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A Two-Sided CUSUM Control Chart with Repetitive Sampling with Industrial Application

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Recently, the practice of employing control charts to monitor manufacturing processes has gained significant interest in the field of statistical process control (SPC). A properly designed control chart is capable of promptly detecting any shifts in the process. In this article, a novel two-sided CUSUM control chart with repetitive sampling is introduced for monitoring the process mean. To evaluate the performance of the proposed CUSUM control chart, various statistical measures such as the average, standard deviation, and percentiles (including the median) of run lengths are utilized. These measures are assessed under different distribution scenarios. Furthermore, a comparison is made between the performance of the proposed chart and several existing control charts. By presenting the new two-sided CUSUM control chart with repetitive sampling, this article contributes to the advancement of process monitoring techniques in SPC. The evaluation reveals that while the proposed chart is highly effective for detecting very small process shifts, there is a trade-off in efficiency as the shift magnitude increases. These findings provide insights into the specific scenarios where the CUSUM-RS chart is most beneficial compared to existing tools.

Key Words: control chart, CUSUM; run-length; repetitive sampling; simulation

Mathematical Subject Classification: 97K80

1. Introduction

One of the most vital statistical quality/process control charts have been expected to identify the assignable causes in manufacturing industries, engineering processes, and medical processes, etc. for quality improvement of the products, see (Montgomery, 2009). The statistical process control (SPC) is a strategy that can be applied to any process to reduce the variation and it contains SPC tool kits like histograms, Pareto charts, check sheets, defect intensity diagrams, cause and effect diagrams, scatter diagrams and control charts. The control charts are the most important tool kit of SPC that classified into two categories, namely memory type and memoryless type control charts. Shewhart-type control charts are the type of memoryless control charts and they're less sensitive to small and moderate shifts in the location and dispersion parameters. The cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts are labeled as memory type charts. This scheme was introduced by (Page, 2006) as the substitution of the traditional Shewhart control chart. The CUSUM chart used the past information along with current information, which makes them very sensitive to small and moderate shifts in the process parameters (location and dispersion) as compared to the traditional Shewhart control chart.

Variation may occur by chance in the process data due to some incorrect specifications of instruments or due to human reporting error. The variations in process data may harmfully disturb the parametric calculations. Thus, the variation detection ability is a major key to sufficiently estimating process parameters. A lot of literature is available related to the CUSUM charts, where some of the parametric, nonparametric, and robust outlier-revealing procedures have been proposed to improve the presentation of control charts in the existence of outliers under a simple random sampling scheme. For example, see (Aslam et al., 2021), (Mahmood et al., 2017), (Nazir et al., 2013), (Liu et al., 2015), (Wang et al., 2017), (Abid et al., 2017), (Zhang et al., 2012), (Amdouni et al., 2017) and the references therein.

(Zhao et al., 2005) developed a dual CUSUM scheme which combines the two CUSUM chart to detect outliers in a process. Some studies like (Hawkins & Olwell, 2012) and (Barnett & Lewis, 1994) developed the different types of outlier detectors control chart. The student-type and Grubbs-type outliners detector commonly used in the regression residuals when data is normally distributed [(Grubbs, 1969) and (Tietjen & Moore, 1972)]. For non-normal data, the repetitive sampling scheme and Tukey's outlier detection model are more robust since their independence of the location and dispersion (Lim et al., 2017).

Another way to improve the efficiency of the control chart in detecting the outliers in the process is to modify the sampling scheme. Conventionally, a lot of sampling schemes have been studied for several outlier detectors underlying the quality process. For example, there have been simple random sampling (Roberts, 1959), repetitive sampling (RS) scheme designed by (Aslam, Khan, et al., 2015), variable sampling interval scheme proposed by (Saccucci et al., 1992) and double sampling scheme introduced by (CROASDALE, 1974). The repetitive sampling scheme is easier to operate than the double sampling scheme because it has two parameters but the double sampling scheme is based on four parameters. Repetitive sampling is different from the variable sampling interval scheme because the variable sampling interval scheme adjusts the sampling interval but repetitive sampling adjusts the control limits.

(Sherman, 1965) first designed the repetitive sampling scheme in acceptance sampling, (Balamurali & Jun., 2006) applied repetitive sampling scheme in variable acceptance sampling plan and found that the repetitive sampling scheme performs efficiently than the single sampling scheme. The repetitive sampling scheme idea in statistical process control monitoring was introduced by (Shafqat, Huang, Aslam, et al., 2020b) and (Ahmad et al., 2014). They showed that the repetitive sampling scheme increases the efficiency of control charts in the process shift detection. More applications on control chart using repetitive sampling can be seen in Nagaraju et al. (2025)

Most of the literature mentioned above is designed to monitor the process using a CUSUM chart under a simple random sampling scheme and a EWMA control chart under simple random sampling and repetitive sampling. After reading the literature, we investigate the CUSUM chart for the monitoring process there is no study available under the repetitive sampling scheme. In this study, we study the effect of outliers on the performance of the CUSUM chart under repetitive sampling for monitoring process location using the run-length properties.

2. The Structure of Classical CUSUM Control Chart:

The classical CUSUM chart for process means monitoring with small to large shifts have been introduced by Page (Page, 2006). The two-sided CUSUM chart plots two statistics C_i^+ and C_i^- against the single control limit H, where i represents the sample number. This can be used to monitor upwards and downwards shifts in the process. Let X be a quality characteristic that follows an independent normal distribution with mean μ_0 and variance σ_0^2. The initial values of the plotted statistics of the classical CUSUM chart are zero (i.e. $C_i^+ = C_i^- = 0$). These charting statistics are mathematically expressed by:

$$\begin{cases} C_i^+ = max\{0, \overline{X}_\iota - (\mu_0 + K) + C_{i-1}^+\} \\ C_i^- = min\{0, (\mu_0 - K) - \overline{X}_\iota + C_{i-1}^-\} \end{cases} \quad (1)$$

where $\overline{X}_\iota$ is the sample mean i.e. $\frac{\sum_i^n x_i}{n}$, and K is the reference value. The process is declared to move upwards when $C_i^+ > H$ and to move downwards when $C_i^- < -H$.

The CUSUM chart depends on the two designed parameters, i.e. K and H, according to the (Montgomery, 2012). These parameters are selected very carefully because the sensitivity of the CUSUM chart depends on them. The designed parameters are usually selected such that:

$$K = k\sqrt{Var(\overline{X}_\iota)} \text{ and } H = h\sqrt{Var(\overline{X}_\iota)} \quad (2)$$

where $Var(\overline{X}_i) = \sigma_X^2/n$, σ_X^2 is the variance of X, h is generally selected to 5 times the standard deviation units, and k is selected to be half of the shift amount δ (i.e. $k = \delta/2$).

3. The Proposed CUSUM Control Chart based on Repetitive Sampling:

In this section, the proposed idea of repetitive sampling (RS) is applied in order to develop a new parametric CUSUM chart based on the normal environment. The complete area of the newly developed CUSUM chart is divided into six regions: I) one extending above the upper control limit (H_1) that is called out-of-control region (O), II) one extending between the upper control limit (H_1) and upper repetitive limit (H_2) that is called resampling region (R), III) one extending between upper repetitive limit (H_2) and central limit that is called in-control region (I), IV) one extending between the central limits and lower repetitive limit ($-H_2$) that is also called in-control region (I), V) one extending between lower repetitive limit ($-H_2$) and lower control limit ($-H_1$) that is also called resampling region (R), and VI) one extending the below lower control limit ($-H_1$) that is also called out-of-control region (O) as shown in figure 1. The proposed CUSUM statistic is plotted against the two control limits H_1 and H_2 in repetitive sampling. If the charting point falls in resampling region R, then we don't take a decision of in-control or out-of-control, but we repeat sampling to obtain a new CUSUM statistics until the charting point either falls in the in-control region (I) or out-of-control region (O). Therefore, we propose the following CUSUM charting process using RS:

- Step 1. Take a sample of size n and compute the CUSUM statistics, C_i^+ and C_i^-, using Eq. (1).
- Step 2. Declare the process as out-of-control if the plotting statistics of the CUSUM chart outside the outer control limit H_1, that is, when $C_i^+ > H_1$ or $C_i^- < -H_1$. Declare the process as in-control if $0 < C_i^+ < H_2$ and $-H_2 < C_i^- < 0$, where H_2 represent the inner control limit. If $H_2 < C_i^+ < H_1$ or $H_2 < C_i^- < H_1$, then go to Step 1 and repeat the process again.

The six regions of the CUSUM chart under RS scheme are separated by upper control limit (H_1), upper repetitive limit (H_2), central limit, lower repetitive limit ($-H_2$), and lower control limit (-H1), with $H_1 \geq H_2 > CL > -H_2 \geq -H_1$.

Then, Figure 1 shows the proposed control scheme using CUSUM with RS.

The three control parameters in the operation of the proposed control chart are given as:

$$K = k\sqrt{Var(\bar{X}_\iota)}\ , H_1 = h_1\sqrt{Var(\bar{X}_\iota)} \text{ and } H_2 = h_2\sqrt{Var(\bar{X}_\iota)} \quad (3)$$

Figure 1: Proposed CUSUM Chart using RS Scheme

When $H_1 = H_2$ the proposed chart becomes the classical CUSUM chart proposed by Page [1954]. We will refer to the proposed chart as the CUSUM-RS chart The Monte Carlo simulation algorithm is used to determine the control chart coefficients h_1 and h_2 for different values of n. These coefficients are chosen such that the average run length value ARL is closer to the fixed value of the $ARL_0 = 500$.

3.1. Performance Measures of the Proposed CUSUM-RS Control Chart

The average run length (ARL) is one of the most popularly used metrics in statistical process control, which is the average number of subgroups before an out-of-control signal is declared. There are two types of ARLs - one is called in-control ARL denoted by ARL_0 and the other one is an out-of-control ARL denoted by ARL_1. The Monte Carlo simulation algorithm is developed in the R programming language for evaluating ARLs. We have determined the proposed control chart ARL values from the normal environment with different values of control limits. This process is repeated 50,000 times so that we can average out the run length to get the fixed ARL (500, say) denoted ARL_0. When a shift (δ) is given, the out-of-control ARL, denoted by ARL_1, is evaluated similarly. Here δ is the deviation in mean from the target mean μ_0 .

It is recommended to have a large in-control ARL (ARL_0) ($ARL_0 = 500$, say) value and a small out-of-control ARL (ARL_1) value with different shift values δ in the location parameter. Other than ARL metric, the standard deviation of the run-length and the percentiles of the run-length distribution (i.e. 5^{th}, 25^{th}, 50^{th},75^{th}, and 95^{th}) are also used to measure the performance of the proposed chart.

To evaluate the sampling effort of the proposed CUSUM-RS chart, the Average Sample Number (ASN) is utilized. In a repetitive sampling scheme, the number of samples is not fixed but depends on the probability of falling into the resampling region. The ASN is defined as: $ASN = \frac{n}{P_I + P_O} = \frac{n}{1-P_R}$, where n is the initial sample size, P_I is the probability of the process being in-control, P_O is the probability of an out-of-control signal, and P_R is the probability of resampling. For classical charts, $ASN = n$, whereas for the CUSUM-RS chart, $ASN > n$.

3.2. Experiment Setting of the Proposed Chart

In this study, there are four distributions, namely, standard normal distribution with mean zero and variance 1 ($N \sim (0,1)$), the student's t(v=4) distribution, the Logistic Distribution $(0, \sqrt{3}/\pi)$, and Laplace distribution with parameter 0 and 1, denoted by Laplace(0,1), using a repetitive sampling scheme. So, the coefficients of control limits are chosen when the process is in-control (when $\delta = 0$) using the process described in Figure 2:

The performance of the CUSUM-RS chart is evaluated under the following environments:

- Case 1 (normal): all observations are generated from the $N \sim (0,1)$.
- Case 2 (mixed normal): 99% of observations are generated from $N \sim (0,1)$ and 1% of observations are from $N \sim (0,9)$.
- Case 3 (non-normal): All observations are generated from a non-normal distribution for investing the effect of a different distribution. We use the student t-distribution with 4 degree of freedom (T_4 ; Case 3-1) as well as t-distribution with 8 degree of freedom (T_8; Case 3-2), Laplace (0,1) (Case 3-3), and logistic distribution ($logis\ (0,3/\sqrt{\pi})$; Case 3-4).

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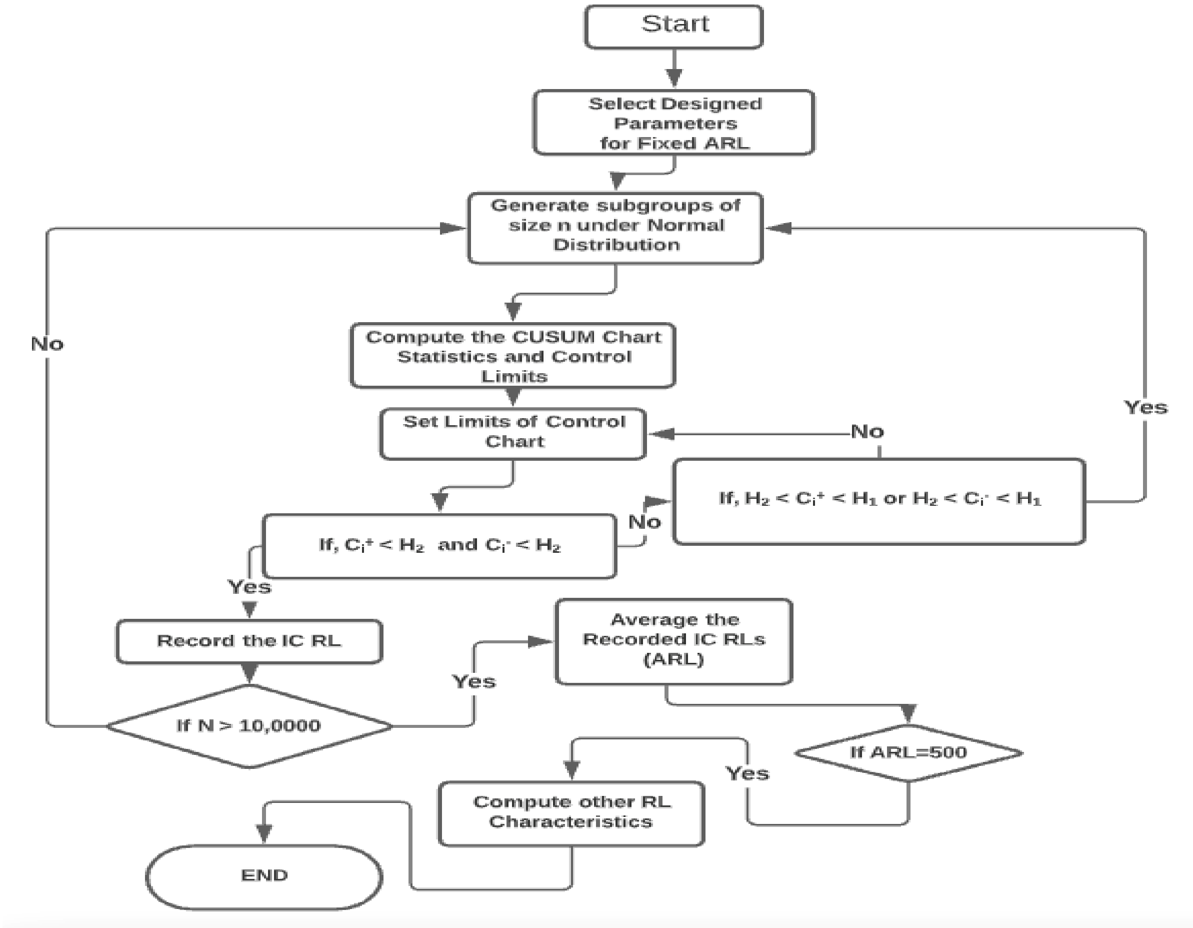

Figure 2. The Flow Chart based on Repetitive sampling Scheme Monte Carlo Simulation of the Proposed CUSUM-RS Chart.

Table 1. The Control Coefficients of the proposed CUSUM-RS control chart under normal environments

ARL_0	k=0.05		k=0.25		k=0.65		k=0.75		k=1.65	
	h_1	h_2	h_1	h_2	h_1	h_2	h_1	h_2	h_1	h_2
100	10.863	8.162	5.763	4.162	2.883	1.962	2.571	1.346	1.003	0.206
200	14.193	12.962	6.967	5.362	3.373	2.142	2.973	1.746	1.229	0.316
300	16.498	14.969	7.723	5.352	3.665	2.552	3.233	1.986	1.369	0.306
370	17.898	16.569	8.083	6.022	3.863	2.092	3.363	1.777	1.439	0.366
500	19.698	18.691	8.623	6.862	4.078	2.003	3.568	1.977	1.533	0.347

Table 2. The ARL, SDRL and MRL values of the CUSUM-RS chart under Normal Environment with different shifts and k when nominal ARL=500

	k														
	0.05			0.25			0.65			0.75			1.65		
δ	ARL	SDRL	MRL	ARL	SDRL	MRL	ARL	SDRL	MRL	ARL	SDRL	MRL	ARL	SDRL	MRL
0.00	499.5	416.0	366	502.5	485.2	341	500.3	500.0	336	500.1	500.6	327	499.2	493.2	341
0.05	362.6	271.8	286	432.1	414.8	310	457.6	436.4	324	475.6	450.3	320	476.6	490.6	331
0.10	227.3	152.1	185	305.1	288.9	211	404.3	409.1	275	425.3	431.3	289	441.3	434.8	327
0.25	89.9	42.4	81	89.9	75.6	67	165.0	162.4	113	182.1	176.9	125	325.7	336.9	220

0.5	42.7	14.3	40	27.4	16.6	23	39.0	36.9	28	47.7	47.1	35	163.0	152.6	121
0.75	27.5	7.4	26	15.2	7.3	13	13.6	11.5	10	18.4	16.7	14	66.8	65.4	47
1.00	20.6	4.8	20	10.2	3.9	10	7.1	5.0	6	8.5	6.4	6	26.4	26.7	18
1.25	16.2	3.2	16	7.8	2.7	7	4.5	2.9	4	5.1	3.2	4	12.6	12.5	9
1.5	13.5	2.5	13	6.3	1.8	6	3.3	1.8	3	3.6	2.0	3	6.3	5.9	4
2.0	10.3	1.6	10	4.5	1.2	4	2.2	1.0	2	2.3	1.1	2	2.5	1.9	2
2.5	8.2	1.1	8	3.6	0.8	4	1.6	0.6	2	1.7	0.7	2	1.5	0.8	1
3.0	6.8	0.9	7	3.0	0.6	3	1.4	0.5	1	1.4	0.5	1	1.2	0.5	1
5.0	4.2	0.4	4	2.0	0.1	2	1.0	0.1	1	1.0	0.1	1	1.0	0.0	1
10.0	2.1	0.3	2	1.0	0.0	1	1.0	0.0	1	1.0	0.0	1	1.0	0.0	1
12.0	2.0	0.0	2	1.0	0.0	1	1.0	0.0	1	1.0	0.0	1	1.0	0.0	1

Table 3. Performance of CUSUM-RS chart under Normal Environment with different shifts, and k when nominal *ARL*=500

	k														
	0.05			0.25			0.65			0.75			1.65		
δ	P_{25}	P_{75}	P_{90}	P_{25}	P_{75}	P_{90}	P_{25}	P_{75}	P_{90}	P_{25}	P_{75}	P_{90}	P_{25}	P_{75}	P_{90}
0.00	201	680	1085	156	702	1170	135	677	1156	139	652	1151	146	698	1152
0.05	166	473	750	132	593	981	137	641	1065	130	650	1096	133	669	1058
0.10	116	296	429	102	417	685	114	553	906	127	580	943	130	683	1021
0.25	59	112	147	36	120	187	51	229	370	57	261	403	92	441	753
0.5	32	51	61	15	35	49	12	54	85	15	64	102	46	227	363
0.75	22	32	37	10	19	25	5	19	29	7	25	41	19	97	154
1.0	17	23	27	7	12	15	3	9	14	4	12	17	8	36	59
1.25	14	18	21	6	9	11	3	6	8	3	6	9	4	17	28
1.5	12	15	17	5	7	9	2	4	6	2	5	6	2	8	15
2.0	9	11	13	4	5	6	2	3	4	2	3	4	1	3	5
2.5	7	9	10	3	4	5	1	2	2	1	2	3	1	2	3
3.0	6	7	8	3	3	4	1	2	2	1	2	2	1	1	2
5.0	4	5	5	2	2	2	1	1	1	1	1	1	1	1	1
10	2	2	3	1	1	1	1	1	1	1	1	1	1	1	1
12	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1

4. **Discussion of the Experiment Results**

4.1 Control Coefficients of Proposed CUSUM-RS control chart

For the accomplishment of the new proposed CUSUM-RS chart, the SPC engineers and specialists need the values of designed parameters such as decision parameters h_1 and h_2 according to the sensitivity parameter k. Table 1 contains these coefficients at several nominal values of ARL_0 when all observations are from a normal distribution (Case 1). So, Table 1 is very important to the engineers and quality managers of the manufacturing industries for the implementation of the proposed chart.

4.2 Proposed Chart Performance under Normal Environments (Case 1): For the control coefficients and ARL_0 values for the proposed CUSUM-RS chart given in Table 1, we have evaluated the performance of the proposed CUSUM-RS chart under various process shifts. Table 2 shows the average (ARL), standard deviation (SDRL) and median (MRL) of the out-of-control run lengths at different shift values. Table 3 reports the percentiles of RLs (Q_{25}, Q_{75}, and Q_{90}).

Tables 2-3 are around here. For observing the robustness of the proposed CUSUM-RS chart under normal environments we observe the following behavioral points from Tables 2-3:

- TAs the shift value increases, the ARL values are showing the decreasing trend at all values of coefficient level (k).
- The proposed CUSUM-RS chart is really good in detecting small to moderate shifts for small values of k and also good in detecting large shifts with large values of k.
- For a fixed value of δ, the ARL decline speed for the proposed chart is decreased as values of k increase. Similar behavior is shown for SDRL values. For example, when k=δ=0.25 the SDRL value is 75.6 but when k=0.65 with δ=0.25 the SDRL value is 176.9.

- The distribution of RL of the CUSUM-RS chart is positively skewed (cf. Table 3). When δ increases, the quantiles values are also decreasing. For example, when δ=0.25 with k= 0.05 the RL quantiles (25,75, and 90) are 59, 112, and 147, while when δ=1.5 the RL quantiles with same k value are 12, 15, and 17.
- For fixed values of δ, the CUSUM-RS chart is more efficient for smaller values of k. For example, where δ= 0.1 the ARL_1 when k= 0.05, 0.25, 0.65, 0.75, and 1.65 were 227.3, 305.1, 404.3, 425.3, and 441.3, respectively. Thus, the proposed chart detects a shift in the process mean faster when a small value of k.
- As δ increases, the ARL_1, $SDRL_1$, MRL_1 and percentiles values approach to 1, 0 and 1, respectively, especially for large values of k. This means that the chart identifies large shifts quickly.

4.3 Proposed Chart Performance under Different Environments:

To investigate the effect of using different distributions of the proposed CUSUM-RS chart, we use a mixed normal distribution (Case 2) student t distributions with 4 (Case 3-1) and 8 (Case 3-2) degrees of freedom, Laplace distribution (Case 3-3), and logistic distribution (Case 3-4). Tables 4-5 contain the ARL values for proposed CUSUM-RS chart under these cases, where the ARL_0 is kept fixed at 500. Figure 3 shows the ARLs under different distributions according to the shift values.

We observe from Tables 4-5 and Figure 3 the following:

- The proposed CUSUM-RS chart still performs well under the different environments in detecting the out-of-control signal quickly with small to large shifts.
- For the fixed value of k with different environments, we have seen that the proposed CUSUM-RS chart performed efficiently under the symmetric variance disturbances (Case 2). The performance of the proposed CUSUM-RS chart under student t distributions perform efficiently at a large degree of freedom (Case 3-2) as compared to a small degree of freedom (Case 3-1). For example, when $k = 0.25$ and $\delta = 0.75$, the value of ARL_1 is 21.4 for 4 degrees of freedom but ARL_1 is 18.1 for 8 degree of freedom.
- The ASN values reported in Tables 4 and 5 demonstrate the efficiency of the CUSUM-RS chart. The sampling effort starts at 1.0 (in-control), peaks at 1.2 for moderate shifts (δ between 1.0 and 2.5), and returns to 1.0 for large shifts. This indicates that even at its 'costliest' point, the chart only requires an average of 1.2 samples to reach a decision, which is a very small overhead for the significant gain in detection speed (ARL) provided.

In Table 5 and Figure 3 (B), we showed the proposed CUSUM-RS chart performance under different environments distributed with fixed values of decision interval values of $H_1 = 8.623$, $H_2 = 6.862$,

Table 4. Performance of CUSUM-RS chart under different environments when $n = 1, k = 0.25$ and nominal *ARL*=500

					δ													
Environments	H_1	H_2	Metric	0.00	0.05	0.10	0.25	0.50	0.75	1.00	1.25	1.5	2.00	2.5	3.0	5.0	10.0	12.0
Case 1	8.623	6.862	ARL	502.5	432.1	305.1	89.9	27.4	15.2	10.2	7.8	6.3	4.5	3.6	3.0	2.0	1.0	1.0
			SDRL	485.2	414.8	288.9	75.6	16.6	7.3	3.9	2.7	1.8	1.2	0.8	0.6	0.1	0.0	0.0
			ASN	1.0	1.0	1.0	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.0
			MRL	341	310	211	67	23	13	10	7	6	4	4	3	2	1	1
			P_{25}	156	132	102	36	15	10	7	6	5	4	3	3	2	1	1
			P_{75}	702	593	417	120	35	19	12	9	7	5	4	3	2	1	1
			P_{90}	1170	981	685	187	49	25	15	11	9	6	5	4	2	1	1
Case 2	21.193	19.262	ARL	501.5	307.7	134.2	38.5	16.2	10.4	7.6	6.1	5.1	3.8	3.1	2.7	1.9	1.0	1.0
			SDRL	457.3	273.1	107.1	19.2	5.4	2.8	1.7	1.2	0.9	0.6	0.4	0.4	0.0	0.0	0.0
			ASN	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.0
			MRL	363	226	104	34	15	10	8	6	5	4	3	3	2	1	1
			P_{25}	175	115	60	25	12	8	6	5	4	3	3	3	2	1	1
			P_{75}	677	406	177	48	19	12	9	7	6	4	3	3	2	1	1
			P_{90}	1083	682	260	63	23	14	10	8	6	5	4	3	2	1	1
Case 3-1	15.253	13.487	ARL	499.6	423.0	298.2	102.9	37.6	21.4	14.5	11.3	9.2	6.8	5.5	4.6	2.9	1.7	1.5
			SDRL	472.9	395.9	283.5	81.6	20.3	9.5	5.3	3.8	2.9	1.9	1.5	1.2	0.7	0.5	0.5
			ASN	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.0	1.0
			MRL	360	313	211	77	33	20	14	11	9	7	5	5	3	2	2
			P_{25}	159	138	100	45	23	15	11	9	7	6	5	4	2	1	1
			P_{75}	676	584	400	135	49	26	17	13	11	8	7	5	3	2	2
			P_{90}	1138	906	655	212	66	34	22	17	13	9	8	6	4	2	2

Case 3-2	10.973	9.687	ARL	501.5	433.0	301.7	96.2	31.2	18.1	12.3	9.6	7.7	5.5	4.4	3.8	2.4	1.4	1.1
			SDRL	479.7	397.4	290.0	76.3	16.9	8.3	4.6	3.2	2.3	1.4	1.1	0.9	0.5	0.4	0.3
			ASN	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.3	1.2
			MRL	363	308	203	75	27	17	12	9	7	5	4	4	2	1	1
			P_{25}	167	142	99	42	19	12	9	7	6	5	4	3	2	1	1
			P_{75}	704	603	404	128	40	22	15	12	9	6	5	4	3	2	1
			P_{90}	1103	939	688	198	54	29	18	14	11	7	6	5	3	2	2
Case 3-3	15.151	12.089	ARL	501.8	459.0	335.5	139.5	48.4	26.4	17.4	13.0	10.5	7.7	6.1	4.9	3.1	1.9	1.6
			SDRL	484.1	433.5	338.9	114.1	32.3	12.9	7.4	4.7	3.4	2.1	1.5	1.1	0.5	0.2	0.4
			ASN	1.0	1.0	1.0	1.0	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.0	1.2
			MRL	347	311	235	107	40	24	16	12	10	7	6	5	3	2	2
			P_{25}	148	147	118	57	26	17	12	10	8	6	5	4	3	2	1
			P_{75}	706	632	438	191	62	33	21	15	12	9	7	6	3	2	2
			P_{90}	1098	1010	727	292	92	43	27	19	15	11	8	6	4	2	2
Case 3-4	8.691	6.289	ARL	500.4	439.0	294.5	87.3	26.2	13.5	9.4	7.1	5.7	4.2	3.4	2.8	1.9	1.0	1.0
			SDRL	483.1	445.6	278.1	75.2	17.3	6.4	3.7	2.4	1.7	1.1	0.8	0.6	0.2	0.0	0.0
			ASN	1.1	1.1	1.1	1.2	1.2	1.2	1.2	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.0
			MRL	348	299	215	64	21	12	9	7	6	4	3	3	2	1	1
			P_{25}	152	140	98	34	14	9	7	5	4	3	3	3	2	1	1
			P_{75}	692	589	379	114	33	16	11	8	7	5	4	3	2	1	1
			P_{90}	1111	974	656	192	47	22	14	10	8	6	4	4	2	1	1

Table 5. Performance of the CUSUM-RS chart under different Environment when $H_1 = 8.623$, $H_2 = 6.862$, $n = 1, k = 0.25$, and nominal *ARL*=500

			δ													
Environments	Metric	0.00	0.05	0.10	0.25	0.50	0.75	1.00	1.25	1.50	2.00	2.5	3.00	5.0	10.0	12.0
Case 1	ARL	502.5	432.1	305.1	89.9	27.4	15.2	10.2	7.8	6.3	4.5	3.6	3.0	2.0	1.0	1.0
	SDRL	485.2	414.8	288.9	75.6	16.6	7.3	3.9	2.7	1.8	1.2	0.8	0.6	0.1	0.0	0.0
	ASN	1.0	1.0	1.0	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.2	1.0
	MRL	341	310	211	67	23	13	10	7	6	4	4	3	2	1	1
	P_{25}	156	132	102	36	15	10	7	6	5	4	3	3	2	1	1
	P_{75}	702	593	417	120	35	19	12	9	7	5	4	3	2	1	1
	P_{90}	1170	981	685	187	49	25	15	11	9	6	5	4	2	1	1
Case 2	ARL	33.9	31.4	28.5	13.2	6.4	4.2	3.1	2.5	2.2	1.7	1.4	1.1	1.0	1.0	1.0
	SDRL	26.5	26.2	22.2	9.2	3.2	1.7	1.1	0.7	0.5	0.4	0.4	0.3	0.0	0.0	0.0
	ASN	1.1	1.1	1.1	1.1	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.2	1.1
	MRL	26	25	22	11	6	4	3	2	2	2	1	1	1	1	1
	P_{25}	15	13	13	7	4	3	2	2	2	2	1	1	1	1	1
	P_{75}	46	41	39	17	8	5	4	3	3	2	2	1	1	1	1
	P_{90}	69	63	57	26	11	6	5	4	3	2	2	2	1	1	1
Case 3-1	ARL	79.9	83.9	70.9	41.1	18.5	11.0	8.1	6.3	5.2	3.8	3.1	2.6	1.7	1.1	1.0
	SDRL	72.0	74.3	63.4	32.3	11.9	5.9	3.8	2.6	2.0	1.3	1.0	0.8	0.5	0.3	0.1
	ASN	1.0	1.0	1.1	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.0
	MRL	61	61	52	32	16	10	7	6	5	4	3	3	2	1	1
	P_{25}	28	29	25	18	10	7	5	4	4	3	2	2	1	1	1
	P_{75}	107	115	98	56	24	14	10	8	6	5	4	3	2	1	1
	P_{90}	169	188	146	87	34	19	13	10	8	5	4	4	2	2	1
Case 3-2	ARL	198.5	173.4	149.7	63.8	23.0	13.2	9.3	7.0	5.7	4.2	3.4	2.8	1.8	1.0	1.0
	SDRL	186.3	159.8	136.4	53.1	13.9	6.7	3.8	2.7	2.0	1.3	0.9	0.7	0.4	0.2	0.0
	ASN	1.0	1.0	1.0	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.0
	MRL	143	127	110	49	20	12	9	7	5	4	3	3	2	1	1
	P_{25}	66	56	52	27	13	8	7	5	4	3	3	2	2	1	1
	P_{75}	262	242	202	84	29	17	11	8	7	5	4	3	2	1	1
	P_{90}	439	388	319	127	42	23	15	11	8	6	5	4	2	1	1
Case 3-3	ARL	74.3	75.8	69.7	50.8	23.8	14.4	10.2	7.5	6.3	4.6	3.7	3.1	2.0	1.0	1.0
	SDRL	64.7	66.8	61.8	40.6	17.0	8.3	5.0	3.4	2.6	1.5	1.1	0.8	0.4	0.1	0.0
	ASN	1.0	1.1	1.1	1.1	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.0
	MRL	55	57	49	41	19	13	9	7	6	4	4	3	2	1	1
	P_{25}	27	28	26	21	12	8	7	5	5	4	3	3	2	1	1
	P_{75}	100	99	95	70	31	19	13	9	8	6	4	4	2	1	1
	P_{90}	165	166	158	105	46	25	17	12	10	7	5	4	2	1	1

Case 3-4	ARL	489.0	429.9	296.2	93.6	26.9	15.0	10.3	7.5	6.2	4.4	3.6	3.0	2.0	1.0	1.0
	SDRL	478.8	431.6	276.9	82.4	15.8	7.2	4.1	2.4	1.9	1.2	0.8	0.6	0.2	0.0	0.0
	ASN	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.0
	MRL	329	299	218	70	23	13	10	7	6	4	4	3	2	1	1
	P_{25}	148	138	102	36	16	10	7	6	5	4	3	3	2	1	1
	P_{75}	680	584	384	118	34	18	12	9	7	5	4	3	2	1	1
	P_{90}	1117	956	659	202	47	25	16	11	9	6	5	4	2	1	1

Table 6. Comparison between Classical CUSUM(Page, 2006), EWMA (Saccucci et al., 1992), EWMA-RS (Shafqat et al. 2020), HWMA (Riaz et al., 2021), and proposed CUSUM charts when ARL0=500

δ	$k = \lambda = 0.25$					$k = \lambda = 0.1$					$k = \lambda = 0.05$				
	CUSUM	EWMA	EWMA RS	HWMA	Proposed CUSUM	CUSUM	EWMA	EWMA RS	HWMA	Proposed CUSUM	CUSUM	EWMA	EWMA RS	HWMA	Proposed CUSUM
0.00	502.8	501.1	509.55	498.4	502.5	501.58	501.6	499.93	500.3	502.1	499.4	487.8	495.22	497.5	502.0
0.03	490.34	484.36	480.51	476.54	450.92	488.21	476.35	456.05	432.98	412.46	476.57	465.79	445.93	436.79	414.57
0.05	473.65	464.8	442.84	440.2	432.1	471.3	437.9	417.30	397.6	380.8	425.3	412.1	352.31	380.6	365.4
0.075	447.2	433.7	395.43	382.8	361.5	438.6	380.9	376.46	318.8	297.7	355.8	336.9	327.51	297.0	290.2
0.1	411.5	390.4	362.41	324.2	288.3	397.6	318.9	306.26	249.8	238.2	291.2	266.2	263.06	230.7	235.9
0.15	335.0	300.2	279.07	224.1	201.4	312.6	215.5	168.53	161.0	151.2	189.4	168.7	117.14	146.0	156.3
0.175	296.5	263.3	280.22	187.6	154.5	273.9	178.6	140.32	133.1	124.7	154.7	136.3	91.87	119.9	133.8
0.2	260.2	225.4	228.56	157.8	129.9	237.2	146.1	115.44	111.7	107.0	128.8	110.5	78.83	100.4	115.1
0.25	199.4	168.9	184.08	113.1	86.4	179.1	102.9	90.36	81.6	75.8	91.2	77.7	54.57	73.0	89.4
0.5	57.1	47.4	38.56	33.7	26.8	48.8	28.7	13.76	28.5	33.1	31.1	23.7	13.46	24.8	41.8
0.75	22.1	19.3	14.43	16.2	14.9	19.6	13.6	7.65	14.8	21.1	18.0	11.9	7.91	12.8	27.9
1.0	11.5	10.4	7.89	9.7	10.2	10.9	8.2	4.81	9.3	15.2	12.6	7.2	5.39	8.0	20.8
1.5	5.4	4.7	3.30	4.9	6.3	5.5	4.1	2.99	4.9	10.0	7.9	3.7	3.42	4.4	13.8
2	3.5	2.9	2.36	3.1	3.6	3.6	2.6	2.13	3.3	7.4	5.8	2.4	2.60	2.9	10.3

k=0.25. The proposed chart performs efficiently in detection the shifts earlier under a mixed normal distribution (Case 2). For example, when δ=0.75, the A〖RL〗_1 is 15.2 with normal distribution, 4.2 with a mixed distribution, 8.1 with T4, 9.3 with T8, 14.4 with Laplace, and 15.0 with the logistic distribution.

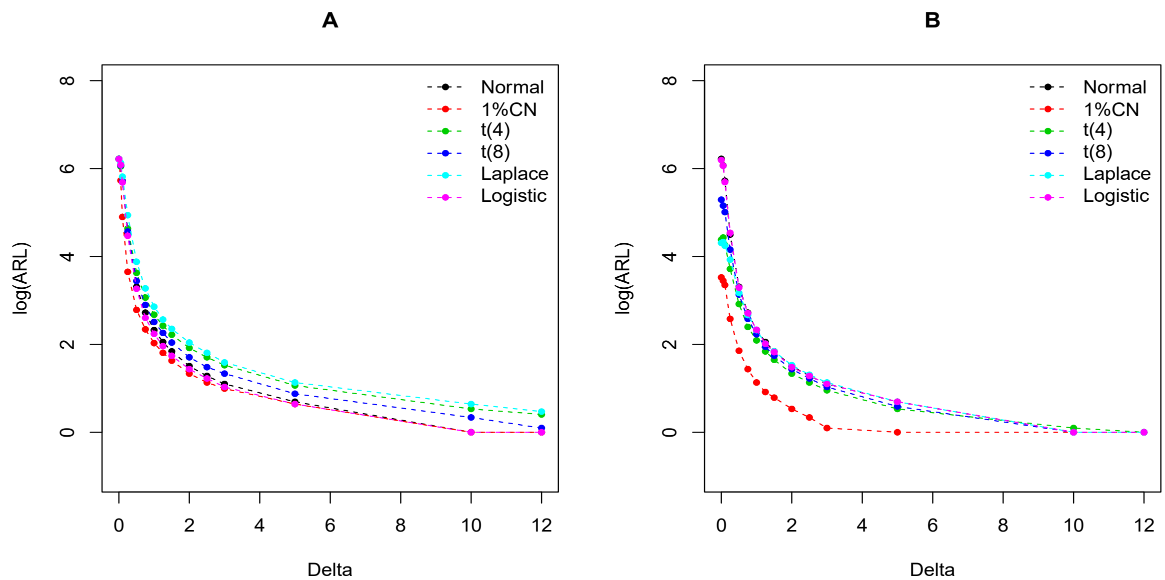

Figure 3: ARLs of the proposed chart under different distributions. A-from Table 4, B-from Table 5.

4.4 Comparisons with Some Existing Charts

This section described the efficiency of the proposed chart from the comparison with several existing control charts. For best proficiency of the control chart, the smallest ARL_1 is desired for specified values of ARL_0. For comparison with existing charts, we considered three different values of reference parameter $k = 0.05, 0.1$ and 0.25. In all cases, the charts' parameters were prearranged to fix ARL_0 at 500. We provide the charts' ARL results that optimized shift at different designed parameters values. Here, we made a comparison with classical CUSUM, EWMA, EWMA-RS, and Homogenously weighted moving average (HWMA) charts proposed by (Riaz et al., 2021) with the proposed CUSUM-RS chart and results are provided in Table 6 and graphical presentation is described in Figure 4. The comparison between the existing charts and proposed chart showed that the proposed chart worked best for the large value of the parameters.

Proposed CUSUM-RS vs Classical CUSUM Chart: The ARL values of the classical CUSUM chart and the proposed CUSUM-RS chart are given in Table 6 where the specified ARL_0=500. Table 6 indicates that the classical CUSUM chart performs worse than the proposed CUSUM-RS chart for the same shift values δ.

Proposed CUSUM-RS Chart vs. Classical EWMA Chart: The ARL values of the classical EWMA chart are also given in Table 6. The proposed CUSUM-RS chart is efficient to classical EWMA for the specific value of δ with similar sensitivity parameters and fixed ARL_0 values. Table 6 clearly shows that the superiority of the proposed CUSUM-RS chart lies where the value of δ is smaller and moderate, while as for large δ values the classical EWMA chart performs better as compared to the proposed chart.

Proposed CUSUM-RS Chart vs. EWMA RS Chart: The EWMA RS chart's ARL values are lager then the proposed chart for specific values of shift with various sensitivity parameters' values. The proposed chart showed the high efficiency in detection shift as compared to the proposed chart (cf. Table 6). The proposed chart performs higher efficiency in detection for a small shift and lower efficiency in the large shift as compared to the EWMA RS chart.

Proposed CUSUM-RS Chart vs Classical HWMA Chart: The classical HWMA chart's ARL values are larger and showed low efficiency in detection shift (cf. Table 6). The proposed chart performs superiorly for a small shift for all values of sensitivity parameter but performs low efficiency in detection for a large shift as compared to the classical HWMA chart.

5. Illustrative Example

An illustrative example is presented in this section for the efficiency and the accomplishment of the proposed CUSUM-RS chart based on a simulated data set, from the (Shafqat et al., 2021), (Aslam et al., 2020), and Abbas et al. (Riaz et al., 2021). For this purpose, we generate 30 observations, in-control state data, from a standard normal distribution with zero mean and one variance, i.e. $N{\sim}(0{,}1)$. Next 20 observations are generated from a normal distribution with a shift mean 1.0 and one variance, i.e. $N{\sim}(1.0{,}1)$.

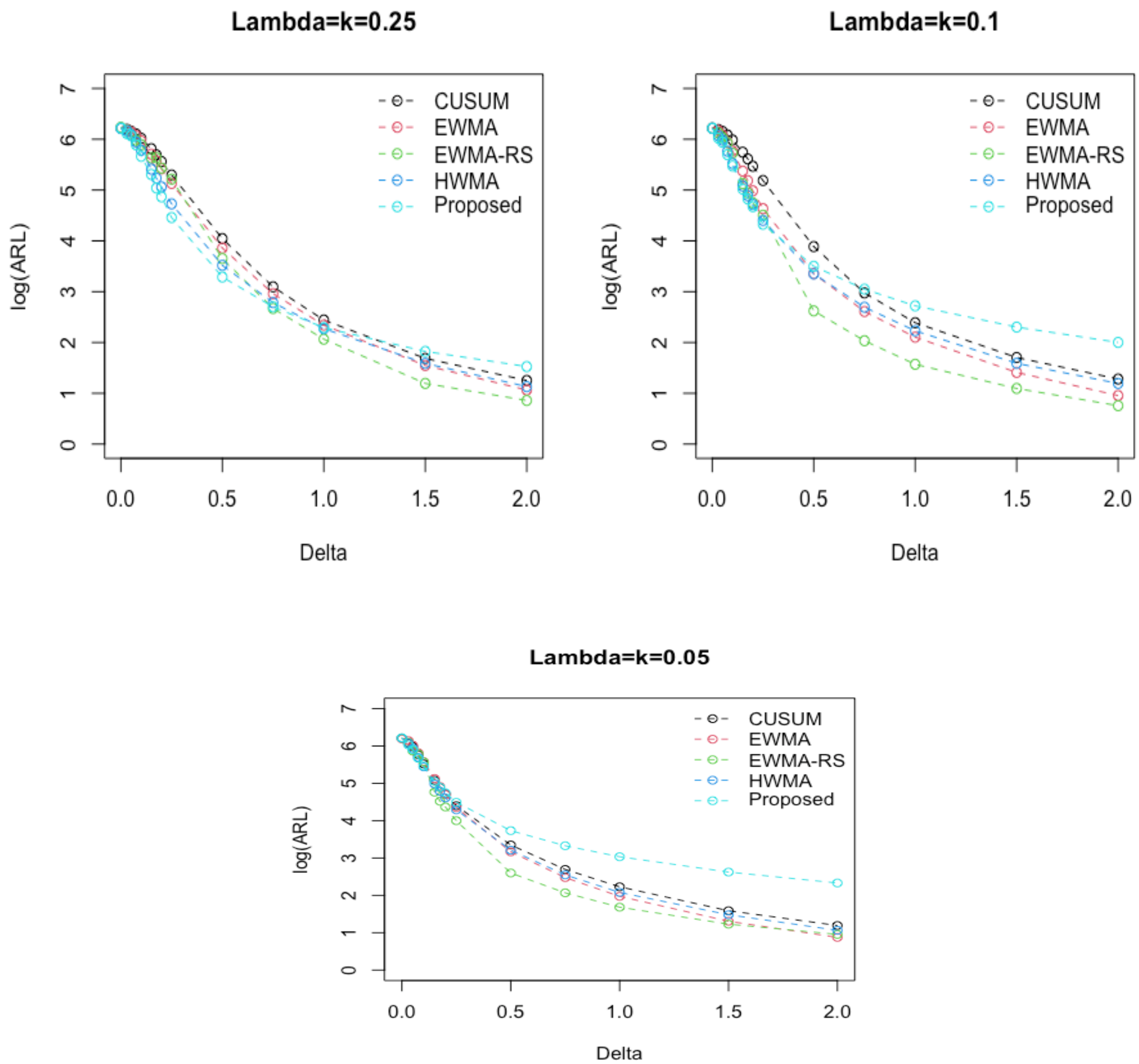

Figure 4: Comparison of Proposed CUSUM-RS Chart with Existing Charts

The parameters assumed for the proposed CUSUM-RS are K=0.25, H_1=8.623, and H_2=6.862. Similarly, the parameters of other existing charts are set as follows: i.e. classical CUSUM with H=13.15029 and K=0.25, classical EWMA with λ=0.25 and k=3.983, and HWMA with λ=0.25 and k=3.465, EWMA-RS with λ=0.25 and k1=3.011, k2=2.015. These coefficients and decision interval values for all charts are calculated such that the in-control ARL becomes 500. The value of the control limit of the proposed chart is given in Table 1.

Figure 5 shows the control charts applying the above simulated data set. The proposed CUSUM-RS chart detects the shift at the 40^{th} sample or 10^{th} sample when the process is actually shifted, whereas the classical CUSUM chart detects the shift at 49^{th} sample. It can be confirmed from Figure 5 (A and B). From Figure 5 (C and D), we see that the EWMA and HWMA charts don't detect the shift at any sample but EWMA-RS (Figure 5(E)) detect the shift at

42th sample or 12th when the process is shifted. This evidence shows that the proposed CUSUM-RS detects a shift earlier compared to other existing charts.

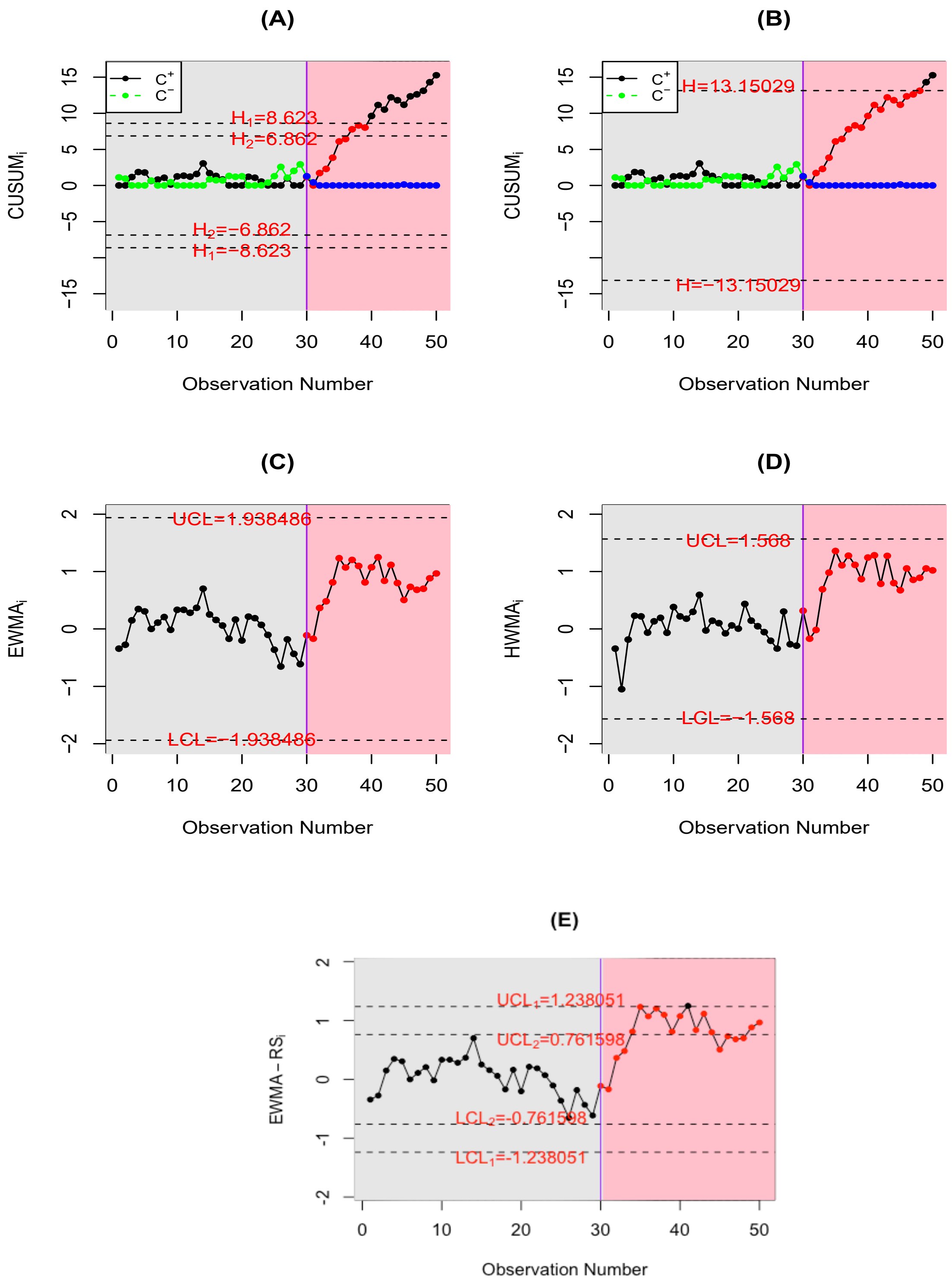

Figure 5: The Proposed and Existing Control Charts using simulated data

6. Practical Implementation Challenges and Industrial Advantages

The implementation of the proposed CUSUM control chart using repetitive sampling may face several practical challenges related to inspection procedures, decision-making, and operational complexity. When the chart statistics falls within the indecision region, the sampling procedure continues until a clear conclusion about the state of the process is reached. In such situations, practitioners may define a maximum number of repetitions to avoid excessive inspection and operational burden. In addition, automated and computerized monitoring systems are recommended for managing the repetitive sampling loop efficiently, as these systems can reduce computational error and support real-time process monitoring. Another practical concern is the possible increase in inspection cost and production delay caused by repeated sampling. However, the average sample number of the proposed chart is close to the sample size of one, indicating that, in most cases, a decision can be made based on the first sample, similar to the CUSUM chart using single sampling. Therefore, the repetition of the sampling process is expected to occur only rarely. On the other hand, the proposed repetitive sampling control chart is particularly useful in applications where high-quality monitoring and early detection of process shifts are essential. An additional advantage is the reduction of false decisions because repeated sampling allows further investigation in situations of indecision before making a final conclusion. Furthermore, in modern smart industries and Industry 4.0 environments, the proposed repetitive sampling control chart can integrated with artificial intelligence and automated monitoring systems. Due to its flexibility, the chart can also be effectively applied in uncertain and indeterminate environments.

7. Conclusion

This study proposed a new CUSUM control chart using a repetitive sampling scheme. The performance of the proposed chart is evaluated under the normal environment and non-normal environments. The proposed chart performs generally well even under different distributions other than normal. Also, the proposed chart performs robustly with fixed values of sensitivity parameter k and decision parameters h_1 and h_2. Finally, the inclusion of the ASN confirms the practical feasibility of the proposed chart. With ASN values ranging between 1.0 and 1.3 across all evaluated shifts, the CUSUM-RS chart provides a high-performance monitoring solution with minimal additional sampling cost. This makes it an ideal candidate for industrial applications where early detection of small shifts is critical, but sampling resources are not unlimited.

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