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The Flexible Nadarajah–Haghighi Distribution: Properties, Inference, and Applications

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Abstract

This article explores a flexible extension of the Nadarajah–Haghighi (NH) model, referred to as the odd inverse Pareto Nadarajah–Haghighi (OIPNH) distribution. We derive the mathematical properties of the probability density function of the OIPNH distribution, which exhibits a variety of behavior shapes, including decreasing, increasing, J-shaped, reversed J-shaped, bathtub, upside-down bathtub, and decreasing-increasing-decreasing hazard rates. Additionally, the distribution can display right-skewed, symmetrical, and concave-down densities. The parameters of the OIPNH distribution are examined using eight classical estimation approaches. We present extensive simulation results to evaluate the performance of these methods for both small and large sample sizes. Furthermore, we analyze three real-life datasets from engineering, medicine and agricultural sciences, demonstrating the flexibility of the OIPNH distribution compared to existing NH distributions.

Key Words: Nadarajah–Haghighi Distribution, Maximum Likelihood, Parameter Estimation, Medicine Data, Simulations.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

A variety of continuous univariate distributions are available; however, applications in fields such as environmental science, finance, biomedical science, and engineering have demonstrated that classical distributions are not always suitable. This has created a demand for more advanced forms of these distributions, leading to significant progress in generalizing well-known models. These extended distributions have proven effective in addressing problems across engineering, finance, economics, and biomedical sciences.

The Nadarajah–Haghighi (NH) distribution, developed by Nadarajah and Haghighi (26), aims to expand and generalize the exponential distribution, enhancing its flexibility. This distribution can effectively replace other models such as the gamma, exponentiated exponential, Burr III, and Weibull distributions. The NH model provides increasing, constant, and decreasing failure rates. The NH distribution is limited in its ability to model data with non-monotonic failure rate patterns.

Several researchers have thoroughly explored various extensions of the NH distribution, including the power inverted

NH (PINH) distribution (2), the NH generalized power Weibull (NHGPW) distribution (27), the transmuted inverted NH (TINH) distribution (31), the inverted NH (INH) distribution (30), the exponentiated generalized NH (EGNH) distribution (34), the exponentiated NH (ENH) model (22), Topp–leone NH distribution (32), odd Lindley NH distribution (33), beta NH distribution (14), skewed NH distribution (12), Poisson NH distribution (6), truncated NH distribution (18), Weibull inverse NH distribution (1), and heavy-tailed NH (8), among others.

To enhance flexibility and broaden its applicability, this research introduces a new variant of the NH distribution, the odd inverse Pareto Nadarajah–Haghighi (OIPNH) distribution, specifically designed to model a wider range of non-monotonic HRs. The OIPNH distribution provides an improved and more flexible approach for modeling diverse lifetime data, making it a valuable tool for researchers and practitioners in survival analysis, and reliability theory. The OIPNH distribution is developed by integrating the NH distribution with the odd inverse Pareto (OIP-G) family, as proposed by (5). We outline its mathematical properties and specific models, emphasizing the key attributes and motivations behind the OIPNH distribution. The NH model accommodates increasing, constant, and decreasing failure rates. However, it is limited in its ability to model data with non-monotonic failure rate patterns. In contrast, the OIPNH distribution can accommodate both monotonic and non-monotonic failure rate patterns. Additionally, the OIPNH distribution can be considered a comprehensive model that encompasses several lifetime models as special cases, including the exponentiated NH model (22), the Marshall–Olkin NH model (29), the exponentiated exponential model (17), and the Marshall–Olkin exponential model (24), among others. Therefore, the OIPNH distribution retains all the key properties of these models while offering several advantages over existing NH models for the following reasons:

- The hazard rate function (HRF) of the OIPNH can represent a diverse array of significant shapes, such as increasing, bathtub, modified bathtub, decreasing, unimodal, reversed J-shaped, J-shaped, and decreasing-increasing-decreasing patterns.
- The OIPNH distribution can be considered a comprehensive model that encompasses several lifetime models as special cases.
- The OIPNH model produces density functions that can be right-skewed, symmetric, reversed J-shaped, or concave-down.
- It consistently outperforms other competing distributions, such as the heavy-tailed NH (HTNH) (8), PINH (2), logistic NH (LNH) (28), Sin NH (SNH) (7), and the standard NH (NH) (26).

Moreover, eight frequentist estimation methods are utilized to determine the parameters of the OIPNH, offering valuable insights for engineers, applied statisticians, and practitioners in choosing the most appropriate estimation technique for this model. These methods are evaluated through comprehensive simulation experiments, focusing on their accuracy using metrics like absolute bias (|BIAS|), mean squared errors (MSE), and mean relative errors (MRE). Additionally, the performance of these metrics is ranked both partially and overall to identify the most effective estimation method for the OIPNH parameters. The application of classical methods for parameter estimation in generalized distributions has been discussed in the literature by several researchers, including works by Hussein et al. (19) and Bandar et al. (10).

This paper is organized as follows: Section 2 and 3 introduces the OIPNH distribution and its properties. Section 4 presents different estimators of the OIPNH's unknown parameters. Comprehensive simulation studies are discussed in Section 5. Section 6 illustrates the relevance of OIPNH model for modeling real lifetime datasets. Section 7 summarizes the present study.

2. The OIPNH Distribution

We present the new OIPNH distribution, utilizing the NH distribution as a foundational model within the OIP-G family. The cumulative distribution function (CDF) of the NH distribution is expressed in the following form:

$$G(x;\tau,\eta) = 1 - \exp\left\{1 - (1+\tau x)^{\eta}\right\}, \qquad x > 0. \tag{1}$$

The corresponding probability density function (PDF) derived from (1) simplifies to:

$$g(x;\tau,\eta) = \eta\,\tau\,(1+\tau x)^{\eta-1}\exp\left\{1 - (1+\tau x)^{\eta}\right\}. \tag{2}$$

Note that the exponential distribution is obtained when $\eta = 1$. The HRF corresponding to (1)

$$h(x;\tau,\eta) = \eta\tau(1+\tau x)^{\eta-1}. \tag{3}$$

Consider the baseline CDF represented by $G(x;\psi)$ with a parameter vector ψ. The CDF of the OIP-G family, as defined by (5), is expressed as follows:

$$F(x;\alpha,\beta,\psi) = \frac{G(x;\psi)^{\alpha}}{[1-G(x;\psi)]^{\alpha}}\left[\beta + \frac{G(x;\psi)}{1-G(x;\psi)}\right]^{-\alpha}. \tag{4}$$

The density function of the OIP-G family has the form

$$f(x;\alpha,\beta,\psi) = \alpha\beta g(x;\psi)G(x;\psi)^{\alpha-1}\left[\beta+(1-\beta)\,G(x_i;\psi)\right]^{-\alpha-1}, \tag{5}$$

where α and β are two positive shape parameters, and $g(x;\psi)$ denotes the baseline density.
The CDF and PDF of the OIPNH distribution can be expressed as

$$F(x;\underline{\varphi}) = \left\{\frac{1-\exp\left\{1-(1+\tau x)^{\eta}\right\}}{\beta+(1-\beta)\left[1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right]}\right\}^{\alpha}, \qquad x>0 \tag{6}$$

and

$$\begin{aligned} f(x;\underline{\varphi}) &= \alpha\beta\tau\eta\,(1+\tau x)^{\eta-1}\exp\left\{1-(1+\tau x)^{\eta}\right\}\left\{1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right\}^{\alpha-1} \\ &\quad \left\{\beta+(1-\beta)\left[1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right]\right\}^{-\alpha-1}, \end{aligned} \tag{7}$$

where the quantities $\alpha > 0$, $\beta > 0$, $\tau > 0$ and $\eta > 0$ are the shape and the scale parameters and $\underline{\varphi} = (\alpha,\beta,\tau,\eta)^{\intercal}$.
Note that the OIPNH distribution includes several special sub-models as follows:

- When $\beta = 1$, the OIPNH distribution reduces to the exponentiated NH distribution (22).
- When $\alpha = 1$, the OIPNH model becomes the Marshall–Olkin NH model (29).
- The exponentiated exponential distribution (17) is obtained when $\eta = \beta = 1$.
- The Marshall–Olkin exponential model (24) is obtained when $\eta = \alpha = 1$.
- The NH distribution follows when $\alpha = \beta = 1$.
- The exponential distribution is obtained when $\alpha = \eta = \beta = 1$.

The HRF of the OIPNH distribution can be expressed as

$$h(x;\underline{\varphi}) = \frac{\alpha\beta\tau\eta\,(1+\tau x)^{\eta-1}\exp\left\{1-(1+\tau x)^{\eta}\right\}\left\{1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right\}^{\alpha-1}}{\left\{\beta+(1-\beta)\left[1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right]\right\}^{\alpha+1}\left[1-\left\{\frac{1-\exp\left\{1-(1+\tau x)^{\eta}\right\}}{\beta+(1-\beta)\left[1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right]}\right\}^{\alpha}\right]}. \tag{8}$$

Figure 1 showcases different shapes of the OIPNH PDF for various parameter values. The plots reveal that the OIPNH density can be right-skewed, symmetric, reversed J-shaped, or concave down.
Meanwhile, Figure 2 presents several shapes of the HRF for the OIPNH model at selected values of α, β, τ and η. These plots demonstrate that the OIPNH HRF can take on forms such as increasing, bathtub, modified bathtub, decreasing, unimodal, reversed J-shaped, J-shaped, and decreasing-increasing-decreasing hazard rates.

3. Properties of The OIPNH Distribution

In this section, we present and discuss various mathematical quantities associated with the OIPNH distribution in detail.

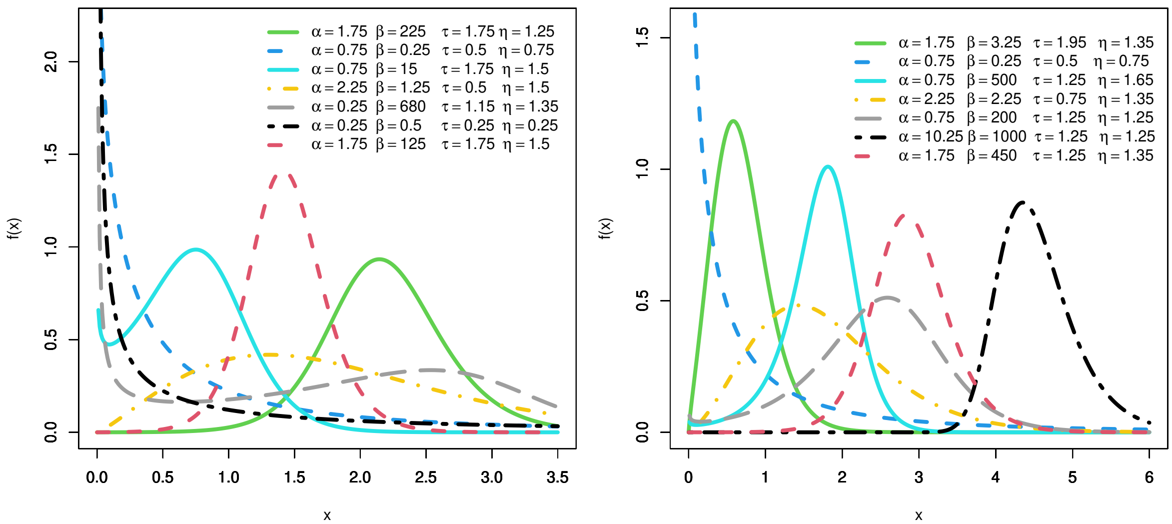

Figure 1: Plots of the OIPNH density function for various parameter values.

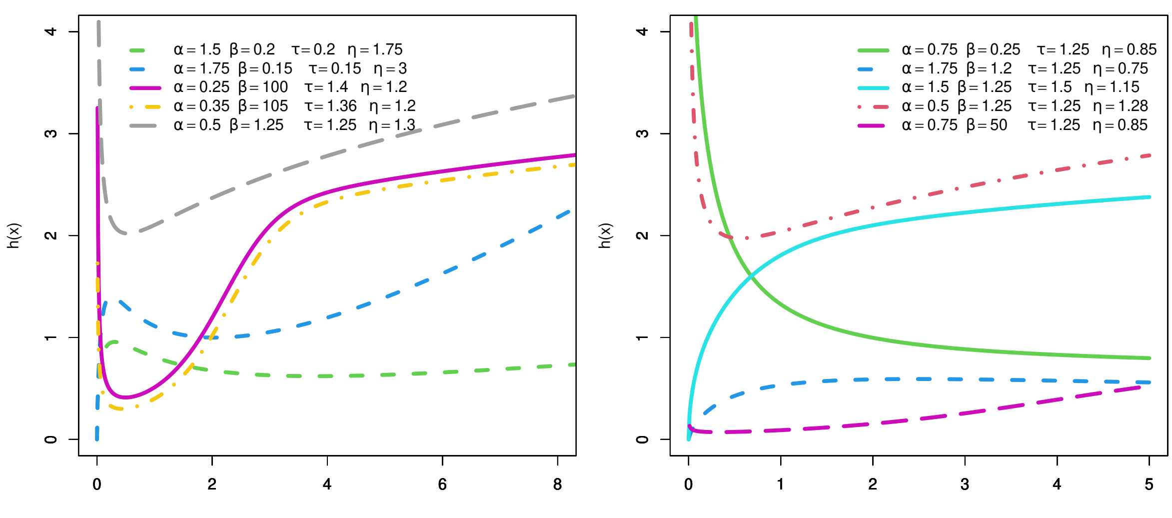

Figure 2: Plots of the OIPNH hazard function for various parameter values.

3.1. Linear representation

This section explores another form of the OIPNH density as a mixture of exponentiated-NH (ENH) densities. Using the following power series

$$(\xi+\delta D)^{-\tau}=\sum_{i=0}^{\infty}\binom{-\tau}{i}\xi^{-\tau-i}\delta^{i}D^{i}, \tag{9}$$

the expression $\left\{\beta+(1-\beta)\left[1-\exp\{1-(1+\tau x)\}^{\eta}\right]\right\}^{-(\alpha+1)}$, in Equation (7) can be written as

$$\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\beta^{-\alpha-i-1}(1-\beta)^{i}\left[1-\exp\{1-(1+\tau x)\}^{\eta}\right]^{i}. \tag{10}$$

By inserting Equation (10) into Equation (7), we get

$$\begin{aligned} f(x;\underline{\varphi}) &= \alpha\tau\eta\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\frac{(1-\beta)^{i}(\alpha+i)}{\beta^{\alpha+i}(\alpha+i)}(1+\tau x)^{\eta-1} \\ &\quad \exp\left\{1-(1+\tau x)^{\eta}\right\}\left\{1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right\}^{\alpha+i-1}. \end{aligned} \tag{11}$$

Then, the PDF of the OIPNH distribution can be expressed as

$$f(x;\underline{\varphi})=\sum_{i=0}^{\infty}L_{i}\,h_{\alpha+i}(x;\tau,\eta), \tag{12}$$

where $h_{\alpha+i}(x;\tau,\eta)$ is the PDF of the ENH distribution with power parameter $(\alpha+i)$ and

$$L_{i}=\frac{\alpha}{(\alpha+i)\beta^{\alpha+i}}\binom{-\alpha-1}{i}(1-\beta)^{i}.$$

3.2. Quantile Function

The quantile function (QF) of the OIPNH distribution can be obtained by solving the equation $F(Q(u))=u$ in (6) for $Q(u)$ in terms of u, and this implies

$$Q(u)=\tau^{-1}\left\{\left[1-\log\left\{1-\left(\frac{\beta u^{\frac{1}{\alpha}}}{1-(1-\beta)u^{\frac{1}{\alpha}}}\right)\right\}\right]^{\frac{1}{\eta}}-1\right\}. \tag{13}$$

The first quartile, median, and third quartile can be obtained by substituting $u=0.25,0.50$ and 0.75 into (13), respectively. Additionally, the QF (13) can be utilized to explore the relationships between the parameters α, β, τ, and η through Galton's skewness (GS) (Galton, (16)) and Moors' kurtosis (MK) (Moors, (25)), both of which rely on the QF. The expressions for GS and MK are given by

$$GS=\frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)}$$

and

$$MK=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}. \tag{14}$$

The plots of the GS and MK for the OIPNH model are displayed in Figure 3. These plots provides flexible shapes of the two measures of the OIPNH distribution.

3.3. Some Moments

In this section, we present the rth moment, mean (μ_X), variance (σ_X^2), skewness ($\xi_1(X)$), kurtosis ($\xi_2(X)$), and the moment generating function (MGF) for the OIPNH distribution, highlighting the significance of moments in statistical theory.

Theorem 1: Based on (12), the rth moment of the OIPNH distribution is derived as

$$\mu_r' = \frac{\alpha}{\tau^r}\sum_{i,j,k=0}^{\infty}\binom{-\alpha-1}{i}\binom{\alpha+i-1}{j}\binom{r}{k}\frac{(-1)^{j+k}(1-\beta)^i e^{j+1}}{(j+1)^{\frac{r-k}{\eta}+1}\beta^{\alpha+i}(\alpha+i)}\Gamma\left(\frac{r-k}{\eta}+1, j+1\right). \tag{15}$$

Setting $r=1$ in Equation (15), we get the mean of x, i.e., μ_X.

Proof: If X has the PDF in Equation (7), then we can write

$$\begin{aligned}\mu_r' &= \sum_{i=0}^{\infty} L_i \int_0^{\infty} x^r h_{\alpha+i}(x;\tau,\eta)\,dx \\ &= \alpha\tau\eta\sum_{i,j=0}^{\infty}\binom{-\alpha-1}{i}\binom{\alpha+i-1}{j}\frac{(-1)^j(1-\beta)^i e^{j+1}}{\beta^{\alpha+i}}\int_0^{\infty} x^r\,(1+\tau x)^{\eta-1}\,e^{-(j+1)(1+\tau x)^{\eta}}\,dx.\end{aligned}$$

Setting $m=(j+1)(1+\tau x)^{\eta}$, we get

$$\mu_r' = \frac{\alpha}{\tau^r}\sum_{i,j=0}^{\infty}\binom{-\alpha-1}{i}\binom{\alpha+i-1}{j}\frac{(1-\beta)^i(-1)^j e^{j+1}}{(j+1)(\alpha+i)\beta^{\alpha+i}}\int_{j+1}^{\infty}\left[\left(\frac{m}{j+1}\right)^{\frac{1}{\eta}}-1\right]^r e^{-m}\,dm. \tag{16}$$

The generalized binomial expansion can be expressed as follows:

$$(y+b)^{\rho} = \sum_{k=0}^{\infty}\binom{\rho}{k} y^k b^{\rho-k}, \tag{17}$$

where $\binom{\rho}{k}$ is a binomial coefficient and ρ is a real number. This power series converges when $\rho \geq 0$ is an integer or $|\frac{y}{b}| < 1$.

Applying (17) to Equation (16), since $|\left(\frac{m}{j+1}\right)^{\frac{1}{\eta}}| < 1$, it follows by interchanging the sum and the integral

$$\mu_r' = \frac{\alpha}{\tau^r}\sum_{i,j,k=0}^{\infty}\binom{-\alpha-1}{i}\binom{\alpha+i-1}{j}\binom{r}{k}\frac{(1-\beta)^i(-1)^{j+k}e^{j+1}}{(\alpha+i)\beta^{\alpha+i}(j+1)^{\frac{r-k}{\eta}+1}}\int_{j+1}^{\infty} m^{\frac{r-k}{\eta}+1-1}e^{-m}\,dm,$$

where $\int_{j+1}^{\infty} m^{\frac{r-k}{\eta}+1-1}e^{-m}\,dm = \Gamma\left(\frac{r-k}{\eta}+1, j+1\right)$ is the complementary IGF. This completes the proof.

The rth incomplete moment of the OIPNH distribution is given by

$$\begin{aligned}m_r(s) &= \sum_{i=0}^{\infty} L_i\int_0^{s} x^r h_{\alpha+i}(x;\tau,\eta)\,dx \\ &= \sum_{i=0}^{\infty} L_i\int_0^{\infty} x^r h_{\alpha+i}(x;\tau,\eta)\,dx - \sum_{i=0}^{\infty} L_i\int_s^{\infty} x^r h_{\alpha+i}(x;\tau,\eta)\,dx \\ &= \mu_r' - \sum_{i=0}^{\infty} L_i\int_s^{\infty} x^r h_{\alpha+i}(x;\tau,\eta)\,dx.\end{aligned}$$

Hence, the rth incomplete moment of the OIPNH distribution follows as

$$\begin{aligned} m_r(s) &= \frac{\alpha}{\tau^r}\sum_{i,j,k=0}^{\infty}\binom{-\alpha-1}{i}\binom{\alpha+i-1}{j}\binom{r}{k}\frac{(-1)^{j+k}(1-\beta)^i e^{j+1}}{(j+1)^{\frac{r-k}{\eta}+1}\beta^{\alpha+i}(\alpha+i)} \\ &\times\left\{\Gamma\left(\frac{r-k}{\eta}+1,j+1\right)-\Gamma\left(\frac{r-k}{\eta}+1,(j+1)(1+\tau s)^{\eta}\right)\right\}. \end{aligned}$$

The MGF has the form $M_X(t)=\sum_{r=0}^{\infty}E(X^r)\frac{t^r}{r!}$. Using Equation (15), the MGF of the OIPNH distribution follows as

$$M_X(t)=\alpha\sum_{i,j,k,r=0}^{\infty}\binom{-\alpha-1}{i}\binom{\alpha+i-1}{j}\binom{r}{k}\frac{(-1)^{j+k}t^r(1-\beta)^i e^{j+1}}{r!\tau^r(j+1)^{\frac{r-k}{\eta}+1}\beta^{\alpha+i}(\alpha+i)}\Gamma\left(\frac{r-k}{\eta}+1,j+1\right).$$

The μ_X, σ_X^2, $\xi_1(X)$, and $\xi_2(X)$ for the OIPNH distribution are determined numerically with different values of α, β, τ and η. Table 1 presents these numerical results. Table 1 shows that the OIPNH model can be right-skewed and it can be leptokurtic (i.e., $\xi_2(X)>3$).

3.4. Rényi and η Entropies

The Rényi entropy of a random variable X is defined by

$$I_{\Im}(X)=\frac{1}{1-\Im}\log\left((J_{\Im}(X)\right),\ \Im>0 \text{ and } \Im\neq 1,$$

where

$$J_{\Im}(X)=\int_{-\infty}^{\infty}f^{\Im}(x)dx.$$

Using Equation (7), we can write

$$\begin{aligned} f^{\Im}(x;\underline{\varphi}) &= \alpha^{\Im}\beta^{\Im}\tau^{\Im}\eta^{\Im}(1+\tau x)^{\Im(\eta-1)}\exp\left\{\Im\left(1-(1+\tau x)^{\eta}\right)\right\}\left\{1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right\}^{\Im(\alpha-1)} \\ & \left\{\beta+(1-\beta)\left[1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right]\right\}^{-\Im(\alpha+1)} \end{aligned}$$

After some simplifications, the Rényi entropy becomes

$$I_{\Im}(X)=\frac{1}{1-\Im}\log\left[\alpha^{\Im}\tau^{\Im-1}\eta^{\Im-1}\sum_{i,j=0}^{\infty}\Re_{i,j}\Gamma\left(\frac{\Im(\eta-1)}{\eta}+1,\Im+j\right)\right],$$

where

$$\Re_{i,j}=\binom{-\Im(\alpha+1)}{i}\binom{\Im(\alpha-1)+i}{j}\frac{(-1)^j(1-\beta)^i e^{\Im+j}}{(\Im+j)^{\frac{\Im(\eta-1)}{\eta}+1}\beta^{\Im\alpha+i}}.$$

The $\Im$–entropy, say $H_{\Im}(X)$, follows as

$$\begin{aligned} H_{\Im}(X) &= \frac{1}{\Im-1}\log\left[1-J_{\Im}(X)\right],\ \Im>0 \\ &= \frac{1}{\Im-1}\log\left(1-\sum_{i,j=0}^{\infty}\Re_{i,j}\right). \end{aligned}$$

3.5. Order Statistics

Let $X_1,X_2,\ldots X_n$ be a random sample from the distribution specified by (7) and $X_{(1)}\leq X_{(2)}\leq\cdots\leq X_{(n)}$ denote the corresponding order statistics. The PDF and the CDF of $X_{(i)}$ are given by

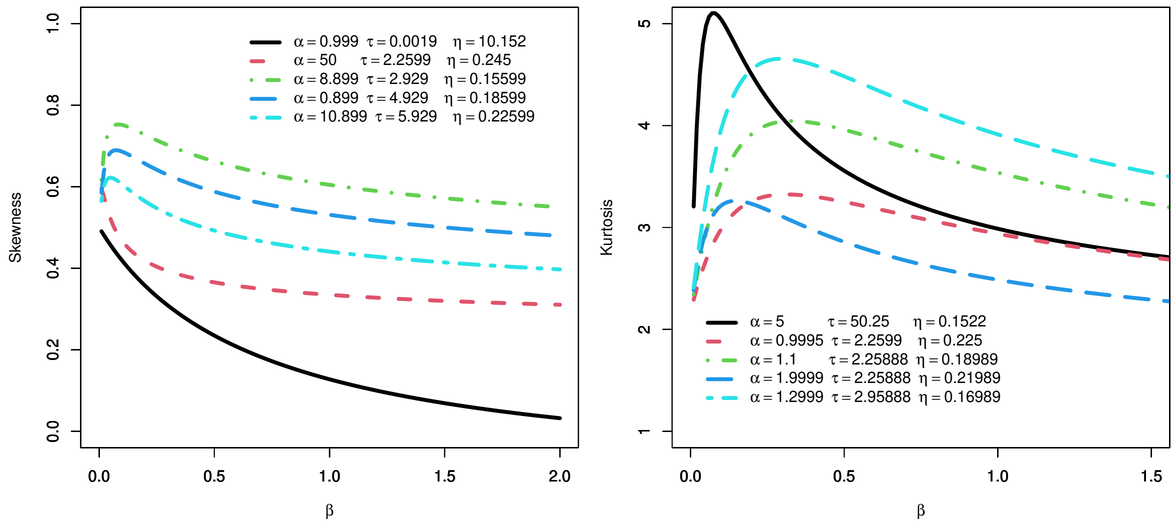

Figure 3: Plots of the GS and MK measures for the OIPNH distribution for various parameter values.

Table 1: Values of the four measures of the OIPNH distribution for varying parameters α, β, τ, and η.

$\underline{\varphi}^{\intercal}$	μ_X	σ_X^2	$\xi_1(X)$	$\xi_2(X)$
$(\alpha = 0.75,\ \beta = 0.25,\ \tau = 1.25,\ \eta = 1.75)$	0.1432609	0.03981548	2.785484	13.548330
$(\alpha = 0.5,\ \beta = 0.5,\ \tau = 1.25,\ \eta = 1.5)$	0.1901782	0.07582861	2.685973	12.752860
$(\alpha = 1.5,\ \beta = 0.75,\ \tau = 1.75,\ \eta = 1.25)$	0.4531931	0.14213340	1.611979	6.696471
$(\alpha = 2.5,\ \beta = 1.75,\ \tau = 2.75,\ \eta = 3.25)$	0.1436916	0.00354886	0.452451	3.147144
$(\alpha = 2.5,\ \beta = 20,\ \tau = 4.75,\ \eta = 5.25)$	0.0772132	0.00021547	-0.010271	-245.2570
$(\alpha = 25,\ \beta = 20,\ \tau = 2,\ \eta = 1.5)$	1.4563680	0.04561030	0.889897	4.472528
$(\alpha = 0.25,\ \beta = 200,\ \tau = 1.20,\ \eta = 5)$	0.2199766	0.02077510	-0.192495	1.768263
$(\alpha = 2.25,\ \beta = 20,\ \tau = 3.25,\ \eta = 6.25)$	0.0905951	0.00031670	-0.103961	-3.704227
$(\alpha = 0.40,\ \beta = 30,\ \tau = 4,\ \eta = 12)$	0.0222221	0.00017637	4.479942	-17.66641

$$
\begin{aligned}
f_{X_{(i)}}(x) &= \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x) \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{u=0}^{n-i}(-1)^u\binom{n-i}{u}[F(x)]^{i-1+u}f(x)
\end{aligned} \tag{18}
$$

and

$$
F_{X_{(i)}}(x) = \sum_{l=k}^{n}\binom{n}{l}[F(x)]^{l}[1-F(x)]^{n-l} = \sum_{l=k}^{n}\sum_{u=0}^{n-i}(-1)^u\binom{n}{l}\binom{n-i}{u}[F(x)]^{l+u}, \tag{19}
$$

for $k = 1,2,\ldots,n$.

Using Equations (18) and (19), the PDF and CDF of the ith order statistic of the OIPNH distribution are given as follows

$$
\begin{aligned}
f_{X_{(i)}}(x) =& \frac{n!\alpha\beta\tau\eta}{(i-1)!(n-i)!}\sum_{u=0}^{n-i}(-1)^u\binom{n-i}{u}(1+\tau x)^{\eta-1}\exp\left\{1-(1+\tau x)^{\eta}\right\} \\
& \left\{1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right\}^{\alpha(i+u)-1} \\
& \left\{\beta+(1-\beta)\left[1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right]\right\}^{\alpha(i-2+u)-1}
\end{aligned}
$$

and

$$
F_{x_{(i)}}(x) = \sum_{l=k}^{n}\sum_{u=0}^{n-i}(-1)^u\binom{n}{l}\binom{n-i}{u}\left\{\frac{1-\exp\left\{1-(1+\tau x)^{\eta}\right\}}{\beta+(1-\beta)\left[1-\exp\left\{1-(1+\tau x)^{\eta}\right\}\right]}\right\}^{\alpha(u+1)}.
$$

4. Estimation Approaches

In this section, the parameters of the OIPNH distribution are estimated using eight classical estimators: least-squares estimators (LSEs), maximum likelihood estimators (MLEs), weighted LSEs (WLSEs), maximum product of spacing estimators (MPSEs), percentile estimators (PCEs), Anderson–Darling estimators (ADEs), Cramér–von Mises estimators (CVRMEs), and right-tail ADEs (RADEs).

The LSEs of the OIPNH parameters minimize the function

$$
V(\underline{\varphi}) = \sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2.
$$

They can also be determined by solving the non-linear equations (for $s = 1,2,3,4$)

$$
\sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]\Delta_s(x_{(i)}) = 0,
$$

where

$$
\Delta_1(x_{(i)}) = \frac{\partial F(x_{(i)})}{\partial\alpha} = \left(\frac{Q_{(i)}}{W_{(i)}}\right)^{\alpha}\ln\left(\frac{Q_{(i)}}{W_{(i)}}\right),
$$

$$
\Delta_2(x_{(i)}) = \frac{\partial F(x_{(i)})}{\partial\beta} = -\frac{\alpha Q_{(i)}^{\alpha}\exp\left\{1-\left(1+\tau x_{(i)}\right)\right\}^{\eta}}{W_{(i)}^{\alpha+1}},
$$

$$\Delta_3(x_{(i)}) = \frac{\partial F(x_{(i)})}{\partial \tau} = \frac{\alpha\eta x_{(i)}\left(1+\tau x_{(i)}\right)^{\eta-1}\exp\left\{1-(1+\tau x)\right\}^{\eta} W_{(i)} Q_{(i)}^{\alpha-1}}{W_{(i)}^{2\alpha}} - \frac{\alpha\eta x_{(i)}(1-\beta)\left(1+\tau x_{(i)}\right)^{\eta-1} W_{(i)}^{\alpha-1}\exp\left\{1-(1+\tau x)\right\}^{\eta} Q_{(i)}^{\alpha}}{W_{(i)}^{2\alpha}}$$

and

$$\Delta_4(x_{(i)}) = \frac{\partial F(x_{(i)})}{\partial \eta} = \frac{\alpha W_{(i)}^{\alpha}\exp\left\{1-\left(1+\tau x_{(i)}\right)\right\}^{\eta}\left(1+\tau x_{(i)}\right)^{\eta}\ln\left(1+\tau x_{(i)}\right) Q_{(i)}^{\alpha-1}}{W_{(i)}^{2\alpha}} - \frac{\alpha(1-\beta)\left(1+\tau x_{(i)}\right)^{\eta}\ln\left(1+\tau x_{(i)}\right) Q_{(i)}^{\alpha} W_{(i)}^{\alpha-1}}{W_{(i)}^{2\alpha}},$$

where $Q_{(i)} = \left[1-\exp\left\{1-\left(1+\tau x_{(i)}\right)^{\eta}\right\}\right]$ and
$W_{(i)} = \left\{\beta+(1-\beta)\left[1-\exp\left\{1-\left(1+\tau x_{(i)}\right)^{\eta}\right\}\right]\right\}$.

The WLSEs of the OIPNH parameters minimize

$$W(\underline{\varphi}) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i)})-\frac{i}{n+1}\right]^2,$$

Additionally, the WLSEs can be determined by solving

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i)})-\frac{i}{n+1}\right]\Delta_s(x_{(i)}) = 0, \quad s=1,2,3,4.$$

The MLEs of the OIPNH parameters maximize the following log–likelihood

$$\begin{aligned}\ell &= n\log(\alpha)+n\log(\beta)+n\log(\tau)+n\log(\eta)+(\eta-1)\sum_{i=1}^{n}\log(1+\tau x_i) \\ &+\sum_{i=1}^{n}\left\{1-(1+\tau x_i)^{\eta}\right\}+(\alpha-1)\sum_{i=1}^{n}\log\left[1-\exp\left\{1-(1+\tau x_i)^{\eta}\right\}\right] \\ &-(\alpha+1)\sum_{i=1}^{n}\log\left\{\beta+(1-\beta)\left[1-\exp\left\{1-(1+\tau x_i)^{\eta}\right\}\right]\right\},\end{aligned} \tag{20}$$

where $R_i = \exp\left\{1-(1+\tau x_i)^{\eta}\right\}$. The likelihood equations are

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha}+\sum_{i=1}^{n}\log(1-R_i)-\sum_{i=1}^{n}\log\left\{\beta+(1-\beta)\left[1-R_i\right]\right\}=0,$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta}-(\alpha+1)\sum_{i=1}^{n}\frac{R_i}{\left\{\beta+(1-\beta)\left[1-R_i\right]\right\}}=0,$$

$$\begin{aligned}\frac{\partial \ell}{\partial \tau} &= (\eta-1)\sum_{i=1}^{n}\frac{x_i}{(1+\tau x_i)}-\eta\sum_{i=1}^{n}x_i(1+\tau x_i)^{\eta-1}+\eta(\alpha+1)\sum_{i=1}^{n}\frac{x_i(1+\tau x_i)^{\eta-1}R_i}{[1-R_i]}\\ &\frac{n}{\tau}-\eta(\alpha+1)\sum_{i=1}^{n}(1-\beta)\sum_{i=1}^{n}\frac{x_i(1+\tau x_i)^{\eta-1}R_i}{\{\beta+(1-\beta)[1-R_i]\}}=0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \eta} &= \frac{n}{\eta}\sum_{i=1}^{n}\log(1+\tau x_i)-\sum_{i=1}^{n}(1+\tau x_i)^{\eta}\ln(1+\tau x_i)+(\alpha-1)\sum_{i=1}^{n}\frac{R_i(1+\tau x_i)^{\eta}\ln(1+\tau x_i)}{[1-R_i]}\\ &-(\alpha+1)(1-\beta)\sum_{i=1}^{n}\frac{R_i(1+\tau x_i)^{\eta}\ln(1+\tau x_i)}{\{\beta+(1-\beta)[1-R_i]\}}=0.\end{aligned}$$

Let $D_i = D_i(\underline{\varphi}) = F(x_{(i)}) - F(x_{(i-1)})$ be the uniform spacings (for $i = 1,\ldots,n+1$), where $F(x_{(0)}) = 0$ and $F(x_{(n+1)}) = 1$. Furthermore, $\sum_{i=1}^{n+1} D_i = 1$.

The MPSEs of the OIPNH parameters maximize the following quantity

$$H\left(\underline{\varphi}\right)=\sum_{i=1}^{n+1}\log[D_i].$$

These estimators can also be determined from the non–linear equations (for $s = 1,2,3,4$)

$$\sum_{i=1}^{n+1}\frac{1}{D_i}\left[\Delta_s(x_{(i)})-\Delta_s(x_{(i-1)})\right]=0.$$

The CRVMEs of the OIPNH parameters minimize

$$C(\underline{\varphi})=-\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{(i)})-\frac{2i-1}{2n}\right]^2,$$

which also follow by solving (for $s = 1,2,3,4$)

$$\sum_{i=1}^{n}\left[F(x_{(i)})-\frac{2i-1}{2n}\right]\Delta_s(x_{(i)})=0.$$

The PCEs are originally suggested by (20, 21). Suppose that $u_i = i/(n+1)$ is an unbiased estimator of $F\left(x_{(i)}|\underline{\varphi}\right)$. Then, the PCEs of the OIPNH parameters minimize the following quantity

$$P(\underline{\varphi})=\sum_{i=1}^{n}\left[x_{(i)}+\tau^{-1}\left\{\left[1-\log\left\{1-\left(\frac{\beta\,u_i^{\frac{1}{\alpha}}}{1-(1-\beta)\,u_i^{\frac{1}{\alpha}}}\right)\right\}\right]^{\frac{1}{\eta}}-1\right\}\right]^2,$$

with respect to its parameters.

The ADEs of the OIPNH parameters minimize

$$A(\underline{\varphi})=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{(i)})+\log S(x_{(i)})\right],$$

which can be found as solutions of the following system (for $s = 1,2,3,4$)

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s(x_{(i)})}{F(x_{(i)})}-\frac{\Delta_i(x_{(n+1-i)})}{S(x_{(n+1-i)})}\right]=0.$$

The RADEs of the OIPNH parameters minimize

$$R(\underline{\varphi}) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{(n+1-i)}).$$

The RADEs are also be obtained from the non–linear equations (for $s = 1,2,3,4$)

$$-2\sum_{i=1}^{n}\Delta_s(x_{(i)}) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_s(x_{(n+1-i)})}{S(x_{(n+1-i)})} = 0.$$

5. Simulation Analysis

This section introduces a simulation study that evaluates estimates from eight different estimation methods by comparing three key metrics: (|BIAS|), MSE, and MRE. The formulas for these quantities are as follows:

$$|\mathrm{Bias}(\widehat{\varphi})| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\varphi} - \varphi|,$$

$$\mathrm{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\varphi} - \varphi)^2$$

and

$$\mathrm{MRE} = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\varphi} - \varphi|/\varphi.$$

Observations from the OIPNH distribution are simulated using Equation 13. We generate $N = 5,000$ random samples $x_1,\ldots,x_n$ for $n =$ 50, 150, 350, and 500, with parameters $\alpha = \{0.50, 0.55, 0.75, 1.50\}$, $\beta = \{0.50, 0.75, 1.25\}$, $\tau = \{0.15, 0.50, 0.75\}$ and $\eta = \{1.50, 2.25, 3.50\}$. The simulations are conducted using R software (R Core Team, 2023, version 4.3.10).

We estimate the OIPNH parameters for various combinations and sample sizes, calculating |BIAS|, MSE, and MRE for each estimate. The results for the eight methods are summarized in Tables 2–9, with superscripts indicating the ranks of the estimates, and $\sum Ranks$ representing the cumulative ranks. Notably, all estimation methods demonstrate consistency, with both MSE and MRE decreasing as the sample size n increases across all parameter combinations.

Table 10 provides a summary of the partial and overall ranks of the eight estimation methods. The estimation method with the lowest overall score is considered the most effective approach. According to Table 10, the eight methods can be ranked from best to worst as follows: WLS, CRVM, LS, MPS, AD, RAD, PC, and ML. Notably, the results from the comprehensive simulation study show that the WLS method, with the lowest overall rank of 35, is the most efficient estimation technique. This lower rank reflects the MPS method's consistent performance, yielding superior results in terms of MSE, BIAS, and MRE across various sample sizes and parameter values. Therefore, with an overall score of 35, the WLS method outperforms all other methods. Our findings confirm the WLS method's superiority for estimating the OIPNH parameters.

6. Data Applications

In this section, we showcase three applications to real-life datasets to highlight the significance and applicability of the OIPNH distribution. The first dataset comprises the remission times (in months) of 128 bladder cancer patients (23). This dataset was analyzed by (4). The observations are as follows: 4.50, 3.25, 0.08, 4.87, 2.09, 3.48, 8.66, 6.94, 13.11, 0.20, 23.63, 2.23, 4.98, 3.52, 6.97, 13.29, 9.02, 0.40, 3.57, 2.26, 6.93, 5.06, 9.22, 7.09, 13.80, 0.50, 25.74, 2.46, 5.09, 3.64, 7.26, 14.24, 9.47, 25.82, 2.54, 0.51, 3.70, 7.28, 5.17, 9.74, 26.31, 14.76, 0.81, 3.82, 2.62, 5.32, 10.06, 7.32, 14.77, 2.64, 32.15, 3.88, 7.39, 5.32, 10.34, 34.26, 14.83, 0.90, 4.18, 2.69, 5.34, 10.66, 7.59, 15.96, 1.05, 36.66, 2.69, 22.69, 5.41, 4.23, 7.62, 16.62, 10.75, 43.01, 2.75, 1.19, 4.26, 7.63, 5.41, 17.12, 1.26, 46.12, 2.83, 5.49, 4.33, 7.66, 17.14, 11.25, 79.05, 2.87, 1.35, 5.62, 11.64, 7.87, 17.36, 3.02, 1.40, 5.71, 4.34, 7.93, 18.10, 11.79, 1.46, 5.85, 4.40, 8.26, 3.36, 19.13, 12.63, 11.98, 1.76, 6.25, 12.02, 8.37, 2.02, 4.51, 3.31, 6.54, 12.03, 8.53, 20.28, 3.36, 2.02, 6.76, 21.73, 12.07, 8.65, 2.07.

The second dataset presents the tensile strength, measured in gigapascals (GPa), of 69 carbon fibers tested under ten-

Table 2: Simulation results for $\underline{\varphi} = (\alpha = 0.55, \beta = 0.5, \tau = 0.15, \eta = 2.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.11587 $^{\{3\}}$	0.13362 $^{\{5\}}$	0.11046 $^{\{2\}}$	0.16723 $^{\{7\}}$	0.11013 $^{\{1\}}$	0.80629 $^{\{8\}}$	0.11852 $^{\{4\}}$	0.15862 $^{\{6\}}$
		$\hat{\beta}$	1.66455 $^{\{8\}}$	1.24822 $^{\{5\}}$	0.64902 $^{\{2\}}$	1.28536 $^{\{6\}}$	0.21760 $^{\{1\}}$	0.72604 $^{\{3\}}$	1.29881 $^{\{7\}}$	1.07314 $^{\{4\}}$
		$\hat{\tau}$	0.92758 $^{\{7\}}$	0.26402 $^{\{3\}}$	0.20550 $^{\{1\}}$	0.27212 $^{\{4\}}$	1.28416 $^{\{8\}}$	0.22813 $^{\{2\}}$	0.45554 $^{\{6\}}$	0.37662 $^{\{5\}}$
		$\hat{\eta}$	6.58702 $^{\{8\}}$	0.60893 $^{\{2\}}$	1.00023 $^{\{4\}}$	0.67353 $^{\{3\}}$	0.41593 $^{\{1\}}$	1.66696 $^{\{7\}}$	1.35688 $^{\{5\}}$	1.35983 $^{\{6\}}$
	MSE	$\hat{\alpha}$	0.02715 $^{\{3\}}$	0.03884 $^{\{5\}}$	0.02626 $^{\{2\}}$	0.56570 $^{\{7\}}$	0.02186 $^{\{1\}}$	12.60524 $^{\{8\}}$	0.03018 $^{\{4\}}$	0.11183 $^{\{6\}}$
		$\hat{\beta}$	77.54419 $^{\{7\}}$	19.08974 $^{\{4\}}$	5.72978 $^{\{2\}}$	21.16835 $^{\{5\}}$	243.97928 $^{\{8\}}$	2.71323 $^{\{1\}}$	22.20616 $^{\{6\}}$	9.45702 $^{\{3\}}$
		$\hat{\tau}$	16.08374 $^{\{7\}}$	0.51882 $^{\{3\}}$	0.31769 $^{\{2\}}$	0.61557 $^{\{4\}}$	54.22418 $^{\{8\}}$	0.29984 $^{\{1\}}$	1.23540 $^{\{6\}}$	0.77852 $^{\{5\}}$
		$\hat{\eta}$	68.29488 $^{\{8\}}$	0.75604 $^{\{1\}}$	1.78832 $^{\{3\}}$	0.91071 $^{\{2\}}$	2.73001 $^{\{6\}}$	6.79688 $^{\{7\}}$	2.56895 $^{\{4\}}$	2.59366 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.21067 $^{\{3\}}$	0.24294 $^{\{5\}}$	0.20084 $^{\{2\}}$	0.30405 $^{\{7\}}$	0.20024 $^{\{1\}}$	1.46598 $^{\{8\}}$	0.21550 $^{\{4\}}$	0.28840 $^{\{6\}}$
		$\hat{\beta}$	3.32910 $^{\{7\}}$	2.49643 $^{\{4\}}$	1.29803 $^{\{1\}}$	2.57071 $^{\{5\}}$	5.96851 $^{\{8\}}$	1.45208 $^{\{2\}}$	2.59762 $^{\{6\}}$	2.14628 $^{\{3\}}$
		$\hat{\tau}$	6.18386 $^{\{7\}}$	1.76012 $^{\{3\}}$	1.37001 $^{\{1\}}$	1.81414 $^{\{4\}}$	8.56105 $^{\{8\}}$	1.52089 $^{\{2\}}$	3.03693 $^{\{6\}}$	2.51081 $^{\{5\}}$
		$\hat{\eta}$	2.92756 $^{\{8\}}$	0.27064 $^{\{2\}}$	0.44455 $^{\{4\}}$	0.29935 $^{\{3\}}$	0.18486 $^{\{1\}}$	0.74087 $^{\{7\}}$	0.60306 $^{\{5\}}$	0.60437 $^{\{6\}}$
		$\sum Ranks$	76 $^{\{8\}}$	42 $^{\{2\}}$	26 $^{\{1\}}$	57 $^{\{5\}}$	52 $^{\{3\}}$	56 $^{\{4\}}$	63 $^{\{7\}}$	60 $^{\{6\}}$
150	|BIAS|	$\hat{\alpha}$	0.06046 $^{\{2\}}$	0.07254 $^{\{5\}}$	0.05744 $^{\{1\}}$	0.07322 $^{\{6\}}$	0.06101 $^{\{3\}}$	0.32455 $^{\{8\}}$	0.06256 $^{\{4\}}$	0.07733 $^{\{7\}}$
		$\hat{\beta}$	0.52377 $^{\{7\}}$	0.40655 $^{\{6\}}$	0.22997 $^{\{1\}}$	0.38848 $^{\{4\}}$	0.70821 $^{\{8\}}$	0.36590 $^{\{2\}}$	0.40094 $^{\{5\}}$	0.38163 $^{\{3\}}$
		$\hat{\tau}$	0.37371 $^{\{7\}}$	0.15098 $^{\{3\}}$	0.10046 $^{\{1\}}$	0.14597 $^{\{2\}}$	0.41339 $^{\{8\}}$	0.19723 $^{\{4\}}$	0.22595 $^{\{6\}}$	0.20154 $^{\{5\}}$
		$\hat{\eta}$	3.75150 $^{\{8\}}$	0.78770 $^{\{3\}}$	0.65883 $^{\{2\}}$	0.83432 $^{\{4\}}$	0.15860 $^{\{1\}}$	1.15941 $^{\{5\}}$	1.30261 $^{\{7\}}$	1.28309 $^{\{6\}}$
	MSE	$\hat{\alpha}$	0.00624 $^{\{3\}}$	0.00877 $^{\{5\}}$	0.00543 $^{\{1\}}$	0.00930 $^{\{6\}}$	0.00576 $^{\{2\}}$	2.59368 $^{\{8\}}$	0.00643 $^{\{4\}}$	0.01038 $^{\{7\}}$
		$\hat{\beta}$	2.56872 $^{\{7\}}$	0.59606 $^{\{5\}}$	0.12899 $^{\{1\}}$	0.65028 $^{\{6\}}$	6.62902 $^{\{8\}}$	0.40312 $^{\{2\}}$	0.49373 $^{\{4\}}$	0.45214 $^{\{3\}}$
		$\hat{\tau}$	1.38383 $^{\{7\}}$	0.05774 $^{\{3\}}$	0.02978 $^{\{1\}}$	0.05750 $^{\{2\}}$	3.16883 $^{\{8\}}$	0.14206 $^{\{6\}}$	0.12953 $^{\{5\}}$	0.09850 $^{\{4\}}$
		$\hat{\eta}$	26.43247 $^{\{8\}}$	0.92619 $^{\{3\}}$	0.91723 $^{\{2\}}$	1.03472 $^{\{4\}}$	0.40880 $^{\{1\}}$	2.33111 $^{\{7\}}$	2.25474 $^{\{6\}}$	2.16860 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.10993 $^{\{2\}}$	0.13190 $^{\{5\}}$	0.10443 $^{\{1\}}$	0.13313 $^{\{6\}}$	0.11092 $^{\{3\}}$	0.59009 $^{\{8\}}$	0.11374 $^{\{4\}}$	0.14059 $^{\{7\}}$
		$\hat{\beta}$	1.04754 $^{\{7\}}$	0.81311 $^{\{6\}}$	0.45994 $^{\{1\}}$	0.77697 $^{\{4\}}$	1.41642 $^{\{8\}}$	0.73180 $^{\{2\}}$	0.80188 $^{\{5\}}$	0.76327 $^{\{3\}}$
		$\hat{\tau}$	2.49140 $^{\{7\}}$	1.00656 $^{\{3\}}$	0.66971 $^{\{1\}}$	0.97314 $^{\{2\}}$	2.75596 $^{\{8\}}$	1.31484 $^{\{4\}}$	1.50636 $^{\{6\}}$	1.34359 $^{\{5\}}$
		$\hat{\eta}$	1.66733 $^{\{8\}}$	0.35009 $^{\{3\}}$	0.29281 $^{\{2\}}$	0.37081 $^{\{4\}}$	0.07049 $^{\{1\}}$	0.51530 $^{\{5\}}$	0.57894 $^{\{7\}}$	0.57026 $^{\{6\}}$
		$\sum Ranks$	73 $^{\{8\}}$	50 $^{\{2.5\}}$	15 $^{\{1\}}$	50 $^{\{2.5\}}$	59 $^{\{4\}}$	61 $^{\{5.5\}}$	63 $^{\{7\}}$	61 $^{\{5.5\}}$
350	|BIAS|	$\hat{\alpha}$	0.03960 $^{\{3\}}$	0.04712 $^{\{6\}}$	0.03691 $^{\{1\}}$	0.04693 $^{\{5\}}$	0.03851 $^{\{2\}}$	0.11028 $^{\{8\}}$	0.04030 $^{\{4\}}$	0.04909 $^{\{7\}}$
		$\hat{\beta}$	0.26358 $^{\{7\}}$	0.23759 $^{\{5\}}$	0.13245 $^{\{1\}}$	0.22934 $^{\{3\}}$	0.21760 $^{\{2\}}$	0.28855 $^{\{8\}}$	0.24031 $^{\{6\}}$	0.23158 $^{\{4\}}$
		$\hat{\tau}$	0.19064 $^{\{8\}}$	0.11833 $^{\{4\}}$	0.05736 $^{\{1\}}$	0.11456 $^{\{3\}}$	0.10656 $^{\{2\}}$	0.16502 $^{\{7\}}$	0.15844 $^{\{6\}}$	0.14585 $^{\{5\}}$
		$\hat{\eta}$	2.39247 $^{\{8\}}$	0.83226 $^{\{3\}}$	0.42095 $^{\{2\}}$	0.85184 $^{\{4\}}$	0.06316 $^{\{1\}}$	0.89075 $^{\{5\}}$	1.14515 $^{\{7\}}$	1.11572 $^{\{6\}}$
	MSE	$\hat{\alpha}$	0.00315 $^{\{4\}}$	0.00366 $^{\{6\}}$	0.00220 $^{\{1\}}$	0.00365 $^{\{5\}}$	0.00233 $^{\{2\}}$	0.19548 $^{\{8\}}$	0.00263 $^{\{3\}}$	0.00397 $^{\{7\}}$
		$\hat{\beta}$	0.26532 $^{\{7\}}$	0.12502 $^{\{5\}}$	0.03544 $^{\{1\}}$	0.11503 $^{\{2\}}$	0.61268 $^{\{8\}}$	0.24393 $^{\{6\}}$	0.12465 $^{\{4\}}$	0.11593 $^{\{3\}}$
		$\hat{\tau}$	0.17302 $^{\{7\}}$	0.02605 $^{\{3\}}$	0.00969 $^{\{1\}}$	0.02490 $^{\{2\}}$	0.40163 $^{\{8\}}$	0.08971 $^{\{6\}}$	0.05073 $^{\{5\}}$	0.04132 $^{\{4\}}$
		$\hat{\eta}$	12.75264 $^{\{8\}}$	0.90911 $^{\{3\}}$	0.45569 $^{\{2\}}$	0.96112 $^{\{4\}}$	0.07798 $^{\{1\}}$	1.46372 $^{\{5\}}$	1.75234 $^{\{7\}}$	1.68125 $^{\{6\}}$
	MRE	$\hat{\alpha}$	0.07201 $^{\{3\}}$	0.08567 $^{\{6\}}$	0.06711 $^{\{1\}}$	0.08533 $^{\{5\}}$	0.07001 $^{\{2\}}$	0.20051 $^{\{8\}}$	0.07327 $^{\{4\}}$	0.08925 $^{\{7\}}$
		$\hat{\beta}$	0.52716 $^{\{7\}}$	0.47518 $^{\{5\}}$	0.26489 $^{\{1\}}$	0.45867 $^{\{3\}}$	0.43520 $^{\{2\}}$	0.57711 $^{\{8\}}$	0.48061 $^{\{6\}}$	0.46315 $^{\{4\}}$
		$\hat{\tau}$	1.27093 $^{\{8\}}$	0.78889 $^{\{4\}}$	0.38239 $^{\{1\}}$	0.76376 $^{\{3\}}$	0.71040 $^{\{2\}}$	1.10012 $^{\{7\}}$	1.05625 $^{\{6\}}$	0.97231 $^{\{5\}}$
		$\hat{\eta}$	1.06332 $^{\{8\}}$	0.36989 $^{\{3\}}$	0.18709 $^{\{2\}}$	0.37859 $^{\{4\}}$	0.02807 $^{\{1\}}$	0.39589 $^{\{5\}}$	0.50896 $^{\{7\}}$	0.49587 $^{\{6\}}$
		$\sum Ranks$	78 $^{\{7\}}$	53 $^{\{4\}}$	15 $^{\{1\}}$	43 $^{\{3\}}$	33 $^{\{2\}}$	81 $^{\{8\}}$	65 $^{\{6\}}$	64 $^{\{5\}}$
500	|BIAS|	$\hat{\alpha}$	0.03344 $^{\{4\}}$	0.03875 $^{\{5\}}$	0.03023 $^{\{1\}}$	0.03883 $^{\{6\}}$	0.03194 $^{\{2\}}$	0.07290 $^{\{8\}}$	0.03320 $^{\{3\}}$	0.04026 $^{\{7\}}$
		$\hat{\beta}$	0.21584 $^{\{7\}}$	0.20132 $^{\{5\}}$	0.10870 $^{\{1\}}$	0.19253 $^{\{3\}}$	0.14274 $^{\{2\}}$	0.24805 $^{\{8\}}$	0.20174 $^{\{6\}}$	0.19387 $^{\{4\}}$
		$\hat{\tau}$	0.15871 $^{\{8\}}$	0.10777 $^{\{4\}}$	0.04668 $^{\{1\}}$	0.10283 $^{\{3\}}$	0.05706 $^{\{2\}}$	0.14103 $^{\{7\}}$	0.13839 $^{\{6\}}$	0.12790 $^{\{5\}}$
		$\hat{\eta}$	1.94552 $^{\{8\}}$	0.79971 $^{\{4\}}$	0.34748 $^{\{2\}}$	0.80989 $^{\{5\}}$	0.04117 $^{\{1\}}$	0.79955 $^{\{3\}}$	1.06283 $^{\{7\}}$	1.04658 $^{\{6\}}$
	MSE	$\hat{\alpha}$	0.00258 $^{\{6\}}$	0.00240 $^{\{4\}}$	0.00146 $^{\{1\}}$	0.00243 $^{\{5\}}$	0.00159 $^{\{2\}}$	0.02073 $^{\{8\}}$	0.00177 $^{\{3\}}$	0.00267 $^{\{7\}}$
		$\hat{\beta}$	0.15988 $^{\{6\}}$	0.08371 $^{\{4\}}$	0.02229 $^{\{1\}}$	0.07306 $^{\{2\}}$	0.23907 $^{\{8\}}$	0.18572 $^{\{7\}}$	0.08375 $^{\{5\}}$	0.07471 $^{\{3\}}$
		$\hat{\tau}$	0.11433 $^{\{7\}}$	0.02051 $^{\{3\}}$	0.00669 $^{\{1\}}$	0.01846 $^{\{2\}}$	0.16261 $^{\{8\}}$	0.06674 $^{\{6\}}$	0.03706 $^{\{5\}}$	0.03041 $^{\{4\}}$
		$\hat{\eta}$	8.88087 $^{\{8\}}$	0.82386 $^{\{3\}}$	0.34630 $^{\{2\}}$	0.86097 $^{\{4\}}$	0.04529 $^{\{1\}}$	1.25517 $^{\{5\}}$	1.51753 $^{\{7\}}$	1.49205 $^{\{6\}}$
	MRE	$\hat{\alpha}$	0.06080 $^{\{4\}}$	0.07045 $^{\{5\}}$	0.05497 $^{\{1\}}$	0.07061 $^{\{6\}}$	0.05808 $^{\{2\}}$	0.13255 $^{\{8\}}$	0.06035 $^{\{3\}}$	0.07321 $^{\{7\}}$
		$\hat{\beta}$	0.43167 $^{\{7\}}$	0.40263 $^{\{5\}}$	0.21740 $^{\{1\}}$	0.38506 $^{\{3\}}$	0.28547 $^{\{2\}}$	0.49610 $^{\{8\}}$	0.40349 $^{\{6\}}$	0.38775 $^{\{4\}}$
		$\hat{\tau}$	1.05810 $^{\{8\}}$	0.71850 $^{\{4\}}$	0.31118 $^{\{1\}}$	0.68554 $^{\{3\}}$	0.38037 $^{\{2\}}$	0.94020 $^{\{7\}}$	0.92262 $^{\{6\}}$	0.85265 $^{\{5\}}$
		$\hat{\eta}$	0.86467 $^{\{8\}}$	0.35543 $^{\{4\}}$	0.15443 $^{\{2\}}$	0.35995 $^{\{5\}}$	0.01830 $^{\{1\}}$	0.35535 $^{\{3\}}$	0.47237 $^{\{7\}}$	0.46514 $^{\{6\}}$
		$\sum Ranks$	81 $^{\{8\}}$	50 $^{\{4\}}$	15 $^{\{1\}}$	47 $^{\{3\}}$	33 $^{\{2\}}$	78 $^{\{7\}}$	64 $^{\{5.5\}}$	64 $^{\{5.5\}}$

Table 3: Simulation results for $\underline{\varphi} = (\alpha = 0.75, \beta = 0.75, \tau = 0.15, \eta = 2.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.20161 {4}	0.28259 {5}	0.16745 {1}	0.29667 {6}	0.17832 {2}	1.38134 {8}	0.19050 {3}	0.35943 {7}
		$\hat{\beta}$	1.95666 {8}	1.45373 {5}	0.72277 {2}	1.35562 {4}	0.41060 {1}	0.86404 {3}	1.49829 {6}	1.58024 {7}
		$\hat{\tau}$	0.62324 {7}	0.18451 {3}	0.15659 {1}	0.17918 {2}	1.32538 {8}	0.29133 {4}	0.32718 {5}	0.33104 {6}
		$\hat{\eta}$	5.47557 {8}	0.83816 {1}	0.98077 {4}	0.90205 {3}	0.85830 {2}	1.50145 {7}	1.39956 {6}	1.33000 {5}
	MSE	$\hat{\alpha}$	0.46849 {4}	1.39680 {6}	0.08250 {2}	1.04649 {5}	0.06480 {1}	39.82618 {8}	0.08834 {3}	4.67522 {7}
		$\hat{\beta}$	71.48480 {7}	16.06459 {5}	3.23582 {2}	13.92752 {3}	249.60220 {8}	2.73403 {1}	19.90663 {6}	15.90368 {4}
		$\hat{\tau}$	4.89217 {7}	0.15799 {3}	0.09864 {1}	0.14290 {2}	19.49626 {8}	0.47612 {5}	0.40089 {4}	0.51683 {6}
		$\hat{\eta}$	48.30312 {8}	1.06502 {1}	1.77439 {3}	1.22993 {2}	4.49876 {7}	4.00018 {6}	2.74476 {5}	2.53621 {4}
	MRE	$\hat{\alpha}$	0.26881 {4}	0.37678 {5}	0.22326 {1}	0.39556 {6}	0.23775 {2}	1.84179 {8}	0.25401 {3}	0.47924 {7}
		$\hat{\beta}$	2.60888 {7}	1.93831 {4}	0.96369 {1}	1.80749 {3}	6.38934 {8}	1.15206 {2}	1.99771 {5}	2.10699 {6}
		$\hat{\tau}$	4.15494 {7}	1.23008 {3}	1.04394 {1}	1.19451 {2}	8.83587 {8}	1.94220 {4}	2.18119 {5}	2.20693 {6}
		$\hat{\eta}$	2.43359 {8}	0.37252 {1}	0.43590 {4}	0.40091 {3}	0.38147 {2}	0.66731 {7}	0.62203 {6}	0.59111 {5}
		$\sum Ranks$	79 {8}	42 {3}	23 {1}	41 {2}	57 {4.5}	63 {6}	57 {4.5}	70 {7}
150	|BIAS|	$\hat{\alpha}$	0.09566 {2}	0.11901 {5}	0.08299 {1}	0.12164 {6}	0.09585 {3}	0.27576 {8}	0.09880 {4}	0.12542 {7}
		$\hat{\beta}$	0.73562 {7}	0.50803 {3}	0.28822 {1}	0.49091 {2}	1.28233 {8}	0.51385 {4}	0.52381 {5}	0.54353 {6}
		$\hat{\tau}$	0.29926 {7}	0.12099 {3}	0.08034 {1}	0.11840 {2}	0.46629 {8}	0.19862 {6}	0.18473 {5}	0.17663 {4}
		$\hat{\eta}$	3.41200 {8}	0.86390 {3}	0.59018 {2}	0.90300 {4}	0.41120 {1}	1.02634 {5}	1.22456 {7}	1.21173 {6}
	MSE	$\hat{\alpha}$	0.01632 {3}	0.02510 {5}	0.01175 {1}	0.02698 {6}	0.01461 {2}	2.39402 {8}	0.01663 {4}	0.02810 {7}
		$\hat{\beta}$	3.33311 {7}	0.99253 {5}	0.22532 {1}	0.89959 {4}	13.62292 {8}	0.79450 {3}	0.73375 {2}	1.02722 {6}
		$\hat{\tau}$	0.53337 {7}	0.02922 {3}	0.02137 {1}	0.02814 {2}	1.92420 {8}	0.14155 {6}	0.07843 {5}	0.07480 {4}
		$\hat{\eta}$	21.55382 {8}	1.01376 {2}	0.83470 {1}	1.09752 {3}	1.35128 {4}	1.88736 {5}	2.10476 {7}	2.04650 {6}
	MRE	$\hat{\alpha}$	0.12754 {2}	0.15868 {5}	0.11065 {1}	0.16219 {6}	0.12780 {3}	0.36768 {8}	0.13173 {4}	0.16722 {7}
		$\hat{\beta}$	0.98083 {7}	0.67737 {3}	0.38429 {1}	0.65455 {2}	1.70977 {8}	0.68513 {4}	0.69841 {5}	0.72471 {6}
		$\hat{\tau}$	1.99508 {7}	0.80659 {3}	0.53562 {1}	0.78932 {2}	3.10860 {8}	1.32415 {6}	1.23152 {5}	1.17751 {4}
		$\hat{\eta}$	1.51644 {8}	0.38396 {3}	0.26230 {2}	0.40133 {4}	0.18275 {1}	0.45615 {5}	0.54425 {7}	0.53855 {6}
		$\sum Ranks$	73 {8}	43 {2.5}	14 {1}	43 {2.5}	62 {5}	68 {6}	60 {4}	69 {7}
350	|BIAS|	$\hat{\alpha}$	0.06039 {2}	0.07633 {5}	0.05173 {1}	0.07698 {6}	0.06103 {3}	0.11214 {8}	0.06402 {4}	0.07986 {7}
		$\hat{\beta}$	0.40281 {6}	0.31409 {3}	0.16284 {1}	0.30637 {2}	0.41060 {7}	0.41261 {8}	0.32199 {4}	0.32878 {5}
		$\hat{\tau}$	0.17550 {8}	0.10134 {3}	0.04372 {1}	0.09885 {2}	0.13638 {6}	0.15609 {7}	0.12929 {5}	0.12688 {4}
		$\hat{\eta}$	2.18438 {8}	0.82160 {3}	0.34366 {2}	0.83762 {4}	0.15394 {1}	0.83812 {5}	1.01087 {7}	1.00783 {6}
	MSE	$\hat{\alpha}$	0.00597 {2}	0.00976 {5}	0.00442 {1}	0.01006 {6}	0.00598 {3}	0.19321 {8}	0.00676 {4}	0.01058 {7}
		$\hat{\beta}$	0.65500 {7}	0.22095 {4}	0.06223 {1}	0.20362 {2}	1.45346 {8}	0.55462 {6}	0.21803 {3}	0.23098 {5}
		$\hat{\tau}$	0.13080 {7}	0.01692 {3}	0.00709 {1}	0.01602 {2}	0.27657 {8}	0.07775 {6}	0.03265 {5}	0.03023 {4}
		$\hat{\eta}$	9.88190 {8}	0.86574 {3}	0.37854 {2}	0.90319 {4}	0.21163 {1}	1.30152 {5}	1.45344 {7}	1.41328 {6}
	MRE	$\hat{\alpha}$	0.08052 {2}	0.10178 {5}	0.06897 {1}	0.10264 {6}	0.08138 {3}	0.14952 {8}	0.08536 {4}	0.10648 {7}
		$\hat{\beta}$	0.53708 {6}	0.41879 {3}	0.21712 {1}	0.40850 {2}	0.54747 {7}	0.55015 {8}	0.42932 {4}	0.43837 {5}
		$\hat{\tau}$	1.16998 {8}	0.67563 {3}	0.29147 {1}	0.65897 {2}	0.90920 {6}	1.04062 {7}	0.86193 {5}	0.84586 {4}
		$\hat{\eta}$	0.97084 {8}	0.36515 {3}	0.15274 {2}	0.37228 {4}	0.06842 {1}	0.37250 {5}	0.44927 {7}	0.44792 {6}
		$\sum Ranks$	72 {7}	43 {3}	15 {1}	42 {2}	54 {4}	81 {8}	59 {5}	66 {6}
500	|BIAS|	$\hat{\alpha}$	0.05117 {3}	0.06336 {5.5}	0.04343 {1}	0.06336 {5.5}	0.05084 {2}	0.08414 {8}	0.05321 {4}	0.06571 {7}
		$\hat{\beta}$	0.33181 {7}	0.26897 {4}	0.13409 {1}	0.26325 {3}	0.23635 {2}	0.35844 {8}	0.27591 {5}	0.27950 {6}
		$\hat{\tau}$	0.14146 {8}	0.09386 {4}	0.03427 {1}	0.09194 {3}	0.06061 {2}	0.13087 {7}	0.11287 {6}	0.11042 {5}
		$\hat{\eta}$	1.74866 {8}	0.79992 {4}	0.27500 {2}	0.81157 {5}	0.08884 {1}	0.75096 {3}	0.92641 {7}	0.92636 {6}
	MSE	$\hat{\alpha}$	0.00432 {3}	0.00651 {5}	0.00312 {1}	0.00657 {6}	0.00408 {2}	0.01879 {8}	0.00457 {4}	0.00716 {7}
		$\hat{\beta}$	0.34637 {6}	0.13626 {3}	0.04231 {1}	0.13049 {2}	0.48301 {8}	0.40869 {7}	0.15409 {4}	0.15791 {5}
		$\hat{\tau}$	0.06805 {7}	0.01372 {3}	0.00463 {1}	0.01315 {2}	0.09047 {8}	0.05400 {6}	0.02455 {5}	0.02204 {4}
		$\hat{\eta}$	6.93089 {8}	0.81191 {3}	0.28238 {2}	0.83934 {4}	0.09056 {1}	1.09611 {5}	1.24529 {7}	1.20728 {6}
	MRE	$\hat{\alpha}$	0.06823 {3}	0.08448 {5.5}	0.05790 {1}	0.08448 {5.5}	0.06779 {2}	0.11218 {8}	0.07094 {4}	0.08761 {7}
		$\hat{\beta}$	0.44241 {7}	0.35863 {4}	0.17879 {1}	0.35100 {3}	0.31513 {2}	0.47791 {8}	0.36788 {5}	0.37266 {6}
		$\hat{\tau}$	0.94306 {8}	0.62573 {4}	0.22849 {1}	0.61296 {3}	0.40409 {2}	0.87250 {7}	0.75246 {6}	0.73613 {5}
		$\hat{\eta}$	0.77718 {8}	0.35552 {4}	0.12222 {2}	0.36070 {5}	0.03949 {1}	0.33376 {3}	0.41174 {7}	0.41172 {6}
		$\sum Ranks$	76 {7}	49 {4}	15 {1}	47 {3}	33 {2}	78 {8}	64 {5}	70 {6}

Table 4: Simulation results for $\underline{\varphi} = (\alpha = 0.5, \beta = 0.75, \tau = 0.5, \eta = 3.5)^\intercal$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.09975$^{\{3\}}$	0.11761$^{\{5\}}$	0.09830$^{\{2\}}$	0.12554$^{\{6\}}$	0.09355$^{\{1\}}$	0.52869$^{\{8\}}$	0.10239$^{\{4\}}$	0.13783$^{\{7\}}$
		$\hat{\beta}$	1.42153$^{\{8\}}$	1.13273$^{\{6\}}$	0.83547$^{\{2\}}$	1.05976$^{\{4\}}$	0.16469$^{\{1\}}$	1.08358$^{\{5\}}$	1.20278$^{\{7\}}$	1.05081$^{\{3\}}$
		$\hat{\tau}$	1.99280$^{\{8\}}$	0.42536$^{\{3\}}$	0.61724$^{\{4\}}$	0.40252$^{\{2\}}$	0.87233$^{\{6\}}$	0.34467$^{\{1\}}$	1.01990$^{\{7\}}$	0.72788$^{\{5\}}$
		$\hat{\eta}$	8.32983$^{\{8\}}$	0.62659$^{\{2\}}$	1.76341$^{\{5\}}$	0.67842$^{\{3\}}$	0.12238$^{\{1\}}$	1.66087$^{\{4\}}$	2.03778$^{\{7\}}$	1.86026$^{\{6\}}$
	MSE	$\hat{\alpha}$	0.02062$^{\{3\}}$	0.02990$^{\{5\}}$	0.01959$^{\{2\}}$	0.03724$^{\{6\}}$	0.01542$^{\{1\}}$	3.68334$^{\{8\}}$	0.02076$^{\{4\}}$	0.10949$^{\{7\}}$
		$\hat{\beta}$	20.01844$^{\{7\}}$	15.81661$^{\{5\}}$	6.18965$^{\{1\}}$	12.09876$^{\{4\}}$	40.56294$^{\{8\}}$	9.11500$^{\{3\}}$	18.80899$^{\{6\}}$	7.14811$^{\{2\}}$
		$\hat{\tau}$	43.51079$^{\{8\}}$	0.71119$^{\{3\}}$	1.39313$^{\{4\}}$	0.68276$^{\{2\}}$	35.14738$^{\{7\}}$	0.27938$^{\{1\}}$	5.58205$^{\{6\}}$	1.79647$^{\{5\}}$
		$\hat{\eta}$	103.95115$^{\{8\}}$	1.17380$^{\{2\}}$	4.88373$^{\{6\}}$	1.40618$^{\{3\}}$	0.26927$^{\{1\}}$	4.22161$^{\{4\}}$	5.90911$^{\{7\}}$	4.86640$^{\{5\}}$
	MRE	$\hat{\alpha}$	0.19951$^{\{3\}}$	0.23521$^{\{5\}}$	0.19659$^{\{2\}}$	0.25108$^{\{6\}}$	0.18711$^{\{1\}}$	1.05739$^{\{8\}}$	0.20478$^{\{4\}}$	0.27566$^{\{7\}}$
		$\hat{\beta}$	1.89538$^{\{8\}}$	1.51031$^{\{5\}}$	1.11396$^{\{1\}}$	1.41301$^{\{3\}}$	1.68535$^{\{7\}}$	1.44478$^{\{4\}}$	1.60371$^{\{6\}}$	1.40108$^{\{2\}}$
		$\hat{\tau}$	3.98561$^{\{8\}}$	0.85072$^{\{3\}}$	1.23448$^{\{4\}}$	0.80504$^{\{2\}}$	1.74465$^{\{6\}}$	0.68934$^{\{1\}}$	2.03980$^{\{7\}}$	1.45575$^{\{5\}}$
		$\hat{\eta}$	2.37995$^{\{8\}}$	0.17903$^{\{2\}}$	0.50383$^{\{5\}}$	0.19383$^{\{3\}}$	0.03496$^{\{1\}}$	0.47454$^{\{4\}}$	0.58222$^{\{7\}}$	0.53150$^{\{6\}}$
		$\sum Ranks$	80$^{\{8\}}$	46$^{\{4\}}$	38$^{\{1\}}$	44$^{\{3\}}$	41$^{\{2\}}$	51$^{\{5\}}$	72$^{\{7\}}$	60$^{\{6\}}$
150	|BIAS|	$\hat{\alpha}$	0.05457$^{\{4\}}$	0.06251$^{\{5\}}$	0.05109$^{\{2\}}$	0.06347$^{\{6\}}$	0.05094$^{\{1\}}$	0.17808$^{\{8\}}$	0.05394$^{\{3\}}$	0.06652$^{\{7\}}$
		$\hat{\beta}$	0.59169$^{\{8\}}$	0.45944$^{\{6\}}$	0.32642$^{\{1\}}$	0.44745$^{\{4\}}$	0.33145$^{\{2\}}$	0.47583$^{\{7\}}$	0.45313$^{\{5\}}$	0.41477$^{\{3\}}$
		$\hat{\tau}$	1.09692$^{\{8\}}$	0.35651$^{\{4\}}$	0.33356$^{\{2\}}$	0.34559$^{\{3\}}$	0.18462$^{\{1\}}$	0.50380$^{\{6\}}$	0.61542$^{\{7\}}$	0.49883$^{\{5\}}$
		$\hat{\eta}$	5.15126$^{\{8\}}$	1.05335$^{\{2\}}$	1.18389$^{\{4\}}$	1.11285$^{\{3\}}$	0.04793$^{\{1\}}$	1.91490$^{\{6\}}$	1.97904$^{\{7\}}$	1.88911$^{\{5\}}$
	MSE	$\hat{\alpha}$	0.00557$^{\{4\}}$	0.00653$^{\{5\}}$	0.00435$^{\{2\}}$	0.00687$^{\{6\}}$	0.00411$^{\{1\}}$	0.66132$^{\{8\}}$	0.00479$^{\{3\}}$	0.00759$^{\{7\}}$
		$\hat{\beta}$	2.07898$^{\{8\}}$	0.56475$^{\{6\}}$	0.24184$^{\{1\}}$	0.53822$^{\{5\}}$	1.26797$^{\{7\}}$	0.52614$^{\{4\}}$	0.52450$^{\{3\}}$	0.47971$^{\{2\}}$
		$\hat{\tau}$	8.49227$^{\{8\}}$	0.27975$^{\{3\}}$	0.27101$^{\{2\}}$	0.27021$^{\{1\}}$	2.61505$^{\{7\}}$	0.53105$^{\{5\}}$	0.86007$^{\{6\}}$	0.49620$^{\{4\}}$
		$\hat{\eta}$	44.84870$^{\{8\}}$	2.02698$^{\{2\}}$	2.58094$^{\{4\}}$	2.23357$^{\{3\}}$	0.05778$^{\{1\}}$	4.66745$^{\{6\}}$	5.19253$^{\{7\}}$	4.62163$^{\{5\}}$
	MRE	$\hat{\alpha}$	0.10914$^{\{4\}}$	0.12501$^{\{5\}}$	0.10218$^{\{2\}}$	0.12694$^{\{6\}}$	0.10188$^{\{1\}}$	0.35616$^{\{8\}}$	0.10787$^{\{3\}}$	0.13304$^{\{7\}}$
		$\hat{\beta}$	0.78892$^{\{8\}}$	0.61259$^{\{6\}}$	0.43522$^{\{1\}}$	0.59659$^{\{4\}}$	0.44193$^{\{2\}}$	0.63444$^{\{7\}}$	0.60417$^{\{5\}}$	0.55303$^{\{3\}}$
		$\hat{\tau}$	2.19384$^{\{8\}}$	0.71302$^{\{4\}}$	0.66712$^{\{2\}}$	0.69118$^{\{3\}}$	0.36924$^{\{1\}}$	1.00760$^{\{6\}}$	1.23085$^{\{7\}}$	0.99766$^{\{5\}}$
		$\hat{\eta}$	1.47179$^{\{8\}}$	0.30096$^{\{2\}}$	0.33825$^{\{4\}}$	0.31796$^{\{3\}}$	0.01369$^{\{1\}}$	0.54711$^{\{6\}}$	0.56544$^{\{7\}}$	0.53975$^{\{5\}}$
		$\sum Ranks$	84$^{\{8\}}$	50$^{\{4\}}$	27$^{\{2\}}$	47$^{\{3\}}$	26$^{\{1\}}$	77$^{\{7\}}$	63$^{\{6\}}$	58$^{\{5\}}$
350	|BIAS|	$\hat{\alpha}$	0.03896$^{\{4\}}$	0.04044$^{\{5\}}$	0.03227$^{\{1\}}$	0.04068$^{\{6\}}$	0.03285$^{\{2\}}$	0.07502$^{\{8\}}$	0.03462$^{\{3\}}$	0.04270$^{\{7\}}$
		$\hat{\beta}$	0.30868$^{\{7\}}$	0.27820$^{\{6\}}$	0.17618$^{\{2\}}$	0.27203$^{\{5\}}$	0.16469$^{\{1\}}$	0.32418$^{\{8\}}$	0.26909$^{\{4\}}$	0.25723$^{\{3\}}$
		$\hat{\tau}$	0.59872$^{\{8\}}$	0.30556$^{\{4\}}$	0.16366$^{\{2\}}$	0.29603$^{\{3\}}$	0.05404$^{\{1\}}$	0.50215$^{\{7\}}$	0.42754$^{\{6\}}$	0.39703$^{\{5\}}$
		$\hat{\eta}$	3.60680$^{\{8\}}$	1.20834$^{\{3\}}$	0.66335$^{\{2\}}$	1.23647$^{\{4\}}$	0.02180$^{\{1\}}$	1.85602$^{\{7\}}$	1.74527$^{\{6\}}$	1.68074$^{\{5\}}$
	MSE	$\hat{\alpha}$	0.00410$^{\{7\}}$	0.00268$^{\{4\}}$	0.00172$^{\{1.5\}}$	0.00274$^{\{5\}}$	0.00172$^{\{1.5\}}$	0.03249$^{\{8\}}$	0.00196$^{\{3\}}$	0.00299$^{\{6\}}$
		$\hat{\beta}$	0.30472$^{\{8\}}$	0.15277$^{\{6\}}$	0.05759$^{\{1\}}$	0.14536$^{\{5\}}$	0.13328$^{\{3\}}$	0.21586$^{\{7\}}$	0.14432$^{\{4\}}$	0.13062$^{\{2\}}$
		$\hat{\tau}$	1.45815$^{\{8\}}$	0.16608$^{\{3\}}$	0.07256$^{\{1\}}$	0.15611$^{\{2\}}$	0.31479$^{\{5\}}$	0.45921$^{\{7\}}$	0.34006$^{\{6\}}$	0.26995$^{\{4\}}$
		$\hat{\eta}$	23.94823$^{\{8\}}$	2.14933$^{\{3\}}$	1.15099$^{\{2\}}$	2.25507$^{\{4\}}$	0.01025$^{\{1\}}$	4.33618$^{\{7\}}$	4.04408$^{\{6\}}$	3.62044$^{\{5\}}$
	MRE	$\hat{\alpha}$	0.07793$^{\{4\}}$	0.08089$^{\{5\}}$	0.06455$^{\{1\}}$	0.08136$^{\{6\}}$	0.06570$^{\{2\}}$	0.15004$^{\{8\}}$	0.06924$^{\{3\}}$	0.08539$^{\{7\}}$
		$\hat{\beta}$	0.41157$^{\{7\}}$	0.37094$^{\{6\}}$	0.23491$^{\{2\}}$	0.36271$^{\{5\}}$	0.21959$^{\{1\}}$	0.43224$^{\{8\}}$	0.35878$^{\{4\}}$	0.34298$^{\{3\}}$
		$\hat{\tau}$	1.19744$^{\{8\}}$	0.61113$^{\{4\}}$	0.32733$^{\{2\}}$	0.59207$^{\{3\}}$	0.10808$^{\{1\}}$	1.00430$^{\{7\}}$	0.85507$^{\{6\}}$	0.79405$^{\{5\}}$
		$\hat{\eta}$	1.03051$^{\{8\}}$	0.34524$^{\{3\}}$	0.18953$^{\{2\}}$	0.35328$^{\{4\}}$	0.00623$^{\{1\}}$	0.53029$^{\{7\}}$	0.49865$^{\{6\}}$	0.48021$^{\{5\}}$
		$\sum Ranks$	85$^{\{7\}}$	52$^{\{3.5\}}$	19.5$^{\{1\}}$	52$^{\{3.5\}}$	20.5$^{\{2\}}$	89$^{\{8\}}$	57$^{\{5.5\}}$	57$^{\{5.5\}}$
500	|BIAS|	$\hat{\alpha}$	0.03607$^{\{7\}}$	0.03423$^{\{4\}}$	0.02766$^{\{1\}}$	0.03433$^{\{5\}}$	0.02853$^{\{2\}}$	0.06275$^{\{8\}}$	0.02972$^{\{3\}}$	0.03508$^{\{6\}}$
		$\hat{\beta}$	0.25999$^{\{7\}}$	0.23538$^{\{6\}}$	0.14318$^{\{2\}}$	0.23306$^{\{5\}}$	0.13132$^{\{1\}}$	0.29135$^{\{8\}}$	0.22973$^{\{4\}}$	0.21706$^{\{3\}}$
		$\hat{\tau}$	0.50728$^{\{8\}}$	0.28969$^{\{4\}}$	0.13491$^{\{2\}}$	0.28555$^{\{3\}}$	0.03078$^{\{1\}}$	0.47023$^{\{7\}}$	0.38964$^{\{6\}}$	0.35411$^{\{5\}}$
		$\hat{\eta}$	3.04525$^{\{8\}}$	1.21690$^{\{3\}}$	0.54335$^{\{2\}}$	1.23899$^{\{4\}}$	0.01578$^{\{1\}}$	1.75557$^{\{7\}}$	1.65132$^{\{6\}}$	1.57413$^{\{5\}}$
	MSE	$\hat{\alpha}$	0.00494$^{\{7\}}$	0.00185$^{\{4\}}$	0.00124$^{\{1\}}$	0.00188$^{\{5\}}$	0.00127$^{\{2\}}$	0.00653$^{\{8\}}$	0.00142$^{\{3\}}$	0.00203$^{\{6\}}$
		$\hat{\beta}$	0.17586$^{\{8\}}$	0.10412$^{\{6\}}$	0.03663$^{\{2\}}$	0.10168$^{\{4\}}$	0.02946$^{\{1\}}$	0.16798$^{\{7\}}$	0.10318$^{\{5\}}$	0.08985$^{\{3\}}$
		$\hat{\tau}$	0.78597$^{\{8\}}$	0.13772$^{\{4\}}$	0.05086$^{\{2\}}$	0.13412$^{\{3\}}$	0.00261$^{\{1\}}$	0.37972$^{\{7\}}$	0.26610$^{\{6\}}$	0.20747$^{\{5\}}$
		$\hat{\eta}$	17.47868$^{\{8\}}$	2.07946$^{\{3\}}$	0.85518$^{\{2\}}$	2.15270$^{\{4\}}$	0.00150$^{\{1\}}$	3.90103$^{\{7\}}$	3.64273$^{\{6\}}$	3.20241$^{\{5\}}$
	MRE	$\hat{\alpha}$	0.07214$^{\{7\}}$	0.06846$^{\{4\}}$	0.05533$^{\{1\}}$	0.06865$^{\{5\}}$	0.05707$^{\{2\}}$	0.12550$^{\{8\}}$	0.05945$^{\{3\}}$	0.07015$^{\{6\}}$
		$\hat{\beta}$	0.34665$^{\{7\}}$	0.31384$^{\{6\}}$	0.19090$^{\{2\}}$	0.31074$^{\{5\}}$	0.17510$^{\{1\}}$	0.38846$^{\{8\}}$	0.30630$^{\{4\}}$	0.28941$^{\{3\}}$
		$\hat{\tau}$	1.01455$^{\{8\}}$	0.57938$^{\{4\}}$	0.26982$^{\{2\}}$	0.57110$^{\{3\}}$	0.06156$^{\{1\}}$	0.94046$^{\{7\}}$	0.77929$^{\{6\}}$	0.70823$^{\{5\}}$
		$\hat{\eta}$	0.87007$^{\{8\}}$	0.34768$^{\{3\}}$	0.15524$^{\{2\}}$	0.35400$^{\{4\}}$	0.00451$^{\{1\}}$	0.50159$^{\{7\}}$	0.47180$^{\{6\}}$	0.44975$^{\{5\}}$
		$\sum Ranks$	91$^{\{8\}}$	51$^{\{4\}}$	21$^{\{2\}}$	50$^{\{3\}}$	15$^{\{1\}}$	89$^{\{7\}}$	58$^{\{6\}}$	57$^{\{5\}}$

Table 5: Simulation results for $\underline{\varphi} = (\alpha = 0.5, \beta = 1.25, \tau = 0.75, \eta = 1.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	$0.10553^{\{4\}}$	$0.12431^{\{5\}}$	$0.10271^{\{2\}}$	$0.13491^{\{6\}}$	$0.10026^{\{1\}}$	$1.07685^{\{8\}}$	$0.10336^{\{3\}}$	$0.13745^{\{7\}}$
		$\hat{\beta}$	$6.39312^{\{8\}}$	$1.51974^{\{3\}}$	$1.90868^{\{4\}}$	$1.37280^{\{2\}}$	$1.07511^{\{1\}}$	$1.91217^{\{5\}}$	$2.56221^{\{7\}}$	$2.49220^{\{6\}}$
		$\hat{\tau}$	$5.32408^{\{7\}}$	$0.85282^{\{2\}}$	$1.30650^{\{3\}}$	$0.80485^{\{1\}}$	$14.30765^{\{8\}}$	$2.31219^{\{6\}}$	$2.03620^{\{5\}}$	$1.89729^{\{4\}}$
		$\hat{\eta}$	$4.26085^{\{8\}}$	$0.80458^{\{1\}}$	$0.83033^{\{2\}}$	$0.85957^{\{3\}}$	$1.11405^{\{6\}}$	$1.13042^{\{7\}}$	$1.00933^{\{4\}}$	$1.01501^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.02389^{\{4\}}$	$0.05248^{\{5\}}$	$0.02328^{\{3\}}$	$0.10624^{\{6\}}$	$0.01741^{\{1\}}$	$30.29045^{\{8\}}$	$0.02181^{\{2\}}$	$0.34831^{\{7\}}$
		$\hat{\beta}$	$976.10071^{\{7\}}$	$7.27804^{\{2\}}$	$25.62925^{\{4\}}$	$5.64895^{\{1\}}$	$4940.10421^{\{8\}}$	$13.56651^{\{3\}}$	$45.16420^{\{6\}}$	$37.57869^{\{5\}}$
		$\hat{\tau}$	$500.17402^{\{7\}}$	$2.46479^{\{2\}}$	$7.62486^{\{3\}}$	$2.42790^{\{1\}}$	$4265.18820^{\{8\}}$	$30.75273^{\{6\}}$	$20.63388^{\{5\}}$	$17.74600^{\{4\}}$
		$\hat{\eta}$	$32.95563^{\{8\}}$	$1.31981^{\{2\}}$	$1.22990^{\{1\}}$	$1.46157^{\{3\}}$	$4.81348^{\{7\}}$	$2.26323^{\{6\}}$	$1.77326^{\{4\}}$	$1.79392^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.21107^{\{4\}}$	$0.24863^{\{5\}}$	$0.20543^{\{2\}}$	$0.26981^{\{6\}}$	$0.20053^{\{1\}}$	$2.15371^{\{8\}}$	$0.20673^{\{3\}}$	$0.27491^{\{7\}}$
		$\hat{\beta}$	$5.11450^{\{7\}}$	$1.21579^{\{2\}}$	$1.52694^{\{3\}}$	$1.09824^{\{1\}}$	$13.30806^{\{8\}}$	$1.52974^{\{4\}}$	$2.04977^{\{6\}}$	$1.99376^{\{5\}}$
		$\hat{\tau}$	$7.09878^{\{7\}}$	$1.13710^{\{2\}}$	$1.74199^{\{3\}}$	$1.07313^{\{1\}}$	$19.07686^{\{8\}}$	$3.08291^{\{6\}}$	$2.71494^{\{5\}}$	$2.52972^{\{4\}}$
		$\hat{\eta}$	$2.84057^{\{8\}}$	$0.53639^{\{1\}}$	$0.55355^{\{2\}}$	$0.57305^{\{3\}}$	$0.74270^{\{6\}}$	$0.75362^{\{7\}}$	$0.67289^{\{4\}}$	$0.67667^{\{5\}}$
		$\sum Ranks$	$79^{\{8\}}$	$32^{\{1.5\}}$	$32^{\{1.5\}}$	$34^{\{3\}}$	$63^{\{5\}}$	$74^{\{7\}}$	$54^{\{4\}}$	$64^{\{6\}}$
150	|BIAS|	$\hat{\alpha}$	$0.05565^{\{3\}}$	$0.06269^{\{5\}}$	$0.05290^{\{1\}}$	$0.06350^{\{6\}}$	$0.05643^{\{4\}}$	$0.23068^{\{8\}}$	$0.05541^{\{2\}}$	$0.06593^{\{7\}}$
		$\hat{\beta}$	$1.81404^{\{7\}}$	$0.88200^{\{3\}}$	$0.80052^{\{1\}}$	$0.83460^{\{2\}}$	$3.99461^{\{8\}}$	$1.67410^{\{6\}}$	$1.20263^{\{5\}}$	$1.15209^{\{4\}}$
		$\hat{\tau}$	$1.90759^{\{7\}}$	$0.71541^{\{3\}}$	$0.70370^{\{2\}}$	$0.68147^{\{1\}}$	$3.82120^{\{8\}}$	$1.86129^{\{6\}}$	$1.18944^{\{5\}}$	$1.10211^{\{4\}}$
		$\hat{\eta}$	$2.23117^{\{8\}}$	$0.62740^{\{3\}}$	$0.60229^{\{2\}}$	$0.66075^{\{4\}}$	$0.54169^{\{1\}}$	$0.88234^{\{7\}}$	$0.83786^{\{6\}}$	$0.81783^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.00511^{\{4\}}$	$0.00658^{\{5\}}$	$0.00461^{\{1\}}$	$0.00693^{\{6\}}$	$0.00487^{\{2\}}$	$2.68046^{\{8\}}$	$0.00499^{\{3\}}$	$0.00739^{\{7\}}$
		$\hat{\beta}$	$27.09138^{\{7\}}$	$1.53949^{\{2\}}$	$1.72934^{\{3\}}$	$1.37613^{\{1\}}$	$164.81856^{\{8\}}$	$9.56162^{\{6\}}$	$3.64110^{\{5\}}$	$3.61831^{\{4\}}$
		$\hat{\tau}$	$32.30790^{\{7\}}$	$1.00118^{\{2\}}$	$1.30918^{\{3\}}$	$0.90896^{\{1\}}$	$140.55666^{\{8\}}$	$11.74903^{\{6\}}$	$3.52232^{\{5\}}$	$3.03016^{\{4\}}$
		$\hat{\eta}$	$11.16252^{\{8\}}$	$0.60762^{\{1\}}$	$0.61861^{\{2\}}$	$0.68357^{\{3\}}$	$1.52743^{\{7\}}$	$1.19133^{\{6\}}$	$1.07659^{\{5\}}$	$1.02488^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.11129^{\{3\}}$	$0.12538^{\{5\}}$	$0.10580^{\{1\}}$	$0.12699^{\{6\}}$	$0.11287^{\{4\}}$	$0.46137^{\{8\}}$	$0.11081^{\{2\}}$	$0.13185^{\{7\}}$
		$\hat{\beta}$	$1.45123^{\{7\}}$	$0.70560^{\{3\}}$	$0.64042^{\{1\}}$	$0.66768^{\{2\}}$	$3.19569^{\{8\}}$	$1.33928^{\{6\}}$	$0.96211^{\{5\}}$	$0.92168^{\{4\}}$
		$\hat{\tau}$	$2.54346^{\{7\}}$	$0.95388^{\{3\}}$	$0.93827^{\{2\}}$	$0.90863^{\{1\}}$	$5.09493^{\{8\}}$	$2.48172^{\{6\}}$	$1.58592^{\{5\}}$	$1.46949^{\{4\}}$
		$\hat{\eta}$	$1.48745^{\{8\}}$	$0.41827^{\{3\}}$	$0.40153^{\{2\}}$	$0.44050^{\{4\}}$	$0.36113^{\{1\}}$	$0.58823^{\{7\}}$	$0.55857^{\{6\}}$	$0.54522^{\{5\}}$
		$\sum Ranks$	$76^{\{7\}}$	$38^{\{3\}}$	$21^{\{1\}}$	$37^{\{2\}}$	$67^{\{6\}}$	$80^{\{8\}}$	$54^{\{4\}}$	$59^{\{5\}}$
350	|BIAS|	$\hat{\alpha}$	$0.03563^{\{3\}}$	$0.04119^{\{5\}}$	$0.03393^{\{1\}}$	$0.04139^{\{6\}}$	$0.03517^{\{2\}}$	$0.08236^{\{8\}}$	$0.03612^{\{4\}}$	$0.04278^{\{7\}}$
		$\hat{\beta}$	$0.90490^{\{6\}}$	$0.64409^{\{3\}}$	$0.46366^{\{1\}}$	$0.62210^{\{2\}}$	$1.07511^{\{7\}}$	$1.61699^{\{8\}}$	$0.79405^{\{5\}}$	$0.76829^{\{4\}}$
		$\hat{\tau}$	$0.97482^{\{6\}}$	$0.62338^{\{3\}}$	$0.42699^{\{1\}}$	$0.60370^{\{2\}}$	$1.08158^{\{7\}}$	$1.50618^{\{8\}}$	$0.83791^{\{5\}}$	$0.78189^{\{4\}}$
		$\hat{\eta}$	$1.24625^{\{8\}}$	$0.58690^{\{3\}}$	$0.42173^{\{2\}}$	$0.60523^{\{4\}}$	$0.27053^{\{1\}}$	$0.73547^{\{7\}}$	$0.71023^{\{6\}}$	$0.68253^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.00212^{\{4\}}$	$0.00276^{\{5\}}$	$0.00188^{\{1\}}$	$0.00282^{\{6\}}$	$0.00194^{\{2\}}$	$0.22356^{\{8\}}$	$0.00210^{\{3\}}$	$0.00300^{\{7\}}$
		$\hat{\beta}$	$4.38992^{\{6\}}$	$0.72991^{\{3\}}$	$0.46020^{\{1\}}$	$0.67441^{\{2\}}$	$12.53202^{\{8\}}$	$7.84965^{\{7\}}$	$1.32527^{\{5\}}$	$1.25099^{\{4\}}$
		$\hat{\tau}$	$5.72609^{\{6\}}$	$0.65884^{\{3\}}$	$0.43634^{\{1\}}$	$0.61163^{\{2\}}$	$15.32340^{\{8\}}$	$6.53769^{\{7\}}$	$1.49168^{\{5\}}$	$1.24532^{\{4\}}$
		$\hat{\eta}$	$4.07577^{\{8\}}$	$0.48063^{\{2\}}$	$0.32401^{\{1\}}$	$0.51732^{\{4\}}$	$0.51164^{\{3\}}$	$0.81629^{\{7\}}$	$0.75523^{\{6\}}$	$0.69603^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.07125^{\{3\}}$	$0.08237^{\{5\}}$	$0.06786^{\{1\}}$	$0.08278^{\{6\}}$	$0.07034^{\{2\}}$	$0.16471^{\{8\}}$	$0.07223^{\{4\}}$	$0.08556^{\{7\}}$
		$\hat{\beta}$	$0.72392^{\{6\}}$	$0.51527^{\{3\}}$	$0.37093^{\{1\}}$	$0.49768^{\{2\}}$	$0.86009^{\{7\}}$	$1.29359^{\{8\}}$	$0.63524^{\{5\}}$	$0.61463^{\{4\}}$
		$\hat{\tau}$	$1.29976^{\{6\}}$	$0.83117^{\{3\}}$	$0.56932^{\{1\}}$	$0.80493^{\{2\}}$	$1.44210^{\{7\}}$	$2.00823^{\{8\}}$	$1.11721^{\{5\}}$	$1.04251^{\{4\}}$
		$\hat{\eta}$	$0.83083^{\{8\}}$	$0.39127^{\{3\}}$	$0.28115^{\{2\}}$	$0.40348^{\{4\}}$	$0.18035^{\{1\}}$	$0.49031^{\{7\}}$	$0.47348^{\{6\}}$	$0.45502^{\{5\}}$
		$\sum Ranks$	$70^{\{7\}}$	$41^{\{2\}}$	$14^{\{1\}}$	$42^{\{3\}}$	$55^{\{4\}}$	$91^{\{8\}}$	$59^{\{5\}}$	$60^{\{6\}}$
500	|BIAS|	$\hat{\alpha}$	$0.02988^{\{3\}}$	$0.03434^{\{5\}}$	$0.02814^{\{1\}}$	$0.03446^{\{6\}}$	$0.02891^{\{2\}}$	$0.06333^{\{8\}}$	$0.03006^{\{4\}}$	$0.03532^{\{7\}}$
		$\hat{\beta}$	$0.67894^{\{7\}}$	$0.56717^{\{3\}}$	$0.37673^{\{1\}}$	$0.55295^{\{2\}}$	$0.57597^{\{4\}}$	$1.41130^{\{8\}}$	$0.67737^{\{6\}}$	$0.65102^{\{5\}}$
		$\hat{\tau}$	$0.71413^{\{6\}}$	$0.57186^{\{4\}}$	$0.34623^{\{1\}}$	$0.55691^{\{3\}}$	$0.54051^{\{2\}}$	$1.26033^{\{8\}}$	$0.72445^{\{7\}}$	$0.67048^{\{5\}}$
		$\hat{\eta}$	$1.05541^{\{8\}}$	$0.54396^{\{3\}}$	$0.35593^{\{2\}}$	$0.55597^{\{4\}}$	$0.16898^{\{1\}}$	$0.67165^{\{7\}}$	$0.64674^{\{6\}}$	$0.62358^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.00146^{\{4\}}$	$0.00186^{\{5\}}$	$0.00127^{\{1\}}$	$0.00188^{\{6\}}$	$0.00130^{\{2\}}$	$0.00778^{\{8\}}$	$0.00143^{\{3\}}$	$0.00203^{\{7\}}$
		$\hat{\beta}$	$1.79839^{\{6\}}$	$0.54224^{\{3\}}$	$0.30163^{\{1\}}$	$0.51683^{\{2\}}$	$3.90017^{\{7\}}$	$5.70530^{\{8\}}$	$0.93343^{\{5\}}$	$0.83758^{\{4\}}$
		$\hat{\tau}$	$2.34747^{\{6\}}$	$0.52706^{\{3\}}$	$0.29190^{\{1\}}$	$0.50159^{\{2\}}$	$5.25871^{\{8\}}$	$4.44817^{\{7\}}$	$1.09809^{\{5\}}$	$0.88788^{\{4\}}$
		$\hat{\eta}$	$3.01017^{\{8\}}$	$0.40244^{\{3\}}$	$0.24297^{\{2\}}$	$0.42419^{\{4\}}$	$0.14738^{\{1\}}$	$0.68579^{\{7\}}$	$0.63138^{\{6\}}$	$0.58452^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.05975^{\{3\}}$	$0.06868^{\{5\}}$	$0.05629^{\{1\}}$	$0.06893^{\{6\}}$	$0.05782^{\{2\}}$	$0.12666^{\{8\}}$	$0.06011^{\{4\}}$	$0.07065^{\{7\}}$
		$\hat{\beta}$	$0.54315^{\{7\}}$	$0.45373^{\{3\}}$	$0.30138^{\{1\}}$	$0.44236^{\{2\}}$	$0.46078^{\{4\}}$	$1.12904^{\{8\}}$	$0.54189^{\{6\}}$	$0.52082^{\{5\}}$
		$\hat{\tau}$	$0.95217^{\{6\}}$	$0.76247^{\{4\}}$	$0.46164^{\{1\}}$	$0.74255^{\{3\}}$	$0.72068^{\{2\}}$	$1.68043^{\{8\}}$	$0.96593^{\{7\}}$	$0.89397^{\{5\}}$
		$\hat{\eta}$	$0.70360^{\{8\}}$	$0.36264^{\{3\}}$	$0.23729^{\{2\}}$	$0.37064^{\{4\}}$	$0.11266^{\{1\}}$	$0.44777^{\{7\}}$	$0.43116^{\{6\}}$	$0.41572^{\{5\}}$
		$\sum Ranks$	$72^{\{7\}}$	$44^{\{3.5\}}$	$15^{\{1\}}$	$44^{\{3.5\}}$	$36^{\{2\}}$	$92^{\{8\}}$	$65^{\{6\}}$	$64^{\{5\}}$

Table 6: Simulation results for $\underline{\varphi} = (\alpha = 0.75, \beta = 0.5, \tau = 0.15, \eta = 2.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.19935 {4}	0.27797 {5}	0.17785 {1}	0.29714 {6}	0.17875 {2}	1.75742 {8}	0.19644 {3}	0.36472 {7}
		$\hat{\beta}$	1.50369 {8}	1.11819 {6}	0.54231 {2}	1.12528 {7}	0.25476 {1}	0.64496 {3}	1.09329 {5}	1.07774 {4}
		$\hat{\tau}$	0.76591 {7}	0.22518 {2}	0.17558 {1}	0.23002 {3}	1.35035 {8}	0.29410 {4}	0.37819 {6}	0.36788 {5}
		$\hat{\eta}$	5.91366 {8}	0.71716 {2}	0.93057 {4}	0.80291 {3}	0.61568 {1}	1.58057 {7}	1.35700 {6}	1.32489 {5}
	MSE	$\hat{\alpha}$	0.15411 {3}	0.83955 {5}	0.20647 {4}	1.16591 {6}	0.07541 {1}	49.77718 {8}	0.14694 {2}	4.81974 {7}
		$\hat{\beta}$	58.57073 {7}	12.54544 {5}	3.23989 {2}	11.12080 {4}	174.54526 {8}	1.68703 {1}	12.56846 {6}	8.40927 {3}
		$\hat{\tau}$	10.00539 {7}	0.30598 {3}	0.20421 {1}	0.28255 {2}	38.30602 {8}	0.61936 {4}	0.73148 {5}	0.77375 {6}
		$\hat{\eta}$	55.44175 {8}	0.88663 {1}	1.60806 {3}	1.05214 {2}	3.79399 {6}	4.58910 {7}	2.54511 {5}	2.47088 {4}
	MRE	$\hat{\alpha}$	0.26581 {4}	0.37063 {5}	0.23713 {1}	0.39619 {6}	0.23833 {2}	2.34323 {8}	0.26193 {3}	0.48629 {7}
		$\hat{\beta}$	3.00738 {7}	2.23638 {5}	1.08462 {1}	2.25056 {6}	6.17595 {8}	1.28992 {2}	2.18658 {4}	2.15547 {3}
		$\hat{\tau}$	5.10608 {7}	1.50118 {2}	1.17054 {1}	1.53344 {3}	9.00236 {8}	1.96069 {4}	2.52126 {6}	2.45256 {5}
		$\hat{\eta}$	2.62829 {8}	0.31874 {2}	0.41359 {4}	0.35685 {3}	0.27363 {1}	0.70248 {7}	0.60311 {6}	0.58884 {5}
		$\sum Ranks$	78 {8}	43 {2}	25 {1}	51 {3}	54 {4}	63 {7}	57 {5}	61 {6}
150	|BIAS|	$\hat{\alpha}$	0.09593 {2}	0.11738 {5}	0.08288 {1}	0.11969 {6}	0.09617 {3}	0.53430 {8}	0.09847 {4}	0.12731 {7}
		$\hat{\beta}$	0.47988 {7}	0.36722 {5}	0.18977 {1}	0.35101 {2}	0.81068 {8}	0.35560 {3}	0.36402 {4}	0.37325 {6}
		$\hat{\tau}$	0.32105 {7}	0.13165 {3}	0.07546 {1}	0.12968 {2}	0.46029 {8}	0.20836 {6}	0.19743 {5}	0.18546 {4}
		$\hat{\eta}$	3.58795 {8}	0.81903 {3}	0.51344 {2}	0.86832 {4}	0.23908 {1}	1.10303 {5}	1.23399 {7}	1.22710 {6}
	MSE	$\hat{\alpha}$	0.01562 {3}	0.02453 {5}	0.01175 {1}	0.02601 {6}	0.01442 {2}	11.75159 {8}	0.01637 {4}	0.03070 {7}
		$\hat{\beta}$	1.31869 {7}	0.57722 {6}	0.08558 {1}	0.49814 {5}	6.78255 {8}	0.38813 {3}	0.37745 {2}	0.45363 {4}
		$\hat{\tau}$	0.66405 {7}	0.04182 {3}	0.01863 {1}	0.03879 {2}	2.74359 {8}	0.18072 {6}	0.09430 {5}	0.08181 {4}
		$\hat{\eta}$	23.75446 {8}	0.94508 {3}	0.67133 {2}	1.04614 {4}	0.59798 {1}	2.14421 {7}	2.06965 {6}	2.04467 {5}
	MRE	$\hat{\alpha}$	0.12791 {2}	0.15650 {5}	0.11051 {1}	0.15958 {6}	0.12823 {3}	0.71239 {8}	0.13129 {4}	0.16974 {7}
		$\hat{\beta}$	0.95975 {7}	0.73445 {5}	0.37954 {1}	0.70202 {2}	1.62135 {8}	0.71120 {3}	0.72804 {4}	0.74650 {6}
		$\hat{\tau}$	2.14035 {7}	0.87770 {3}	0.50306 {1}	0.86454 {2}	3.06859 {8}	1.38907 {6}	1.31617 {5}	1.23643 {4}
		$\hat{\eta}$	1.59465 {8}	0.36401 {3}	0.22819 {2}	0.38592 {4}	0.10626 {1}	0.49023 {5}	0.54844 {7}	0.54538 {6}
		$\sum Ranks$	73 {8}	49 {3}	15 {1}	45 {2}	59 {5}	68 {7}	57 {4}	66 {6}
350	|BIAS|	$\hat{\alpha}$	0.05947 {2}	0.07579 {5}	0.05270 {1}	0.07595 {6}	0.06048 {3}	0.17139 {8}	0.06282 {4}	0.07988 {7}
		$\hat{\beta}$	0.26915 {7}	0.21584 {3}	0.11167 {1}	0.21249 {2}	0.25476 {6}	0.29996 {8}	0.22130 {4}	0.22406 {5}
		$\hat{\tau}$	0.18664 {8}	0.10593 {3}	0.04411 {1}	0.10428 {2}	0.13132 {4}	0.16716 {7}	0.14005 {6}	0.13425 {5}
		$\hat{\eta}$	2.12872 {8}	0.82284 {3}	0.33007 {2}	0.82805 {4}	0.10290 {1}	0.86617 {5}	1.06789 {7}	1.05971 {6}
	MSE	$\hat{\alpha}$	0.00578 {3}	0.00962 {5}	0.00465 {1}	0.00982 {6}	0.00574 {2}	1.65494 {8}	0.00645 {4}	0.01070 {7}
		$\hat{\beta}$	0.29023 {7}	0.09705 {2}	0.02723 {1}	0.09834 {3}	0.69288 {8}	0.26741 {6}	0.10521 {4}	0.10962 {5}
		$\hat{\tau}$	0.16099 {7}	0.01935 {3}	0.00683 {1}	0.01918 {2}	0.38436 {8}	0.09365 {6}	0.03953 {5}	0.03428 {4}
		$\hat{\eta}$	10.09945 {8}	0.87654 {3}	0.36303 {2}	0.88906 {4}	0.18945 {1}	1.39581 {5}	1.58986 {7}	1.55411 {6}
	MRE	$\hat{\alpha}$	0.07929 {2}	0.10106 {5}	0.07026 {1}	0.10127 {6}	0.08064 {3}	0.22852 {8}	0.08376 {4}	0.10650 {7}
		$\hat{\beta}$	0.53830 {7}	0.43168 {3}	0.22334 {1}	0.42497 {2}	0.50951 {6}	0.59992 {8}	0.44259 {4}	0.44811 {5}
		$\hat{\tau}$	1.24429 {8}	0.70623 {3}	0.29408 {1}	0.69519 {2}	0.87550 {4}	1.11441 {7}	0.93369 {6}	0.89503 {5}
		$\hat{\eta}$	0.94610 {8}	0.36571 {3}	0.14670 {2}	0.36802 {4}	0.04573 {1}	0.38497 {5}	0.47462 {7}	0.47098 {6}
		$\sum Ranks$	75 {7}	41 {2}	15 {1}	43 {3}	47 {4}	81 {8}	62 {5}	68 {6}
500	|BIAS|	$\hat{\alpha}$	0.04917 {2}	0.06241 {5}	0.04313 {1}	0.06251 {6}	0.04969 {3}	0.09571 {8}	0.05179 {4}	0.06500 {7}
		$\hat{\beta}$	0.21123 {7}	0.18117 {3}	0.09133 {1}	0.18167 {4}	0.15341 {2}	0.24274 {8}	0.18484 {5}	0.18626 {6}
		$\hat{\tau}$	0.14447 {8}	0.09572 {3}	0.03520 {1}	0.09615 {4}	0.06328 {2}	0.13495 {7}	0.12118 {6}	0.11635 {5}
		$\hat{\eta}$	1.76979 {8}	0.77832 {4}	0.26677 {2}	0.80527 {5}	0.05987 {1}	0.75633 {3}	0.97526 {7}	0.97395 {6}
	MSE	$\hat{\alpha}$	0.00387 {3}	0.00624 {5}	0.00301 {1}	0.00641 {6}	0.00385 {2}	0.03104 {8}	0.00427 {4}	0.00704 {7}
		$\hat{\beta}$	0.14181 {6}	0.06404 {2}	0.01723 {1}	0.06484 {3}	0.21773 {8}	0.19657 {7}	0.07033 {5}	0.06996 {4}
		$\hat{\tau}$	0.08181 {7}	0.01504 {2}	0.00459 {1}	0.01521 {3}	0.12670 {8}	0.06202 {6}	0.02902 {5}	0.02506 {4}
		$\hat{\eta}$	7.31760 {8}	0.78440 {3}	0.26828 {2}	0.83003 {4}	0.07129 {1}	1.16662 {5}	1.35640 {7}	1.33809 {6}
	MRE	$\hat{\alpha}$	0.06556 {2}	0.08321 {5}	0.05751 {1}	0.08335 {6}	0.06625 {3}	0.12761 {8}	0.06905 {4}	0.08667 {7}
		$\hat{\beta}$	0.42246 {7}	0.36233 {3}	0.18267 {1}	0.36334 {4}	0.30682 {2}	0.48549 {8}	0.36969 {5}	0.37252 {6}
		$\hat{\tau}$	0.96315 {8}	0.63812 {3}	0.23470 {1}	0.64101 {4}	0.42183 {2}	0.89967 {7}	0.80788 {6}	0.77566 {5}
		$\hat{\eta}$	0.78657 {8}	0.34592 {4}	0.11856 {2}	0.35790 {5}	0.02661 {1}	0.33615 {3}	0.43345 {7}	0.43287 {6}
		$\sum Ranks$	74 {7}	42 {3}	15 {1}	54 {4}	35 {2}	78 {8}	65 {5}	69 {6}

Table 7: Simulation results for $\underline{\varphi} = (\alpha = 0.5, \beta = 0.75, \tau = 0.5, \eta = 1.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.10370 {4}	0.12248 {5}	0.10048 {2}	0.13790 {7}	0.09809 {1}	1.08165 {8}	0.10314 {3}	0.13216 {6}
		$\hat{\beta}$	4.74309 {8}	1.16693 {4}	1.15911 {3}	1.08330 {2}	0.81248 {1}	1.25249 {5}	1.76811 {7}	1.76517 {6}
		$\hat{\tau}$	4.53547 {7}	0.62606 {2}	0.86288 {3}	0.60712 {1}	9.93087 {8}	1.26958 {4}	1.58629 {6}	1.48514 {5}
		$\hat{\eta}$	5.16103 {8}	0.69157 {1}	0.78959 {3}	0.75667 {2}	1.07124 {6}	1.20838 {7}	1.03145 {4}	1.05107 {5}
	MSE	$\hat{\alpha}$	0.02266 {4}	0.04601 {5}	0.02106 {2}	0.35161 {7}	0.01666 {1}	26.25023 {8}	0.02202 {3}	0.06670 {6}
		$\hat{\beta}$	900.78250 {7}	8.74056 {3}	15.80857 {4}	7.53589 {1}	2445.03749 {8}	7.55345 {2}	26.98405 {6}	24.43380 {5}
		$\hat{\tau}$	405.72386 {7}	1.69407 {1}	6.25587 {3}	1.74698 {2}	1861.10312 {8}	9.22966 {4}	13.71764 {6}	12.39081 {5}
		$\hat{\eta}$	43.49341 {8}	0.90064 {1}	1.11646 {3}	1.04464 {2}	5.87579 {7}	2.71367 {6}	1.76506 {4}	1.83862 {5}
	MRE	$\hat{\alpha}$	0.20741 {4}	0.24497 {5}	0.20096 {2}	0.27581 {7}	0.19617 {1}	2.16330 {8}	0.20629 {3}	0.26431 {6}
		$\hat{\beta}$	6.32412 {7}	1.55591 {3}	1.54548 {2}	1.44439 {1}	13.66107 {8}	1.66999 {4}	2.35748 {6}	2.35356 {5}
		$\hat{\tau}$	9.07094 {7}	1.25212 {2}	1.72575 {3}	1.21423 {1}	19.86174 {8}	2.53915 {4}	3.17258 {6}	2.97028 {5}
		$\hat{\eta}$	3.44068 {8}	0.46105 {1}	0.52639 {3}	0.50445 {2}	0.71416 {6}	0.80559 {7}	0.68764 {4}	0.70072 {5}
		$\sum Ranks$	79 {8}	33 {1.5}	33 {1.5}	35 {3}	63 {5}	67 {7}	58 {4}	64 {6}
150	|BIAS|	$\hat{\alpha}$	0.05535 {4}	0.06232 {5}	0.05195 {1}	0.06311 {6}	0.05498 {3}	0.43609 {8}	0.05450 {2}	0.06606 {7}
		$\hat{\beta}$	1.19438 {7}	0.58079 {3}	0.45982 {1}	0.55648 {2}	2.45383 {8}	0.96054 {6}	0.74632 {5}	0.72390 {4}
		$\hat{\tau}$	1.63981 {7}	0.50388 {3}	0.44016 {1}	0.48546 {2}	3.18439 {8}	1.27515 {6}	0.86331 {5}	0.81500 {4}
		$\hat{\eta}$	2.77871 {8}	0.61480 {3}	0.53767 {2}	0.65210 {4}	0.52021 {1}	0.94899 {7}	0.86448 {5}	0.86814 {6}
	MSE	$\hat{\alpha}$	0.00534 {4}	0.00650 {5}	0.00445 {1}	0.00684 {6}	0.00463 {2}	6.55158 {8}	0.00488 {3}	0.00745 {7}
		$\hat{\beta}$	11.34797 {7}	0.73682 {3}	0.52716 {1}	0.68380 {2}	51.35412 {8}	3.34749 {6}	1.51747 {4}	1.69603 {5}
		$\hat{\tau}$	25.72928 {7}	0.50055 {2}	0.53980 {3}	0.46436 {1}	81.41686 {8}	5.94990 {6}	1.92943 {5}	1.74761 {4}
		$\hat{\eta}$	15.28283 {8}	0.55256 {2}	0.54416 {1}	0.63038 {3}	1.89605 {7}	1.35156 {6}	1.12517 {5}	1.12137 {4}
	MRE	$\hat{\alpha}$	0.11070 {4}	0.12465 {5}	0.10391 {1}	0.12623 {6}	0.10996 {3}	0.87219 {8}	0.10900 {2}	0.13212 {7}
		$\hat{\beta}$	1.59251 {7}	0.77439 {3}	0.61310 {1}	0.74197 {2}	3.27178 {8}	1.28072 {6}	0.99509 {5}	0.96520 {4}
		$\hat{\tau}$	3.27962 {7}	1.00775 {3}	0.88032 {1}	0.97091 {2}	6.36879 {8}	2.55030 {6}	1.72663 {5}	1.62999 {4}
		$\hat{\eta}$	1.85248 {8}	0.40986 {3}	0.35845 {2}	0.43473 {4}	0.34680 {1}	0.63266 {7}	0.57632 {5}	0.57876 {6}
		$\sum Ranks$	78 {7}	40 {2.5}	16 {1}	40 {2.5}	65 {6}	80 {8}	51 {4}	62 {5}
350	|BIAS|	$\hat{\alpha}$	0.03775 {4}	0.04073 {5}	0.03328 {1}	0.04094 {6}	0.03471 {2}	0.14022 {8}	0.03519 {3}	0.04237 {7}
		$\hat{\beta}$	0.58834 {6}	0.40350 {3}	0.26357 {1}	0.39216 {2}	0.81248 {7}	0.95730 {8}	0.48672 {5}	0.47325 {4}
		$\hat{\tau}$	0.80025 {6}	0.43681 {3}	0.25524 {1}	0.42434 {2}	1.07766 {7}	1.09349 {8}	0.60536 {5}	0.58398 {4}
		$\hat{\eta}$	1.55433 {8}	0.59373 {3}	0.35790 {2}	0.61038 {4}	0.26742 {1}	0.80919 {7}	0.73692 {6}	0.73384 {5}
	MSE	$\hat{\alpha}$	0.00336 {7}	0.00270 {4}	0.00180 {1}	0.00275 {5}	0.00188 {2}	1.23923 {8}	0.00201 {3}	0.00292 {6}
		$\hat{\beta}$	1.96344 {6}	0.29895 {3}	0.14939 {1}	0.28396 {2}	5.65336 {8}	2.82053 {7}	0.51136 {5}	0.47630 {4}
		$\hat{\tau}$	4.45362 {7}	0.32108 {3}	0.18154 {1}	0.30306 {2}	12.72570 {8}	3.51337 {6}	0.79345 {5}	0.70765 {4}
		$\hat{\eta}$	5.83853 {8}	0.47379 {2}	0.27424 {1}	0.50524 {3}	0.66012 {4}	0.95328 {7}	0.79823 {6}	0.78083 {5}
	MRE	$\hat{\alpha}$	0.07551 {4}	0.08147 {5}	0.06657 {1}	0.08188 {6}	0.06943 {2}	0.28044 {8}	0.07038 {3}	0.08473 {7}
		$\hat{\beta}$	0.78445 {6}	0.53801 {3}	0.35143 {1}	0.52287 {2}	1.08330 {7}	1.27641 {8}	0.64896 {5}	0.63100 {4}
		$\hat{\tau}$	1.60051 {6}	0.87361 {3}	0.51049 {1}	0.84867 {2}	2.15532 {7}	2.18699 {8}	1.21072 {5}	1.16795 {4}
		$\hat{\eta}$	1.03622 {8}	0.39582 {3}	0.23860 {2}	0.40692 {4}	0.17828 {1}	0.53946 {7}	0.49128 {6}	0.48923 {5}
		$\sum Ranks$	76 {7}	40 {2.5}	14 {1}	40 {2.5}	56 {4}	90 {8}	57 {5}	59 {6}
500	|BIAS|	$\hat{\alpha}$	0.03281 {4}	0.03386 {5}	0.02732 {1}	0.03396 {6}	0.02870 {2}	0.07955 {8}	0.02912 {3}	0.03486 {7}
		$\hat{\beta}$	0.44960 {6}	0.35191 {3}	0.20933 {1}	0.34446 {2}	0.48011 {7}	0.87812 {8}	0.41094 {5}	0.39878 {4}
		$\hat{\tau}$	0.60418 {7}	0.39909 {3}	0.19926 {1}	0.39037 {2}	0.60073 {6}	0.95993 {8}	0.52396 {5}	0.49833 {4}
		$\hat{\eta}$	1.27047 {8}	0.55440 {3}	0.29481 {2}	0.56694 {4}	0.16397 {1}	0.75001 {7}	0.68055 {6}	0.67593 {5}
	MSE	$\hat{\alpha}$	0.00321 {7}	0.00182 {4}	0.00121 {1}	0.00184 {5}	0.00127 {2}	0.04414 {8}	0.00135 {3}	0.00198 {6}
		$\hat{\beta}$	0.87193 {6}	0.21972 {3}	0.08842 {1}	0.20915 {2}	2.10906 {7}	2.24284 {8}	0.34783 {5}	0.31412 {4}
		$\hat{\tau}$	2.02400 {6}	0.25733 {3}	0.11608 {1}	0.24736 {2}	5.60584 {8}	2.60169 {7}	0.57493 {5}	0.48961 {4}
		$\hat{\eta}$	3.97945 {8}	0.40372 {3}	0.20183 {1}	0.42709 {4}	0.21626 {2}	0.83159 {7}	0.68168 {6}	0.66333 {5}
	MRE	$\hat{\alpha}$	0.06562 {4}	0.06773 {5}	0.05464 {1}	0.06792 {6}	0.05740 {2}	0.15910 {8}	0.05825 {3}	0.06973 {7}
		$\hat{\beta}$	0.59947 {6}	0.46922 {3}	0.27911 {1}	0.45928 {2}	0.64014 {7}	1.17082 {8}	0.54792 {5}	0.53170 {4}
		$\hat{\tau}$	1.20835 {7}	0.79819 {3}	0.39851 {1}	0.78075 {2}	1.20146 {6}	1.91985 {8}	1.04791 {5}	0.99666 {4}
		$\hat{\eta}$	0.84698 {8}	0.36960 {3}	0.19654 {2}	0.37796 {4}	0.10931 {1}	0.50001 {7}	0.45370 {6}	0.45062 {5}
		$\sum Ranks$	77 {7}	41 {2.5}	14 {1}	41 {2.5}	51 {4}	92 {8}	57 {5}	59 {6}

Table 8: Simulation results for $\underline{\varphi} = (\alpha = 1.5, \beta = 0.5, \tau = 0.15, \eta = 3.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	1.30574 {4}	1.41684 {5}	0.81717 {1}	1.57430 {6}	0.89106 {3}	4.31252 {8}	0.87808 {2}	2.57874 {7}
		$\hat{\beta}$	0.80995 {5}	0.86310 {7}	0.46129 {2}	0.84878 {6}	0.12571 {1}	0.58972 {3}	0.77369 {4}	0.88369 {8}
		$\hat{\tau}$	0.38570 {8}	0.13338 {1}	0.13887 {2}	0.13933 {3}	0.34044 {7}	0.25691 {6}	0.22772 {4}	0.25005 {5}
		$\hat{\eta}$	6.01767 {8}	0.89906 {2}	1.38436 {4}	0.96298 {3}	0.35011 {1}	2.25139 {7}	1.77940 {6}	1.71464 {5}
	MSE	$\hat{\alpha}$	46.99992 {6}	19.31439 {3}	10.69018 {1}	21.32503 {4}	39.55717 {5}	163.79060 {8}	11.78598 {2}	66.37845 {7}
		$\hat{\beta}$	5.46735 {7}	4.34174 {5}	1.43855 {2}	4.14290 {4}	12.27168 {8}	1.14421 {1}	4.02941 {3}	4.56618 {6}
		$\hat{\tau}$	1.22860 {7}	0.08348 {1}	0.10765 {3}	0.09053 {2}	1.68178 {8}	0.25709 {5}	0.18166 {4}	0.25793 {6}
		$\hat{\eta}$	57.95668 {8}	1.39930 {2}	4.87386 {6}	1.57116 {3}	0.96492 {1}	7.35393 {7}	4.34551 {5}	4.11270 {4}
	MRE	$\hat{\alpha}$	0.87050 {4}	0.94456 {5}	0.54478 {1}	1.04953 {6}	0.59404 {3}	2.87501 {8}	0.58539 {2}	1.71916 {7}
		$\hat{\beta}$	1.61991 {4}	1.72621 {6}	0.92258 {1}	1.69755 {5}	2.17834 {8}	1.17945 {2}	1.54738 {3}	1.76738 {7}
		$\hat{\tau}$	2.57136 {8}	0.88923 {1}	0.92579 {2}	0.92884 {3}	2.26958 {7}	1.71274 {6}	1.51811 {4}	1.66701 {5}
		$\hat{\eta}$	1.71934 {8}	0.25687 {2}	0.39553 {4}	0.27514 {3}	0.10003 {1}	0.64325 {7}	0.50840 {6}	0.48990 {5}
		$\sum Ranks$	77 {8}	40 {2}	29 {1}	48 {4}	53 {5}	68 {6}	45 {3}	72 {7}
150	|BIAS|	$\hat{\alpha}$	0.28995 {3}	0.43574 {5}	0.21477 {1}	0.45284 {6}	0.27292 {2}	1.15874 {8}	0.30353 {4}	0.51378 {7}
		$\hat{\beta}$	0.34251 {8}	0.30649 {5}	0.16178 {1}	0.30129 {4}	0.29839 {3}	0.31608 {6}	0.27122 {2}	0.33236 {7}
		$\hat{\tau}$	0.22540 {8}	0.07621 {3}	0.06142 {1}	0.07522 {2}	0.09342 {4}	0.17882 {7}	0.12903 {6}	0.12595 {5}
		$\hat{\eta}$	4.24903 {8}	0.96447 {3}	0.82214 {2}	0.99920 {4}	0.13386 {1}	1.91412 {7}	1.62136 {6}	1.54033 {5}
	MSE	$\hat{\alpha}$	0.29009 {4}	0.67428 {5}	0.14307 {1}	0.74035 {6}	0.14407 {2}	19.09569 {8}	0.19669 {3}	1.76506 {7}
		$\hat{\beta}$	0.52343 {7}	0.31052 {5}	0.06113 {1}	0.30019 {4}	0.63196 {8}	0.20588 {3}	0.19202 {2}	0.36041 {6}
		$\hat{\tau}$	0.22289 {8}	0.01158 {3}	0.01050 {1}	0.01138 {2}	0.21228 {7}	0.07756 {6}	0.03194 {4}	0.03572 {5}
		$\hat{\eta}$	32.40851 {8}	1.34949 {2}	1.67410 {4}	1.43644 {3}	0.28445 {1}	5.13892 {7}	3.48710 {6}	3.09800 {5}
	MRE	$\hat{\alpha}$	0.19330 {3}	0.29049 {5}	0.14318 {1}	0.30190 {6}	0.18194 {2}	0.77249 {8}	0.20235 {4}	0.34252 {7}
		$\hat{\beta}$	0.68501 {8}	0.61298 {5}	0.32356 {1}	0.60257 {4}	0.59678 {3}	0.63216 {6}	0.54243 {2}	0.66473 {7}
		$\hat{\tau}$	1.50266 {8}	0.50809 {3}	0.40949 {1}	0.50144 {2}	0.62277 {4}	1.19214 {7}	0.86017 {6}	0.83967 {5}
		$\hat{\eta}$	1.21401 {8}	0.27556 {3}	0.23490 {2}	0.28548 {4}	0.03825 {1}	0.54689 {7}	0.46325 {6}	0.44009 {5}
		$\sum Ranks$	81 {8}	47 {3.5}	17 {1}	47 {3.5}	38 {2}	80 {7}	51 {5}	71 {6}
350	|BIAS|	$\hat{\alpha}$	0.17222 {3}	0.25245 {5}	0.11615 {1}	0.25571 {6}	0.17087 {2}	0.31562 {8}	0.18763 {4}	0.27374 {7}
		$\hat{\beta}$	0.20685 {7}	0.18396 {5}	0.08898 {1}	0.18102 {4}	0.12571 {2}	0.21595 {8}	0.16613 {3}	0.19465 {6}
		$\hat{\tau}$	0.15208 {8}	0.06574 {4}	0.03404 {2}	0.06471 {3}	0.01840 {1}	0.13222 {7}	0.09746 {6}	0.09414 {5}
		$\hat{\eta}$	2.97854 {8}	0.97521 {3}	0.47898 {2}	0.99054 {4}	0.03461 {1}	1.42149 {7}	1.38710 {6}	1.38001 {5}
	MSE	$\hat{\alpha}$	0.05553 {3}	0.12475 {5}	0.03268 {1}	0.13009 {6}	0.04986 {2}	0.73160 {8}	0.06457 {4}	0.16523 {7}
		$\hat{\beta}$	0.13219 {8}	0.07564 {6}	0.01804 {1}	0.07035 {4}	0.05850 {3}	0.10439 {7}	0.05456 {2}	0.07493 {5}
		$\hat{\tau}$	0.07525 {8}	0.00685 {3}	0.00384 {1}	0.00656 {2}	0.01704 {6}	0.03943 {7}	0.01615 {5}	0.01507 {4}
		$\hat{\eta}$	17.43789 {8}	1.26158 {3}	0.80840 {2}	1.30180 {4}	0.04666 {1}	3.11986 {7}	2.58743 {6}	2.45535 {5}
	MRE	$\hat{\alpha}$	0.11481 {3}	0.16830 {5}	0.07743 {1}	0.17047 {6}	0.11391 {2}	0.21041 {8}	0.12509 {4}	0.18250 {7}
		$\hat{\beta}$	0.41371 {7}	0.36791 {5}	0.17796 {1}	0.36205 {4}	0.25142 {2}	0.43190 {8}	0.33226 {3}	0.38930 {6}
		$\hat{\tau}$	1.01385 {8}	0.43830 {4}	0.22695 {2}	0.43139 {3}	0.12269 {1}	0.88147 {7}	0.64974 {6}	0.62761 {5}
		$\hat{\eta}$	0.85101 {8}	0.27863 {3}	0.13685 {2}	0.28301 {4}	0.00989 {1}	0.40614 {7}	0.39631 {6}	0.39429 {5}
		$\sum Ranks$	79 {7}	51 {4}	17 {1}	50 {3}	24 {2}	89 {8}	55 {5}	67 {6}
500	|BIAS|	$\hat{\alpha}$	0.14165 {3}	0.20590 {6}	0.09192 {1}	0.20762 {7}	0.14053 {2}	0.20457 {5}	0.15309 {4}	0.22002 {8}
		$\hat{\beta}$	0.16961 {7}	0.15496 {5}	0.07191 {1}	0.15350 {4}	0.09832 {2}	0.18252 {8}	0.13946 {3}	0.16117 {6}
		$\hat{\tau}$	0.12396 {8}	0.06209 {4}	0.02501 {2}	0.06147 {3}	0.01080 {1}	0.11018 {7}	0.08464 {6}	0.08406 {5}
		$\hat{\eta}$	2.61086 {8}	0.96617 {3}	0.35378 {2}	0.98053 {4}	0.02042 {1}	1.23419 {5}	1.26682 {6}	1.28772 {7}
	MSE	$\hat{\alpha}$	0.03561 {3}	0.07538 {5}	0.01961 {1}	0.07771 {6}	0.03297 {2}	0.18353 {8}	0.04129 {4}	0.08542 {7}
		$\hat{\beta}$	0.08136 {8}	0.04426 {5}	0.01123 {1}	0.04331 {4}	0.02583 {2}	0.07954 {7}	0.03653 {3}	0.04824 {6}
		$\hat{\tau}$	0.04554 {8}	0.00564 {3}	0.00230 {1}	0.00553 {2}	0.00642 {4}	0.02956 {7}	0.01206 {6}	0.01147 {5}
		$\hat{\eta}$	13.92643 {8}	1.21202 {3}	0.53989 {2}	1.25018 {4}	0.01954 {1}	2.54929 {7}	2.21805 {6}	2.14532 {5}
	MRE	$\hat{\alpha}$	0.09444 {3}	0.13727 {6}	0.06128 {1}	0.13841 {7}	0.09369 {2}	0.13638 {5}	0.10206 {4}	0.14668 {8}
		$\hat{\beta}$	0.33923 {7}	0.30991 {5}	0.14383 {1}	0.30699 {4}	0.19664 {2}	0.36505 {8}	0.27893 {3}	0.32233 {6}
		$\hat{\tau}$	0.82637 {8}	0.41392 {4}	0.16674 {2}	0.40979 {3}	0.07198 {1}	0.73456 {7}	0.56429 {6}	0.56041 {5}
		$\hat{\eta}$	0.74596 {8}	0.27605 {3}	0.10108 {2}	0.28015 {4}	0.00583 {1}	0.35263 {5}	0.36195 {6}	0.36792 {7}
		$\sum Ranks$	79 {7.5}	52 {3.5}	17 {1}	52 {3.5}	21 {2}	79 {7.5}	57 {5}	75 {6}

Table 9: Simulation results for $\underline{\varphi} = (\alpha = 0.5, \beta = 0.5, \tau = 0.15, \eta = 1.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.10004$^{\{3\}}$	0.11622$^{\{5\}}$	0.09450$^{\{2\}}$	0.12396$^{\{6\}}$	0.09433$^{\{1\}}$	0.65959$^{\{8\}}$	0.10081$^{\{4\}}$	0.13408$^{\{7\}}$
		$\hat{\beta}$	2.81920$^{\{8\}}$	1.79084$^{\{6\}}$	0.84079$^{\{3\}}$	1.75753$^{\{5\}}$	0.54762$^{\{1\}}$	0.75611$^{\{2\}}$	1.90109$^{\{7\}}$	1.63668$^{\{4\}}$
		$\hat{\tau}$	1.32847$^{\{7\}}$	0.39019$^{\{3\}}$	0.26204$^{\{2\}}$	0.39303$^{\{4\}}$	2.38950$^{\{8\}}$	0.24139$^{\{1\}}$	0.62095$^{\{6\}}$	0.54765$^{\{5\}}$
		$\hat{\eta}$	5.28805$^{\{8\}}$	0.51837$^{\{1\}}$	0.68061$^{\{3\}}$	0.58203$^{\{2\}}$	0.81410$^{\{4\}}$	1.01153$^{\{7\}}$	0.95101$^{\{5\}}$	0.99549$^{\{6\}}$
	MSE	$\hat{\alpha}$	0.02025$^{\{4\}}$	0.03033$^{\{5\}}$	0.01748$^{\{2\}}$	0.03776$^{\{6\}}$	0.01519$^{\{1\}}$	10.24507$^{\{8\}}$	0.02019$^{\{3\}}$	0.08959$^{\{7\}}$
		$\hat{\beta}$	639.03002$^{\{7\}}$	51.22913$^{\{4\}}$	19.32979$^{\{2\}}$	60.72460$^{\{5\}}$	826.23494$^{\{8\}}$	3.33352$^{\{1\}}$	97.33014$^{\{6\}}$	24.76837$^{\{3\}}$
		$\hat{\tau}$	65.51855$^{\{7\}}$	1.44520$^{\{3\}}$	1.12331$^{\{2\}}$	1.57520$^{\{4\}}$	150.66817$^{\{8\}}$	0.41808$^{\{1\}}$	4.72169$^{\{6\}}$	1.79151$^{\{5\}}$
		$\hat{\eta}$	45.44411$^{\{8\}}$	0.49734$^{\{1\}}$	0.87788$^{\{3\}}$	0.58206$^{\{2\}}$	4.98542$^{\{7\}}$	2.62529$^{\{6\}}$	1.36213$^{\{4\}}$	1.54441$^{\{5\}}$
	MRE	$\hat{\alpha}$	0.20007$^{\{3\}}$	0.23245$^{\{5\}}$	0.18900$^{\{2\}}$	0.24792$^{\{6\}}$	0.18866$^{\{1\}}$	1.31919$^{\{8\}}$	0.20162$^{\{4\}}$	0.26817$^{\{7\}}$
		$\hat{\beta}$	5.63841$^{\{7\}}$	3.58168$^{\{5\}}$	1.68158$^{\{2\}}$	3.51505$^{\{4\}}$	11.06558$^{\{8\}}$	1.51222$^{\{1\}}$	3.80218$^{\{6\}}$	3.27337$^{\{3\}}$
		$\hat{\tau}$	8.85645$^{\{7\}}$	2.60123$^{\{3\}}$	1.74694$^{\{2\}}$	2.62019$^{\{4\}}$	15.92997$^{\{8\}}$	1.60930$^{\{1\}}$	4.13970$^{\{6\}}$	3.65098$^{\{5\}}$
		$\hat{\eta}$	3.52537$^{\{8\}}$	0.34558$^{\{1\}}$	0.45374$^{\{3\}}$	0.38802$^{\{2\}}$	0.54273$^{\{4\}}$	0.67436$^{\{7\}}$	0.63401$^{\{5\}}$	0.66366$^{\{6\}}$
		$\sum Ranks$	77$^{\{8\}}$	42$^{\{2\}}$	28$^{\{1\}}$	50$^{\{3\}}$	59$^{\{5\}}$	51$^{\{4\}}$	62$^{\{6\}}$	63$^{\{7\}}$
150	|BIAS|	$\hat{\alpha}$	0.05336$^{\{4\}}$	0.06184$^{\{5\}}$	0.04993$^{\{1\}}$	0.06280$^{\{6\}}$	0.05195$^{\{2\}}$	0.38527$^{\{8\}}$	0.05310$^{\{3\}}$	0.06519$^{\{7\}}$
		$\hat{\beta}$	0.80258$^{\{7\}}$	0.57269$^{\{6\}}$	0.30950$^{\{1\}}$	0.54937$^{\{4\}}$	1.60091$^{\{8\}}$	0.43205$^{\{2\}}$	0.56504$^{\{5\}}$	0.49565$^{\{3\}}$
		$\hat{\tau}$	0.53525$^{\{7\}}$	0.20458$^{\{3\}}$	0.12959$^{\{1\}}$	0.19870$^{\{2\}}$	1.04049$^{\{8\}}$	0.23118$^{\{4\}}$	0.29315$^{\{6\}}$	0.25895$^{\{5\}}$
		$\hat{\eta}$	2.82880$^{\{8\}}$	0.57015$^{\{3\}}$	0.46516$^{\{1\}}$	0.59719$^{\{4\}}$	0.48757$^{\{2\}}$	0.68189$^{\{5\}}$	0.86609$^{\{6\}}$	0.89465$^{\{7\}}$
	MSE	$\hat{\alpha}$	0.00528$^{\{4\}}$	0.00633$^{\{5\}}$	0.00413$^{\{1\}}$	0.00668$^{\{6\}}$	0.00421$^{\{2\}}$	4.15297$^{\{8\}}$	0.00466$^{\{3\}}$	0.00723$^{\{7\}}$
		$\hat{\beta}$	5.26249$^{\{7\}}$	2.59191$^{\{6\}}$	0.28347$^{\{1\}}$	2.17147$^{\{5\}}$	24.20654$^{\{8\}}$	0.68753$^{\{2\}}$	1.05849$^{\{4\}}$	0.91185$^{\{3\}}$
		$\hat{\tau}$	2.69795$^{\{7\}}$	0.14444$^{\{3\}}$	0.05215$^{\{1\}}$	0.13069$^{\{2\}}$	9.08598$^{\{8\}}$	0.28128$^{\{6\}}$	0.25593$^{\{5\}}$	0.18579$^{\{4\}}$
		$\hat{\eta}$	16.32209$^{\{8\}}$	0.45336$^{\{2\}}$	0.42632$^{\{1\}}$	0.49816$^{\{3\}}$	2.26769$^{\{7\}}$	0.91256$^{\{4\}}$	1.05403$^{\{5\}}$	1.15063$^{\{6\}}$
	MRE	$\hat{\alpha}$	0.10673$^{\{4\}}$	0.12369$^{\{5\}}$	0.09987$^{\{1\}}$	0.12560$^{\{6\}}$	0.10390$^{\{2\}}$	0.77053$^{\{8\}}$	0.10620$^{\{3\}}$	0.13038$^{\{7\}}$
		$\hat{\beta}$	1.60515$^{\{7\}}$	1.14539$^{\{6\}}$	0.61900$^{\{1\}}$	1.09874$^{\{4\}}$	3.20182$^{\{8\}}$	0.86411$^{\{2\}}$	1.13008$^{\{5\}}$	0.99129$^{\{3\}}$
		$\hat{\tau}$	3.56834$^{\{7\}}$	1.36387$^{\{3\}}$	0.86390$^{\{1\}}$	1.32465$^{\{2\}}$	6.93657$^{\{8\}}$	1.54122$^{\{4\}}$	1.95431$^{\{6\}}$	1.72632$^{\{5\}}$
		$\hat{\eta}$	1.88586$^{\{8\}}$	0.38010$^{\{3\}}$	0.31011$^{\{1\}}$	0.39813$^{\{4\}}$	0.32504$^{\{2\}}$	0.45459$^{\{5\}}$	0.57740$^{\{6\}}$	0.59643$^{\{7\}}$
		$\sum Ranks$	78$^{\{8\}}$	50$^{\{3\}}$	12$^{\{1\}}$	48$^{\{2\}}$	65$^{\{7\}}$	58$^{\{5\}}$	57$^{\{4\}}$	64$^{\{6\}}$
350	|BIAS|	$\hat{\alpha}$	0.03557$^{\{4\}}$	0.03998$^{\{5\}}$	0.03218$^{\{1\}}$	0.04023$^{\{6\}}$	0.03347$^{\{2\}}$	0.13757$^{\{8\}}$	0.03442$^{\{3\}}$	0.04215$^{\{7\}}$
		$\hat{\beta}$	0.40330$^{\{7\}}$	0.31198$^{\{4\}}$	0.17078$^{\{1\}}$	0.30476$^{\{2\}}$	0.54762$^{\{8\}}$	0.37101$^{\{6\}}$	0.33150$^{\{5\}}$	0.31042$^{\{3\}}$
		$\hat{\tau}$	0.26173$^{\{7\}}$	0.14405$^{\{3\}}$	0.07215$^{\{1\}}$	0.14085$^{\{2\}}$	0.36577$^{\{8\}}$	0.18736$^{\{6\}}$	0.18654$^{\{5\}}$	0.18020$^{\{4\}}$
		$\hat{\eta}$	1.65038$^{\{8\}}$	0.56361$^{\{4\}}$	0.31324$^{\{2\}}$	0.58079$^{\{5\}}$	0.20126$^{\{1\}}$	0.54689$^{\{3\}}$	0.75004$^{\{7\}}$	0.74691$^{\{6\}}$
	MSE	$\hat{\alpha}$	0.00299$^{\{7\}}$	0.00260$^{\{4\}}$	0.00168$^{\{1\}}$	0.00266$^{\{5\}}$	0.00176$^{\{2\}}$	1.01522$^{\{8\}}$	0.00194$^{\{3\}}$	0.00289$^{\{6\}}$
		$\hat{\beta}$	1.05131$^{\{7\}}$	0.23100$^{\{4\}}$	0.06160$^{\{1\}}$	0.22374$^{\{3\}}$	2.54887$^{\{8\}}$	0.55816$^{\{6\}}$	0.25347$^{\{5\}}$	0.21908$^{\{2\}}$
		$\hat{\tau}$	0.57496$^{\{7\}}$	0.04105$^{\{3\}}$	0.01481$^{\{1\}}$	0.04009$^{\{2\}}$	1.58922$^{\{8\}}$	0.16339$^{\{6\}}$	0.07627$^{\{5\}}$	0.07056$^{\{4\}}$
		$\hat{\eta}$	6.47514$^{\{8\}}$	0.40553$^{\{2\}}$	0.22707$^{\{1\}}$	0.43365$^{\{3\}}$	0.50743$^{\{4\}}$	0.61849$^{\{5\}}$	0.78713$^{\{6\}}$	0.79089$^{\{7\}}$
	MRE	$\hat{\alpha}$	0.07114$^{\{4\}}$	0.07996$^{\{5\}}$	0.06435$^{\{1\}}$	0.08045$^{\{6\}}$	0.06694$^{\{2\}}$	0.27513$^{\{8\}}$	0.06885$^{\{3\}}$	0.08429$^{\{7\}}$
		$\hat{\beta}$	0.80660$^{\{7\}}$	0.62396$^{\{4\}}$	0.34156$^{\{1\}}$	0.60952$^{\{2\}}$	1.09525$^{\{8\}}$	0.74203$^{\{6\}}$	0.66301$^{\{5\}}$	0.62084$^{\{3\}}$
		$\hat{\tau}$	1.74486$^{\{7\}}$	0.96036$^{\{3\}}$	0.48102$^{\{1\}}$	0.93898$^{\{2\}}$	2.43846$^{\{8\}}$	1.24907$^{\{6\}}$	1.24357$^{\{5\}}$	1.20134$^{\{4\}}$
		$\hat{\eta}$	1.10025$^{\{8\}}$	0.37574$^{\{4\}}$	0.20883$^{\{2\}}$	0.38719$^{\{5\}}$	0.13417$^{\{1\}}$	0.36459$^{\{3\}}$	0.50003$^{\{7\}}$	0.49794$^{\{6\}}$
		$\sum Ranks$	81$^{\{8\}}$	45$^{\{3\}}$	14$^{\{1\}}$	43$^{\{2\}}$	60$^{\{6\}}$	71$^{\{7\}}$	59$^{\{4.5\}}$	59$^{\{4.5\}}$
500	|BIAS|	$\hat{\alpha}$	0.03212$^{\{4\}}$	0.03394$^{\{6\}}$	0.02769$^{\{1\}}$	0.03392$^{\{5\}}$	0.02886$^{\{2\}}$	0.08488$^{\{8\}}$	0.02961$^{\{3\}}$	0.03507$^{\{7\}}$
		$\hat{\beta}$	0.30479$^{\{6\}}$	0.26769$^{\{3\}}$	0.14305$^{\{1\}}$	0.26244$^{\{2\}}$	0.33946$^{\{8\}}$	0.33689$^{\{7\}}$	0.28813$^{\{5\}}$	0.27066$^{\{4\}}$
		$\hat{\tau}$	0.18880$^{\{7\}}$	0.13221$^{\{3\}}$	0.06054$^{\{1\}}$	0.12981$^{\{2\}}$	0.20827$^{\{8\}}$	0.16131$^{\{5\}}$	0.16733$^{\{6\}}$	0.15860$^{\{4\}}$
		$\hat{\eta}$	1.31214$^{\{8\}}$	0.54826$^{\{4\}}$	0.26513$^{\{2\}}$	0.55848$^{\{5\}}$	0.13250$^{\{1\}}$	0.47508$^{\{3\}}$	0.69997$^{\{7\}}$	0.69361$^{\{6\}}$
	MSE	$\hat{\alpha}$	0.00314$^{\{7\}}$	0.00181$^{\{4\}}$	0.00122$^{\{1\}}$	0.00183$^{\{5\}}$	0.00129$^{\{2\}}$	0.08355$^{\{8\}}$	0.00140$^{\{3\}}$	0.00202$^{\{6\}}$
		$\hat{\beta}$	0.40318$^{\{6\}}$	0.15360$^{\{3\}}$	0.04354$^{\{1\}}$	0.14845$^{\{2\}}$	1.02941$^{\{8\}}$	0.44963$^{\{7\}}$	0.17897$^{\{5\}}$	0.15845$^{\{4\}}$
		$\hat{\tau}$	0.21432$^{\{7\}}$	0.03129$^{\{3\}}$	0.01074$^{\{1\}}$	0.03037$^{\{2\}}$	0.65673$^{\{8\}}$	0.10861$^{\{6\}}$	0.05667$^{\{5\}}$	0.05114$^{\{4\}}$
		$\hat{\eta}$	4.35997$^{\{8\}}$	0.38223$^{\{3\}}$	0.17457$^{\{1\}}$	0.39866$^{\{4\}}$	0.22179$^{\{2\}}$	0.47891$^{\{5\}}$	0.68683$^{\{7\}}$	0.67874$^{\{6\}}$
	MRE	$\hat{\alpha}$	0.06424$^{\{4\}}$	0.06787$^{\{6\}}$	0.05538$^{\{1\}}$	0.06785$^{\{5\}}$	0.05773$^{\{2\}}$	0.16976$^{\{8\}}$	0.05921$^{\{3\}}$	0.07013$^{\{7\}}$
		$\hat{\beta}$	0.60958$^{\{6\}}$	0.53538$^{\{3\}}$	0.28611$^{\{1\}}$	0.52488$^{\{2\}}$	0.67892$^{\{8\}}$	0.67379$^{\{7\}}$	0.57626$^{\{5\}}$	0.54132$^{\{4\}}$
		$\hat{\tau}$	1.25869$^{\{7\}}$	0.88139$^{\{3\}}$	0.40357$^{\{1\}}$	0.86538$^{\{2\}}$	1.38846$^{\{8\}}$	1.07541$^{\{5\}}$	1.11550$^{\{6\}}$	1.05733$^{\{4\}}$
		$\hat{\eta}$	0.87476$^{\{8\}}$	0.36551$^{\{4\}}$	0.17675$^{\{2\}}$	0.37232$^{\{5\}}$	0.08834$^{\{1\}}$	0.31672$^{\{3\}}$	0.46665$^{\{7\}}$	0.46241$^{\{6\}}$
		$\sum Ranks$	78$^{\{8\}}$	45$^{\{3\}}$	14$^{\{1\}}$	41$^{\{2\}}$	58$^{\{4\}}$	72$^{\{7\}}$	62$^{\{5.5\}}$	62$^{\{5.5\}}$

Table 10: Partial and overall ranks of all methods of estimation for several parametric values.

$\varphi^{\intercal}$	n	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
$(\hat{\alpha}=0.55,\hat{\beta}=0.5,\hat{\tau}=0.15,\hat{\eta}=2.25)$	50	8	2	1	5	3	4	7	6
	150	8	2.5	1	2.5	4	5.5	7	5.5
	350	7	4	1	3	2	8	6	5
	500	8	4	1	3	2	7	5.5	5.5
$(\hat{\alpha}=0.75,\hat{\beta}=0.75,\hat{\tau}=0.15,\hat{\eta}=2.25)$	50	8	3	1	2	4.5	6	4.5	7
	150	8	2.5	1	2.5	5	6	4	7
	350	7	3	1	2	4	8	5	6
	500	7	4	1	3	2	8	5	6
$(\hat{\alpha}=0.5,\hat{\beta}=0.5,\hat{\tau}=0.5,\hat{\eta}=3.5)$	50	8	4	1	3	2	5	7	6
	150	8	4	2	3	1	7	6	5
	350	7	3.5	1	3.5	2	8	5.5	5.5
	500	8	4	2	3	1	7	6	5
$(\hat{\alpha}=0.5,\hat{\beta}=1.25,\hat{\tau}=0.75,\hat{\eta}=1.5)$	50	8	1.5	1.5	3	5	7	4	6
	150	7	3	1	2	6	8	4	5
	350	7	2	1	3	4	8	5	6
	500	7	3.5	1	3.5	2	8	6	5
$(\hat{\alpha}=0.75,\hat{\beta}=0.5,\hat{\tau}=0.15,\hat{\eta}=2.25)$	50	8	2	1	3	4	7	5	6
	150	8	3	1	2	5	7	4	6
	350	7	2	1	3	4	8	5	6
	500	7	3	1	4	2	8	5	6
$(\hat{\alpha}=0.5,\hat{\beta}=0.75,\hat{\tau}=0.5,\hat{\eta}=1.5)$	50	8	1.5	1.5	3	5	7	4	6
	150	7	2.5	1	2.5	6	8	4	5
	350	7	2.5	1	2.5	4	8	5	6
	500	7	2.5	1	2.5	4	8	5	6
$(\hat{\alpha}=1.5,\hat{\beta}=0.5,\hat{\tau}=0.15,\hat{\eta}=3.5)$	50	8	2	1	4	5	6	3	7
	150	8	3.5	1	3.5	2	7	5	6
	350	7	4	1	3	2	8	5	6
	500	7.5	3.5	1	3.5	2	7.5	5	6
$(\hat{\alpha}=0.5,\hat{\beta}=0.5,\hat{\tau}=0.15,\hat{\eta}=1.5)$	50	8	2	1	3	5	4	6	7
	150	8	3	1	2	7	5	4	6
	350	8	3	1	2	6	7	4.5	4.5
	500	8	3	1	2	4	7	5.5	5.5
$\sum Ranks$		242.5	93.5	35	92.5	116.5	223	162.5	186.5
Overall Rank		8	3	1	2	4	7	5	6

sion at a gauge length of 20 millimeters (9). The data are as follows: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800.

The third dataset presents the total milk production from the first calving of 107 SINDI cows (13)). These cows are owned by Carnauba Farm, which is part of Agropecuaria Manoel Dantas Ltda (AMDA), located in Taperoa City, Paraiba, Brazil. The observations are as follows: 0.6768, 0.7131, 0.2303, 0.5853, 0.5350, 0.4151, 0.5707, 0.4576, 0.7687, 0.6789, 0.3259, 0.4371, 0.3891, 0.5529, 0.5113, 0.4530, 0.4752, 0.3134, 0.0168, 0.1167, 0.5447, 0.3175, 0.6750, 0.4143, 0.6012, 0.1479, 0.0671, 0.2356, 0.1525, 0.5483, 0.5394, 0.7261, 0.2361, 0.6927, 0.3323, 0.4800, 0.4564, 0.6114, 0.5481, 0.3480, 0.7804, 0.3406, 0.3383, 0.5912, 0.1131, 0.4823, 0.5744, 0.7290, 0.6907, 0.4260, 0.6058, 0.5140, 0.7471, 0.2605, 0.4365, 0.8781, 0.6891, 0.6196, 0.4990, 0.5770, 0.3945, 0.5150, 0.0650, 0.0776, 0.4553, 0.4470, 0.5627, 0.5232, 0.8492, 0.5285, 0.6465, 0.8147, 0.4694, 0.0854, 0.4517, 0.3821, 0.3635, 0.4111, 0.4332, 0.3751, 0.2681, 0.5349, 0.1546, 0.4049, 0.4612, 0.3906, 0.4675, 0.4438, 0.3188, 0.2160, 0.3627, 0.6220, 0.6844, 0.6707, 0.5629, 0.3413, 0.3598, 0.5878, 0.6488, 0.4741, 0.7629, 0.5941, 0.5553, 0.6860, 0.2747 0.6174, 0.0609.

The fits of the OIPNH distribution will be compared with several rival distributions, including the heavy-tailed NH(HTNH) (8), power inverted NH (PINH) (2), logistic NH (LNH) (28), sin NH (SNH) (7), and NH (26) distributions. To evaluate the fitted distributions, we use criteria such as the Akaike information criterion (AIC), consistent AIC (CAIC), Hannan Quinn information criterion (HQIC)and Bayesian information criterion (BIC), Cramér–Von Mises (W^*), Anderson–Darling (A^*), Kolmogorov–Smirnov (KS) and its p-value.

The R program is utilized to derive the numerical results presented in Tables 11–16, which list the ML estimates, their corresponding standard errors (SEs), and the values of AIC, CAIC, BIC, HQIC, W^*, A^*, KS, and its p-value for the OIPNH distribution in comparison with the other distributions. From Tables 12, 14 and 16, it is evident that the OIPNH distribution has the lowest goodness-of-fit criteria values among all fitted models, suggesting it as the best choice for the datasets. The figures in these tables further demonstrate that the OIPNH distribution consistently exhibits the lowest values for all goodness-of-fit measures when compared to the other distributions.

Many of the fitted functions for the OIPNH distribution based on the three datasets are illustrated in Figures 4 through 6. To further evaluate the performance of the OIPNH distribution, Figures 7 to 9 present probability-probability (PP) plots that compare the OIPNH model against other competing distributions across the three analyzed datasets. Complementing these, Figures 10 to 12 display quantile-quantile (QQ) plots, which provide an additional visual assessment of how well the OIPNH distribution matches the empirical quantiles relative to other models for the same datasets.

Moreover, the fitted densities for the OIPNH distribution alongside those of the competing distributions are shown in Figures 13 to 15 for each dataset. These density plots offer a clear illustration of the OIPNH distribution's ability to capture the underlying data patterns.

Together, these graphical representations strongly corroborate the numerical results reported in Tables 11 through 14, providing consistent and convincing evidence that the OIPNH distribution yields the best fit across all three datasets.

Table 11: The ML estimates (first line) and their SEs (second line) for the first dataset.

OIPNH	1.4373150	62.425741	80.277100	0.2771830
(α,β,τ,η)	0.4854930	175.83329	316.87495	0.1040409
HTNH	100.65590	_	0.1200671	0.9269960
(λ,τ,η)	496.01472	_	0.0349017	0.1544529
PINH	2.9656028	_	8122.3085	0.1422644
(γ,τ,η)	0.6720443	_	18883.730	0.0378357
LNH	9.6578227	_	498.95081	0.0866665
(λ,τ,η)	1.6414226	_	590.54191	0.0129002
SNH	_	_	0.0912151	0.7489400
(τ,η)	_	_	0.0280919	0.1480413
NH	_	_	0.1216680	0.9226620
(τ,η)	_	_	0.0343990	0.1514951

Table 12: Findings of goodness-of-fit measures for the first dataset.

Model	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	p-value
OIPNH	826.7424	827.0677	838.1506	831.3776	0.0151796	0.0955019	0.0304807	0.9997730
HTNH	834.4916	834.6851	843.0477	837.9680	0.1018377	0.6142459	0.0921060	0.2276114
PINH	834.2528	834.4464	842.8089	837.7292	0.0997016	0.6519923	0.0777970	0.4206828
LNH	835.5166	835.7101	844.0727	838.9930	0.1064994	0.7351019	0.0467464	0.9424276
SNH	830.7828	830.8788	838.4868	833.1004	0.0885677	0.5320476	0.0877286	0.2781190
NH	832.4510	832.5470	838.1550	834.7686	0.1018090	0.6139146	0.0919299	0.2294995

Table 13: The ML estimates (first line) and their SEs (second line) for the second dataset.

OIPNH	2.2111106	24.694344	0.2227084	3.7941612
(α,β,τ,η)	4.2922220	165.04955	1.1806860	14.403677
HTNH	104.06280	_	0.0069886	45.754270
(λ,τ,η)	147.92834	_	0.0049474	32.421788
PINH	11.839980	_	490386.30	0.1900832
(γ,τ,η)	0.3499896	_	13757.650	0.0152372
LNH	6.6309013	_	0.0672164	4.5723548
(λ,τ,η)	0.7664805	_	0.0541143	3.4130715
SNH	_	_	0.0064497	31.855619
(τ,η)	_	_	0.0036684	18.190980
NH	_	_	0.0033384	96.240230
(τ,η)	_	_	0.0012495	37.767033

Table 14: Findings of goodness-of-fit measures for the second dataset.

Model	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	p-value
OIPNH	106.1488	106.7738	115.0852	109.6942	0.0148598	0.1224779	0.0424258	0.9996549
HTNH	224.6794	225.0487	231.3818	227.3385	0.0217593	0.1821324	0.4688867	0.0000000
PINH	106.8499	107.2192	118.5523	109.8090	0.0457965	0.3657762	0.0754337	0.8272287
LNH	106.4067	106.7759	117.1090	109.9657	0.0419252	0.2866701	0.0471052	0.9979719
SNH	228.6150	228.7968	233.0832	230.3877	0.0257174	0.2072138	0.4625097	0.0000000
NH	221.6365	221.8183	226.1047	223.4092	0.0213350	0.1794321	0.4703884	0.0000000

Table 15: The ML estimates (first line) and their SEs (second line) for the third dataset.

OIPNH	0.9360511	9.8856950	0.1872304	13.7239937
(α,β,τ,η)	0.2195765	6.8831931	0.4094654	27.7245503
HTNH	372.81602973	_	0.01600257	96.38751934
(λ,τ,η)	992.27481010	_	0.01890742	113.54306884
PINH	28.034179549	_	0.006427972	0.023963148
(γ,τ,η)	15.33736730	_	0.01376849	0.01352333
LNH	2.54332845	_	0.03141282	48.99427943
(λ,τ,η)	0.21922930	_	0.02693107	41.75985075
SNH	_	_	0.01178709	85.04368470
(τ,η)	_	_	0.01037397	74.85452114
NH	_	_	0.008348958	184.6450
(τ,η)	_	_	0.008619105	191.5014

Table 16: Findings of goodness-of-fit measures for the third dataset.

Model	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	p-value
OIPNH	-50.5690	-50.1768	-39.8777	-46.2349	0.0218437	0.1658830	0.0403840	0.9948972
HTNH	5.1304	5.3634	13.1489	8.3810	0.2623777	1.7028640	0.2942976	0.0000000
PINH	-15.4490	-15.2160	-7.4305	-12.1984	0.5349078	3.0401650	0.2255732	0.0000373
LNH	-20.1209	-19.8879	-12.1024	-16.8703	0.4508261	2.8707590	0.0862966	0.4029481
SNH	10.3964	10.5117	15.7420	12.5634	0.3187559	2.0482320	0.2970653	0.0000000
NH	2.7237	2.8391	8.0694	4.8908	0.2598683	1.6875240	0.2938995	0.0000000

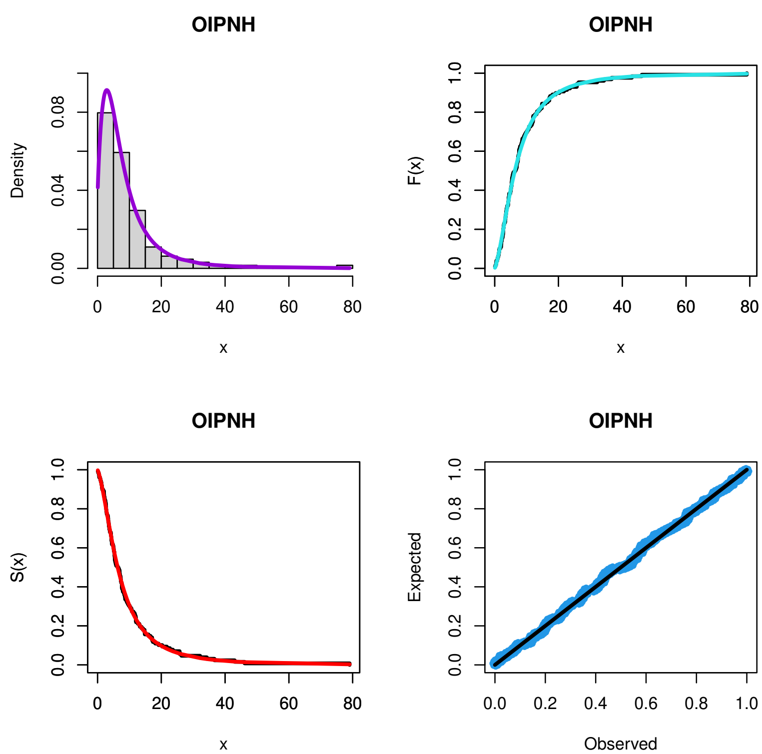

Figure 4: Plots of the fitted functions for the OIPNH distribution based on the first dataset.

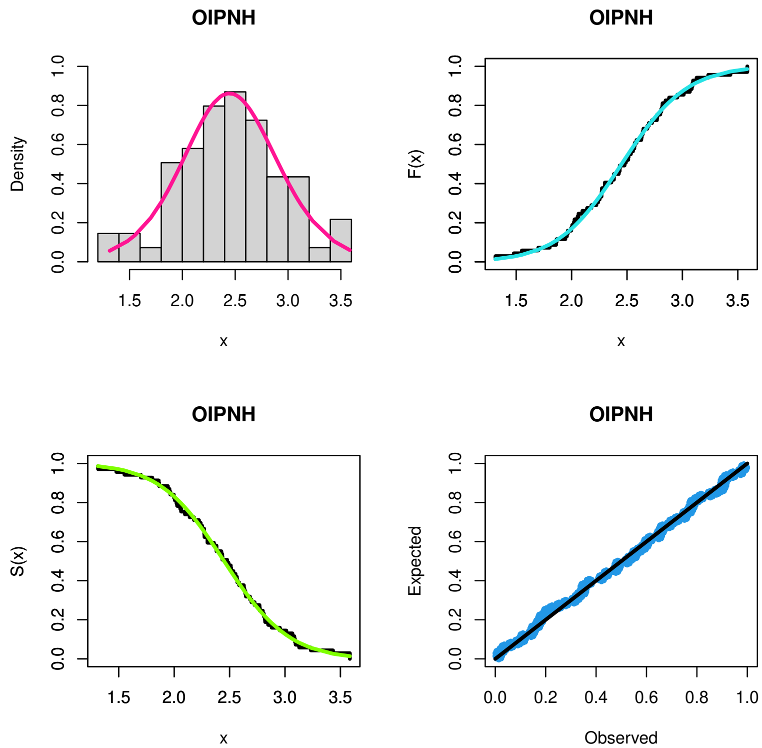

Figure 5: Plots of the fitted functions for the OIPNH distribution based on the second dataset.

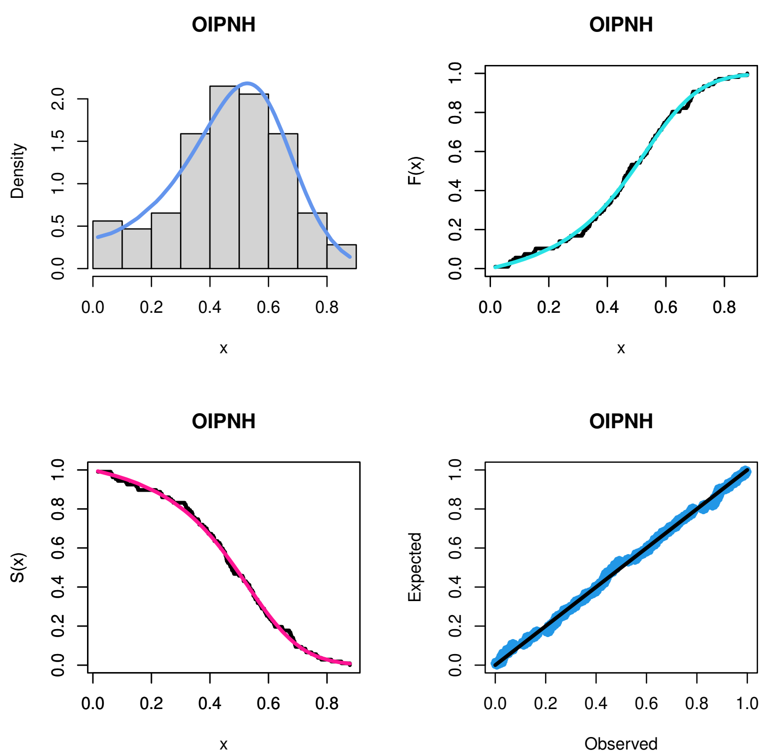

Figure 6: Plots of the fitted functions for the OIPNH distribution based on the third dataset.

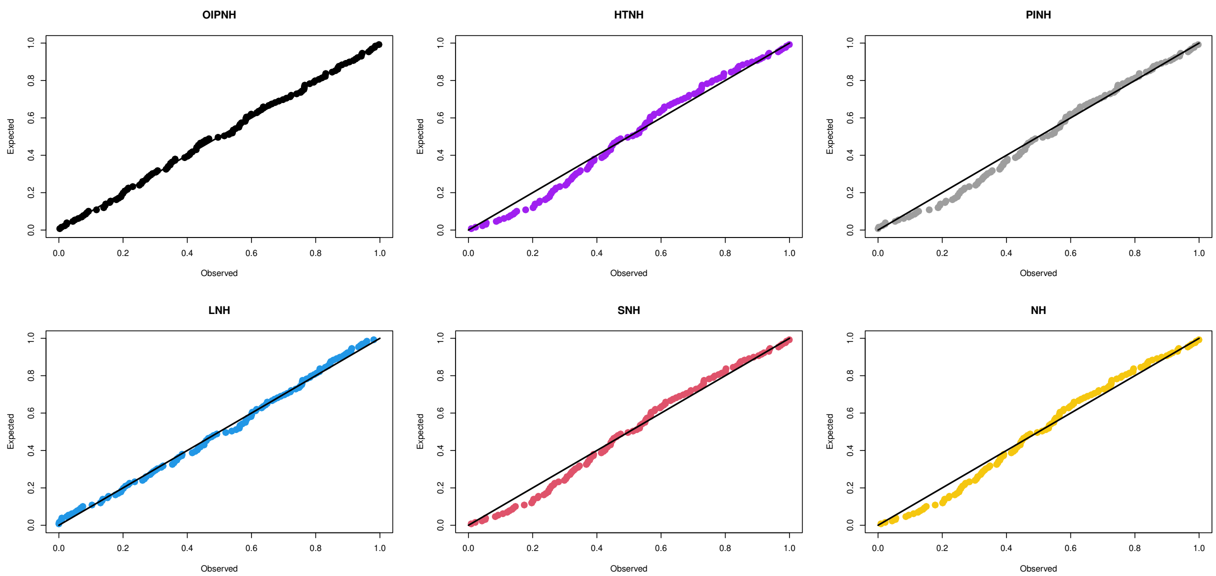

Figure 7: PP plots comparing the OIPNH distribution with other distributions for the first dataset.

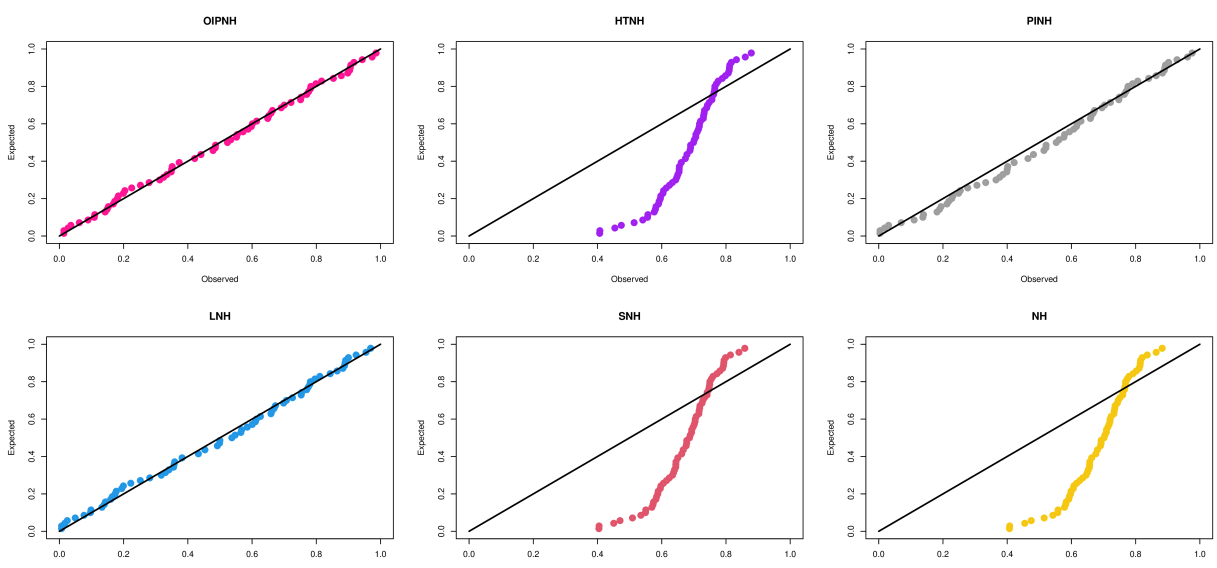

Figure 8: PP plots comparing the OIPNH distribution with other distributions for the second dataset.

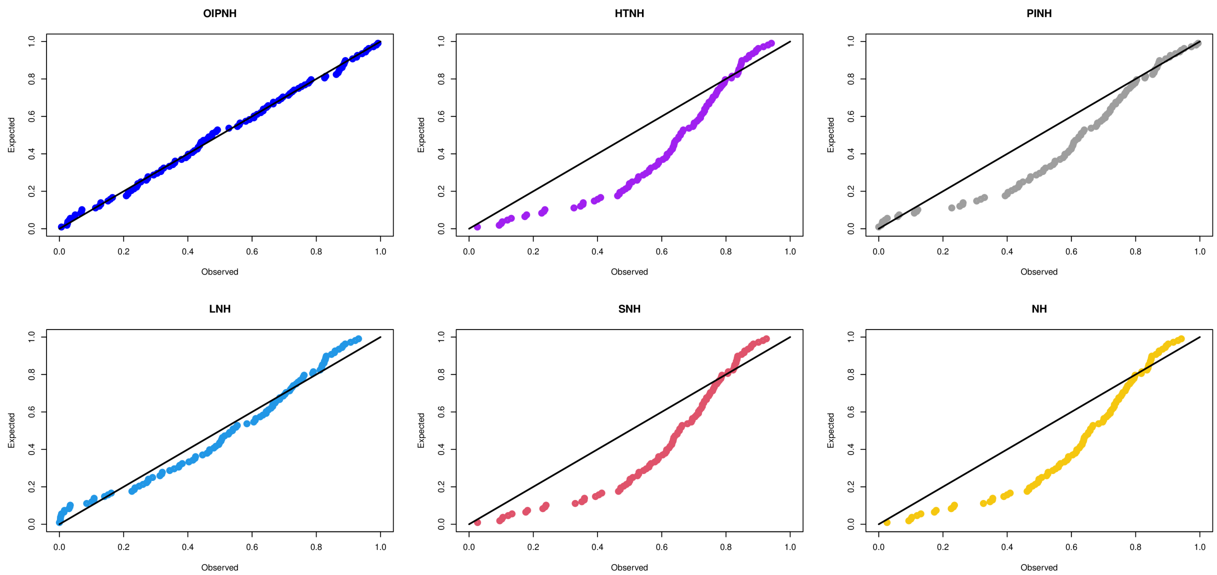

Figure 9: PP plots comparing the OIPNH distribution with other distributions for the third dataset.

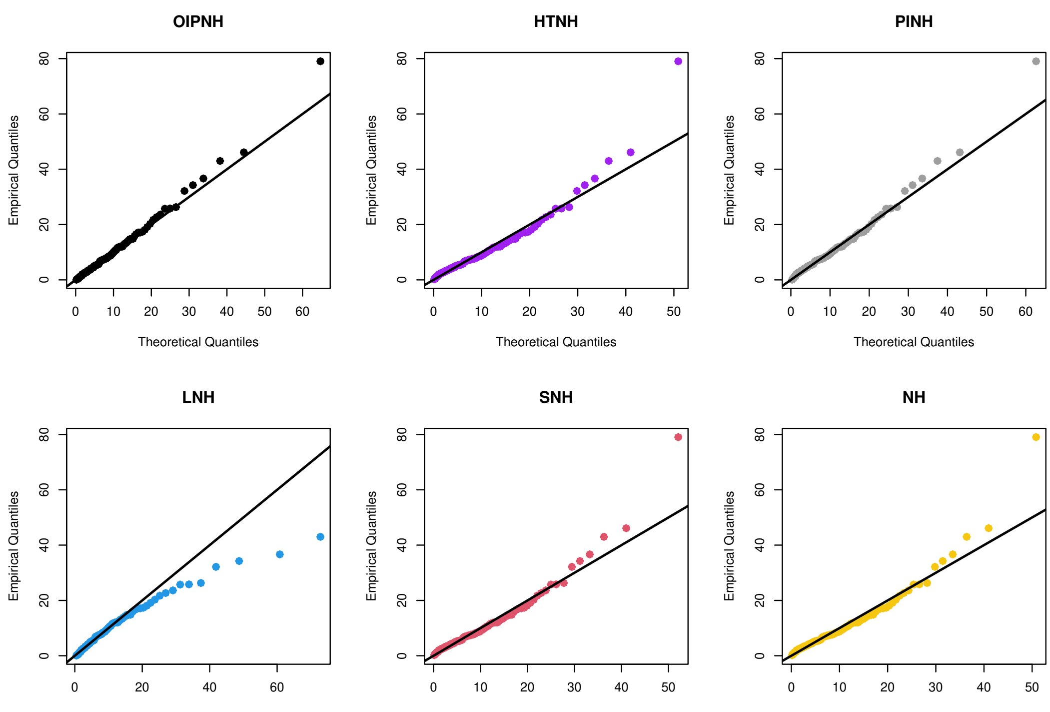

Figure 10: QQ plots comparing the OIPNH distribution with other distributions for the first dataset.

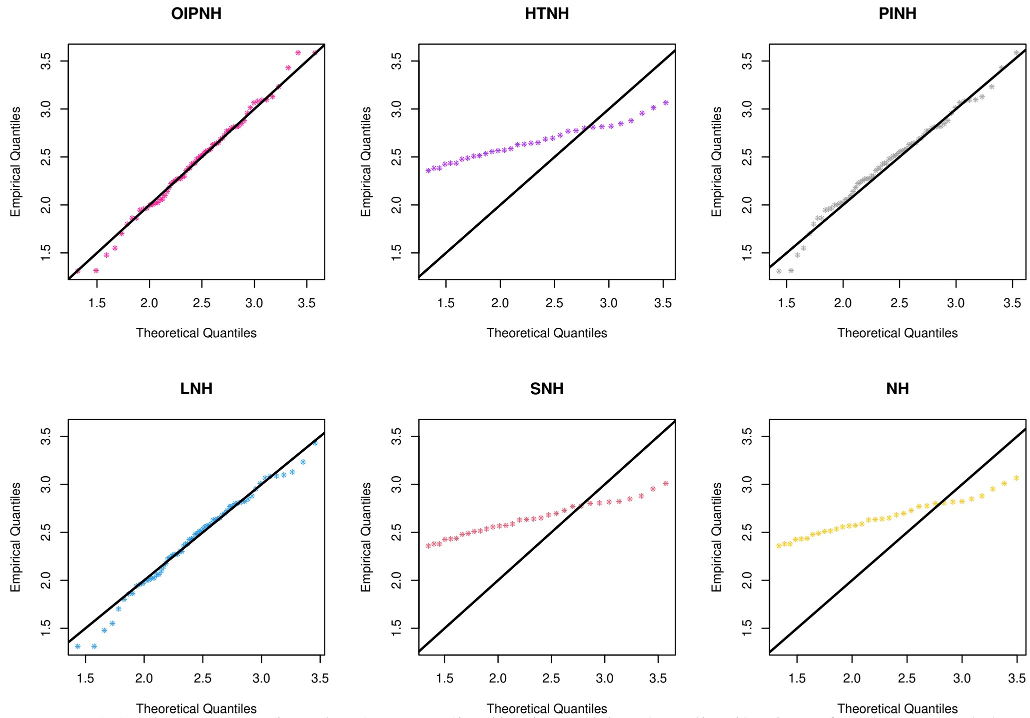

Figure 11: QQ plots comparing the OIPNH distribution with other distributions for the second dataset.

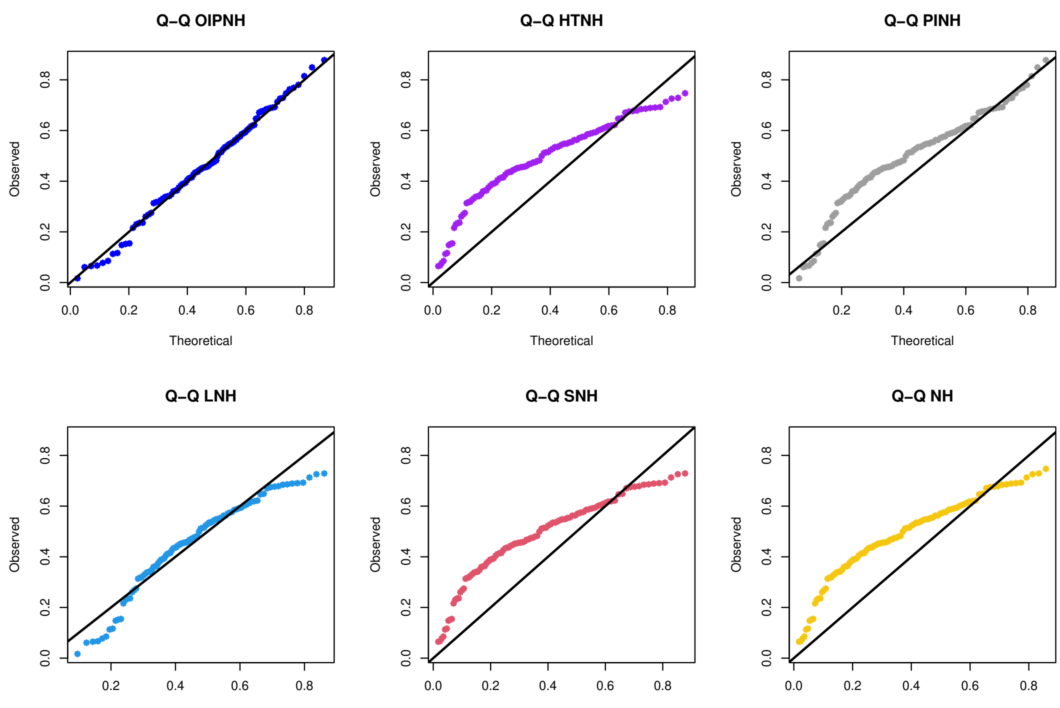

Figure 12: QQ plots comparing the OIPNH distribution with other distributions for the third dataset.

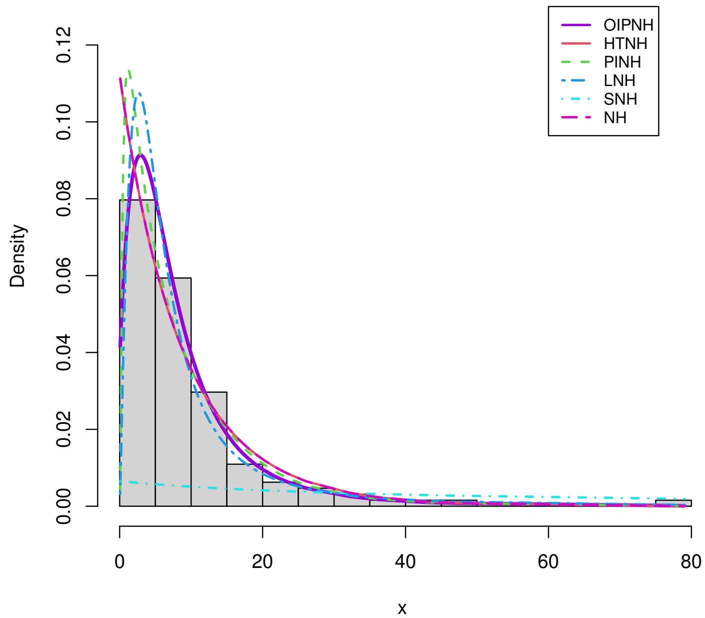

Figure 13: Fitted densities for the OIPNH distribution and other distributions for the first dataset.

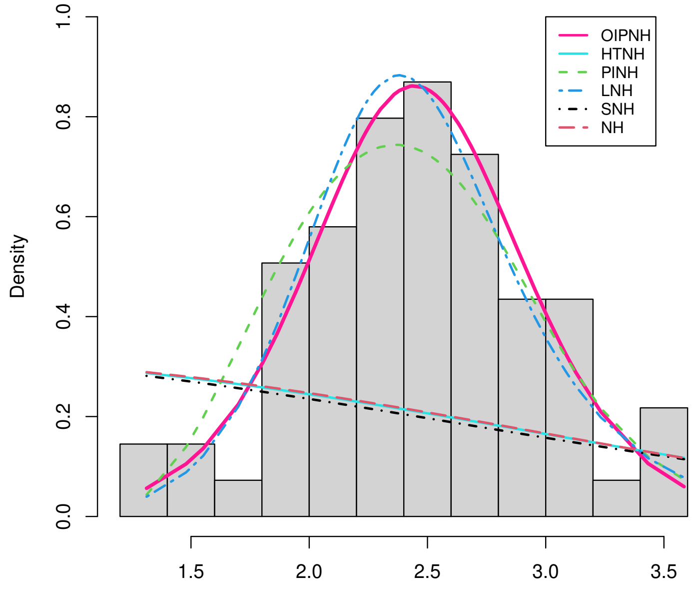

Figure 14: Fitted densities for the OIPNH distribution and other distributions for the second dataset.

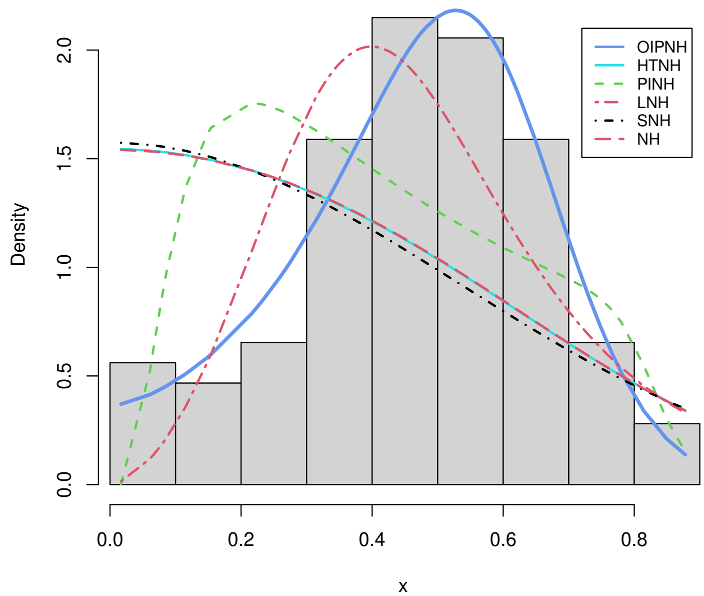

Figure 15: Fitted densities for the OIPNH distribution and other distributions for the third dataset.

7. Conclusions

In this paper, we present a flexible extension of the Nadarajah–Haghighi model, referred to as the odd inverse Pareto Nadarajah–Haghighi (OIPNH) distribution. The OIPNH density can be characterized as a mixture of exponentiated Nadarajah–Haghighi densities, and its failure rate can exhibit various shapes, including increasing, unimodal, decreasing–increasing–decreasing, decreasing, and bathtub patterns. We derive the mathematical properties of the OIPNH model and estimate its parameters using eight different estimation methods. Extensive simulation experiments assess the accuracy of these approaches and facilitate comparisons among them. The results indicate that the weighted least squares method outperforms all others, achieving an overall score of 35. Based on our findings, we recommend this method for estimating OIPNH parameters. This finding has significant implications for increasing the precision of statistical modeling in real-world applications. Finally, we illustrate the practical applicability of the OIPNH distribution by modeling three real-life datasets, with goodness-of-fit metrics showing that it offers a superior fit. The OIPNH distribution outperforms existing Nadarajah–Haghighi distributions, highlighting its potential for improved data processing. The OIPNH model's practical importance stems from its capacity to improve modeling flexibility and accuracy, especially in fields such as survival analysis, where it can provide more reliable insights into failure rates and data behavior.

Future research could enhance the utility and applicability of the OIPNH distribution across various fields. Some possible directions for future research include improving parameter estimation techniques, such as maximum likelihood or Bayesian methods for censored data, to enhance the OIPNH model's robustness and accuracy. Additionally, exploring non-parametric or semi-parametric regression models based on the OIPNH model could broader applicability for modelling survival data. Developing a discrete version of the OIPNH model to facilitate its use in modeling count data in diverse applied fields, as discussed in (3), (15) and (11).

Competing Interests

The authors declare no conflict of interest.

Availability

The data are fully available in the article and the mentioned references.

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