On Introducing Asymmetry into Circular Distributions

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Abstract

We give a brief history of the results which led to the introduction of asymmetry into symmetric circular distributions. This is followed by the presentation of another method of introducing asymmetry. Some properties of the induced distributions are studied. Finally, this new distribution is shown to be a reasonable fit to the Jander ant data as presented in Fisher (1993).

1. Introduction and Main Result

Azzalini (1985, 1986) introduced a method of skewing a symmetric distribution to produce tractable families of non-symmetric distributions. He showed that with f the density of a symmetric distribution and G being the distribution function of a symmetric distribution that

$$2G(\lambda x)f(x) \qquad (1)$$

is a density for each λ and that λ controls the amount of skew. This generated a large number of papers producing skewed models from many of the well-known symmetric distributions. Ali, Pal, and Woo (2008 and 2009) and Nadarajah and Ali (2005) are good examples of such work.

Later, Azzalini and Capitanio (1999) showed more generally that

$$2G(w(x))f(x) \qquad (2)$$

is a density where f and G are as above and w is odd. Along this line, Umbach and Jammalamadaka (2009) introduced this skewing idea to circular distributions. They showed that with f a symmetric (about 0) circular density and G the distribution function of a symmetric (about 0) circular density and w odd and periodic with $|w(\theta)| \leq \pi$ that

$$2f(\theta)G(w(\theta)) \qquad (3)$$

is a circular density with $G(-\pi) = 0$. This typically results in an asymmetric distribution. One can introduce various parameters by judicious choices of G and w.

In this work, we introduce a different method of introducing asymmetry into circular distributions. In particular, we show that

$$p(\theta;\gamma) = \frac{2f(\theta)(1+(\gamma-1)G(w(\theta)))}{\gamma+1} \qquad (4)$$

is a circular density for each $\gamma \geq 0$, with f, G, and w as above. Results and interpretation concerning the introduction of the parameter γ will be presented.

Theorem: Suppose that G is the distribution function of a symmetric (about 0), circular distribution and that f is the density of a symmetric (about 0), circular distribution, and that w is an odd function with $|w(\theta)| \leq \pi$ and $\gamma \geq 0$, then $p(\theta;\gamma)$ from (4) is a circular density.
The proof is immediate from (3) and noting that $(1+(\gamma-1)G(w(\theta)) \geq (1-G(w(\theta)) \geq 0$.

Figure 1 gives some indication for the amount of asymmetry present in the density p. The central symmetric density is that of the von Mises distribution

$$f(\theta) = \frac{e^{\kappa\cos(\theta)}}{2\pi I_0(\kappa)}, \qquad (5)$$

with $I_0(\kappa)$ being the modified Bessel function of the first kind with $\kappa = 3$. The densities graphed are for $\gamma = 1, 0.1, 2$, and 10 and $G(\theta) = (\theta+\pi)/(2\pi)$ (uniform distribution) and $w(\theta) = \pi\sin(\theta)$.

2. Properties of the Distribution

To develop some properties of the distribution p let us write

$$p(\theta;\gamma) = \frac{2}{\gamma+1}f(\theta) + \frac{\gamma-1}{\gamma+1}2f(\theta)G(w(\theta)) \qquad (6)$$

Thus, we can express p as quasi-mixture of the symmetric central density f and the distribution introduced by Umbach and Jammalamadaka (2009) in (3). Note that the coefficients do add to 1, but $(\gamma-1)/(\gamma+1)$ can be negative.

However, we can express p as a proper mixture of different distributions. It is easily seen that

$$(4) = \frac{1}{\gamma+1}\{2f(\theta) + \gamma 2f(\theta)G(w(\theta)) - 2f(\theta))G(w(\theta))\}$$

$$= \frac{1}{\gamma+1}2f(\theta)\bar{G}(w(\theta)) + \frac{\gamma}{\gamma+1}2f(\theta)G(w(\theta)), \qquad (7)$$

with $\bar{G}(\theta) = 1 - G(\theta)$.

Next we show that $f_1(\theta) = 2f(\theta)\bar{G}(w(\theta))$ is also a circular density under the conditions that yielded (3). Note that the symmetry of G implies $G(-\theta) = \bar{G}(\theta)$.
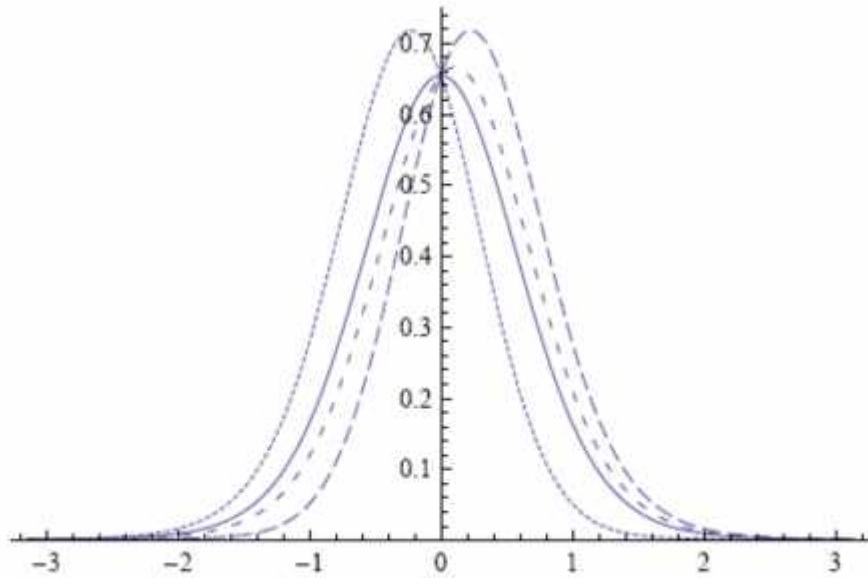
Figure 1: Graphs of the skewed von Mises distribution with $\kappa = 3$ and $\gamma = 1$ (the von Mises itself)(solid line), $\gamma = 0.1$ (dotted line), $\gamma = 2$ (sparse dashing), and $\gamma = 10$ (dense dashing).

Noting that w will be odd, periodic and bounded by π precisely when $-w$ is odd, periodic and bounded by π, we see that p is a proper mixture of the distributions in (7).

In Umbach and Jammalamadaka (2010), they showed that if h is even and periodic that

$$\int_{-\pi}^{\pi} h(\theta)f(\theta)d\theta = \int_{-\pi}^{\pi} h(\theta)2f(\theta)G(w(\theta))d\theta. \tag{8}$$

We note that this property of expectations of even functions is shared in the linear case by all distributions that are skewed in the Azzalini style. Since p is a linear combination of the densities in (6) we see that the common value of the integrals in (6) is also the value of

$$\int_{-\pi}^{\pi} h(\theta)p(\theta,\gamma)d\theta$$

for all $\gamma > 0$.

One can apply the ``odd function" results of Umbach and Jammalamadaka (2010) as well. In particular, note that for h odd and periodic, we have

$$\int_{-\pi}^{\pi} h(\theta)p(\theta,\gamma)d\theta = \frac{\gamma-1}{\gamma+1}\int_{-\pi}^{\pi} h(\theta)2f(\theta)G(w(\theta))d\theta. \tag{9}$$

In the next section, we apply these results to trigonometric moments.

Trigonometric moments

For a circular random variable Θ, the pth trigonometric moment φ_p, is defined by

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$$\varphi_p = E\left(e^{ip\Theta}\right) = E(\cos p\Theta) + iE(\sin p\Theta) = \alpha_p + i\beta_p.$$

Let ρ_p be the length of φ_p and if φ_p is not 0, let μ_p be its direction. μ_1 is typically referred to as the mean (or preferred) direction. We shall compare and contrast the moments of the distributions presented herein. To streamline the results, we shall refer to quantities calculated for the cental symmetric distribution, f, with the superscript ^{o}, as φ_p^o, for example. We shall refer to quantities calculated for the distribution in (3) with the superscript $'$, as μ_p', for example. We shall use the superscript * in relation to quantities calculated for the distribution p in (5) and will use the superscript * in relation to the folded distribution $2f(\theta)$ for $0 \le \theta \le \pi$. Umbach and Jammalamadaka (2010) showed that

$$0 = \beta_1^o \le \beta_1' \le \beta_1^*. \tag{10}$$

Noting that $\beta_1^\star = ((\gamma - 1)/(\gamma + 1))\beta_1^*$, we see that

$$0 = \beta_1^o \le \beta_1^\star \le \beta_1' \le \beta_1^* \, for \gamma \ge 1 and$$
$$\beta_1^\star \le 0 = \beta_1^o \le \beta_1' \le \beta_1^* \, for 0 < \gamma \le 1. \tag{11}$$

The "even function" results above directly yield

$$\alpha_p^o = \alpha_p^\star = \alpha_p' = \alpha_p^* \, for p = \pm 1, \pm 2, \ldots, \tag{12}$$

and thus

$$0 < \rho_1^o \le \rho_1^\star \le \rho_1' \le \rho_1^*.$$

Combining (11) and (12), we see that

$$0 = \mu_1^o \le \mu_1^\star \le \mu_1' \le \mu_1^* \, for \gamma \ge 1 and$$
$$\mu_1^\star \le 0 = \mu_1^o \le \mu_1' \le \mu_1^* \, for 0 < \gamma \le 1. \tag{13}$$

An Application

Rudolf Jander's experiments concerning the direction of ants in response to a stimulus has long provided some interesting problems in modeling of circular distributions. See Jander (1957) for an original description of these experiments. In particular, the data set of Example 4.4 on page 60 of Fisher (1993) has generated much interest. Fisher clearly demonstrates that the von Mises distribution does not fit this data very well. We come closer to an appropriate model using an asymmetric model as presented herein.

We will model the distribution with the density

$$p(\theta; \mu, \kappa, \gamma) = \frac{e^{\kappa \cos(\theta - \mu)}}{(\gamma + 1)2\pi I_0(\kappa)}(1 + \gamma + (\gamma - 1)\sin(\theta - \mu)). \tag{14}$$

This density results from the choices of f being the von Mises distribution of (5), $G(\theta) = (\theta + \pi)/(2\pi)$ (uniform distribution), and $w(\theta) = \pi \sin(\theta)$ and finally introducing a directional parameter μ.

Mathematica was used to numerically carry out maximization of the likelihood with the following results.

$$\hat{\mu} = 2.88018 = 165.053°$$
$$\hat{\kappa} = 1.4361$$
$$\hat{\gamma} = 3.19855.$$

To provide some guidance as to the goodness of the fit, we calculated both the Kuiper and Watson statistics for the model in (14) with the mle's used for the parameters. The value of the Kuiper statistic as described on page 153 of Jammalamadaka and SenGupta (2001) is $V_{100} = 2.33717$. The value of the Watson statistic on page 156 of the same text is 0.287414. Both these indicate a reasonable fit of the model (14) to this data. On the other hand, Fisher (1993, p.84) tried fitting the vonMises model in (5) to this data and found the maximum likelihood estimators to be

$$\hat{\mu} = 3.1957 = 183.1° \;\; and \;\; \hat{\kappa} = 1.54.$$

After doing formal goodness of fit tests as well as Q-Q plots, he concludes that the vonMises distribution is not a suitable model for this data.

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