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On the Uniqueness and Structural Identification of Some Univariate Continuous Probability Distributions

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Abstract

This paper examines the characterizations of the five recent univariate continuous probability distributions (2022-2025) that were proposed relatively recently. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) reverse hazard function. It should be mentioned that for the characterization (i) the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equations.

Key Words: Characterizations, Univariate Distribution, Hazard Function, Truncated Moments, Reverse Hazard Function.

Mathematical Subject Classification: 60E05, 62E15.

1 Introduction

Characterizing probability distributions plays a central role in applied research. Investigators are often concerned with verifying whether a proposed model satisfies the conditions of a specific distribution. To achieve this, one relies on characterizations, which establish the criteria under which a distribution can be uniquely identified. This problem of characterization has long been a fundamental topic in probability and statistics and continues to attract increasing attention from researchers due to its wide-ranging applications. In the construction of different models based on a probabilistic approach, determining whether a new distribution appropriately represents the underlying data is a critical task. Although traditional probability distributions are powerful in theory, they often fail to capture the irregular behaviors and diverse shapes present in real-world datasets. For instance, data arising in fields such as reliability engineering, risk assessment, economics, environmental sciences, and medical research commonly exhibit features such as skewness, heavy tails, or multimodal characteristics that classical models may not adequately represent. To address these limitations, statisticians and applied researchers have developed new families of distributions that extend or generalize existing models. These advanced distributions provide greater flexibility and improve the accuracy of data fitting, making them invaluable for practical applications.

These innovative models are not only of theoretical interest but also serve as essential tools for applied scientists who require robust and adaptable frameworks for analysis and decision-making. A fundamental aspect of introducing any new distribution lies in its characterization. Such results ensure that the distribution can be uniquely distinguished, its properties can be formally established, and its relevance to real-world phenomena can be verified. Characterizations based on truncated moments, hazard functions, reverse hazard functions, conditional expectations, and other statistical measures offer deep insights into the behavior of these models and strengthen their theoretical foundations. Through these characterizations, important distributional properties such as reliability measures, stochastic orders, and inferential procedures can be derived, thereby enhancing both the understanding and utility of the model in practical contexts.

Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. For example, Hamedani (2016) discussed characterization results of some of the distributions that are infinitely divisible via Bondesson's 1979 classifications. Mohammad (2024) proposed characterizations and infinite divisibility of distributions introduced in 2016-2017. Characterizations of some selected distributions like Amoroso, SSK (Shakil–Singh–Kibria), SKS (Shakil–Kibria–Singh), SK (Shakil–Kibria), and SKStype are discussed by Hamedani (2012). Yousof et al. (2017) discussed the characterization results of a new family of continuous distributions called the transmuted Topp-Leone G family with some other properties. Afify et al. (2017) also mentioned some results of characterizations odd exponentiated halflogistic-G family of distribution. The study of Hamedani et al. (2024) deals with various characterizations of certain univariate continuous distributions proposed in (2023-2024) while the study of Hamedani et al. (2025) deals with characterizations of some other univariate continuous distributions proposed in (2023-2024). Besides, characterizations of some skew distributions have also been found in the literature. It includes the works of Pathak et al. (2025), Das et al. (2023), Das et al. (2024), Das et al. (2025), Pathak et al. (2023), Alizadeh et al. (2025), Sulewski et al. (2025) etc. Additionally, some other recent studies on characterizations of different distributions include the notable works of Ibrahim et al. (2019), Ahmad et al. (2018), Aryal et al. (2017), Korkmaz et al. (2018), Hamedani (2011), Yousof et al. (2018a), Yousof et al. (2018d), Alizadeh et al. (2017), Reyad et al. (2021), Yousof et al. (2018c), Hamedani (1992), Yousof et al. (2018b), Hamedani et al. (2023), Hamedani and Volkmer (2005), Ahmad et al. (2022), among others.

The present work deals with certain characterizations of some of the recent distributions, including the new two-parameter modified half-logistic (MHL) distribution of Shaheed (2022a), a new mixture of exponential and Weibull distributions (NMEW) of Mohammad (2024), A new weighted Topp-Leone family of distributions (NWTL-G) of Shaheed (2023), Novel Weighted G family of Probability Distributions (NWG) of Shaheed (2022b) and A Weighted Exponentiated class of Distributions (WExp-G) of Shaheed (2025). These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) reverse hazard function. It should be mentioned that for the characterization (i) the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation. Below, we present the cumulative distribution functions (CDFs), probability density functions (PDFs) and Hazard rate function (HRFs) of these distributions in the same order as listed above.

1. The CDF, PDF and HRF of MHL (Shaheed, 2022a) are given, respectively, by

$$F(x;\alpha,\beta)=\frac{(1-e^{-x})^{\alpha}}{1+e^{-\beta x}}, \tag{1.1}$$

$$f(x;\alpha,\beta)=e^{-x}\left(1-e^{-x}\right)^{\alpha-1}P(x), \tag{1.2}$$

$$h(x;\alpha,\beta)=\frac{\mathrm{e}^{-\mathrm{x}}(1-\mathrm{e}^{-\mathrm{x}})^{\alpha-1}\left[\alpha+\beta\,\mathrm{e}^{-(\beta-1)\mathrm{x}}+(\alpha-\beta)\mathrm{e}^{-\beta\,\mathrm{x}}\right]}{(1+\mathrm{e}^{-\beta\,\mathrm{x}})\left[1+\mathrm{e}^{-\beta\,\mathrm{x}}-(1-\mathrm{e}^{-\mathrm{x}})^{\alpha}\right]}, \tag{1.3}$$

where $\alpha>0,\beta>0$ are parameters and $P(x)=\frac{\left[\alpha+\beta e^{-(\beta-1)x}+(\alpha-\beta)e^{-\beta x}\right]}{(1+e^{-\beta x})^{2}}$.

2. The CDF, PDF and HRF of NMEW (Mohammad, 2024) are given, respectively, by

$$F(x;\delta,\xi)=\frac{1-e^{-\delta x^{\xi}}}{1+e^{-\delta x}}, \tag{2.1}$$

$$f\left(x;\delta,\xi\right)=\delta x^{\xi-1}e^{-\delta x^{\xi}}P\left(x\right), \tag{2.2}$$

$$h(x;\delta,\xi)=\frac{\delta\eta\, x^{\eta-1}\, e^{-\delta x^{\eta}}(1+e^{-\delta x})+\delta\, e^{-\delta x}(1-e^{-\delta x^{\eta}})}{(1+e^{-\delta x})(e^{-\delta x}+e^{-\delta x^{\eta}})}. \tag{2.3}$$

where $x>0,\delta>0,\xi>0$ are parameters and $P\left(x\right)=\frac{\xi\left(1+e^{-\delta x}\right)+x^{1-\xi}e^{-\delta x+\delta x^{\xi}}\left(1-e^{-\delta x^{\xi}}\right)}{(1+e^{-\delta x})^{2}}$.

3. The CDF, PDF and HRF of NWTL-G (Shaheed, 2023) are given, respectively, by

$$F\left(x;\alpha,\beta\right)=\frac{\left[1-\overline{G}\left(x\right)^{2}\right]^{\alpha}}{\left[1-\overline{G}\left(x\right)^{2}\right]^{\alpha}+\overline{G}\left(x\right)^{2\beta}}, \tag{3.1}$$

$$f\left(x;\alpha,\beta\right)=g\left(x\right)\overline{G}\left(x\right)^{2\beta-1}P\left(x\right), \tag{3.2}$$

$$h(x;\alpha,\beta)=\frac{g(x)\left[1-\bar{G}(x)^{2}\right]^{\alpha-1}\left[\alpha+(\beta-\alpha)\left[1-\bar{G}(x)^{2}\right]\right]}{\left\{\left[1-\bar{G}(x)^{2}\right]^{\alpha}+\bar{G}(x)^{2\beta}\right\}\bar{G}(x)}. \tag{3.3}$$

where $x\in\mathbb{R}$,$\alpha>0,\beta>0$ are parameters, $P\left(x\right)=\frac{\left[1-\overline{G}(x)^{2}\right]^{\alpha-1}\left[\alpha+(\beta-\alpha)\left[1-\overline{G}(x)^{2}\right]\right]}{\left\{\left[1-\overline{G}(x)^{2}\right]^{\alpha}+\overline{G}(x)^{2\beta}\right\}^{2}}$ and $G\left(x\right)$ is a baseline CDF with the corresponding PDF $g\left(x\right)$.

4. The CDF, PDF and HRF of NWG (Shaheed, 2022b) are given, respectively, by

$$F\left(x;\alpha,\beta\right)=\frac{G\left(x\right)^{\alpha}}{2-G\left(x\right)^{\beta}}, \tag{4.1}$$

$$f\left(x;\alpha,\beta\right)=g\left(x\right)G\left(x\right)^{\alpha-1}P\left(x\right), \tag{4.2}$$

$$h(x;\alpha,\beta)=\frac{g(x)\,G(x)^{\alpha-1}\left[2\alpha+(\beta-\alpha)G(x)^{\beta}\right]}{\left[2-G(x)^{\beta}\right]\left[2-G(x)^{\beta}-G(x)^{\alpha}\right]}. \tag{4.3}$$

where $x\in\mathbb{R},\alpha>0,\beta>0$ are parameters, $P\left(x\right)=\frac{\left[2\alpha+(\beta-\alpha)G(x)^{\beta}\right]}{\left[2-G(x)^{\beta}\right]^{2}}$ and $G\left(x\right)$ is a baseline CDF with the corresponding PDF $g\left(x\right)$.

5 The CDF, PDF and HRF of WExp-G (Shaheed, 2025) are given, respectively, by

$$F\left(x;\alpha,\beta\right)=\frac{G\left(x\right)^{\alpha}}{2-\overline{G}\left(x\right)^{\alpha}}, \tag{5.1}$$

$$f\left(x;\alpha,\beta\right)=\alpha g\left(x\right)G\left(x\right)^{\alpha-1}P\left(x\right), \tag{5.2}$$

$$h(x;\alpha,\beta)=\frac{\alpha\, g(x)G(x)^{\alpha-1}\left[1+\bar{G}(x)^{\alpha-1}\right]}{\left[1+\bar{G}(x)^{\alpha}\right]\left[1+\bar{G}(x)^{\alpha}-G(x)^{\alpha}\right]}. \tag{5.3}$$

where $x\in\mathbb{R},\alpha>0$ is a parameter and $P\left(x\right)=\frac{\left[1+\overline{G}(x)^{\alpha-1}\right]}{\left[1+\overline{G}(x)^{\alpha}\right]^{2}}$ and $G\left(x\right)$ is a baseline CDF with the corresponding PDF $g\left(x\right)$.

2 Characterization of Distributions

This section deals with various characterizations of 5 recently proposed univariate continuous distributions. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) reverse hazard function. It should be mentioned that for the characterization (i) the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

2.1 Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of the five distributions, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem G below. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the CDF F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem G. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right) \mid X \geq x\right] = \mathbf{E}\left[q_1\left(X\right) \mid X \geq x\right]\eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F\left(x\right) = \int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right)q_1\left(u\right) - q_2\left(u\right)}\right|\exp\left(-s\left(u\right)\right)\,du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\,q_1}{\eta\,q_1\,-\,q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Remark 2.1.1. The goal is to have $\eta\left(x\right)$ as simple as possible.

Proposition 2.1.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1\left(x\right) = \left[P\left(x\right)\right]^{-1}$ and $q_2\left(x\right) = q_1\left(x\right)\left(1 - e^{-x}\right)^{\alpha}$ for $x > 0$. The random variable X has PDF (1.2) if and only if the function η defined in Theorem G has the form

$$\eta\left(x\right) = \frac{1}{2}\left\{1 + \left(1 - e^{-x}\right)^{\alpha}\right\}, \qquad x > 0.$$

Proof. Let X be a random variable with PDF (1.2), then

$$\begin{aligned}\left(1 - F\left(x\right)\right)E\left[q_1\left(X\right) \mid X \geq x\right] &= \int_x^{\infty} e^{-u}\left(1 - e^{-u}\right)^{\alpha - 1}du \\ &= \frac{1}{\alpha}\left\{1 - \left(1 - e^{-x}\right)^{\alpha}\right\}, \qquad x > 0,\end{aligned}$$

and

$$
\begin{aligned}
(1-F(x))\,E\left[q_2(X)\ \mid X\geq x\right] &= \int_x^{\infty} e^{-u}\left(1-e^{-u}\right)^{2\alpha-1}du \\
&= \frac{1}{2\alpha}\left\{1-\left(1-e^{-x}\right)^{2\alpha}\right\}, \qquad x>0,
\end{aligned}
$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\}>0 \quad for \quad x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}}{1-\left(1-e^{-x}\right)^{\alpha}}, \quad x>0,$$

and hence

$$s(x)=-\log\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\}, \qquad x>0.$$

Now, in view of Theorem G, X has density (1.2).

Corollary 2.1.1. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.1. The PDF of X is (1.2) if and only if there exist functions q_2 and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}}{1-\left(1-e^{-x}\right)^{\alpha}}, \quad x>0.$$

Corollary 2.1.2. The general solution of the differential equation in Corollary 2.1.1 is

$$\eta(x)=\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\}^{-1}\left[-\int \alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. If X has PDF (1.2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x)=\left(\frac{\alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}}{1-\left(1-e^{-x}\right)^{\alpha}}\right)\eta(x)-\left(\frac{\alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}}{1-\left(1-e^{-x}\right)^{\alpha}}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$
\begin{aligned}
&\eta'(x)-\left(\frac{\alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}}{1-\left(-e^{-x}\right)^{\alpha}}\right)\eta(x) \\
&= -\left(\frac{\alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}}{1-\left(1-e^{-x}\right)^{\alpha}}\right)\left(q_1(x)\right)^{-1}q_2(x),
\end{aligned}
$$

or

$$\frac{d}{dx}\left\{\left(1-\left(1-e^{-x}\right)^{\alpha}\right)\eta(x)\right\}=-\left(\alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

from which we arrive at

$$\eta(x) = \left\{1 - \left(1 - e^{-x}\right)^{\alpha}\right\}^{-1} \left[-\int \alpha e^{-x} \left(1 - e^{-x}\right)^{\alpha - 1} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right].$$

Note that a set of functions satisfying the differential equation in Corollary 2.1.1, is given in Proposition 2.1.1 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem G.

Proposition 2.1.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x) e^{-\delta x^{\xi}}$ for $x > 0$. The random variable X has PDF (2.2) if and only if the function η defined in Theorem G has the form

$$\eta(x) = \frac{1}{2} e^{-\delta x^{\xi}}, \qquad x > 0.$$

Proof. Let X be a random variable with PDF (2.2), then

$$\begin{aligned}(1 - F(x)) E[q_1(X) \mid X \geq x] &= \int_x^{\infty} \delta u^{\xi - 1} e^{-\delta u^{\xi}} du \\ &= \frac{1}{\xi} e^{-\delta x^{\xi}}, \qquad x > 0,\end{aligned}$$

and

$$\begin{aligned}(1 - F(x)) E[q_2(X) \mid X \geq x] &= \int_x^{\infty} \delta u^{\xi - 1} e^{-2\delta u^{\xi}} du \\ &= \frac{1}{2\xi} e^{-2\delta x^{\xi}}, \qquad x > 0,\end{aligned}$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\delta x^{\xi}} < 0 \quad for \quad x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \delta \xi x^{\xi - 1}, \quad x > 0,$$

and hence

$$s(x) = \delta x^{\xi}, \quad x > 0.$$

Now, in view of Theorem G, X has density (2.2).

Corollary 2.1.3. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.2. The PDF of X is (2.2) if and only if there exist functions q_2 and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \delta \xi x^{\xi - 1}, \quad x > 0.$$

Corollary 2.1.4. The general solution of the differential equation in Corollary 2.1.3 is

$$\eta(x) = e^{\delta x^{\xi}} \left[-\int \delta \xi x^{\xi - 1} e^{-\delta x^{\xi}} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. Is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.3, is given in Proposition 2.1.2 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem G.

Proposition 2.1.3. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\,\overline{G}(x)^{2\beta}$ for $x \in \mathbb{R}$. The random variable X has PDF (3.2) if and only if the function η defined in Theorem G has the form

$$\eta(x) = \frac{1}{2}\overline{G}(x)^{2\beta}, \qquad x \in \mathbb{R}.$$

Proof. Let X be a random variable with PDF (3.2), then

$$\begin{aligned}(1 - F(x))\, E[q_1(X) \mid X \geq x] &= \int_x^{\infty} g(u)\,\overline{G}(u)^{2\beta-1}\, du \\ &= \frac{1}{2\beta}\overline{G}(x)^{2\beta}, \qquad x \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(1 - F(x))\, E[q_2(X) \mid X \geq x] &= \int_x^{\infty} g(u)\,\overline{G}(u)^{4\beta-1}\, du \\ &= \frac{1}{4\beta}\overline{G}(x)^{4\beta}, \qquad x \in \mathbb{R},\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\overline{G}(x)^{2\beta} < 0 \quad for \;\; x \in \mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{2\beta g(x)}{\overline{G}(x)}, \quad x \in \mathbb{R},$$

and hence

$$s(x) = -2\beta \log\left[\overline{G}(x)\right], \quad x \in \mathbb{R}.$$

Now, in view of Theorem G, X has density (3.2).

Corollary 2.1.5. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.3. The PDF of X is (3.2) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{2\beta g(x)}{\overline{G}(x)}, \quad x \in \mathbb{R}.$$

Corollary 2.1.6. The general solution of the differential equation in Corollary 2.1.5 is

$$\eta(x) = \left[\overline{G}(x)^{2\beta}\right]^{-1}\left[-\int 2\beta g(x)\,\overline{G}(x)^{2\beta-1}\,(q_1(x))^{-1}\, q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. Is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.5, is given in Proposition 2.1.3 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem G.

Proposition 2.1.4. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, G(x)^{\alpha}$ for $x \in \mathbb{R}$. The random variable X has PDF (4.2) if and only if the function η defined in Theorem G has the form

$$\eta(x) = \frac{1}{2}\left\{1 + G(x)^{\alpha}\right\}, \qquad x \in \mathbb{R}.$$

Proof. Let X be a random variable with PDF (4.2), then

$$\begin{aligned}(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] &= \int_x^{\infty} g(u)\, G(u)^{\alpha-1}\, du \\ &= \frac{1}{\alpha}\left\{1 - G(x)^{\alpha}\right\}, \qquad x \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(1 - F(x))\, E\left[q_2(X) \mid X \geq x\right] &= \int_x^{\infty} g(u)\, G(u)^{2\alpha-1}\, du \\ &= \frac{1}{2\alpha}\left\{1 - G(x)^{2\alpha}\right\}, \qquad x \in \mathbb{R},\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left\{1 - G(x)^{\alpha}\right\} < 0 \quad for \quad x \in \mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha g(x)\, G(x)^{\alpha-1}}{1 - G(x)^{\alpha}}, \quad x \in \mathbb{R},$$

and hence

$$s(x) = -\log\left\{1 - G(x)^{\alpha}\right\}, \quad x \in \mathbb{R}.$$

Now, in view of Theorem G, X has density (3.2).

Corollary 2.1.7. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.4. The PDF of X is (4.2) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha g(x)\, G(x)^{\alpha-1}}{1 - G(x)^{\alpha}}, \quad x \in \mathbb{R}.$$

Corollary 2.1.8. The general solution of the differential equation in Corollary 2.1.7 is

$$\eta(x) = \left\{1 - G(x)^{\alpha}\right\}^{-1}\left[-\int \alpha g(x)\, G(x)^{\alpha-1}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. Is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.7, is given in Proposition 2.1.4 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem G.

Proposition 2.1.5. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x) G(x)^{\alpha}$ for $x \in \mathbb{R}$. The random variable X has PDF (5.2) if and only if the function η defined in Theorem G has the form

$$\eta(x) = \frac{1}{2}\{1 + G(x)^{\alpha}\}, \qquad x \in \mathbb{R}.$$

Proof. Is the same as that of Proposition 2.1.4.

2.2 Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in support\ of\ F.$$

In this subsection we present characterizations of five distributions in terms of the reverse hazard function.

Proposition 2.2.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has PDF (1.2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) + r_F(x) = e^{-x}\frac{d}{dx}\left\{\frac{P(x)\left(1 + e^{-\beta x}\right)}{1 - e^{-x}}\right\}, \quad x > 0,$$

with boundary condition $\lim_{x\to\infty} r_F(x) = 0$.

Proof. Multiplying both sides of the above equation by e^x, we obtain

$$\frac{d}{dx}\{e^x r_F(x)\} = \frac{d}{dx}\left\{\frac{P(x)\left(1 + e^{-\beta x}\right)}{1 - e^{-x}}\right\},$$

or

$$r_F(x) = e^{-x}\left\{\frac{P(x)\left(1 + e^{-\beta x}\right)}{1 - e^{-x}}\right\},$$

which is the reverse hazard function corresponding to the PDF (1.2).

Proposition 2.2.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has PDF (2.2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) - (\xi - 1) r_F(x) = \delta x^{\xi-1}\frac{d}{dx}\left\{\frac{e^{-\delta x^{\xi}} P(x)\left(1 + e^{-\delta x}\right)}{1 - e^{-\delta x^{\xi}}}\right\}, \quad x > 0,$$

with boundary condition $\lim_{x\to\infty} r_F(x) = 0$.

Proof. Is similar to that of Corollary 2.1.2.

Proposition 2.2.3. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The random variable X has PDF (3.2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) - \frac{g'(x)}{g(x)} r_F(x)$$
$$= g(x) \frac{d}{dx} \left\{ \frac{\overline{G}(x)^{2\beta-1} P(x) \left[\left[1 - \overline{G}(x)^2 \right]^{\alpha} + \overline{G}(x)^{2\beta} \right]}{\left[1 - \overline{G}(x)^2 \right]^{\alpha}} \right\}, \quad x \in \mathbb{R},$$

with boundary condition $\lim_{x\to\infty} r_F(x) = 0$ for $\beta > 1/2$.

Proof. Is similar to that of Corollary 2.1.2.

Proposition 2.2.4. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The random variable X has PDF (4.2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) - \frac{g'(x)}{g(x)} r_F(x) = g(x) \frac{d}{dx} \left\{ \frac{2\alpha + (\beta - \alpha) G(x)^{\beta}}{\left[2 - G(x)^{\beta} \right]^2} \right\}, \quad x \in \mathbb{R},$$

with boundary condition $\lim_{x\to\infty} r_F(x) = 0$ for $\beta > 1/2$.

Proof. Is similar to that of Corollary 2.1.2.

Proposition 2.2.5. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The random variable X has PDF (5.2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) - \frac{g'(x)}{g(x)} r_F(x) = \alpha g(x) \frac{d}{dx} \left\{ \frac{\left[1 + \overline{G}(x)^{\alpha-1} \right]}{\left[1 + \overline{G}(x)^{\alpha} \right]} \right\}, \quad x \in \mathbb{R},$$

with boundary condition $\lim_{x\to\infty} r_F(x) = \alpha \lim_{x\to\infty} g(x)$.

Proof. Is similar to that of Corollary 2.1.2.

3 Conclusion

In this study, we explored the characterizations of five recently proposed univariate continuous distributions using two complementary approaches: relationships between truncated moments and reverse hazard functions. These characterizations not only provide some conditions for uniquely identifying the distributions but also highlight their theoretical significance by linking probabilistic structures with functional equations and differential systems. Such results ensure that these models are not only mathematically well-founded but also practically reliable for applied contexts.

The significance of such characterizations lies in their dual contribution to theory and practice. On the theoretical side, they demonstrate relationships between probability distributions, functional equations, and differential systems, thereby providing the role of characterizations on the probability theory. On the applied side, the characterized distributions prove to be highly adaptable to real-world data scenarios where traditional models may fail, particularly in the presence of skewness, heavy tails, multimodality, and other forms of irregular behavior. This makes them highly valuable tools for researchers and practitioners working in fields such as reliability engineering, financial risk analysis, environmental sciences, biomedical research, and economics, where accurate modeling of uncertainty is essential.

In summary, the present work contributes to the expanding literature on the characterization of five recent probability distributions. The results demonstrate that characterizations are not merely abstract constructs, but essential tools that enhance the credibility, applicability, and interpretability of new models. Future research may extend these ideas to multivariate distributions, mixtures, and further generalized families, thereby providing the bridge between mathematical theory and practical modeling. The present work thus contributes to both the theoretical advancement and the applied utility of probability distributions.

References

Afify, A. Z., Altun, E., Alizadeh, M., Ozel, G., and Hamedani, G. (2017). The odd exponentiated half-logistic-g family: properties, characterizations and applications. *Chilean Journal of Statistics*, 8(2):65–91.

Ahmad, Z., Elgarhy, M., and Hamedani, G. (2018). A new weibull-x family of distributions: properties, characterizations and applications. *Journal of Statistical Distributions and Applications*, 5(1):1–18.

Ahmad, Z., Hamedani, G., and Elgarhy, M. (2022). The weighted exponentiated family of distributions: Properties, applications and characterizations. *Journal of the Iranian Statistical Society*, 19(1):209–228.

Alizadeh, M., Ghosh, I., Yousof, H. M., Rasekhi, M., and Hamedani, G. (2017). The generalized odd generalized exponential family of distributions: properties, characterizations and applications. *Journal of Data Science*, 15(3):443–465.

Alizadeh, M., Hazarika, P. J., Das, J., Contreras-Reyes, J. E., Hamedani, G., Sulewski, P., and Yousof, H. M. (2025). Reliability and risk analysis under peaks over a random threshold value-at-risk method based on a new flexible skew-logistic distribution. *Life Cycle Reliability and Safety Engineering*, pages 1–28.

Aryal, G. R., Ortega, E. M., Hamedani, G., and Yousof, H. M. (2017). The topp-leone generated weibull distribution: regression model, characterizations and applications. *International Journal of Statistics and Probability*, 6(1):126–141.

Das, J., Hazarika, P. J., Alizadeh, M., Contreras-Reyes, J. E., Mohammad, H. H., and Yousof, H. M. (2025). Economic peaks and value-at-risk analysis: A novel approach using the laplace distribution for house prices. *Mathematics and Computers in Applications*, 30(1):4.

Das, J., Hazarika, P. J., Chakraborty, S., Pathak, D., Hamedani, G. G., and Karamikabir, H. (2024). A bimodal extension of the tanh skew normal distribution: Properties and applications. *Pakistan Journal of Statistics and Operation Research*, 20(3):533–551.

Das, J., Pathak, D., Hazarika, P. J., Chakraborty, S., and Hamedani, G. (2023). A new flexible alpha skew normal distribution. *Journal of the Indian Society for Probability and Statistics*, 24:485–507.

Hamedani, G. (1992). Bivariate and multivariate normal characterizations: a brief survey. *Communications in Statistics-Theory and Methods*, 21(9):2665–2688.

Hamedani, G. (2011). Characterizations of the shakil-kibria-singh distribution. *Austrian Journal of Statistics*, 40(3):201–207.

Hamedani, G. (2012). Characterizations of certain continuous distributions. In *Multiscale Signal Analysis and Modeling*, pages 297–316. Springer.

Hamedani, G., Goual, H., Emam, W., Tashkandy, Y., Ahmad Bhatti, F., Ibrahim, M., and Yousof, H. M. (2023). A new right-skewed one-parameter distribution with mathematical characterizations, distributional validation, and actuarial risk analysis, with applications. *Symmetry*, 15(7):1297.

Hamedani, G., Roshani, A., and Butt, N. S. (2024). Characterizations of certain (2023-2024) introduced univariate continuous distributions. *Pakistan Journal of Statistics and Operation Research*, 20(4):661–692.

Hamedani, G., Roshani, A., and Hazarika, P. J. (2025). Characterizations of certain (2023-2024) introduced univariate continuous distributions ii. *Pakistan Journal of Statistics and Operation Research*, 21(2):237–271.

Hamedani, G. and Volkmer, H. (2005). Certain characterizations of the uniform dstribution. *Metrika*, 61(2):117–125.

Hamedani, G. G. (2016). On characterizations and infinite divisibility of recently introduced distributions. *Studia Scientiarum Mathematicarum Hungarica*, 53(4):467–511.

Ibrahim, M., Yadav, A. S., Yousof, H. M., Goual, H., and Hamedani, G. (2019). A new extension of lindley distribution: modified validation test, characterizations and different methods of estimation. *Communications for Statistical Applications and Methods*, 26:475–493.

Korkmaz, M. C., Yousof, H. M., and Hamedani, G. G. (2018). The exponential lindley odd log-logistic-g family: properties, characterizations and applications. *Journal of Statistical Theory and Applications*, 17(3):554–571.

Mohammad, G. S. (2024). A new mixture of exponential and weibull distributions: properties, estimation and relibilty modelling. *São Paulo Journal of Mathematical Sciences*, 18(1):438–458.

Pathak, D., Hazarika, P. J., Chakraborty, S., Das, J., and Hamedani, G. (2023). Modeling tri-model data with a new skew logistic distribution. *Pakistan Journal of Statistics and Operation Research*, 19(3):585–602.

Pathak, D., Hazarika, P. J., Shah, S., and Hamedani, G. G. (2025). Properties and application of trimodal skew normal distribution. *Pakistan Journal of Statistics and Operation Research*, 21(1):15–31.

Reyad, H., Korkmaz, M. Ç., Afify, A. Z., Hamedani, G., and Othman, S. (2021). The fréchet topp leone-g family of distributions: Properties, characterizations and applications. *Annals of Data Science*, 8(2):345–366.

Shaheed, G. (2022a). A new two-parameter modified half-logistic distribution: properties and applications. *Statistics, Optimization & Information Computing*, 10(2):589–605.

Shaheed, G. (2022b). Novel weighted g family of probability distributions with properties, modelling and different methods of estimation. *Statistics, Optimization & Information Computing*, 10(4):1143–1161.

Shaheed, G. (2023). A new weighted topp-leone family of distributions. *Statistics, Optimization & Information Computing*, 11(3):615–628.

Shaheed, G. (2025). A weighted exponentiated class of distributions: Properties with applications for modelling reliability data. *Statistics, Optimization & Information Computing*, 13(3):1144–1161.

Sulewski, P., Alizadeh, M., Das, J., Hamedani, G. G., Hazarika, P. J., Contreras-Reyes, J. E., and Yousof, H. M. (2025). A new logistic distribution and its properties, applications and port-var analysis for extreme financial claims. *Mathematical and Computational Applications*, 30(3):62.

Yousof, H., Afify, A. Z., Alizadeh, M., Hamedani, G., Jahanshahi, S., and Ghosh, I. (2018a). The generalized transmuted poisson-g family of distributions: Theory, characterizations and applications. *Pakistan Journal of Statistics and Operation Research*, 14(4):759–779.

Yousof, H., Jahanshahi, S., Ramires, T., Aryal, G., and Hamedani, G. (2018b). A new distribution for extreme values: regression model, characterizations and applications. *Journal of Data Science*, 16(4):677–706.

Yousof, H. M., Alizadeh, M., Jahanshahi, S., Ghosh, T. G. R. I., and Hamedani, G. (2017). The transmuted topp-leone g family of distributions: theory, characterizations and applications. *Journal of Data Science*, 15(4):723–740.

Yousof, H. M., Altun, E., and Hamedani, G. (2018c). A new extension of frechet distribution with regression models, residual analysis and characterizations. *Journal of Data Science*, 16(4):743–770.

Yousof, H. M., Majumder, M., Jahanshahi, S., Ali, M. M., and Hamedani, G. (2018d). A new weibull class of distributions: theory, characterizations and applications. *Journal of Statistical Research of Iran*, 15(1):45–82.