^2}(t^HR^{-1}t+n)\right\}\ dt_1dt_2\cdots dt_pds.$$

Again if the transformation

$$Z=\frac{s^2}{\sigma^2}(t^HR^{-1}t+n),$$

$$2sds=\sigma^2(t^HR^{-1}t+n)^{-1}dz,$$

is made and s is integrated out, the joint density of $t=(t_1,t_2,\cdots,t_p)'$ reduces to (3.1).

Example 3.1 Univariate Complex t-Distribution

If $p=1,$ then $R=(1),|R|=1,R^{-1}=1,$and $P(t)=\frac{1}{\pi}\left(1+\frac{|t|^2}{n}\right)^{-(n+1)}$

where $|t|^2=tt^*=t_R^2+t_I^2.$

Example 3.2 The Bivariate Complex t-Distribution

If $p=2,$ then $t=(t_1,t_2)'=(t_{1R}+it_{1I},t_{2R}+it_{2I})',$and

$$R=\begin{pmatrix}1 & \alpha_{12}+i\beta_{12}\\ \alpha_{12}-i\beta_{12} & 1\end{pmatrix},$$

$$|R|=1-\alpha_{12}^2-\beta_{12}^2,\text{ and}$$

$$R^{-1}=\frac{1}{|R|}\begin{pmatrix}1 & -(\alpha_{12}+i\beta_{12})\\ -(\alpha_{12}-i\beta_{12}) & 1\end{pmatrix}.$$

Thus, the joint density of $t_1 t_2$ is given as

$$P(t) = \frac{n+1}{n\pi^2(1-\alpha_{12}^2-\beta_{12}^2)}\left(1+\frac{|t_1|^2+|t_2|^2-2RP(\alpha_{12}+i\beta_{12}t_1^* t_2)}{n(1-\alpha_{12}^2-\beta_{12}^2)}\right)^{-(n+2)}.$$

Example 3.3 A Test of Hypothesis

Let $Z \sim CN_1(\mu,\sigma^2)$ and let $z_1, z_2, \cdots, z_n$ be a random sample from this population (Z), where z_j is of the form $x_j + iy_j, j = 1,2,\cdots,n.$ Situations may arise where it is desirable to test the hypothesis $H_0 : |\mu| = 0,$ i.e., μ has a representation $\mu = 0 + i\,0,$ versus the alternative hypothesis $H_A : |\mu| \neq 0,$ if there is an ordered sample of the form $[(x_1, y_1)(x_2, y_2), \cdots, (x_n, y_n)],$ and the null hypothesis is that this sample is centered around the origin of the 2-dimensional Euclidean space. The usual tests of hypothesis in this situation do not seem appropriate. Instead, a test statistic based on complex distribution theory is derived.

The likelihood function of $(z_1, z_2, \cdots, z_n)$ is

$$P(z_1, z_2, \cdots, z_n) = \left(\frac{1}{n\sigma^2}\right)^2 \exp\left\{-\frac{1}{\sigma^2}\sum_{j=1}^{n}|z_j-\mu|^2\right\}, \text{ where } \mu = E(Z) = E(X) + i\,E(Y).$$

The maximum likelihood estimates of μ and σ^2 are $\hat{\mu} = \bar{z} = \frac{1}{n}\sum_{j=1}^{n} z_j$, and $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}|z_j - z|^2$ and are independently distributed. An unbiased estimate of $\hat{\sigma}^2$ is $s^2 = \frac{1}{n-1}\sum_{j=1}^{n}|z_j - \bar{z}|^2$. Also, $\bar{z}$ and s^2 are independently distributed, and $\bar{z} \sim CN_1\left(\mu, \frac{\sigma^2}{n}\right)$ and $\frac{2(n-1)s^2}{\sigma^2} \sim \chi^2_{2(n-1)}.$ Thus, for testing $H_0 : |\mu| = 0,$ using likelihood ratio test, a statistic $t = \sqrt{n}\, z/s$ is arrived at which has univariate complex t-distribution under the null hypothesis with $(n-1)$ degrees of freedom. The percentage points of the above t-statistics can be obtained using

$$P(t \le t_0) = P(t_R \le t_{R_0}, t_I \le t_{I_0}),$$

where $t_R = \frac{\sqrt{2n}\bar{z}}{s}$, $t_I = \frac{\sqrt{2n}z_I}{s}$, and t_R, t_I are bivariate t-distributed with each having $2(n-1)$ degrees of freedom.

Example 3.4 Correlation Between Two Complex Random Variables

Let $\xi_1, \xi_2, \cdots, \xi_n$ be a random sample of size n from a $CN_2(\underline{\mu}, \sum)$

$$\xi_k = \begin{pmatrix} z_{1k} \\ z_{2k} \end{pmatrix}, \quad \underline{\mu} = \begin{pmatrix} \mu_1 + i\nu_1 \\ \mu_2 + i\nu_2 \end{pmatrix}, \quad \sum = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \; k = 1,2,\cdots,n.$$

Let the matrix of sum of squares and products be

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{ij} = \sum_{k=1}^{n} (z_{ik} - \bar{z}_i)(z_{jk} - \bar{z}_j)^*$, $i, j = 1, 2$ and $\bar{z}_i = \frac{1}{n}\sum_{k=1}^{n} z_{ik}$, $i = 1, 2.$

The correlation between z_1 and z_2 is given by $\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$ and an estimate of ρ, that is, the sample covariance, is given by $r = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}$. For testing the null hypothesis $H_0 : \rho = 0,$ it can be shown following Young (1971) that the statistic $\tilde{r} = \frac{r}{(1 - rr^*)}$ has univariate complex t-distribution under H_0.

4. Mean Vector and Covariance Matrix of a p-variate Complex t-Distribution

For simplicity, let $T = (T_1, T_2, \cdots, T_p)'$ be a p-variate complex T (denoted as CT_p); then, by definition $E(T) = \underline{0}$. Because p-variate complex random vector T is a 2p-variate real random variable t whose variance-covariance matrix is $\frac{n}{n-1} R_t$, $n > 1$, and because $2R_t \cong R_T$, the Hermitian variance-covariance matrix of CT_p is $\frac{n}{2(n-1)} R_T$, $n > 1$. This leads to the following definition.

Definition 4.1 A p-variate complex random vector $T = (T_1, T_2, \cdots, T_p)'$ is to have a non-singular complex multivariate T distribution with complex mean vector $\mu' = (\mu_1, \mu_2, \cdots, \mu_p)$ and Hermitian covariance matrix $\frac{n}{2(n-1)} R$, $n > 1$, if it has the following p.d.f.

$$P(T) = \frac{\Gamma(n+p)}{(n\pi)^p |R| \Gamma(n)} \left(1 + \frac{T - \mu^H R^{-1} (T - \mu)}{n}\right)^{-(n+p)},$$

Thus, let $T \sim CT_p(\mu, R, n)$; then, T may be written as $T = S^{-1} z$ where $z \sim CN_p(\mu, R)$ and $2nS^2 \sim \chi^2_{2n}$, independent of z. This implies that $T \mid S^2 = S^2 \sim CN_p(\mu, S^{-2} R)$. Here $T \mid S^2$ denotes the conditional distribution of T given that the random variable S^2 assumes the value s^2.

Theorem 4.1 A Characterization Theorem for $CT_p(\mu, R, n)$

$T \sim CT_p(\mu, R, n)$ iff for any nondegenerate $p \times 1$ complex vector a

$$\frac{a^H(T-\mu)}{\sqrt{a^H R a}} \sim CT_1(0,1,n).$$

Proof Let $2nS^2 \sim \chi^2_\nu$; then, $T \big| S^2 = S^2 \sim CN_p(\mu, s^{-2}R)$ iff

$$\frac{a^H(T-\mu)}{\sqrt{a^H R a}} = s^2 \sim CN_1(0, s^{-2})$$

(by Theorem 2.2), i.e., iff $\frac{a^H(T-\mu)}{\sqrt{a^H R a}} \sim CT_1(0,1,n)$.

5. Conclusions

In this article we have shown that complex multivariate random variable of Z is a complex multivariate normal variable of dimension *n* if all nondegenerate complex linear combination of Z have a complex univariate normal distribution. The characteristic function of Z has been derived. An extension of complex multivariate t-distribution has been proposed and few examples are suggested.

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