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On Estimating Reliability of Multicomponent Stress–Strength Model for Kumaraswamy Inverse Weibull Distribution

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Abstract
In this paper, we have discussed the reliability estimation for a multicomponent stress-strength (MCSS) model when stress and strength follow the Kumaraswamy inverse Weibull distribution, given by Shahbaz et al. (2012). We have obtained the maximum likelihood estimate of the reliability alongside the asymptotic distribution of the parameters involved. Also, the asymptotic confidence intervals have been obtained. An extensive simulation study has been conducted to assess the performance of the estimates. A real data application has also been given. It is found that the reliability increases with an increase in one of the shape parameters of stress distribution.

Key Words: Kumaraswamy inverse Weibull distribution, multi-component system, reliability.

1. Introduction

The stress–strength systems are widely used in many areas of engineering. The system is based upon two components, stress, and strength, and both are assumed to be random. The reliability of a system depends upon the fact that the strength exceeds stress. Statistically, the reliability of a stress–strength system is given by $R = P(X > Y)$, where the random variable X denotes the strength and the random variable Y denotes the stress. The system works if strength exceeds stress otherwise the system will fail. Estimation of reliability has been an area of study by various authors assuming various distributions for the random variables X and Y. A general method for estimation of system reliability is discussed by Hanagal (1999) and after that people have studied the reliability of stress–strength models for various distributions. A thorough discussion about the mathematical treatment of reliability estimation is discussed by Cao and Cheng (2006). The reliability of a three-parameter generalized exponential distribution is studied by Raqab et al. (2008). The reliability of a stress–strength model, when both components follow the Kumaraswamy distribution, is studied by Nadar et al. (2014) whereas Ghitany et al. (2015) have studied reliability when the components follow the power Lindley distribution.

Generally, a system is a collection of several components and in such cases reliability of the system depends upon these multicomponents. A multi-component system depends upon k independent strengths and a common stress. The system works if s ($1 \le s \le k$) of these components work and hence is also known as *s–out of–k* system. Various examples of such systems can be given. A simple example of such a system is a V8 engine which operates if at least 4 of its cylinders work and hence is known as 4–out of–8 system. The reliability of an *s–out of–k* system is denoted by $R_{s:k}$. A general method to estimate the reliability of a multicomponent system was introduced by Bhattacharyya and Johnson (1974). The reliability of a multicomponent stress–strength model is defined by Bhattacharyya and Johnson (1974) as

$$\begin{aligned} R_{s:k} &= P\left[At\ least\ s\ of\ \left(X_1, X_2, \ldots, X_k \right) exceed\ Y \right] \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} \left[1 - F_X\left(y \right) \right]^i \left[F_X\left(y \right) \right]^{k-i} dF_Y\left(y \right), \end{aligned} \tag{1}$$

where $F_X\left(x\right)$ is the distribution associated with the strength component *X* and $F_Y\left(y\right)$ is the distribution associated with the stress component *Y*.

The multicomponent stress–strength (MCSS) reliability has been an area of interest by several authors. The reliability of an MCSS model when the stress and strength components follow the generalized exponential distribution has been studied by Rao (2012). The reliability of the MCSS model under Rayleigh and generalized Rayleigh distributions has been discussed by Rao et al. (2013) and Rao (2014). The reliability of the multicomponent stress–strength model for Kumaraswamy distributions Nadar et al. (2014), Dey et al. (2016) Kizilaslan and Nadar (2018). The reliability of an MCSS model under the Weibull distribution has been studied by Kizilaslan and Nadar (2015) whereas the reliability of a multicomponent stress–strength model for Marshall–Olkin Weibull distributions has been discussed by Nadar and Kizilaslan (2016). The reliability of the MCSS model for power Lindley distribution was studied by Pak et al. (2018) and the reliability of the MCSS model under Topp–Leone distribution has been studied by Akgül (2019). The reliability of the MCSS model using progressive censored samples for Kumaraswamy distribution has been studied by Kohansal (2019). The ranked set sampling has also been effectively used to estimate the reliability of MCSS models. The MCSS reliability for the inverse Weibull distribution has been studied by Shawki and Khan (2022). Some other notable references in this regard are Akgül and Şenoğlu (2017), Akgül et al. (2018), Safariyan et al. (2019), Mahto et al. (2020), and Jha et al. (2022), among others.

In this paper, we will study the reliability of the MCSS model when the stress and strength components follow the Kumaraswamy inverse Weibull (KIW) distribution. In Section 2, the reliability of the MCSS model is computed for KIW distribution. Maximum likelihood estimation of the reliability is given in Section 3 alongside the asymptotic confidence intervals. Simulation and real data applications are given in Section 4 followed by conclusions and recommendations in Section 5.

2. Reliability of MCSS Model for Kumaraswamy Inverse Weibull Distribution

The Kumaraswamy inverse Weibull (KIW) distribution has been proposed by Shahbaz et al. (2012) by using the inverse Weibull distribution as a baseline distribution in the Kumaraswamy–G family of distributions, proposed by Cordeiro and de Castro (2011). The density and distribution function of KIW distribution are

$$f_X\left(x\right) = ab\alpha\beta\left(\beta x\right)^{-(\alpha+1)} \mathrm{e}^{-a(\beta x)^{-\alpha}} \left[1 - \mathrm{e}^{-a(\beta x)^{-\alpha}}\right]^{b-1}, \left(x, a, b, \alpha, \beta\right) > 0 \tag{2}$$

and

$$F_X\left(x\right) = 1 - \left[1 - \mathrm{e}^{-a(\beta x)^{-\alpha}}\right]^{b}, \left(x, a, b, \alpha, \beta\right) > 0, \tag{3}$$

and is written as $KIW(a,b,\alpha,\beta)$.

Now, we assume that the *k* strength components in a multicomponent system are independently and identically distributed each having $KIW(c,b_1,\alpha,1)$. The distribution function is

$$F_X\left(x\right) = 1 - \left(1 - \mathrm{e}^{-cx^{-\alpha}}\right)^{b_1}, \left(x, c, b_1, \alpha\right) > 0$$

We will also assume that the stress components *Y* follow $KIW(c,b_2,\alpha,1)$. The density and distribution functions of *Y* are

$$F_Y\left(y\right) = 1 - \left(1 - \mathrm{e}^{-cy^{-\alpha}}\right)^{b_2}, \left(y, c, b_2, \alpha\right) > 0$$

$$f_Y(y)=cb_2\alpha y^{-(\alpha+1)}e^{-cy^{-\alpha}}\left(1-e^{-cy^{-\alpha}}\right)^{b_2-1},\ (y,c,b_2,\alpha)>0.$$

Now under these specifications, the reliability of such an MCSS system is

$$R_{s:k}=P\left[\text{At least } s \text{ of } \left(X_1,X_2,\ldots,X_k\right) \text{ exceed } Y\right]$$
$$=\sum_{p=s}^{k}\binom{k}{p}\int_{-\infty}^{\infty}\left[1-F_X(y)\right]^p\left[F_X(y)\right]^{k-p}dF_Y(y),$$

or

$$R_{s:k}=\sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left(1-e^{-cy^{-\alpha}}\right)^{ib_1}\left[1-\left(1-e^{-cy^{-\alpha}}\right)^{b_1}\right]^{k-i}\alpha cb_2 y^{-(\alpha+1)}e^{-cy^{-\alpha}}\left(1-e^{-cy^{-\alpha}}\right)^{b_2-1}dy.$$

Using the transformation $\left(1-e^{-cy^{-\alpha}}\right)^{b_1}=w$, we have $\left(1-e^{-cy^{-\alpha}}\right)^{b_2}=w^{b_2/b_1}$ and

$$\left|-\alpha cb_2 y^{-(\alpha+1)}e^{-cy^{-\alpha}}\left(1-e^{-cy^{-\alpha}}\right)^{b_2-1}dy\right|=\left|\frac{b_2}{b_1}w^{(b_2/b_1)-1}dw\right|.$$

The reliability is

$$R_{s:k}=\sum_{i=s}^{k}\binom{k}{i}\frac{b_2}{b_1}\int_0^1 w^{(b_2/b_1)+i-1}(1-w)^{k-i}dw=\sum_{p=s}^{k}\binom{k}{i}\frac{b_2}{b_1}B\left(\frac{b_2}{b_1}+i,k-i+1\right).$$

This can be further simplified to

$$R_{s:k}=\sum_{p=s}^{k}\frac{k!}{i!(k-i)!}\frac{b_2}{b_1}\frac{\Gamma\left(b_2/b_1+i\right)\Gamma(k-i+1)}{\Gamma\left(k+b_2/b_1+1\right)}=\frac{b_2}{b_1}\sum_{i=s}^{k}\frac{k!}{i!}\left[\prod_{j=i}^{k}\left(\frac{b_2}{b_1}+j\right)\right]^{-1} \tag{4}$$

The reliability can also be written as

$$R_{s:k}=1-\frac{\Gamma(k+1)\Gamma(\gamma+s)}{\Gamma(k+\gamma+1)\Gamma(s)};\ \gamma=\frac{b_2}{b_1}. \tag{5}$$

Some specific values of $R_{s:k}$ are

$$R_{1:3}=\gamma\sum_{i=1}^{3}\frac{k!}{i!}\left[\prod_{j=i}^{3}(\gamma+j)\right]^{-1}=\frac{\gamma\left(\gamma^2+6\gamma+11\right)}{(\gamma+1)(\gamma+2)(\gamma+3)};\ \gamma=\frac{b_2}{b_1}$$
$$R_{1:4}=\gamma\sum_{i=1}^{4}\frac{k!}{i!}\left[\prod_{j=i}^{4}(\gamma+j)\right]^{-1}=\frac{\gamma\left(\gamma^3+10\gamma^2+35\gamma+50\right)}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}$$
$$R_{2:4}=\gamma\sum_{i=2}^{4}\frac{k!}{i!}\left[\prod_{j=i}^{4}(\gamma+j)\right]^{-1}=\frac{\gamma\left(\gamma^2+9\gamma+26\right)}{(\gamma+2)(\gamma+3)(\gamma+4)}$$
$$R_{3:5}=\gamma\sum_{i=3}^{5}\frac{k!}{i!}\left[\prod_{j=i}^{5}(\gamma+j)\right]^{-1}=\frac{\gamma\left(\gamma^2+12\gamma+47\right)}{(\gamma+3)(\gamma+4)(\gamma+5)}.$$

The estimation of the reliability of the MCSS model for KIW distribution, given in (4) or (5), can be done by using estimates of the parameters b_1 and b_2. We can estimate these parameters by using the maximum likelihood method. In the following section, we have discussed the estimation of the reliability of the MCSS model for KIW when independent samples on strength and stress are given.

3. Estimation of Reliability

The reliability estimation of the MCSS model for KIW distribution requires estimates of the unknown parameters. These parameters can be estimated by using the maximum likelihood method of estimation. For this, suppose we have a random sample of size n from the strength component and a random sample of size m is available from the stress component. The likelihood function for the combined sample of size $(n + m)$ is

$$L\left(b_1,b_2,c,\alpha;\mathbf{x},\mathbf{y}\right)=\prod_{i=1}^{n}\frac{\alpha c b_1}{x_i^{(\alpha+1)}}\mathrm{e}^{-cx_i^{-\alpha}}\left(1-\mathrm{e}^{-cx_i^{-\alpha}}\right)^{b_1-1}\prod_{j=1}^{m}\frac{\alpha c b_2}{y_j^{(\alpha+1)}}\mathrm{e}^{-cy_j^{-\alpha}}\left(1-\mathrm{e}^{-cy_j^{-\alpha}}\right)^{b_2-1}$$
$$=\frac{\alpha^{n+m}c^{n+m}b_1^n b_2^m}{\prod_{i=1}^{n}x_i^{(\alpha+1)}\prod_{j=1}^{m}y_i^{(\alpha+1)}}\exp\left(-c\sum_{i=1}^{n}x_i^{-\alpha}\right)\exp\left(-c\sum_{j=1}^{n}y_j^{-\alpha}\right)$$
$$\times\prod_{i=1}^{n}\left(1-\mathrm{e}^{-cx_i^{-\alpha}}\right)^{b_1-1}\prod_{j=1}^{m}\left(1-\mathrm{e}^{-cy_j^{-\alpha}}\right)^{b_2-1}.$$

The log-likelihood function is

$$\begin{aligned}\ell\left(\boldsymbol{\theta};\mathbf{x},\mathbf{y}\right)&=\left(n+m\right)\ln\alpha+\left(n+m\right)\ln c+n\ln b_1+m\ln b_2-c\sum_{i=1}^{n}x_i^{-\alpha}\\&-\left(\alpha+1\right)\left(\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j\right)-c\sum_{j=1}^{m}y_j^{-\alpha}+\left(b_1-1\right)\sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-cx_i^{-\alpha}}\right)\\&+\left(b_2-1\right)\sum_{j=1}^{m}\ln\left(1-\mathrm{e}^{-cy_j^{-\alpha}}\right),\end{aligned}\tag{6}$$

where $\boldsymbol{\theta}=(b_1,b_2,c,\alpha)$ is the vector of unknown parameters.
The derivatives of the log-likelihood function with respect to unknown parameters are

$$\begin{aligned}\frac{\partial}{\partial\alpha}\ell\left(\boldsymbol{\theta};\mathbf{x},\mathbf{y}\right)&=\frac{n+m}{\alpha}-\left(\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j\right)+c\left(\sum_{i=1}^{n}x_i^{-\alpha}\ln x_i+\sum_{j=1}^{m}y_j^{-\alpha}\ln y_j\right)\\&-\left(b_1-1\right)\sum_{i=1}^{n}\frac{x_i^{-\alpha}\ln x_i\mathrm{e}^{-cx_i^{-\alpha}}}{1-\mathrm{e}^{-cx_i^{-\alpha}}}-\left(b_2-1\right)\sum_{j=1}^{m}\frac{y_j^{-\alpha}\ln y_j\mathrm{e}^{-cy_j^{-\alpha}}}{1-\mathrm{e}^{-cy_j^{-\alpha}}}\end{aligned}\tag{7}$$

$$\begin{aligned}\frac{\partial}{\partial c}\ell\left(\boldsymbol{\theta};\mathbf{x},\mathbf{y}\right)&=\frac{n+m}{c}-\left(\sum_{i=1}^{n}x_i^{-\alpha}+\sum_{j=1}^{m}y_j^{-\alpha}\right)+\left(b_1-1\right)\sum_{i=1}^{n}\frac{x_i^{-\alpha}\mathrm{e}^{-cx_i^{-\alpha}}}{1-\mathrm{e}^{-cx_i^{-\alpha}}}\\&-\left(b_2-1\right)\sum_{j=1}^{m}\frac{y_j^{-\alpha}\mathrm{e}^{-cy_j^{-\alpha}}}{1-\mathrm{e}^{-cy_j^{-\alpha}}}\end{aligned}\tag{8}$$

$$\frac{\partial}{\partial b_1}\ell\left(\boldsymbol{\theta};\mathbf{x},\mathbf{y}\right)=\frac{n}{b_1}+\sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-cx_i^{-\alpha}}\right)\tag{9}$$

and

$$\frac{\partial}{\partial b_2}\ell\left(\boldsymbol{\theta};\mathbf{x},\mathbf{y}\right)=\frac{m}{b_2}+\sum_{j=1}^{m}\ln\left(1-\mathrm{e}^{-cy_j^{-\alpha}}\right).\tag{10}$$

The maximum likelihood estimates $\boldsymbol{\theta}=(b_1,b_2,c,\alpha)$ are obtained by setting the derivatives (7)–(10) to zero and simultaneously solving the resulting equations. It is interesting to note that if α and c are known then the maximum likelihood estimates of b_1 and b_2 are easily written from (9) and (10) as

$$\hat{b}_1=-n\left[\sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-cx_i^{-\alpha}}\right)\right]^{-1}\tag{11}$$

and
$$\hat{b}_2=-m\left[\sum_{j=1}^{m}\ln\left(1-\mathrm{e}^{-cy_j^{-\alpha}}\right)\right]^{-1}\tag{12}$$

Now if α and c are known then the estimate of reliability can be obtained by using estimates of b_1 and b_2 from (11) and (12) in (4) and is given as

$$\hat{R}_{s:k}=\frac{\hat{b}_2}{\hat{b}_1}\sum_{i=s}^{k}\frac{k!}{i!}\left[\prod_{j=i}^{k}\left(\frac{\hat{b}_2}{\hat{b}_1}+j\right)\right]^{-1}=\hat{\gamma}\sum_{i=s}^{k}\frac{k!}{i!}\left[\prod_{j=i}^{k}\left(\hat{\gamma}+j\right)\right]^{-1};\ \hat{\gamma}=\frac{\hat{b}_2}{\hat{b}_1}.\tag{13}$$

Then, the reliability can be estimated for any combinations of s and k.

We will now discuss the interval estimation of the reliability.

3.1. *Asymptotic Confidence Intervals*

The estimate of reliability, given in (13) is a point estimate and hence is less reliable. The asymptotic confidence intervals can be constructed for the reliability coefficient under the normality assumption. To construct the asymptotic confidence intervals, we need to compute the asymptotic variance of the reliability coefficient $\hat{R}_{s,k}$. The asymptotic variance of $\hat{R}_{s:k}$ is given as

$$AV\left(\hat{R}_{s:k}\right)=\left(\frac{\partial R_{s,k}}{\partial b_1}\right)^2 V\left(\hat{b}_1\right)+\left(\frac{\partial R_{s,k}}{\partial b_2}\right)V\left(\hat{b}_2\right)\Bigg|_{b_1=\hat{b}_1 \& b_2=\hat{b}_2}. \tag{14}$$

Now

$$Var\left(\hat{b}_1\right)=\left[E\left(-\frac{\partial^2 \ell}{\partial b_1^2}\right)\right]^{-1}=\frac{b_1^2}{n} \text{ and } Var\left(\hat{b}_2\right)=\left[E\left(-\frac{\partial^2 \ell}{\partial b_2^2}\right)\right]^{-1}=\frac{b_2^2}{m}. \tag{15}$$

Also, using (5)

$$\frac{\partial R_{s,k}}{\partial b_1}=\frac{k\gamma}{b_1}\frac{\Gamma(k)\Gamma(\gamma+s)}{\Gamma(s)\Gamma(k+\gamma+1)}\left[H_{s+\gamma-1}-H_{k+\gamma}\right] \tag{16}$$

and

$$\frac{\partial R_{s,k}}{\partial b_2}=\frac{k}{b_1}\frac{\Gamma(k)\Gamma(\gamma+s)}{\Gamma(s)\Gamma(k+\gamma+1)}\left[H_{s+\gamma-1}-H_{k+\gamma}\right], \tag{17}$$

where $H_t=\sum_{h=1}^{t} h^{-1}$ is the Harmonic number. Substituting the values of (15), (16), and (17) in (14), the asymptotic variance of $\hat{R}_{s:k}$ is

$$AV\left(\hat{R}_{s:k}\right)=\frac{\hat{\gamma}^2\Gamma^2(k+1)\Gamma^2(s+\hat{\gamma})}{\Gamma^2(k+\hat{\gamma}+1)\Gamma^2(s)}\left(H_{s+\hat{\gamma}-1}-H_{k+\hat{\gamma}}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right). \tag{18}$$

The asymptotic variance for (s,k) = (1,3), (2,4) and (3,5) are given below.

$$AV\left(\hat{R}_{1:3}\right)=\frac{\hat{\gamma}^2\Gamma^2(3+1)\Gamma^2(1+\hat{\gamma})}{\Gamma^2(3+\hat{\gamma}+1)\Gamma^2(1)}\left(\sum_{h=1}^{\hat{\gamma}+3}h^{-1}-\sum_{h=1}^{\hat{\gamma}}h^{-1}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right)$$

$$=\frac{36\hat{\gamma}^2\Gamma^2(\hat{\gamma}+1)}{\Gamma^2(\hat{\gamma}+4)}\left(\sum_{h=\hat{\gamma}+1}^{\hat{\gamma}+3}h^{-1}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right)$$

or

$$AV\left(\hat{R}_{1:3}\right)=\frac{\left[6\hat{\gamma}\left(3\hat{\gamma}^2+12\hat{\gamma}+11\right)\right]^2}{\left[(\hat{\gamma}+1)(\hat{\gamma}+2)(\hat{\gamma}+3)\right]^4}\left(\frac{1}{n}+\frac{1}{m}\right). \tag{19}$$

$$AV\left(\hat{R}_{2:4}\right)=\frac{\hat{\gamma}^2\Gamma^2(4+1)\Gamma^2(2+\hat{\gamma})}{\Gamma^2(4+\hat{\gamma}+1)\Gamma^2(2)}\left(\sum_{h=1}^{\hat{\gamma}+4}h^{-1}-\sum_{h=1}^{\hat{\gamma}+1}h^{-1}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right)$$

$$=\frac{24^2\hat{\gamma}^2\Gamma^2(\hat{\gamma}+2)}{\Gamma^2(\hat{\gamma}+5)}\left(\sum_{h=\hat{\gamma}+2}^{\hat{\gamma}+4}h^{-1}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right)$$

or

$$AV\left(\hat{R}_{2:4}\right)=\frac{\left[24\hat{\gamma}\left(3\hat{\gamma}^2+18\hat{\gamma}+26\right)\right]^2}{\left[(\hat{\gamma}+2)(\hat{\gamma}+3)(\hat{\gamma}+4)\right]^4}\left(\frac{1}{n}+\frac{1}{m}\right). \tag{20}$$

and

$$AV\left(\hat{R}_{3:5}\right)=\frac{\hat{\gamma}^2\Gamma^2\left(5+1\right)\Gamma^2\left(3+\hat{\gamma}\right)}{\Gamma^2\left(5+\hat{\gamma}+1\right)\Gamma^2\left(3\right)}\left(\sum_{h=1}^{\hat{\gamma}+5}h^{-1}-\sum_{h=1}^{\hat{\gamma}+2}h^{-1}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right)$$
$$=\frac{60^2\hat{\gamma}^2\Gamma^2\left(\hat{\gamma}+3\right)}{\Gamma^2\left(\hat{\gamma}+6\right)}\left(\sum_{h=\hat{\gamma}+3}^{\hat{\gamma}+5}h^{-1}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right)$$

or
$$AV\left(\hat{R}_{3:5}\right)=\frac{\left[60\hat{\gamma}\left(3\hat{\gamma}^2+24\hat{\gamma}+47\right)\right]^2}{\left[\left(\hat{\gamma}+3\right)\left(\hat{\gamma}+4\right)\left(\hat{\gamma}+5\right)\right]^4}\left(\frac{1}{n}+\frac{1}{m}\right). \tag{21}$$

Now as $(n, m)\to\infty$ then

$$\frac{\hat{R}_{s:k}-R_{s:k}}{\sqrt{AV\left(\hat{R}_{s:k}\right)}}\to N\left(0,1\right)$$

and hence a 100(1–α)% confidence interval for the true value of $R_{s:k}$ is

$$\hat{R}_{s:k}\pm Z_{1-\alpha/2}\sqrt{\frac{\hat{\gamma}^2\Gamma^2\left(k+1\right)\Gamma^2\left(s+\hat{\gamma}\right)}{\Gamma^2\left(k+\hat{\gamma}+1\right)\Gamma^2\left(s\right)}\left(H_{s+\hat{\gamma}-1}-H_{k+\hat{\gamma}}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right)}, \tag{22}$$

where $Z_{1-\alpha/2}$ is the $(1-\alpha/2)$th quantile of standard normal distribution. A 100(1–α)% asymptotic confidence intervals for (s,k) = (1,3), (2,4) and (3,5) are

$$\hat{R}_{1:3}\pm Z_{1-\alpha/2}\frac{6\hat{\gamma}\left(3\hat{\gamma}^2+12\hat{\gamma}+11\right)}{\left[\left(\hat{\gamma}+1\right)\left(\hat{\gamma}+2\right)\left(\hat{\gamma}+3\right)\right]^2}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)}$$

$$\hat{R}_{2:4}\pm Z_{1-\alpha/2}\frac{24\hat{\gamma}\left(3\hat{\gamma}^2+18\hat{\gamma}+26\right)}{\left[\left(\hat{\gamma}+2\right)\left(\hat{\gamma}+3\right)\left(\hat{\gamma}+4\right)\right]^2}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)}$$

and
$$\hat{R}_{3:5}\pm Z_{1-\alpha/2}\frac{60\hat{\gamma}\left(3\hat{\gamma}^2+24\hat{\gamma}+47\right)}{\left[\left(\hat{\gamma}+3\right)\left(\hat{\gamma}+4\right)\left(\hat{\gamma}+5\right)\right]^2}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)}.$$

Now a simulation study and real data application will be conducted to estimate the reliability coefficient when the data available is from the KIW distribution.

4. Simulation Study and Real Data Application

In this section, we will give a simulation study and real data application to estimate the reliability coefficient when samples available are from the KIW distribution. These studies are given in the following sub-sections.

4.1. Simulation Study

This section contains the simulation study regarding the reliability coefficient when random samples available are from the KIW distribution. We have first computed the true values of the reliability coefficient for $\left(s,k\right)=$(1,3), (2,4), and (3,5), α=2.0, c = 2.0 and (b_1, b_2) = (3.5,1.5), (3.0,1.5), (2.5,1.5), (2.0,1.5), (1.5,1.5), (1.5,2.0), (1.5,2.5), (1.5,3.0) and (1.5,3.5). These values are given in Appendix A, Table A.1. From this table we can see that, for fixed (s, k), the reliability increases as b_2 increases. This table also shows that for fixed (b_1, b_2), the reliability decreases as both (s, k) increase.

We have next conducted the simulation study by drawing random samples of sizes 10, 15, 20, 30, and 40 from both stress and strength populations for different values of (b_1, b_2) as given above. The simulation has been conducted by obtaining the maximum likelihood estimators of the unknown parameters for samples of various sizes. We have then used these maximum likelihood estimators to compute the reliability coefficient $\hat{R}_{s,k}$ for various combinations of (s, k). The process is repeated 5000 times and we computed the average value and mean square error of the reliability coefficients. The average value of $\hat{R}_{s,k}$ was then used to compute the bias. The results of bias and mean square errors of reliability coefficients are given in Appendix A, Tables A.2 and A.3. The results of Table A.2 show that the bias is small for all the combinations of (b_1, b_2) and for all sample sizes. This indicates that the estimated reliability is close to the true reliability. The mean square errors of $\hat{R}_{s,k}$ for various combinations of parameters and various sample sizes are given in Table A.3. This table shows that the mean square error of the estimated reliability $\hat{R}_{s,k}$ decreases as the sample size increases.

The simulation study also includes the estimated width of a 95% confidence interval to estimate the true reliability alongside the coverage probability of the intervals. These results are given in Tables A.4 and A.5 of Appendix A. Table A.4 contains the width of the estimated 95% confidence interval for the true reliability coefficient, $R_{s;k}$, for various combinations of parameter values and various sample sizes. This table shows that the width of the confidence interval decreases as the sample size increases and hence for a larger sample size the estimation has a smaller error. Table A.5 contains the results of coverage probability which increases as the sample size increases. The graphs of the standard error of estimates for the reliability coefficient and the coverage probability are also constructed and are given in Figures 1 below. The graphs are constructed for different values of (b_1, b_2) and for different sample sizes.

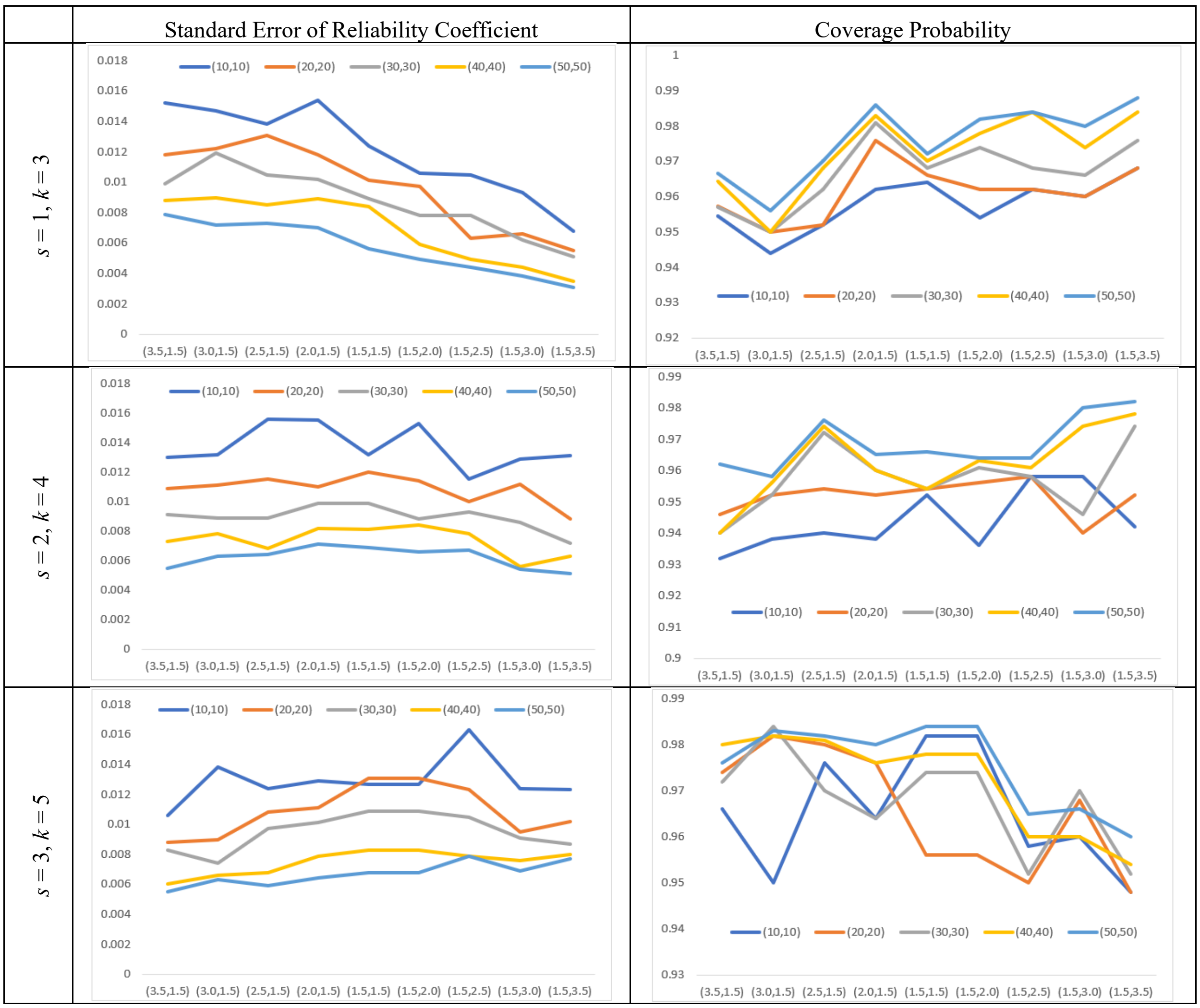

Figure 1: Standard Error of Estimate and Coverage Probability for the Reliability Coefficient

The figure 1 indicates that the standard error of estimate for the reliability coefficient decreases with an increase in the sample size for all values of (s, k) and (b_1, b_2). Also, the coverage probability slightly decreases when s and k are large and when b_1 is much smaller than b_2.

4.2. Real Data Applications

In this section, we will give a real data application to compute the reliability coefficient. The real data application has been done using strength data (X) from Bhaumik et al. (2009) which contains 34 observations and is given below.

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6
0.9	0.4	2.0	0.5	5.3	3.2	2.7	2.9	2.5	2.3	1.0	0.2
0.1	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2		

The stress data (Y) has been used from Chhikara and Folks (1977) and contains 46 observations which are given below.

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8
0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0
2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3	4.0	4.0	4.5	4.7
5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3	22.0	24.5		

Initially, we have fitted KIW to both data sets to see whether the data follow KIW distribution. The summary measures to see whether these data sets follow KIW distribution are given in Table 1 below.

From Table 1, we can see that both variables are positively skewed. The table also shows that both variables follow the KIW distribution as p–value of the KS test for both variables is well over 0.05.

Table 1: Summary Measures for Both Data Sets

	Strength (X)	**Stress (Y)**
Mean	1.879	3.607
Median	1.150	1.750
Standard Deviation	1.953	4.944
Skewness	1.679	2.987
$\hat{\alpha}$	0.1747	0.5403
$\hat{b}$	268.3067	3.8005
$\hat{c}$	6.1521	2.4637
Log-likelihood	–54.8339	–99.7200
KS Test Statistic	0.0778	0.0827
KS Test *p-value*	0.9762	0.8854

We have next computed the reliability coefficient for the MCSS model when data follows the KIW distribution. For this, we have first computed the maximum likelihood estimators of α, c, b_1, and b_2 using the given data sets and the joint log-likelihood function (6). The maximum likelihood estimates are $\hat{\alpha} = 0.2920$, $\hat{c} = 3.9671$, $\hat{b}_1 = 29.8255$ and $\hat{b}_2 = 18.3135$. Using these estimates we have next computed the values of $\hat{R}_{s,k}$ for $(s,k) = (1,3)$, (2,4), and (3,5) alongside the standard error of $\hat{R}_{s,k}$. The results are shown in Table 2 below.

Table 2: Estimated Reliability for the Real Data

(s,k)	(1,3)	(2,4)	(3,5)
$\hat{R}_{s,k}$	0.2038	0.4494	0.3591
$SE\left(\hat{R}_{s,k}\right)$	0.0699	0.0547	0.0598

Using the results of Table 2, the asymptotic 95% confidence interval for $R_{1,3}$ is (0.0668,0.3408), for $R_{2,4}$ is (0.3422,0.5566) and for $R_{3,5}$ is (0.2419,0.4763).

5. Conclusions

In this paper, we have obtained an estimate of MCSS reliability when the strength and stress components follow the KIW distribution. We have computed the true value of the reliability coefficient and its estimate for sample data. The expression for asymptotic variance of reliability coefficient has been obtained for $(s,k) =$ (1,3), (2,4) and (3,5). We have found that for fixed s and k, the reliability increases with an increase in the shape parameter b_2 of the stress component. We have also seen that for fixed values of b_1 and b_2, the reliability decreases with increase in s and k.

We have also computed the reliability for a real data set and we have found that the reliability coefficient has a maximum value at $s = 2$ and $k = 4$ and its asymptotic confidence interval also contains 0.5 which means that for $s = 2$ and $k = 4$, the system can have more than 50% chances of survival.

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Appendix–A

Table A.1: True Values of $\hat{R}_{s:k}$ for Various Combinations of Parameters

(s, k)	(b_1, b_2)								
	(3.5,1.5)	(3.0,1.5)	(2.5,1.5)	(2.0,1.5)	(1.5,1.5)	(1.5,2.0)	(1.5,2.5)	(1.5,3.0)	(1.5,3.5)
(1,3)	0.4956	0.5429	0.5994	0.6675	0.7500	0.8219	0.8685	0.9000	0.9221
(2,4)	0.3491	0.3905	0.4426	0.5100	0.6000	0.6884	0.7525	0.8000	0.8360
(3,5)	0.2721	0.3074	0.3530	0.4142	0.5000	0.5900	0.6597	0.7143	0.7578

Table A.2: Estimated Bias of $\hat{R}_{s:k}$

(s, k)	(n, m)	(b_1, b_2)								
		(3.5,1.5)	(3.0,1.5)	(2.5,1.5)	(2.0,1.5)	(1.5,1.5)	(1.5,2.0)	(1.5,2.5)	(1.5,3.0)	(1.5,3.5)
(1,3)	(10,10)	-0.0416	-0.0106	-0.0098	-0.0166	-0.0159	-0.0193	-0.0050	-0.0240	-0.0135
	(15,15)	-0.0018	-0.0203	-0.0036	-0.0130	-0.0212	-0.0178	-0.0042	-0.0134	-0.0044
	(20,20)	-0.0159	-0.0164	-0.0010	-0.0059	-0.0145	0.0005	0.0036	-0.0050	-0.0105
	(30,30)	-0.0123	-0.0058	-0.0107	-0.0023	-0.0124	0.0046	-0.0002	-0.0004	-0.0065
	(40,40)	0.0065	-0.0156	-0.0129	-0.0012	0.0045	-0.0095	-0.0055	0.0004	-0.0028
(2,4)	(10,10)	-0.0066	-0.0137	0.0146	0.0147	-0.0135	-0.0060	-0.0155	-0.0270	-0.0293
	(15,15)	0.0057	0.0135	-0.0112	-0.0011	0.0015	0.0037	-0.0016	-0.0123	-0.0027
	(20,20)	-0.0029	-0.0065	0.0104	0.0120	0.0179	0.0008	0.0014	-0.0003	-0.0044
	(30,30)	-0.0060	-0.0082	0.0044	0.0093	0.0060	0.0068	-0.0025	0.0132	0.0021
	(40,40)	0.0006	-0.0129	0.0022	-0.0137	-0.0008	-0.0048	0.0014	-0.0004	-0.0027
(3,5)	(10,10)	-0.0181	-0.0028	-0.0086	-0.0106	0.0074	0.0074	-0.0095	-0.0132	0.0133
	(15,15)	0.0172	0.0108	0.0011	-0.0046	-0.0092	-0.0092	0.0092	0.0011	0.0101
	(20,20)	0.0143	-0.0191	0.0120	-0.0012	0.0168	0.0168	0.0108	0.0105	-0.0011
	(30,30)	-0.0059	-0.0046	0.0026	0.0079	0.0159	0.0159	0.0152	-0.0011	-0.0033
	(40,40)	-0.0080	-0.0076	-0.0119	-0.0016	0.0058	0.0058	0.0100	-0.0049	0.0013

Table A.3: Estimated Standard Error of $\hat{R}_{s:k}$

(s, k)	(n, m)	(b_1, b_2)								
		(3.5,1.5)	(3.0,1.5)	(2.5,1.5)	(2.0,1.5)	(1.5,1.5)	(1.5,2.0)	(1.5,2.5)	(1.5,3.0)	(1.5,3.5)
(1,3)	(10,10)	0.0152	0.0147	0.0138	0.0154	0.0124	0.0106	0.0105	0.0093	0.0068
	(15,15)	0.0118	0.0122	0.0131	0.0118	0.0101	0.0097	0.0063	0.0066	0.0055
	(20,20)	0.0099	0.0119	0.0105	0.0102	0.0089	0.0078	0.0078	0.0062	0.0051
	(30,30)	0.0088	0.0090	0.0085	0.0089	0.0084	0.0059	0.0049	0.0044	0.0035
	(40,40)	0.0079	0.0072	0.0073	0.0070	0.0056	0.0049	0.0044	0.0038	0.0031
(2,4)	(10,10)	0.0130	0.0132	0.0156	0.0155	0.0132	0.0153	0.0115	0.0129	0.0131
	(15,15)	0.0109	0.0111	0.0115	0.0110	0.0120	0.0114	0.0100	0.0112	0.0088
	(20,20)	0.0091	0.0089	0.0089	0.0099	0.0099	0.0088	0.0093	0.0086	0.0072
	(30,30)	0.0073	0.0078	0.0068	0.0082	0.0081	0.0084	0.0078	0.0056	0.0063
	(40,40)	0.0055	0.0063	0.0064	0.0071	0.0069	0.0066	0.0067	0.0054	0.0051
(3,5)	(10,10)	0.0106	0.0138	0.0124	0.0129	0.0127	0.0127	0.0163	0.0124	0.0123
	(15,15)	0.0088	0.0090	0.0108	0.0111	0.0131	0.0131	0.0123	0.0095	0.0102
	(20,20)	0.0083	0.0074	0.0097	0.0101	0.0109	0.0109	0.0105	0.0091	0.0087
	(30,30)	0.0060	0.0066	0.0068	0.0079	0.0083	0.0083	0.0079	0.0076	0.0080
	(40,40)	0.0055	0.0063	0.0059	0.0064	0.0068	0.0068	0.0079	0.0069	0.0077

Table A.4: Estimated Width of 95% Confidence Interval of $\hat{R}_{s:k}$

(s, k)	(n, m)	(b_1, b_2)								
		(3.5,1.5)	(3.0,1.5)	(2.5,1.5)	(2.0,1.5)	(1.5,1.5)	(1.5,2.0)	(1.5,2.5)	(1.5,3.0)	(1.5,3.5)
(1,3)	(10,10)	0.1865	0.2195	0.2404	0.1245	0.1319	0.1240	0.1047	0.1021	0.0887
	(15,15)	0.1695	0.1787	0.1991	0.1065	0.1114	0.1043	0.0948	0.0849	0.0709
	(20,20)	0.1431	0.1571	0.1777	0.0951	0.0978	0.0902	0.0770	0.0712	0.0655
	(30,30)	0.1180	0.1325	0.1446	0.0785	0.0805	0.0756	0.0682	0.0593	0.0543
	(40,40)	0.1072	0.1133	0.1260	0.0690	0.0713	0.0680	0.0606	0.0519	0.0462
(2,4)	(10,10)	0.0981	0.1065	0.1192	0.1267	0.1330	0.1202	0.1162	0.1029	0.0908
	(15,15)	0.0842	0.0942	0.0981	0.1087	0.1101	0.1031	0.0943	0.0829	0.0719
	(20,20)	0.0723	0.0798	0.0904	0.0962	0.0973	0.0927	0.0817	0.0719	0.0644
	(30,30)	0.0593	0.0654	0.0743	0.0796	0.0813	0.0754	0.0686	0.0579	0.0516
	(40,40)	0.0528	0.0565	0.0639	0.0682	0.0712	0.0675	0.0596	0.0528	0.0463
(3,5)	(10,10)	0.0938	0.1042	0.1168	0.1267	0.1333	0.1333	0.1099	0.1042	0.0834
	(15,15)	0.0877	0.0942	0.0993	0.1070	0.1083	0.1083	0.0919	0.0846	0.0704
	(20,20)	0.0756	0.0767	0.0889	0.0940	0.0964	0.0964	0.0810	0.0713	0.0644
	(30,30)	0.0590	0.0657	0.0735	0.0792	0.0809	0.0809	0.0669	0.0610	0.0533
	(40,40)	0.0509	0.0565	0.0625	0.0689	0.0711	0.0711	0.0587	0.0537	0.0455

Table A.5: Estimated Coverage Probability of Confidence Interval of $\hat{R}_{s:k}$

(s, k)	(n, m)	(b_1, b_2)								
		(3.5,1.5)	(3.0,1.5)	(2.5,1.5)	(2.0,1.5)	(1.5,1.5)	(1.5,2.0)	(1.5,2.5)	(1.5,3.0)	(1.5,3.5)
(1,3)	(10,10)	0.9546	0.9440	0.9520	0.9620	0.9640	0.9540	0.9620	0.9600	0.9680
	(15,15)	0.9574	0.9500	0.9520	0.9760	0.9660	0.9620	0.9620	0.9600	0.9680
	(20,20)	0.9570	0.9500	0.9620	0.9810	0.9680	0.9740	0.9680	0.9660	0.9760
	(30,30)	0.9644	0.9500	0.9680	0.9830	0.9700	0.9780	0.9840	0.9740	0.9840
	(40,40)	0.9666	0.9560	0.9700	0.9860	0.9720	0.9820	0.9840	0.9800	0.9880
(2,4)	(10,10)	0.9320	0.9380	0.9400	0.9380	0.9520	0.9360	0.9580	0.9580	0.9420
	(15,15)	0.9460	0.9520	0.9540	0.9520	0.9540	0.9560	0.9580	0.9400	0.9520
	(20,20)	0.9400	0.9520	0.9720	0.9600	0.9540	0.9610	0.9580	0.9460	0.9740
	(30,30)	0.9400	0.9560	0.9740	0.9600	0.9540	0.9630	0.9610	0.9740	0.9780
	(40,40)	0.9620	0.9580	0.9760	0.9650	0.9660	0.9640	0.9640	0.9800	0.9820
(3,5)	(10,10)	0.9660	0.9500	0.9760	0.9640	0.9820	0.9820	0.9580	0.9600	0.9480
	(15,15)	0.9740	0.9820	0.9800	0.9760	0.9560	0.9560	0.9500	0.9680	0.9480
	(20,20)	0.9720	0.9840	0.9700	0.9640	0.9740	0.9740	0.9520	0.9700	0.9520
	(30,30)	0.9800	0.9820	0.9810	0.9760	0.9780	0.9780	0.9600	0.9600	0.9540
	(40,40)	0.9760	0.9830	0.9820	0.9800	0.9840	0.9840	0.9650	0.9660	0.9600