Statistics in the Twenty-First Century: Special Volume
In Honour of Distinguished Professor Dr. Mir Masoom Ali
On the Occasion of his 75th Birthday Anniversary
PJSOR, Vol. 8, No. 3, pages 507-510, July 2012

Intersection of a Sure Ellipsoid and a Random Ellipsoid

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Abstract

An Expression for the expected value of the intersection of a sure sphere and a random sphere has been derived by Laurent (1974). In the present paper we derive the expression for the expected intersection volume of a sure ellipsoid and a random ellipsoid.

Keywords and phrases: Expected value; Sure ellipsoid; Random field; Noncentral Chi square distribution; Bessel function

1. Introduction

We follow the notation and terminology of Laurent (1974). Let S_0 be a fixed n-dimensional ellipsoid with the equation $x'\Delta x \leq r^2$, where $n \times n$ Δ is positive definite symmetric and r is known. Let the center C, with coordinates ξ, of an n-dimensional ellipsoid S follow an elliptically symmetric distribution about the fixed center A, with coordinates η, i.e., the density of ξ is of the type $g_n(y'\Delta y)$, $y = (\xi - \eta)$. We wish to obtain an expression for $\mu_n = E[V(S_0 \cap S)]$, the expected intersection volume contained in S_0 and S, where S denotes the volume $y'\Delta y \leq R^2$, R known.

Let m be the coordinates of a random point M inside this intersection. Then from Laurent (1974, p. 183, equation (4)) we have

$$\mu_n = \int_{m'\Delta m \leq r^2} \left[\int\!\!\int_{(\xi-m)'\Delta(\xi-m)\leq R^2} g_n(y'\Delta y) dy \right] dm$$

$$= \int I_{S_0} \left[\int\!\!\int I_{S_0} g_n(y'\Delta y) dy \right] dm \,, \tag{1}$$

where the integral inside the square brackets represents $P\{(\xi - m)'\Delta(\xi - m) \leq R^2\}$. I_S denotes the indicator function of the set $I_S(m; \xi, R^2)$, and I_{S_0} is the indicator function of the set $I_{S_0}(m, r^2)$. Note that $y'\Delta y \leq R^2$ implies $(\xi - m)'\Delta(\xi - m) \leq R^2$. For a fixed m, and normal ξ, we know that $\rho^2 = (\xi - m)'\Delta(\xi - m)$ follows a noncentral χ^2 distribution with n degrees of freedom and noncontrality parameter $\delta^2 = (m - \eta)'\Delta(m - \eta)$ (Muirhead, 2005). This noncentral distribution can be expressed in terms of a Bessel function of first kind and order $(n - 2)/2$. (Abramowitz and Stegun, 1972). Obviously, $P\{(\xi - m)'\Delta(\xi - m) \leq R^2\}$ depends on M only through δ^2. Let us denote the distribution function of ρ^2. i.e. $P\{\rho^2 \le R^2\}$ by $Q_n(R^2;\ \delta^2)$ and its density function by $q_n(\rho^2;\ \delta^2)$. Thus the equation (1) now becomes

$$\mu_n = \int Q_n(R^2;\ \delta^2) I_{S_0} dm = \int_{n'\Delta n \le r^2} Q_n(R^2;\ \delta^2) dm\ . \tag{2}$$

We proceed to evaluate (2) in the next section. We first note that the random variable $(\xi - m)'\Delta(\xi - m)$, for a fixed m, has a noncentral χ^2 type density. This density is necessary to evaluate the first integral in (1). Next the random variable $m'\Delta m$ also has anoncentral χ^2 type density with noncentrality parameter $\eta'\Delta\eta$. Thus the final result depends only on r, R, $\eta'\Delta\eta$, and Δ. The final result involves three Bessel functions of the first kind. The first Bessel function appears in the noncentral χ^2 type density of $(\xi - \mu)'\Delta(\xi - m)$. The second Bessel function appears because the integration involves integration over a certain angle. The third Bessel function appears in the noncentral χ^2 type density of $m'\Delta m$.

2. The squared noncentral radical error distribution

As in Laurent (1974), we get $(y - \eta)'\Delta(y - \eta) = |y|^2$, $|y| = [(y - \eta)'\Delta(y - \eta)]^{1/2}$, $\delta^2 = (m - \eta)'\Delta(m - \eta)$, $\delta = [(m - \eta)'\Delta(m - \eta)]^{1/2}$, $|y| > 0$, $\delta > 0$, and

$$\rho^2 = |y|^2 + \delta^2 - 2(m - \eta)'\Delta(m - \eta) = |y|^2 + \delta^2 - 2|y|\,\delta\cos\varphi\ , \tag{3}$$

where $\cos\varphi = (m - \eta)'\Delta(m - \eta)/|y|\delta$.

The characteristic function of an elliptically symmetric distribution (Gupta and Varga, 1993) is given by

$$\begin{aligned}\int_{-\infty}^{\infty} \exp\{it'\Delta y\} f(y'\Delta y)\,dy &= K|\Delta|^{-\frac{1}{2}} f(u) u^{\frac{n-2}{2}} \int_{-1}^{1} \exp\left\{i\sqrt{u}\sqrt{t'\Delta t}v\right\}(1 - v^2)^{\frac{n-3}{2}} dv \\ &= K|\Delta|^{-\frac{1}{2}} f(u) u^{\frac{n-2}{2}} \int_{-\pi}^{\pi} \exp\left\{i\sqrt{u}\sqrt{t'\Delta t}\cos\varphi\right\} \sin^{n-2}\varphi\, d\varphi \\ &= K|\Delta|^{-\frac{1}{2}} f(u) u^{\frac{n-2}{2}} J_{\frac{n-2}{2}}\left(\sqrt{u}\sqrt{t'\Delta t}\right)\end{aligned} \tag{4}$$

where $\cos\varphi = t'\Delta y/(t'\Delta t)^{1/2}(y'\Delta y)^{1/2}$,

$$J_n(x) = \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n {}_0F_1\left(n + 1, -\frac{1}{4}x^2\right),$$

and K denotes constant terms. Here $J_n(x)$ is the Bessel function of the first kind defined in terms of the confluent hypergeometric function ${}_0F_1$ $(\cdot)$.Thus the integral (4) depends on only one angle φ.

Now Laurent (1974) shows that the evaluation of $Q_n(R^2, \delta^2;\ |y|^2)$, i.e., the conditional distribution function of the noncentral radial error ρ depends on only one angle φ, and is given by

$$Q_n(R^2, \delta^2;\ |y|^2) = \int_0^{\varphi} \sin^{n-2}\varphi d\varphi \Big/ B\left(\frac{n-1}{2}, \frac{1}{2}\right), \tag{5}$$

where $\cos\varphi = (|y|^2 + \delta^2 - R^2)/2|y|\delta$. Further (see Laurent (1974, p. 184, equation (8))) we find (5) to be

$$Q_n(R^2, \delta^2;\ |y|^2) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2R}{|y|\delta}\right)^{\frac{n-2}{2}} R \int_0^{\infty} u^{\frac{-(n-2)}{2}} J_{\frac{n-2}{2}}(|y|u) J_{\frac{n}{2}}(Ru) J_{\frac{n-2}{2}}(\delta u) du\ . \tag{6}$$

The noncentral density of the squared radial error ρ^2 is

$$q_n(\rho^2;\ \delta^2;\ |y|^2) = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)\left(\frac{2\rho}{|y|\delta}\right)^{\frac{n-2}{2}} \int_0^{\infty} u^{\frac{-(n-4)}{2}} J_{\frac{n}{2}}(|y|u) J_{\frac{n-2}{2}}(\delta u) J_{\frac{n-2}{2}}(\rho u) du\ . \tag{7}$$

The result (7) follows by noting that the density of the angle φ, measured in radians, is given by

$$g(\varphi) = \frac{1}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)}(1 - \cos^2\varphi)^{\frac{n-3}{2}}\ , \quad -\pi < \varphi < \pi\ , \tag{8}$$

and that

$$\cos\varphi = \frac{|y|^2 + \delta^2 - \rho^2}{2|y|\delta}, \quad (|y| - \delta)^2 \le \rho^2 \le (|y| + \delta)^2\ . \tag{9}$$

Hence from (8) and (9), we get the density of ρ^2, given $|y|^2$, as

$$q_n(\rho^2, \delta^2;\ |y|^2) = \frac{(2|y|\delta)^{-(n-2)}[\rho^2 - (\delta - |y|)^2]^{\frac{n-3}{2}}[(\delta + |y|)^2 - \rho^2]^{\frac{n-3}{2}}}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)}\ . \tag{10}$$

The result (10) is given by Laurent (1974, p. 185, equation (10)).

Again from Laurent (1974, p. 185, equation (11)), we have that

$$Q_n(R^2, \delta^2) = \int_0^{\infty} Q_n(R^2, \delta^2;\ |y|^2) h_n(|y|^2) d|y|^2\ , \tag{11}$$

where

$$h_n(|y|^2) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}(|y|^2)^{\frac{n-2}{2}}|\Delta|^{-\frac{1}{2}} g_n(|y|^2)\ . \tag{12}$$

It follows that the equation (12) of Laurent (1974, p. 185), now reads as

$$Q_n(R^2,\delta^2) = |\Delta|^{-\frac{1}{2}}\left(\frac{R}{\delta}\right)^{\frac{n-2}{2}} R\int_0^\infty \varphi(u)J_{\frac{n-2}{2}}(\delta u)J_{\frac{n}{2}}(Ru)du\,, \tag{13}$$

where from (4)

$$\varphi(|t|) = \varphi\left(\sqrt{t'\Delta t}\right) = K\int_0^\infty |\Delta|^{-\frac{1}{2}}f(u)u^{\frac{n-2}{2}}J_{\frac{n-2}{2}}\left(\sqrt{u}\sqrt{t'\Delta t}\right)du\,. \tag{14}$$

Thus from Laurent (1974, p. 188, equation (24)), we find that

$$\mu_n = |\Delta|^{-\frac{1}{2}}\int_0^\infty \left(\frac{R}{\delta}\right)^{\frac{n-2}{2}} R\int_{S_0} \varphi(u)J_{\frac{n-2}{2}}(\delta u)J_{\frac{n}{2}}(Ru)du\,dm\,.$$

Now integrating out m over the range $m'\Delta m \le r^2$, in the same way as has been done to integrate out ξ, we find that

$$\mu_n = |\Delta|^{-\frac{1}{2}}\left(\frac{2\pi R}{|\eta|}\right)^{\frac{n}{2}}\int_0^\infty \varphi(u)J_{\frac{n}{2}}(Ru)J_{\frac{n-2}{2}}(|\eta|u)J_{\frac{n}{2}}(ru)du,$$

where $|\eta|^2 = \eta'\Delta\eta$.

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