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Some New Double and Multiple Acceptance Sampling Plans Using the X-gamma Distribution: Theory and Evaluation with Some Applications

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Abstract
This study develops double and multiple (three-stage) acceptance sampling plans based on the X-gamma distribution, a flexible model widely used in reliability engineering. The plans are built around truncated life tests, meaning testing stops either after a predetermined time or once a set number of failures occur. For the two-stage (double) sampling approach, we determine the smallest practical sample sizes and calculate the average sample number (ASN). We then expand the framework to a three-stage plan, outlining how it works and specifying the sample sizes needed at each step. To evaluate performance, we derive operating characteristic (OC) curves for all plan types, showing how well they distinguish between good and poor-quality lots across different quality levels. We also identify the minimum ratio of actual to specified mean life required to keep producer's risk within acceptable limits. Additional efficiency metrics like average total inspection (ATI) and average outgoing quality (AOQ) are calculated to give a complete picture of how each plan performs in practice. Real-life numerical examples are included to walk through how these plans guide lot-acceptance decisions using the X-gamma model. Ultimately, this work gives quality control professionals reliable, distribution-specific tools for designing smarter, more efficient acceptance sampling procedures.

Keywords: Double acceptance sampling plan; Three-stage acceptance sampling plan; Average outgoing quality; X-gamma distribution; Average number of samples; Quality control testing.
MSC: 62D05.

1. Introduction

While Ibrahim et al. (2025) recently developed single acceptance sampling plans for the X-gamma distribution, double sampling plans under the same model have received far less attention. This study fills that gap by developing double acceptance sampling plans (DASPs) within the X-gamma framework. In a double sampling scheme, if the number of nonconforming units in the first sample falls between the acceptance and rejection limits, the result is inconclusive, so a second sample is drawn to reach a final decision. Research on double sampling remains relatively limited compared to single sampling, though several foundational and comparative studies provide important context (e.g., Ibrahim et al., 2025; Aslam et al., 2010, 2012; Aslam & Jun, 2010; Gui & Xu, 2015; Muthulakshmi & Selvi, 2013; Fallahnezhad et al., 2015). Our design follows the operational procedure originally outlined by Aslam et al. (2010). More broadly, lifetime-based acceptance sampling has long been a staple of statistical quality control, with influential early work from Gupta and Groll (1961), Balakrishnan et al. (2007), and others. In industrial settings, applying these statistical methods is essential for maintaining consistent production standards. Ultimately, quality control is about systematically monitoring manufacturing processes to ensure that products reliably meet their specified design and performance requirements.

Acceptance sampling is fundamentally about deciding whether to accept or reject a production lot based on sample data and because we're working with samples rather than full inspection, two kinds of statistical errors can occur. First, a Type I error (often called the producer's risk), happens when a perfectly good lot gets rejected just because the sample happened to look worse than the batch actually is.

The second, a Type II error (the consumer's risk), occurs when a substandard lot slips through and gets accepted because the sample didn't reveal its flaws (Carolino & Barao, 2013). From the producer's side, the goal is straightforward: make sure that lots meeting quality standards have a high chance denoted as ψ of being accepted. On the other side, consumers want to keep the risk of accidentally accepting a poor-quality lot as low as possible, ideally capping that probability at a small value like $1-\varepsilon^{*}$. Balancing these two perspectives is at the heart of designing fair and effective sampling plans.

Due to Sen et al. (2016), the probability density function (PDF) of the X-gamma distribution can be expressed as

$$f_{\Omega}(x)=\frac{1}{\Omega^{2}\left(1+\frac{1}{\Omega}\right)}\left(\frac{1}{2\Omega}x^{2}+1\right)\exp\left(-\frac{1}{\Omega}x\right)|x>0,\Omega>0. \quad (1)$$

Then the cumulative distribution function (CDF) corresponding to (1) can be written as

$$F_{\Omega}(x)=1-\frac{1}{1+\frac{1}{\Omega}}\left[\frac{1}{2}\left(\frac{x}{\Omega}\right)^{2}+\frac{x}{\Omega}+\frac{1}{\Omega}+1\right]\exp\left(-\frac{x}{\Omega}\right)|x>0,\Omega>0. \quad (2)$$

The rth ordinary moment of the X-gamma distribution is

$$E(X^{\boldsymbol{r}})=\frac{\boldsymbol{r}!}{1+\frac{1}{\Omega}}\left(\boldsymbol{r}+c(\boldsymbol{r})+\frac{1}{\Omega}\right)\left(\frac{1}{\Omega}\right)^{-\boldsymbol{r}},$$

where

$$c(\boldsymbol{r})=\boldsymbol{r}+c(\boldsymbol{r}-1)\ \forall\ \boldsymbol{r}=1,2,\ldots,$$

with $c(\boldsymbol{r}=0)=0,$ and $c(\boldsymbol{r}=1)=2$. This approach naturally extends to life testing scenarios (Balakrishnan et al., 2007).

When acceptance sampling is based on truncated life tests, we treat the lifetime of each inspected unit as a random variable and make accept/reject decisions using shortened test procedures meaning we don't wait for every unit to fail. Instead, testing stops either after a pre-specified time or once a certain number of failures occur, and the resulting data guides the final lot decision (Gupta & Groll, 1961). This makes the process far more practical for real-life quality control, where time and resources are limited, while still providing statistically sound judgments about product reliability.

This study is driven by a clear gap in the quality control literature: although the X-gamma distribution has shown real promise as a flexible lifetime model, it has never been applied to double or multi-stage acceptance sampling. Ibrahim et al. (2025) recently developed single sampling plans under this distribution, but to our knowledge, no one has yet extended it to two- or three-stage designs. That's a notable oversight, given that multi-stage sampling is widely recognized for cutting inspection costs and improving decision accuracy, all while maintaining the same level of producer and consumer protection. In reliability testing, running products to complete failure is rarely practical, so experiments are typically truncated to save time and resources. Pairing these shortened tests with a more adaptable distribution like X-gamma can significantly boost sampling efficiency, especially since most existing research still leans heavily on traditional models such as Weibull, exponential, or Burr-type distributions. By developing, optimizing, and numerically testing double and three-stage sampling plans under truncated life tests using the X-gamma framework, this work advances the underlying methodology while giving practitioners practical, statistically grounded tools for modern reliability data. Each plan is evaluated using standard performance metrics—operating characteristic curves, average sample number, average outgoing quality, and average total inspection—so we can assess them thoroughly from both a technical and cost-effectiveness standpoint.

2. The methodology of the DASPs

Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ represent the observed counts of nonconforming units in the first and second samples, respectively. The double acceptance sampling plan is fully specified by the parameter vector $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{a}_{c_1}, \boldsymbol{a}_{c_2}, x/\Omega_0)$, where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ denote the sample sizes at each stage, $\boldsymbol{a}_{c_1} < \boldsymbol{a}_{c_2}$ are the cumulative acceptance thresholds, and x/Ω_0 anchors the reliability assessment to the truncated life test under the X-gamma model. Under this framework, the overall probability of accepting a lot, P_2, decomposes into two mutually exclusive components: immediate acceptance after the first sample, and acceptance after proceeding to the second sample when the initial result is inconclusive. Formally,

$$P_2 = P_{(\boldsymbol{n}_1,\boldsymbol{\pi})} + P_{(\boldsymbol{n}_2,\boldsymbol{\pi})} = B(\boldsymbol{a}_{c_1} \mid \boldsymbol{n}_1, \boldsymbol{\pi}) + \sum_{\boldsymbol{r}_1=\boldsymbol{a}_{c_1}+1}^{\boldsymbol{a}_{c_2}} b(\boldsymbol{r}_1 \mid \mathbf{n}_1, \boldsymbol{\pi})\, B(\boldsymbol{a}_{c_2} - \boldsymbol{r}_1 \mid \boldsymbol{n}_2, \boldsymbol{\pi}), \tag{5}$$

where

$$b(\boldsymbol{r} \mid \boldsymbol{n}, \boldsymbol{\pi}) = \binom{\boldsymbol{n}}{\boldsymbol{r}} \boldsymbol{\pi}^{\boldsymbol{r}} (1-\boldsymbol{\pi})^{\boldsymbol{n}-\boldsymbol{r}}$$

is the binomial probability mass function and

$$B(m \mid \boldsymbol{n}, \boldsymbol{\pi}) = \sum_{i=0}^{m} b(i \mid \boldsymbol{n}, \boldsymbol{\pi})$$

is the corresponding cumulative distribution function, with $\boldsymbol{\pi} = F_{X\text{-gamma}}(x_0; \boldsymbol{\theta})$ denoting the failure probability before truncation time x_0. Expanding Equation (5) yields an equivalent expression that explicitly tracks cumulative failures across stages:

$$P_2 = \sum_{\boldsymbol{r}_1=0}^{\boldsymbol{a}_{c_1}} \binom{\boldsymbol{n}_1}{\boldsymbol{r}_1} \boldsymbol{\pi}^{\boldsymbol{r}_1} (1-\boldsymbol{\pi})^{\boldsymbol{n}_1-\boldsymbol{r}_1} + \sum_{\boldsymbol{r}_1=\boldsymbol{a}_{c_1}+1}^{\boldsymbol{a}_{c_2}} \binom{\boldsymbol{n}_1}{\boldsymbol{r}_1} \boldsymbol{\pi}^{\boldsymbol{r}_1} (1-\boldsymbol{\pi})^{\boldsymbol{n}_1-\boldsymbol{r}_1} \left[\sum_{\boldsymbol{r}_2=0}^{\boldsymbol{a}_{c_2}-\boldsymbol{r}_1} \binom{\boldsymbol{n}_2}{\boldsymbol{r}_2} \boldsymbol{\pi}^{\boldsymbol{r}_2} (1-\boldsymbol{\pi})^{\boldsymbol{n}_2-\boldsymbol{r}_2} \right].$$

When the acceptance thresholds are set to $\boldsymbol{a}_{c_1} = 0$ and $\boldsymbol{a}_{c_2} = 2$— a common choice that balances simplicity with discriminatory power, the acceptance probability simplifies to a compact closed form

$$P_2 = (1-\boldsymbol{\pi})^{\boldsymbol{n}_1} + \boldsymbol{n}_1 \boldsymbol{\pi}^2 (1-\boldsymbol{\pi})^{\boldsymbol{n}_1+\boldsymbol{n}_2-2} [\boldsymbol{n}_2 + \boldsymbol{\pi}^{-1}(1-\boldsymbol{\pi}) + 0.5(\boldsymbol{n}_1 - 1)]$$

To guarantee the desired level of consumer protection, we impose the constraint that lots meeting or exceeding the specified mean life ($\Omega \geq \Omega_0$) must be accepted with probability at least ε^*, which translates to the inequality $P_2 \leq 1 - \varepsilon^*$. Solving this constraint for integer pairs $(\boldsymbol{n}_1, \boldsymbol{n}_2)$ yields feasible designs that control the consumer's risk at the target level.

Efficiency is assessed via the Average Sample Number (ASN), which for a double sampling plan takes the form ASN $= \boldsymbol{n}_1 + \boldsymbol{n}_2(1-\psi)$, where ψ is the probability that a decision is reached after the first sample alone. For the specific thresholds $\boldsymbol{a}_{c_1} = 0$, $\boldsymbol{a}_{c_2} = 1$, this reduces to $\psi = 1 - \boldsymbol{n}_1 \boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1}$, and consequently,

$$\text{ASN} = \boldsymbol{n}_1 \boldsymbol{n}_2 \boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1} \left[1 + \left(\frac{1}{\boldsymbol{n}_2 \boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1}} \right) + \frac{\boldsymbol{n}_1 - 1}{2} \left(\frac{\boldsymbol{\pi}}{1-\boldsymbol{\pi}} \right) \right].$$

Following the widely adopted design principle of minimizing expected inspection effort (Balamurali et al., 2005; Jun et al., 2006), the optimal sample sizes are obtained by solving the constrained integer optimization problem:

$$\begin{aligned} &\text{Minimize } (\boldsymbol{n}_1, \boldsymbol{n}_2) \quad \text{ASN} = \boldsymbol{n}_1 \boldsymbol{n}_2 \boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1} \left[1 + \left(\frac{1}{\boldsymbol{n}_2 \boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1}} \right) + \frac{\boldsymbol{n}_1 - 1}{2} \left(\frac{\boldsymbol{\pi}}{1-\boldsymbol{\pi}} \right) \right] \\ &\text{Subject to} \quad P_2 \leq 1 - \varepsilon^*, \\ &\qquad 1 \leq \boldsymbol{n}_2 \leq \boldsymbol{n}_1, \\ &\qquad \boldsymbol{n}_1, \boldsymbol{n}_2 \in \mathbb{Z}^+. \end{aligned} \tag{6}$$

Numerical solutions to this problem were computed across a grid of consumer confidence levels ($\varepsilon^* = 0.75$, 0.90, 0.0.95, 0.99) and design ratios

x/Ω_0 = 498, 973, 1500, 3522 and 4980

The resulting optimal triples ($\boldsymbol{n}_1, \boldsymbol{n}_2$, ASN)are summarized in Table 1, providing practitioners with ready-to-implement specifications that balance statistical rigor with operational efficiency under the X-gamma lifetime model.

Table 1: Minimum sample size and ASN for X-gamma distribution in double sampling plan

$\frac{x}{\Omega_0}$	ε^*											
	0.75			0.90			0.95			0.99		
	$\boldsymbol{n}_1$	$\boldsymbol{n}_2$	ASN	$\boldsymbol{n}_1$	$\boldsymbol{n}_2$	ASN	$\boldsymbol{n}_1$	$\boldsymbol{n}_2$	ASN	$\boldsymbol{n}_1$	$\boldsymbol{n}_2$	ASN
0.498	261	42	271.3	310	133	331.1	331	148	350.1	620	363	622.4
0.973	83	10	83.2	87	26	87.8	88	29	89.2	110	24	110.6
1.500	18	10	21.1	30	11	30.6	34	12	33.9	48	13	48.0
3.522	4	3	5.2	7	2	6.7	8	2	7.8	9	2	9.1
4.980	5	1	4.8	6	1	6.2	7	1	6.8	8	1	8.2

As shown in Table 1, As shown in Table 1, the required sample sizes for the double sampling plan drop sharply as the ratio of actual to specified mean life (x/Ω_0) increases. When product reliability is high relative to the standard (e.g., $x/\Omega_0 = 4.980$), very small first-stage samples ($\boldsymbol{n}_1$) are sufficient often fewer than 10 units—and the Average Sample Number (ASN) stays close to $\boldsymbol{n}_1$, indicating that most lots are decided in the first stage. Conversely, when reliability is near the threshold (e.g., $x/\Omega_0 = 0.498$), substantially larger samples are needed to maintain the desired protection levels, particularly as the consumer's risk parameter ε^*becomes more stringent (moving from 0.75 to 0.99).

The table also highlights the efficiency advantage of the double sampling approach: even when the combined sample size ($\boldsymbol{n}_1 + \boldsymbol{n}_2$) is large, the ASN remains relatively modest because many decisions are resolved after the first sample. For practitioners, this means the plan adapts intelligently to observed quality, inspecting more only when the initial data are inconclusive while still controlling both producer's and consumer's risk according to the X-gamma model assumptions.

Based on Table 1, a few clear patterns emerge that speak directly to how the double sampling plan performs in practice. First, as the actual mean life of the product improves relative to the specified requirement (i.e., as x/Ω_0increases), the required sample sizes drop dramatically. For instance, when $x/\Omega_0 = 0.498$, meaning the product's reliability, is only about half of what's claimed, the plan demands a large initial sample ($\boldsymbol{n}_1 = 620$ when $\varepsilon^* = 0.99$) to confidently assess quality. But when reliability is nearly five times the specification ($x/\Omega_0 = 4.980$), the same level of consumer protection can be achieved with just $\boldsymbol{n}_1 = 8$ units.

Second, tightening the consumer's risk requirement (raising $\varepsilon^*$from 0.75 to 0.99) naturally increases sample sizes, but this effect is most pronounced when product quality is borderline. At higher reliability levels, even the strictest $\varepsilon^* = 0.99$ requires only modest sampling effort.

Third, and perhaps most importantly for efficiency, the Average Sample Number (ASN) stays remarkably close to the first-stage sample size $\boldsymbol{n}_1$across nearly all scenarios. This tells us that in most cases, especially when quality is clearly good or clearly poor, the decision is made after the first sample, and the second sample ($\boldsymbol{n}_2$) is rarely needed in full. For example, at $x/\Omega_0 = 1.500$ and $\varepsilon^* = 0.95$, $\boldsymbol{n}_1 = 34$and $\boldsymbol{n}_2 = 12$, but the ASN is only 33.9, meaning the second stage is invoked infrequently.

Finally, note that $\boldsymbol{n}_2$is consistently much smaller than $\boldsymbol{n}_1$, reinforcing that the second stage serves as a targeted follow-up rather than a full re-inspection. For quality control practitioners, this means the double sampling plan under the X-gamma model offers a smart balance: it minimizes average inspection effort when quality is stable, while retaining the flexibility to gather more evidence when initial results are ambiguous all without compromising the agreed-upon producer and consumer risk levels.

Table 2 below presents the OC function values for the DASP under the X-gamma distribution, showing how the probability of accepting a lot changes with different true quality levels (Ω/Ω_0), design ratios (x/Ω_0), and consumer's risk specifications (ε^*), given acceptance criteria $\boldsymbol{a}_{c_1} = 0$ and $\boldsymbol{a}_{c_2} = 2$.

Based on Table 2, several practical insights stand out. First, the plans behave exactly as we'd hope when the true mean life is only twice the specified value ($\Omega/\Omega_0 = 2$), acceptance probabilities remain relatively low (ranging from about 0.04 to 0.54 depending on the scenario), meaning substandard lots are likely to be rejected. Conversely, when product reliability is strong ($\Omega/\Omega_0 \geq 7$), the OC values climb toward 0.95 or higher, ensuring that good lots are accepted with high confidence. Second, tightening the consumer's risk requirement (increasing $\varepsilon^*$from 0.75 to 0.99) makes the plan more conservative across the board. For example, at $x/\Omega_0 = 1.500$ and $\Omega/\Omega_0 = 4$, the acceptance probability drops from 0.879 under $\varepsilon^* = 0.75$ to 0.716 under $\varepsilon^* = 0.99$. This trade-off is expected: stronger protection for the consumer comes at the cost of slightly higher rejection rates for borderline lots. Third, the design ratio x/Ω_0influences how sharply the OC curve rises. Plans calibrated for higher x/Ω_0values (e.g., 4.980) tend to show steeper transitions between rejection and acceptance regions, which can be advantageous when the goal is to clearly separate acceptable from unacceptable quality.

However, this also means that if the actual product performance falls just below the design expectation, the lot may face a higher chance of rejection, a consideration practitioners should weigh when selecting plan parameters. The OC values in Table 2 confirm that the proposed double sampling plans under the X-gamma model deliver strong discriminatory power. They reliably protect consumers from poor-quality lots while giving producers a fair chance of acceptance when quality meets or exceeds specifications. For quality engineers, these tables provide a ready reference for choosing $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and acceptance thresholds that align with both risk tolerance and inspection efficiency goals.

Table 2: Values of the OC under the DASP $\left(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{a}_{c_1}, \boldsymbol{a}_{c_2}, \frac{x}{\Omega_0}\right)$ and for a given ε^* when $\boldsymbol{a}_{c_1} = 0$ and $\boldsymbol{a}_{c_2} = 2$.

ε^*	$\frac{x}{\Omega_0}$	Ω/Ω_0							
		2	3	4	5	6	7	8	9
0.75	0.498	0.54	0.806	0.899	0.937	0.957	0.968	0.975	0.979
	0.973	0.418	0.675	0.832	0.903	0.938	0.957	0.968	0.976
	1.500	0.362	0.76	0.879	0.931	0.957	0.971	0.979	0.985
	3.522	0.295	0.606	0.776	0.864	0.912	0.94	0.957	0.969
	4.980	0.257	0.286	0.51	0.67	0.772	0.838	0.882	0.911
0.90	0.498	0.481	0.774	0.881	0.926	0.949	0.962	0.97	0.975
	0.973	0.3	0.662	0.825	0.898	0.935	0.955	0.967	0.975
	1.500	0.284	0.639	0.811	0.89	0.931	0.953	0.967	0.975
	3.522	0.138	0.444	0.663	0.789	0.862	0.905	0.932	0.95
	4.980	0.126	0.195	0.42	0.597	0.717	0.797	0.851	0.887
0.95	0.498	0.458	0.761	0.873	0.921	0.946	0.959	0.968	0.974
	0.973	0.294	0.657	0.822	0.896	0.934	0.954	0.966	0.974
	1.500	0.246	0.607	0.791	0.879	0.923	0.948	0.963	0.972
	3.522	0.096	0.384	0.616	0.757	0.839	0.889	0.92	0.941
	4.980	0.09	0.166	0.385	0.567	0.694	0.779	0.837	0.877
0.99	0.498	0.232	0.599	0.776	0.858	0.9	0.925	0.941	0.951
	0.973	0.217	0.592	0.783	0.872	0.918	0.943	0.958	0.968
	1.500	0.135	0.49	0.716	0.831	0.892	0.927	0.947	0.961

	3.522	0.065	0.327	0.568	0.722	0.815	0.871	0.908	0.931
	4.980	0.041	0.115	0.318	0.506	0.645	0.741	0.808	0.854

The smallest ratio Ω/Ω_0 can be determined by solving the following inequality for the producer's risk of τ, where

$$P_2 \geq 1-\tau, \tag{7}$$

and P_2 is defined before and $\boldsymbol{\pi}$ is given in (7). For a given acceptance sampling plan $\left(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{a}_{c_1}, \boldsymbol{a}_{c_2}, \frac{x}{\Omega_0}\right)$, and ε^*, the minimum ratio of $\frac{\Omega}{\Omega_0}$, satisfying inequality (7) is computed and presented in Table 3.

Table 3 presents the minimum ratio of actual mean life to specified mean life (Ω/Ω_0) required to ensure a batch is accepted with 95% confidence (i.e., under a producer's risk of 0.05), across different design settings (x/Ω_0) and consumer's risk levels (ε^*).

Based on Table 3, a few important patterns emerge for practitioners. First, as the consumer's risk requirement tightens (moving ε^* from 0.75 to 0.99), the required Ω/Ω_0 ratio generally increases meaning producers need to deliver higher true reliability to maintain the same 5% rejection risk. For example, when $x/\Omega_0 = 0.498$, the minimum acceptable Ω/Ω_0 jumps from 5.573 at $\varepsilon^* = 0.75$ to 8.878 at $\varepsilon^* = 0.99$. This reflects the expected trade-off: stronger consumer protection demands higher product performance to keep producer's risk in check.

Second, the influence of the design ratio x/Ω_0isn't linear. At lower ε^*values (e.g., 0.75), increasing x/Ω_0 from 0.498 to 3.522 raises the required Ω/Ω_0(from 5.573 to 7.516), but the jump becomes much steeper at $x/\Omega_0 = 4.980$ (11.304). This suggests that plans calibrated for very high design ratios become considerably more demanding on actual product life—useful when targeting premium reliability, but potentially costly if manufacturing variability is high.

Third, the most "efficient" settings where the required Ω/Ω_0 is relatively modest tend to cluster around mid-range x/Ω_0 values (e.g., 0.973 to 1.500) and moderate ε^*(0.75 – 0.90). For instance, at $x/\Omega_0 = 0.973$and $\varepsilon^* = 0.90$, a product only needs to achieve about 6.7 times the specified mean life to satisfy both risk constraints. This could inform practical plan selection when balancing inspection burden, production capability, and contractual risk allocation.

Table 3: The minimum ratio of the actual mean life to specified mean life for the acceptance of a batch under producer's risk 0.05.

ε^*	$\frac{x}{\Omega_0}$				
	0.498	0.973	1.500	3.522	4.980
0.75	5.573	6.58	5.658	7.516	11.304
0.90	6.067	6.713	6.813	9.024	12.441
0.95	6.268	6.756	7.103	9.605	12.894
0.99	8.878	7.411	8.159	10.182	13.823

3. Multiple acceptance sampling plans

Multiple sampling plans extend the flexibility of double sampling by allowing inspectors to draw additional samples when early test results are inconclusive. Rather than limiting the decision process to two stages, a multiple sampling scheme permits up to $k$sequential samples, providing finer control over inspection effort, risk allocation, and decision accuracy.

Formally, a k-stage multiple acceptance sampling plan is defined by three parameter vectors:

- $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_k)$, where each $\boldsymbol{n}_j \geq 1$represents the sample size drawn at stage j;
- $\mathbf{a}_c = (\boldsymbol{a}_{c_1}, \boldsymbol{a}_{c_2}, \ldots, \boldsymbol{a}_{c_k})$, a non-decreasing sequence of acceptance thresholds ($0 \leq \boldsymbol{a}_{c_1} \leq \boldsymbol{a}_{c_2} \leq \cdots \leq \boldsymbol{a}_{c_k}$);

- $\mathbf{a}_{\boldsymbol{r}} = (\boldsymbol{a}_{\boldsymbol{r}_1}, \boldsymbol{a}_{\boldsymbol{r}_2}, \dots, \boldsymbol{a}_{\boldsymbol{r}_k})$, a non-decreasing sequence of rejection thresholds $(1 \le \boldsymbol{a}_{\boldsymbol{r}_1} \le \boldsymbol{a}_{\boldsymbol{r}_2} \le \cdots \le \boldsymbol{a}_{\boldsymbol{r}_k})$, with the critical condition that $\boldsymbol{a}_{c_j} < \boldsymbol{a}_{\boldsymbol{r}_j}$ at every stage to maintain a clear decision boundary.

The integer k denotes the maximum number of sampling stages. In classical quality control, it is well established that multiple sampling plans consistently outperform single and double schemes in terms of steeper OC curves, lower average sample number (ASN), and reduced average total inspection (ATI) particularly when quality characteristics follow symmetric distributions like the normal. However, these advantages have rarely been translated into reliability engineering, where lifetimes are inherently skewed, censoring is unavoidable, and experimental budgets are tight. To our knowledge, no prior work has developed multiple acceptance sampling plans for the X-gamma distribution, despite its demonstrated flexibility in modeling complex failure-time data. This section closes that gap by introducing a k-stage multiple sampling framework specifically tailored for truncated (amputated) life tests under the X-gamma model.
Under a pre-specified truncation time x_0, the plan operates through the following sequential decision procedure:

Stage 1:
Draw an initial sample of size $\boldsymbol{n}_1$ from the lot and subject it to life testing until time x_0. Let $\boldsymbol{r}_1$ denote the number of units that fail before x_0.

- If $\boldsymbol{r}_1 \le \boldsymbol{a}_{c_1}$, accept the lot immediately.
- If $\boldsymbol{r}_1 \ge \boldsymbol{a}_{\boldsymbol{r}_1}$, terminate testing and reject the lot.
- If $\boldsymbol{a}_{c_1} < \boldsymbol{r}_1 < \boldsymbol{a}_{\boldsymbol{r}_1}$, the result is inconclusive; proceed to Stage 2.

Stage 2:
Draw a second sample of size $\boldsymbol{n}_2$ from the same lot and test it under identical truncation conditions. Let $\boldsymbol{r}_2$ be the new failures observed and compute the cumulative failure count $C_2 = \boldsymbol{r}_1 + \boldsymbol{r}_2$.

- If $C_2 \le \boldsymbol{a}_{c_2}$, accept the lot.
- If $C_2 \ge \boldsymbol{a}_{\boldsymbol{r}_2}$, reject the lot.
- If $\boldsymbol{a}_{c_2} < C_2 < \boldsymbol{a}_{\boldsymbol{r}_2}$, continue to Stage 3.

Stage j (for $3 \le j < k$):
Repeat the process. Draw $\boldsymbol{n}_j$ additional units, record $\boldsymbol{r}_j$ failures before x_0, and update the cumulative count $C_j = \sum_{i=1}^{j} \boldsymbol{r}_i$. Compare C_j against the stage-specific thresholds $\boldsymbol{a}_{c_j}$ and $\boldsymbol{a}_{\boldsymbol{r}_j}$. Accept if $C_j \le \boldsymbol{a}_{c_j}$, reject if $C_j \ge \boldsymbol{a}_{\boldsymbol{r}_j}$, or proceed to the next stage if the cumulative count falls in the indeterminate zone.

Final Stage (k):
After inspecting the k-th sample of size $\boldsymbol{n}_k$, calculate the total cumulative failures $C_k = \sum_{i=1}^{k} \boldsymbol{r}_i$. Since no further sampling is permitted, a definitive decision must be made. In well-constructed plans, the final thresholds are typically set such that $\boldsymbol{a}_{\boldsymbol{r}_k} = \boldsymbol{a}_{c_k} + 1$, forcing a conclusive outcome:

- Accept the lot if $C_k \le \boldsymbol{a}_{c_k}$.
- Reject the lot if $C_k \ge \boldsymbol{a}_{\boldsymbol{r}_k}$.

This sequential structure ensures that inspection effort scales intelligently with observed quality: lots that are clearly good or clearly poor are resolved early, while borderline lots receive additional scrutiny only when necessary. Cumulative failure tracking also aligns naturally with reliability testing, where the primary metric of interest is the time-to-failure distribution rather than simple defect counts.

As a technical note, if the acceptance thresholds are held constant across all stages ($\boldsymbol{a}_{c_1} = \boldsymbol{a}_{c_2} = \cdots = \boldsymbol{a}_{c_k}$) and the rejection boundaries are aligned to permit only a single decision point, the multiple sampling framework collapses into a conventional single sampling plan. This demonstrates how the proposed methodology naturally encompasses simpler schemes as special cases while offering substantially greater adaptability for modern reliability and life-testing applications.

The proposed multiple acceptance sampling plan is built around a truncated (amputated) life test and is fully characterized by the parameter set $(\mathbf{n}_k, \mathbf{a}_{c_k}, \mathbf{a}_{r_k}, x/\Omega_0)$, where:

- $\mathbf{n}_k = (\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_k)$ denotes the sample sizes at each of the $k$stages;
- $\mathbf{a}_{c_k} = (\boldsymbol{a}_{c_1}, \boldsymbol{a}_{c_2}, \ldots, \boldsymbol{a}_{c_k})$is a strictly increasing sequence of cumulative acceptance thresholds ($\boldsymbol{a}_{c_1} < \boldsymbol{a}_{c_2} < \cdots < \boldsymbol{a}_{c_k}$);
- $\mathbf{a}_{r_k} = (\boldsymbol{a}_{r_1}, \boldsymbol{a}_{r_2}, \ldots, \boldsymbol{a}_{r_k})$is a sequence of cumulative rejection thresholds satisfying $\boldsymbol{a}_{c_j} < \boldsymbol{a}_{r_j}$at every stage j;
- $x/\Omega_0$is the ratio of the test truncation time to the specified mean life, which anchors the reliability assessment under the X-gamma model.

Under this framework, the overall probability of accepting a lot., the OC function is derived by accounting for all possible paths through the sequential decision tree. Let $\boldsymbol{\pi}$ denote the probability that a single unit fails before the truncation time x_0, computed from the cumulative distribution function of the X-gamma distribution. Then $b(\boldsymbol{r} \mid \boldsymbol{n}, \boldsymbol{\pi}) = \binom{n}{r}\boldsymbol{\pi}^{\boldsymbol{r}}(1-\boldsymbol{\pi})^{\boldsymbol{n}-\boldsymbol{r}}$is the binomial probability mass function (the chance of observing exactly $\boldsymbol{r}$failures in a sample of size $\boldsymbol{n}$), and $B(c \mid \boldsymbol{n}, \boldsymbol{\pi}) = \sum_{i=0}^{c} b(i \mid \boldsymbol{n}, \boldsymbol{\pi})$is the corresponding cumulative distribution function (the chance of observing $c$or fewer failures).

The total acceptance probability P_{accept} aggregates contributions from every stage at which a lot could be accepted:

1. **Acceptance at Stage 1:** The lot is accepted immediately if the first sample yields $\boldsymbol{r}_1 \leq \boldsymbol{a}_{c_1}$failures. This contributes to the term $B(\boldsymbol{a}_{c_1} \mid \boldsymbol{n}_1, \boldsymbol{\pi})$.
2. **Acceptance at Stage 2:** If the first sample is inconclusive ($\boldsymbol{a}_{c_1} < \boldsymbol{r}_1 < \boldsymbol{a}_{r_1}$), a second sample is drawn. The lot is accepted if the cumulative failures $\boldsymbol{r}_1 + \boldsymbol{r}_2 \leq \boldsymbol{a}_{c_2}$. The probability of this path is:
$$\sum_{\boldsymbol{r}_1=\boldsymbol{a}_{c_1}+1}^{\boldsymbol{a}_{r_1}-1} b(\boldsymbol{r}_1 \mid \boldsymbol{n}_1, \boldsymbol{\pi}) \cdot B(\boldsymbol{a}_{c_2} - \boldsymbol{r}_1 \mid \boldsymbol{n}_2, \boldsymbol{\pi}).$$
3. **Acceptance at Stage 3:** If the first two samples remain inconclusive, a third sample is taken. Acceptance occurs when $\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 \leq \boldsymbol{a}_{c_3}$, contributing a triple-sum term that conditions on the outcomes of the first two stages and evaluates the binomial CDF for the third.
4. **General Stage j ($3 < j \leq k$):** The pattern continues recursively. For any intermediate stage j, the contribution to P_{accept}involves nested summations over all inconclusive failure counts from prior stages, multiplied by the binomial CDF evaluated at the remaining "acceptance budget" for stage j:
$B(\boldsymbol{a}_{c_j} - R_{j-1} \mid \boldsymbol{n}_j, \boldsymbol{\pi})$, where $R_{j-1} = \sum_{i=1}^{j-1} \boldsymbol{r}_i$.

5. **Final Stage k:** At the last permissible stage, no further sampling is allowed. The lot is accepted if the total cumulative failures $R_k = \sum_{i=1}^{k} \boldsymbol{r}_i$ do not exceed $\boldsymbol{a}_{c_k}$. The corresponding term closes the recursive expansion, ensuring that all possible decision paths are accounted for.

Putting these pieces together, the full OC expression can be written compactly as:

$$P_{\text{accept}} = B(\boldsymbol{a}_{c_1} \mid \boldsymbol{n}_1, \boldsymbol{\pi}) + \sum_{j=2}^{k} \left[\left(\prod_{i=1}^{j-1} \sum_{\boldsymbol{r}_i = L_i}^{U_i} b\,(\boldsymbol{r}_i \mid \boldsymbol{n}_i, \boldsymbol{\pi}) \right) \cdot B(\boldsymbol{a}_{c_j} - R_{j-1} \mid \boldsymbol{n}_j, \boldsymbol{\pi}) \right], \tag{8}$$

where $L_i = \boldsymbol{a}_{c_i} - R_{i-1} + 1$and $U_i = \boldsymbol{a}_{r_i} - 1$define the inconclusive region at stage i, and $R_i = \sum_{\ell=1}^{i} \boldsymbol{r}_\ell$. This formulation makes explicit how the acceptance probability accumulates across stages, weighting each possible trajectory by its binomial likelihood under the X-gamma failure model. Practically, it allows quality engineers to compute the OC curve for any chosen parameter set, evaluate the plan's discriminatory power, and optimize $\left(\boldsymbol{n}_j, \boldsymbol{a}_{c_j}, \boldsymbol{a}_{r_j}\right)$to meet target producer's and consumer's risk levels—all while respecting the constraints of truncated life testing. Moreover, because $\boldsymbol{\pi}$is derived from the X-gamma CDF, the entire procedure remains fully aligned with the underlying reliability distribution, offering a statistically coherent alternative to plans based on simpler, less flexible lifetime models.

where $R_{k} = \sum_{i=1}^{k} \boldsymbol{r}_i$ in (8) is defined as a cumulative number of nonconforming units and cumulative probabilities with negative arguments are taken to be zero. According to Schilling and Nubauer (2009), if $\boldsymbol{a}_{c_{k}} = -1$, that means no acceptance is allowed at k stage.

When the multiple acceptance sampling plan is configured for up to three stages ($k = 1,2,3$) using the parameter set $\left(\mathbf{n}_k, \mathbf{a}_{c_k}, \mathbf{a}_{r_k}, x/\Omega_0\right)$, the overall probability of accepting a lot decomposes naturally into three mutually exclusive components, each corresponding to the stage at which a final decision is reached:

$$P_3 = P_{(\boldsymbol{n}_1, \boldsymbol{\pi})} + P_{(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{\pi})} + P_{(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3, \boldsymbol{\pi})}.$$

Here, $\boldsymbol{\pi}$ represents the probability that a single unit fails before the pre-specified truncation time x_0, computed from the cumulative distribution function of the X-gamma lifetime model. The binomial probability mass function $\binom{\boldsymbol{n}}{\boldsymbol{r}} \boldsymbol{\pi}^{\boldsymbol{r}} (1 - \boldsymbol{\pi})^{\boldsymbol{n}-\boldsymbol{r}}$ then describes the likelihood of observing exactly $\boldsymbol{r}$ failures in a sample of size $\boldsymbol{n}$under this truncated testing scheme.

1. Acceptance at Stage 1

The first term, $P_{(\boldsymbol{n}_1, \boldsymbol{\pi})}$, captures the probability that the lot is accepted immediately after inspecting the initial sample of size $\boldsymbol{n}_1$. This occurs when the number of observed failures $\boldsymbol{r}_1$does not exceed the first-stage acceptance threshold $\boldsymbol{a}_{c_1}$:

$$P_{(\boldsymbol{n}_1, \boldsymbol{\pi})} = \sum_{\boldsymbol{r}_1 = 0}^{\boldsymbol{a}_{c_1}} \binom{\boldsymbol{n}_1}{\boldsymbol{r}_1} \boldsymbol{\pi}^{\boldsymbol{r}_1} (1 - \boldsymbol{\pi})^{\boldsymbol{n}_1 - \boldsymbol{r}_1}.$$

This is simply the binomial cumulative distribution evaluated at a_{c_1}, reflecting the chance that early evidence is sufficiently favorable to warrant immediate acceptance.

2. Acceptance at Stage 2

If the first sample yields an inconclusive result i.e., $a_{c_1} < r_1 < a_{r_1}$—a second sample of size $n_2$is drawn. The lot is accepted at this stage if the cumulative number of failures across both samples, $r_1 + r_2$, does not exceed a_{c_2}. The corresponding probability is:

$$P_{(n_1, n_2, \pi)} = \sum_{r_1=a_{c_1}+1}^{a_{r_1}-1} \binom{n_1}{r_1} \pi^{r_1}(1-\pi)^{n_1-r_1} \left[\sum_{r_2=0}^{a_{c_2}-r_1} \binom{n_2}{r_2} \pi^{r_2}(1-\pi)^{n_2-r_2} \right].$$

The outer sum enumerates all inconclusive outcomes from Stage 1, while the inner sum computes the conditional probability that the second sample provides enough additional evidence to bring the cumulative failure count within the acceptance bound.

3. Acceptance at Stage 3 (Final Stage)

When both the first and second samples remain inconclusive—specifically, when $a_{c_1} < r_1 < a_{r_1}$and $a_{c_2} - r_1 < r_2 < a_{r_2} - r_1$—a third and final sample of size $n_3$is inspected. Acceptance occurs if the total cumulative failures $R_2 + r_3 = r_1 + r_2 + r_3$do not exceed a_{c_3}. The probability of contribution is

$$\begin{aligned} P_{(n_1, n_2, n_3, \pi)} = & \left[\sum_{r_1=a_{c_1}+1}^{a_{r_1}-1} \binom{n_1}{r_1} \pi^{r_1}(1-\pi)^{n_1-r_1} \right] \\ \times & \left[\sum_{r_2=a_{c_2}-r_1+1}^{a_{r_2}-1} \binom{n_2}{r_2} \pi^{r_2}(1-\pi)^{n_2-r_2} \right] \\ \times & \left[\sum_{r_3=0}^{a_{c_3}-(r_1+r_2)} \binom{n_3}{r_3} \pi^{r_3}(1-\pi)^{n_3-r_3} \right]. \end{aligned}$$

This triple-nested structure reflects the sequential conditioning inherent in multi-stage sampling: each stage's outcome narrows the range of possibilities for the next, and the final binomial CDF evaluates whether the accumulated evidence meets the ultimate acceptance criterion.

Control Table Framework

To streamline the computation of OC curves and related performance metrics for such plans, the control table approach originally introduced by the Statistical Research Group (1948), provides a compact, tabular representation of the decision rules. For a three-stage plan, a typical control table might appear as follows:

Sample	n_1	n_2	n_3
Acceptance limits	0	1	2
Rejection limits	2	3	3

Two critical design constraints govern the specification of these limits:

1. **Intermediate Stages:** For any stage $j < k$, the gap between the acceptance and rejection thresholds must satisfy $a_{r_j} - a_{c_j} \geq 2$. This ensures a non-empty "continue-sampling" region ($a_{c_j} < R_j < a_{r_j}$) that justifies proceeding to the next stage.

2. **Final Stage:** At the terminal stage k, the rejection limit is set exactly one unit above the acceptance limit ($\boldsymbol{a}_{r_k} = \boldsymbol{a}_{c_k} + 1$). This eliminates ambiguity and forces a definitive accept/reject decision once the maximum allowable sampling effort has been expended.

By embedding these structural rules within the control table, practitioners can systematically enumerate all admissible decision paths, compute the corresponding binomial probabilities under the X-gamma model, and evaluate key performance indicators—such as the OC function, ASN, and ATI—without resorting to simulation. Moreover, because the failure probability $\boldsymbol{\pi}$is derived directly from the X-gamma CDF evaluated at the truncation time x_0, the entire procedure remains fully coherent with the underlying reliability distribution, offering a statistically rigorous alternative to plans based on less flexible lifetime models.

Table 4 presents the structured control table for a three-stage multiple acceptance sampling plan under the X-gamma distribution, detailing how acceptance probabilities are computed at each stage by combining binomial likelihoods with the sequential decision rules defined by the acceptance ($\boldsymbol{a}_{c_j}$) and rejection ($\boldsymbol{a}_{r_j}$) thresholds.

Based on Table 4, the logic of the plan unfolds as follows:

Stage 1: Initial Screening

The first sample of size $\boldsymbol{n}_1$is inspected under the truncated life test, and the number of failures (defective units) $\boldsymbol{r}_1$determines the immediate action:

- If $\boldsymbol{r}_1 = 0$, the lot is accepted with probability $P_{01} = p(0 \mid \boldsymbol{n}_1) = (1-\boldsymbol{\pi})^{\boldsymbol{n}_1}$.
- If $\boldsymbol{r}_1 = 1$, the result is inconclusive ($\boldsymbol{a}_{c_1} < \boldsymbol{r}_1 < \boldsymbol{a}_{r_1}$), so sampling continues; this path carries probability $P_{11} = p(1 \mid \boldsymbol{n}_1) = \boldsymbol{n}_1\boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1}$.
- If $\boldsymbol{r}_1 \geq 2$, the lot is rejected outright, with cumulative probability $P_{21} = 1 - P(\boldsymbol{r}_1 \leq 1 \mid \boldsymbol{n}_1)$.

This stage efficiently filters clearly good or clearly poor lots, minimizing unnecessary inspection when early evidence is decisive.

Stage 2: Conditional Follow-Up

Only lots that were inconclusive at Stage 1 ($\boldsymbol{r}_1 = 1$) proceed to the second sample of size $\boldsymbol{n}_2$. The decision now depends on the cumulative failure count $\boldsymbol{r}_1 + \boldsymbol{r}_2$:

- If the second sample yields $\boldsymbol{r}_2 = 0$failures, the total is 1, which meets the Stage-2 acceptance limit ($\boldsymbol{a}_{c_2} = 1$); the lot is accepted with joint probability $P_{12} = P_{11} \times p(0 \mid \boldsymbol{n}_2)$.
- If $\boldsymbol{r}_2 = 1$, the cumulative count becomes 2, which falls in the continue-sampling region ($\boldsymbol{a}_{c_2} < 2 < \boldsymbol{a}_{r_2} = 3$); the path probability is $P_{22} = P_{11} \times p(1 \mid \boldsymbol{n}_2)$.
- If $\boldsymbol{r}_2 \geq 2$, the cumulative failures reach or exceed the rejection threshold ($\boldsymbol{a}_{r_2} = 3$), and the lot is rejected with probability $P_{32} = P_{11} \times [1 - P(\boldsymbol{r}_2 \leq 1 \mid \boldsymbol{n}_2)]$.

Notice how the Stage-2 probabilities are *joint* probabilities: they condition on having reached this stage (i.e., $\boldsymbol{r}_1 = 1$) and then weigh the likelihood of each possible $\boldsymbol{r}_2$ outcome under the same failure probability $\boldsymbol{\pi}$derived from the X-gamma model.

Stage 3: Final Resolution

Lots that remain inconclusive after Stage 2 (specifically, those with cumulative failures $R_2 = 2$) advance to the third and final sample of size $\boldsymbol{n}_3$. Because this is the terminal stage, the decision rules are tightened to force a conclusion:

- If $\boldsymbol{r}_3 = 0$, the total failures stay at 2, which equals the final acceptance limit ($\boldsymbol{a}_{c_3} = 2$); the lot is accepted with probability $P_{23} = P_{22} \times p(0 \mid \boldsymbol{n}_3)$.
- If $\boldsymbol{r}_3 \geq 1$, the cumulative count reaches 3or more, exceeding $\boldsymbol{a}_{c_3}$and triggering rejection with probability $P_{33} = P_{22} \times [1 - p(0 \mid \boldsymbol{n}_3)]$.

The control table elegantly captures this sequential reasoning: each row represents a distinct decision path, and the associated probability is the product of binomial terms along that path. Summing all "Accept" probabilities across stages yields the overall OC value for the plan at a given $\boldsymbol{\pi}$; similarly, summing "Reject" paths gives the rejection probability, and the expected sample size (ASN) can be computed by weighting each stage's sample size by the probability of reaching that stage.

4. Practical Implications

For quality engineers implementing this framework under the X-gamma distribution:

1. The failure probability $\boldsymbol{\pi} = F_{X\text{-gamma}}(x_0;\theta)$is computed from the model's CDF at the truncation time x_0, ensuring the sampling plan is fully aligned with the assumed lifetime distribution.
2. The control table provides a transparent, auditable blueprint for programming the decision logic in inspection software or standard operating procedures.
3. Because the acceptance and rejection limits satisfy $\boldsymbol{a}_{r_j} - \boldsymbol{a}_{c_j} \geq 2$ for $j < 3$ and $\boldsymbol{a}_{r_3} = \boldsymbol{a}_{c_3} + 1$, the plan guarantees both a well-defined continue-sampling region and a conclusive final decision.
4. The modular structure allows easy extension to more stages or adaptation to different risk specifications (ε^*, producer's risk) by simply updating the control table entries and recomputing the associated binomial probabilities.

In short, Table 4 transforms the abstract mathematics of multi-stage sampling into an actionable, step-by-step inspection protocol one that leverages the flexibility of the X-gamma model while maintaining rigorous control over both producer's and consumer's risk in truncated life-testing scenarios.

Table 4 :Appropriate control table.

The apt probability of the First Stage				
Defective units	The Plan First Stage		probability	Decision
0	0		$p(0\|\boldsymbol{n}_1) = P_{01}$	Accept
1	1		$p(1\|\boldsymbol{n}_1) = P_{11}$	No decision
≥ 2	More than 2		$1 - P(2\|\boldsymbol{n}_1) = P_{21}$	Reject
The apt probability of the second stage as the sum of the joint probabilities of the first and second sampling.				
Defective units	The Plan		probability	Decision
	First	Second		
1	1	0	$p_{11}\, p(0\|\boldsymbol{n}_2) = P_{12}$	Accept
2	1	1	$p_{11}\, p(1\|\boldsymbol{n}_2) = p_{22}$	No decision
3	1	More than 2	$p_{11}\, [1 - P(1\|\boldsymbol{n}_2)] = P_{32}$	Reject
The apt probability of the third stage as the sum of the joint probabilities of the second and third sampling.				
Defective units	The Plan		probability	Decision
	Second	Third		
2	2	0	$p_{22}p(0\|\boldsymbol{n}_3) = P_{23}$	Accept
3	2	More than 1	$p_{22}[1 - P(0\|\boldsymbol{n}_3)] = P_{33}$	Reject

The overall probability of accepting a lot under the proposed three-stage multiple sampling plan can be expressed in closed form as:

$$P_3 \;=\; (1-\boldsymbol{\pi})^{\boldsymbol{n}_1} \;+\; \boldsymbol{n}_1\boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1+\boldsymbol{n}_2-1}[\,1 \;+\; \boldsymbol{n}_2\boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_3-1}], \tag{9}$$

where $\boldsymbol{\pi} = F_{X\text{-gamma}}(x_0;\boldsymbol{\theta})$denotes the probability that a single unit fails before the pre-specified truncation time x_0, computed from the cumulative distribution function of the X-gamma lifetime model (as defined in Equation 2). This expression aggregates the probabilities of all decision paths that culminate in acceptance: immediate acceptance after Stage 1 (zero failures), acceptance after Stage 2 (one failure in Stage 1 followed by zero in Stage 2), and acceptance after Stage 3 (one failure in Stage 1, one in Stage 2, and zero in Stage 3). Each term reflects the binomial likelihood of observing the requisite failure pattern under the assumed reliability model.

To ensure that the sampling plan provides the desired level of consumer protection, we impose the constraint that the acceptance probability for a lot whose true mean life Ωmeets or exceeds the specified value Ω_0 must not exceed ε^*, where ε^* represents the target consumer confidence level. Formally:

$$P_3 \;\leq\; 1 - \varepsilon^*.$$

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This inequality guarantees that a lot of acceptable quality will be rejected with probability no greater than the prescribed consumer's risk $(1-\varepsilon^*)$, thereby aligning the plan with contractual or regulatory quality requirements.
Because Equation (9) is nonlinear in the sample sizes $(\boldsymbol{n_1}, \boldsymbol{n_2}, \boldsymbol{n_3})$, multiple integer triples may satisfy the constraint for a given $(\boldsymbol{\pi}, \varepsilon^*)$. To identify the most efficient solution, the one that minimizes inspection effort—we adopt the Average Sample Number (ASN) as our optimality criterion and impose the natural ordering constraint $\boldsymbol{n_3} \le \boldsymbol{n_2} \le \boldsymbol{n_1}$. This ordering reflects practical inspection logic: later stages serve as targeted follow-ups to resolve ambiguity, so they typically require fewer units than the initial, broader screening sample.
The ASN for a k-stage sequential plan can be derived recursively by weighing each stage's sample size by the probability of reaching that stage. In general form:

$$\text{ASN} = \boldsymbol{n}_1 + \psi_1[\boldsymbol{n}_2 + \psi_2\{\boldsymbol{n}_3 + \psi_3[\ \dots(\boldsymbol{n}_m + 0)\dots\ ]\}], \tag{10}$$

where ψ_j denotes the probability of obtaining an *inconclusive* result at stage j (i.e., the cumulative failure count falls strictly between $\boldsymbol{a}_{c_j}$ and $\boldsymbol{a}_{r_j}$), thereby necessitating progression to stage $j+1$. For the specific three-stage control table adopted here (with $\boldsymbol{a}_{c_1}=0$, $\boldsymbol{a}_{r_1}=2$; $\boldsymbol{a}_{c_2}=1$, $\boldsymbol{a}_{r_2}=3$; $\boldsymbol{a}_{c_3}=2$, $\boldsymbol{a}_{r_3}=3$), the recursion simplifies considerably because only one failure count leads to continuation at each intermediate stage. Consequently:

$$\text{ASN} = \boldsymbol{n}_1 + \psi_1[\boldsymbol{n}_2 + \psi_2\,(\boldsymbol{n}_3)].$$

The continuation probabilities ψ_1 and ψ_2 are derived directly from the binomial model:

- ψ_1 is the probability of observing exactly one failure in the first sample (the sole inconclusive outcome at Stage 1):

$$\psi_1 = \binom{\boldsymbol{n}_1}{1}\boldsymbol{\pi}^1(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1} = \boldsymbol{n}_1\boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1}.$$

- ψ_2 is the joint probability of observing one failure in Stage 1 and exactly one additional failure in Stage 2 (yielding a cumulative count of 2, which remains inconclusive under $\boldsymbol{a}_{c_2}=1$, $\boldsymbol{a}_{r_2}=3$):
-

$$\psi_2 = [\boldsymbol{n}_1\boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1}] \times [\boldsymbol{n}_2\boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_2-1}] = \boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{\pi}^2(1-\boldsymbol{\pi})^{\boldsymbol{n}_1+\boldsymbol{n}_2-2}.$$

Substituting these expressions into the ASN formula yields the explicit objective function for optimization:

$$\text{ASN} = \boldsymbol{n}_1 + [\boldsymbol{n}_1\boldsymbol{\pi}(1-\boldsymbol{\pi})^{\boldsymbol{n}_1-1}][\boldsymbol{n}_2 + (\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{\pi}^2(1-\boldsymbol{\pi})^{\boldsymbol{n}_1+\boldsymbol{n}_2-2})\boldsymbol{n}_3].$$

The design problem is therefore cast as a constrained integer nonlinear program:

$$\begin{aligned} \text{Minimize } (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \quad & \text{ASN}(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \\ \text{Subject to} \quad & P_3(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3; \boldsymbol{\pi}) \le 1-\varepsilon^*, \\ & 1 \le \boldsymbol{n}_3 \le \boldsymbol{n}_2 \le \boldsymbol{n}_1, \\ & \boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3 \in \mathbb{Z}^+. \end{aligned}$$

Because the decision variables are discrete and the constraint surface is non-convex, a systematic grid search or branch-and-bound procedure is typically employed. Starting from plausible initial values (e.g., $\boldsymbol{n_1} = \boldsymbol{n_2} = \boldsymbol{n_3} = 1$), the algorithm incrementally explores feasible triples, evaluates P_3 using Equation (9), computes the corresponding ASN, and retains the combination that minimizes ASN while satisfying the consumer-risk constraint. This search is repeated across the grid of $(x/\Omega_0, \varepsilon^*)$ values of practical interest.

The numerical results summarized in Table 4 reveal several statistically and practically meaningful patterns:

1. As the ratio of truncation time to specified mean life increases, the failure probability $\boldsymbol{\pi}$ decreases (since fewer units are expected to fail within a longer observation window relative to the product's reliability). Consequently, less sampling effort is required to achieve the same discriminatory power. For instance, when x/Ω_0 rises from 0.498 to 4.980, the required first-stage sample size $\boldsymbol{n_1}$ often drops by an order of magnitude, and the ASN follows suit. This underscores the value of extending test duration when feasible: each additional unit of observation time yields more informative data, reducing the need for large samples.
2. Tightening the consumer's risk requirement (increasing ε^* from 0.75 to 0.99) systematically inflates sample sizes and ASN. This is intuitive: higher confidence that a marginal lot will be rejected demands more evidence, which can only be obtained through larger or more numerous samples. The trade-off is most pronounced at lower x/Ω_0 values, where the inherent uncertainty about product reliability is greatest.

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3. For moderate-to-high x/Ω_0 values (e.g., ≥ 3.232), the optimal plans frequently assign very small sizes to the second and third stages ($\boldsymbol{n_2}, \boldsymbol{n}_3 \in \{1,2\}$). This reflects the plan's adaptive intelligence: when early samples provide strong evidence (either clearly acceptable or clearly rejectable), minimal follow-up is needed. Only in borderline cases where the cumulative failure count hovers near the decision thresholds—does the plan invest additional inspection effort. This adaptive behavior is precisely what gives sequential sampling its efficiency advantage over fixed-sample-size designs.
4. The tabulated ($\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, ASN)triples provide ready-to-use design specifications for quality engineers. To apply them:
    - Compute $\boldsymbol{\pi}$ from the X-gamma CDF using the known truncation time $x_0$and the specified mean life Ω_0.
    - Select the row in Table 4 corresponding to the closest (x/Ω_0, ε^*)pair.
    - Implement the sequential inspection protocol using the listed sample sizes and the control table decision rules.
    - Monitor the cumulative failure count at each stage to determine whether to accept, reject, or continue sampling.

Table 4: Minimum sample size and ASN for X-gamma distribution in three-stage sampling plan with three stages.

$\frac{x}{\Omega_0}$	ε^*															
	0.75				0.90				0.95				0.99			
	$\boldsymbol{n}_1$	$\boldsymbol{n}_2$	$\boldsymbol{n}_3$	ASN	$\boldsymbol{n}_1$	$\boldsymbol{n}_2$	$\boldsymbol{n}_3$	ASN	$\boldsymbol{n}_1$	$\boldsymbol{n}_2$	$\boldsymbol{n}_3$	ASN	$\boldsymbol{n}_1$	$\boldsymbol{n}_2$	$\boldsymbol{n}_3$	ASN
0.498	213	10	10	214.5	368	50	50	369.6	414	19	6.676	413.9	514.5	33	32.526	514.6
0.973	50	4	2	50.7	63	5	3	63.2	76	7	4.614	75.7	88.8	10	5.784	88.9
1.500	17	6	3	17.5	25	3	3	25.3	29	5	2.524	29.5	33.6	7	3.973	33.6
3.522	4	2	1	3.7	4	2	1	4.1	7	2	1	7.2	8	2	2	8.0
4.980	2	1	1	2.3	3	2	1	2.7	4	2	1.554	4.0	4	2	1.554	4.0

The computational steps needed to evaluate the OC function using Equation (18) are straightforward and closely parallel with those used for double sampling plans. The foundational methodology traces back to the Statistical Research Group (1948), who not only derived the key formulas but also provided detailed algorithms and worked numerical examples to facilitate practical implementation; see also Wu (1965) for further elaboration on these computational techniques.

A more formal probabilistic framework was later introduced by Wilson and Burgess (1971), who modeled multiple sampling plans as a finite Markov chain. In this representation, the cumulative count of observed failures evolves as a stochastic process across stages, with the acceptance and rejection thresholds acting as *absorbing states*—once the process reaches either boundary, sampling stops and a final decision is locked in. Leveraging this structure, they developed a matrix-based technique for computing the OC: by constructing the transition probability matrix that governs movement between transient states (inconclusive cumulative counts) and absorbing states (accept/reject), the overall acceptance probability can be obtained through standard operations on these matrices, such as solving a system of linear equations or computing fundamental matrices.

While these general methods provide powerful tools for OC evaluation under a wide range of distributional assumptions, the present study focuses specifically on the X-gamma lifetime model. Table 5 reports the resulting OC values computed via Equation (21) for the proposed three-stage multiple sampling plan, across a grid of practically relevant scenarios. Specifically, the table displays acceptance probabilities for varying ratios of true to specified mean life (Ω/Ω_0), different consumer confidence levels (ε^*), and fixed plan parameters $\left(\mathbf{n}_k, \mathbf{a}_{c_k}, \mathbf{a}_{\boldsymbol{r}_k}, x/\Omega_0\right)$. These results allow practitioners to directly assess how well the plan discriminates between acceptable and unacceptable quality levels under the X-gamma assumption, and to verify that the chosen design meets both producer's and consumer's risk requirements in truncated life-testing applications.

Table 5 gives the OC values for the proposed three-stage multiple acceptance sampling plan under the X-gamma distribution, computed for fixed decision rules ($\boldsymbol{a}_{c1} = 0$, $\boldsymbol{a}_{c2} = 1$, $\boldsymbol{a}_{c3} = 2$; $\boldsymbol{a}_{\boldsymbol{r}1} = 2, \boldsymbol{a}_{\boldsymbol{r}2} = \boldsymbol{a}_{\boldsymbol{r}3} = 3$) across a grid of consumer confidence levels ($\varepsilon^* = 0.75, 0.90, 0.95, 0.99$), design ratios ($x/\Omega_0$), and true-to-specified mean life ratios (Ω/Ω_0). Each entry represents the probability that a lot will be accepted given its actual reliability relative to the specification, thereby quantifying the plan's discriminatory power. The table systematically varies Ω/Ω_0 from 2 to 9 to illustrate how acceptance probabilities rise as product quality improves, while holding the truncation time and risk parameters constant. By organizing results first by ε^*, then by x/Ω_0, the layout enables direct comparison of how stricter consumer protection requirements and longer test durations jointly influence sampling performance. These OC values are derived from Equation (21), which aggregates binomial probabilities across all admissible decision paths under the X-gamma failure model. Practitioners can use this table to select plan parameters that achieve target risk levels without resorting to simulation or iterative computation. The consistent structure acceptance probabilities increasing monotonically with Ω/Ω_0 confirm that the plan behaves as theoretically expected. Moreover, the proximity of many OC values to 1.0 for $\Omega/\Omega_0 \geq 7$ indicates strong protection for high-quality lots, while lower values at $\Omega/\Omega_0 = 2$ reflect appropriate rejection of marginal quality. Overall, Table 5 serves as a ready reference for implementing the three-stage X-gamma sampling plan in truncated life-testing applications.

Table 5: The OC values of $\left(\boldsymbol{n}_{\mathcal{k}}, \boldsymbol{a}_{c_{\mathcal{k}}}.\boldsymbol{a}_{\boldsymbol{r}_{\mathcal{k}}}, \frac{x}{\Omega_0}\right)$, $\mathcal{k} = 1, 2$ and 3 for a given ε^* when $\boldsymbol{a}_{c1} = 0$, $\boldsymbol{a}_{c2} = 1$, $\boldsymbol{a}_{c3} = 2$ and $\boldsymbol{a}_{\boldsymbol{r}1} = 2, \boldsymbol{a}_{\boldsymbol{r}2} = \boldsymbol{a}_{\boldsymbol{r}3} = 3$.

ε^*	$\frac{x}{\Omega_0}$	$\frac{\Omega}{\Omega_0}$							
		2	3	4	5	6	7	8	9
0.75	0.498	0.909	0.986	0.996	0.999	0.999	1	1	1
	0.973	0.846	0.976	0.994	0.998	0.999	1	1	1
	1.500	0.834	0.974	0.994	0.998	0.999	1	1	1
	3.522	0.721	0.939	0.984	0.994	0.998	0.999	0.999	1
	4.980	0.664	0.92	0.978	0.992	0.997	0.999	0.999	1
0.90	0.498	0.778	0.962	0.99	0.996	0.998	0.999	1	1
	0.973	0.783	0.964	0.991	0.997	0.999	0.999	1	1
	1.500	0.715	0.946	0.987	0.996	0.998	0.999	1	1
	3.522	0.669	0.924	0.979	0.993	0.997	0.999	0.999	1
	4.980	0.525	0.868	0.961	0.986	0.994	0.997	0.999	0.999
0.95	0.498	0.745	0.953	0.987	0.995	0.998	0.999	1	1
	0.973	0.718	0.949	0.988	0.996	0.998	0.999	1	1
	1.500	0.648	0.929	0.982	0.994	0.998	0.999	0.999	1
	3.522	0.365	0.79	0.932	0.975	0.99	0.995	0.998	0.999
	4.980	0.262	0.712	0.899	0.961	0.983	0.992	0.996	0.998
0.99	0.498	0.655	0.931	0.981	0.993	0.996	0.999	0.999	1
	0.973	0.648	0.933	0.983	0.994	0.998	0.999	0.999	1
	1.500	0.573	0.909	0.977	0.992	0.997	0.999	0.999	1
	3.522	0.283	0.743	0.915	0.969	0.987	0.994	0.997	0.998
	4.980	0.262	0.712	0.899	0.961	0.983	0.992	0.996	0.998

Based on Table 5, several clear and practically important patterns emerge regarding the performance of the three-stage multiple sampling plan under the X-gamma model. First, as the true mean life ratio Ω/Ω_0 increases, indicating better product reliability the OC values rise sharply toward 1.0, confirming that the plan reliably accepts high-quality lots; for instance, when $\Omega/\Omega_0 \geq 7$, acceptance probabilities exceed 0.99 across nearly all scenarios, providing strong assurance to producers of reliable products. Second, tightening the consumer's confidence requirement (raising ε^* from 0.75 to 0.99) systematically lowers OC values at lower Ω/Ω_0 levels, reflecting the intended trade-off: stricter consumer protection makes the plan more conservative, increasing the chance of rejecting borderline lots. For example, at $x/\Omega_0 = 4.980$ and $\Omega/\Omega_0 = 2$, the acceptance probability drops from 0.664 ($\varepsilon^* = 0.75$) to 0.262 ($\varepsilon^* = 0.99$), illustrating how risk allocation directly shapes decision thresholds. Third, the design ratio x/Ω_0 exerts a nuanced influence: at lower truncation ratios (e.g., 0.498), OC curves are steeper, meaning the plan more sharply distinguishes between acceptable and unacceptable quality, whereas higher x/Ω_0 values (e.g., 4.980) yield more gradual transitions, which may be preferable when manufacturing variability is high and abrupt rejections are undesirable. Fourth, the plan demonstrates remarkable efficiency at moderate-to-high reliability levels: even with relatively small sample sizes (as optimized in Table 4), OC values approach unity quickly, indicating that the sequential structure successfully leverages early evidence to minimize inspection effort without compromising decision accuracy. Fifth, the consistency of results across different x/Ω_0values at high Ω/Ω_0suggests robustness: once product quality is sufficiently high, the exact test duration becomes less critical to the acceptance decision. Sixth, for quality engineers, the table provides a practical lookup tool: given a target $\varepsilon^*$and an estimate of x/Ω_0 based on test constraints, one can immediately read off the expected acceptance probability for any assumed Ω/Ω_0, facilitating risk-based contract negotiations or internal quality benchmarks. Seventh, the monotonicity and smoothness of the OC surface support the use of interpolation for intermediate parameter values, reducing the need for exhaustive recomputation. Eighth, the fact that OC values never fall below approximately 0.26—even under the strictest $\varepsilon^* = 0.99$and lowest $\Omega/\Omega_0 = 2$—indicates that the plan avoids excessive producer punishment, maintaining a reasonable balance between stakeholder interests. Ninth, the convergence of OC values to 1.0 at high Ω/Ω_0across all ε^*levels confirms that the plan does not over-penalize excellent quality, a critical feature for incentivizing continuous improvement in manufacturing. Finally, these results collectively validate the theoretical development: the X-gamma-based multiple sampling plan delivers statistically coherent, operationally feasible, and economically efficient acceptance decisions in truncated life-testing contexts, offering a compelling alternative to classical distribution-free or simpler parametric approaches.

Table 6 gives the minimum ratio of actual mean life to specified mean life (Ω/Ω_0) required to ensure that a production lot is accepted with at least 95% probability i.e., under a producer's risk of 0.05 for the proposed three-stage multiple sampling plan under the X-gamma distribution. The table is organized by consumer confidence level (ε^*) and design ratio (x/Ω_0), allowing practitioners to quickly identify the reliability performance a product must achieve to satisfy both stakeholder risk constraints. Each entry represents a threshold: if the true mean life falls below the tabulated Ω/Ω_0value, the producer faces a rejection risk exceeding 5%, even when the sampling plan is optimally configured. These values are derived by solving the nonlinear constraint $P_3 \geq 0.95$(where $P_3$is the OC function from Equation 21) for Ω/Ω_0, holding $\varepsilon^*$and x/Ω_0fixed. The results assume the control table parameters $\boldsymbol{a}_{c1} = 0$, $\boldsymbol{a}_{c2} = 1$, $\boldsymbol{a}_{c3} = 2$, and $\boldsymbol{a}_{r1} = 2, \boldsymbol{a}_{r2} = \boldsymbol{a}_{r3} = 3$, with sample sizes optimized to minimize ASN. By presenting these thresholds in a compact grid, Table 6 serves as a practical design aid for quality engineers negotiating acceptance criteria or setting internal reliability targets. The monotonic increase in required Ω/Ω_0 with higher ε^*reflects the expected trade-off: stronger consumer protection demands higher product performance to keep producer risk in check. Similarly, the influence of x/Ω_0 is non-linear, with mid-range truncation ratios often yielding more favorable (lower) thresholds. Overall, the table translates abstract statistical constraints into actionable reliability benchmarks aligned with the X-gamma lifetime model.

Based on Table 6, several practically significant insights emerge for producers and quality engineers implementing the three-stage X-gamma sampling plan. First, as the consumer's confidence requirement tightens (increasing $\varepsilon^*$from 0.75 to 0.99), the minimum required Ω/Ω_0ratio consistently rises, meaning producers must deliver higher true reliability to maintain the same 5% rejection risk; for example, at $x/\Omega_0 = 0.498$, the threshold climbs from 2.281 to 3.224, a roughly 41% increase in required mean life. Second, the effect of the design ratio $x/\Omega_0$is nuanced: at lower ε^*levels (e.g., 0.75), increasing $x/\Omega_0$from 0.498 to 4.980 raises the required Ω/Ω_0only modestly (2.281 to 3.352), but at higher ε^*(e.g., 0.99), the same change produces a steeper increase (3.224 to 4.816), indicating that plans calibrated for longer test durations become more demanding on actual product life when consumer protection is stringent. Third, the most "producer-friendly" settings where the required $\Omega/\Omega_0$is relatively low tend to cluster around mid-range x/Ω_0values (0.973 to 1.500) and moderate ε^*(0.75–0.90); for instance, at $x/\Omega_0 = 0.973$and $\varepsilon^* = 0.90$, a product needs only about 2.8 times the specified mean life to satisfy both risk constraints, offering a practical target for manufacturing process control. Fourth, the table confirms that the X-gamma-based plan maintains a reasonable balance: even under the strictest $\varepsilon^* = 0.99$, the required Ω/Ω_0never exceeds 4.82, suggesting that the sequential structure prevents excessive producer burden while still delivering strong consumer protection. Fifth, for contractual negotiations, these thresholds provide objective, distribution-based benchmarks: if a supplier can demonstrate that their product's mean life exceeds the tabulated $\Omega/\Omega_0$for the agreed $(x/\Omega_0, \varepsilon^*)$pair, they can confidently expect lot acceptance with $\geq$ 95% probability. Sixth, the relatively smooth progression of values across the grid supports interpolation for intermediate parameter choices, reducing the need for recomputation when test conditions vary slightly. Seventh, the fact that all thresholds exceed 2.0 underscores that truncated life tests inherently require products to outperform specifications to compensate for limited observation time—a critical consideration when setting realistic reliability goals. Eighth, comparing Table 6 with analogous results for single or double sampling (e.g., Tables 1 and 3) would likely reveal that the three-stage plan achieves comparable risk control with lower required Ω/Ω_0ratios, highlighting the efficiency gain from additional sampling flexibility. Ninth, practitioners should note that these thresholds assume optimal sample sizes; using larger-than-necessary samples could relax the Ω/Ω_0requirement slightly, but at the cost of higher inspection effort. Finally, Table 6 transforms the theoretical OC framework into a concrete decision tool: by linking the X-gamma reliability model, truncated testing constraints, and sequential sampling logic, it enables producers to proactively align manufacturing performance with statistically defensible acceptance criteria, fostering trust and efficiency across the supply chain.

Table 6: Minimum ratio of real mean life to prescribed mean life for the acceptance of a batch with producer's risk 0.05.

ε^*	$\frac{x}{\Omega_0}$				
	0.498	0.973	1.500	3.522	4.980
0.75	2.281	2.577	2.625	3.136	3.352
0.90	2.83	2.806	3.049	3.31	3.79
0.95	2.953	3.008	3.238	4.293	4.724
0.99	3.224	3.199	3.418	4.515	4.816

5. Numerical illustration

Tables 1–6 summarize the numerical results for the X-gamma distribution. Table 1 provides the minimum sample sizes and ASN for the DASP to ensure the mean life exceeds a specified value Ω_0 at a given consumer's confidence level ε^*. Table 2 displays the OC values for the DASP with acceptance criteria $\mathbf{a}_{c1} = 0$ and $\boldsymbol{a}_{c2} = 1$. Table 3 gives the minimum Ω/Ω_0 ratios required for batch acceptance at a 5% producer's risk under the double sampling framework.

Similarly, Tables 4–6 present the corresponding results for the three-stage multiple acceptance sampling plan: optimal sample sizes and ASN (Table 4), OC values (Table 5), and minimum Ω/Ω_0thresholds for producer's risk control (Table 6).

Both Tables 1 and 4 show that sample sizes and ASN rise with higher confidence levels (ε^*) but decline as the termination time ratio x/Ω_0increases. Tables 2 and 5 display OC values, which approach 1 as quality improves (Ω/Ω_0 increases) but decreases under stricter confidence requirements. Tables 3 and 6 give minimum Ω/Ω_0 ratios for batch acceptance at a 5% producer's risk, showing these ratios generally grow with both $x/\Omega_0$and confidence level.

5.1 First illustration: Double sampling plan application

Suppose a quality engineer wishes to establish a double sampling plan to confirm that the true mean lifetime of automobile components is at least $\Omega_0 = 1000$hours with a consumer's confidence level of $\varepsilon^* = 0.75$. The engineer decides to truncate the life test at $x_0 = 498$hours, giving a design ratio of $x/\Omega_0 = 0.498$(the closest tabulated value to practical truncation scenarios).

From Table 1, for $\varepsilon^* = 0.75$and $x/\Omega_0 = 0.498$, the optimal double sampling plan parameters are $\mathbf{n}_1 = 261$, $\boldsymbol{n}_2 = 42$, with acceptance criteria $\boldsymbol{a}_{c1} = 0$ and $\boldsymbol{a}_{c2} = 2$, yielding an ASN of 271.3. The inspection procedure is as follows:

- Draw an initial sample of 261 units and test them for 498 hours.
- If zero failures are observed, accept the lot immediately.
- If two or more failures occur, reject the lot.
- If exactly one failure is observed, draw a second sample of 42 units and test them under identical conditions. Accept the lot if the cumulative number of failures across both samples is ≤ 1; otherwise, reject.

The OC values for this plan $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{a}_{c1}, \boldsymbol{a}_{c2}, x/\Omega_0) = (261, 42, 0, 2, 0.498)$with $\varepsilon^* = 0.75$ are reported in Table 2. When the true mean life is twice the specified value ($\Omega/\Omega_0 = 2$), the acceptance probability is 0.540, implying a producer's risk of $1 - 0.540 = 0.460$. As product quality improves, the producer's risk declines: at $\Omega/\Omega_0 = 8$, the OC value reaches 0.975, reducing the producer's risk to 0.025.

From Table 3, the minimum Ω/Ω_0 ratio required to ensure the producer's risk does not exceed 0.05 is 5.573 for this plan configuration ($\varepsilon^* = 0.75$, $x/\Omega_0 = 0.498$). This means the product must achieve a mean lifetime of at least $5.573 \times 1000 = 5{,}573$hours for the batch to be accepted with probability of at least 0.95, thereby protecting the producer while maintaining the agreed consumer confidence level.

5.2 Second illustration: Three-stage sampling plan application

Now consider a scenario where the engineer opts for a three-stage multiple sampling plan under the same reliability requirement: $\Omega_0 = 1000$hours, $\varepsilon^* = 0.75$, and truncation at $x_0 = 498$hours ($x/\Omega_0 = 0.498$).

From Table 4, the optimal three-stage plan parameters are $\boldsymbol{n}_1 = 213, \boldsymbol{n}_2 = 10, \boldsymbol{n}_3 = 10$, with an ASN of 214.5. Using the control table framework with acceptance limits (0, 1, 2) and rejection limits (2, 3, 3) across the three stages, the sequential decision rule operates as follows:

- Stage 1: Test 213 units in 498 hours. Accept if 0 failures; reject if $\geq$ 2failures; continue if exactly 1 failure.
- Stage 2: If continued, test an additional 10 units. Accept if cumulative failures ≤ 1; reject if ≥ 3; continue if cumulative $= 2$.
- Stage 3 (final): If continued, test a final 10 units. Accept if total cumulative failures ≤ 2; otherwise reject.

Table 5 reports the OC values for this three-stage configuration. At $\Omega/\Omega_0 = 2$, the acceptance probability is 0.909, corresponding to a producer's risk of $1 - 0.909 = 0.091$—substantially lower than the double sampling plan's risk at the same quality level. When $\Omega/\Omega_0 = 8$, the OC value exceeds 0.999, effectively eliminating producer's risk for high-quality lots.

From Table 6, the minimum Ω/Ω_0 ratio ensuring producer's risk ≤ 0.05 is 2.281 for this three-stage plan ($\varepsilon^* = 0.75$, $x/\Omega_0 = 0.498$). This threshold is notably lower than the 5.573 required under the double sampling plan, illustrating the efficiency gain from the additional sampling stage: the three-stage design achieves equivalent risk protection with less demanding product performance requirements.

These illustrations demonstrate how the proposed X-gamma based sequential sampling plans provide quality engineers with flexible, statistically grounded tools for designing truncated life tests that balance inspection economy with rigorous risk control.

6. Discussion

The results presented across the numerical evaluations and optimization tables consistently demonstrate that embedding the X-gamma distribution within double and three-stage acceptance sampling frameworks yields inspection protocols that are both statistically rigorous and economically efficient. By anchoring the failure probability directly to the X-gamma CDF at a predetermined truncation time, the proposed methodology captures the right-skewed lifetime behaviors that conventional exponential or Weibull models often oversimplify, producing more realistic risk allocations for time-constrained reliability testing. The optimization outputs clearly reveal that extending the test duration relative to the specified mean life dramatically reduces both initial and follow-up sample requirements, a pattern that aligns with fundamental reliability principles: longer observation windows naturally extract more informative failure data, thereby lowering the need for extensive physical sampling. Conversely, tightening consumer confidence thresholds inevitably inflates sample sizes and average inspection effort, underscoring the inherent trade-off between risk mitigation and resource expenditure. What distinguishes the multi-stage design is its adaptive intelligence; because additional sampling is triggered only when early outcomes fall within the indeterminate zone, the average sample number remains tightly clustered around the first-stage size in the majority of operational scenarios. This sequential efficiency not only curbs unnecessary testing costs but also preserves stringent control over both producer’s and consumer’s risks without disproportionately penalizing high-quality production lots. The control table architecture further bridges theoretical probability with practical implementation, offering quality practitioners a transparent, rule-based decision matrix that can be directly encoded into standard operating procedures or automated inspection software. Moreover, the smooth monotonicity of the operating characteristic surfaces and the consistent progression of producers’ thresholds across parameter grids confirm that the plans are robust, easily interposable, and readily scalable to varying contractual requirements. When benchmarked against existing single-stage approaches, the three-stage configuration consistently achieves comparable or superior discriminatory power with substantially lower expected inspection burdens, validating the strategic value of controlled sampling flexibility. Ultimately, these findings position the X-gamma-based sequential plans as a statistically defensible yet operationally lean alternative for modern manufacturing quality assurance and reliability validation programs.

7. Conclusions

This study successfully extends the application of the X-gamma distribution from single-stage inspection to fully optimized double and three-stage acceptance sampling plans specifically designed for truncated life testing scenarios. By deriving closed-form expressions for operating characteristic curves, average sample numbers, and risk-constrained sample allocations, the research delivers a complete methodological framework that balances distribution-specific reliability modeling with industrial cost constraints. The integer nonlinear optimization routine, which minimizes expected inspection effort while strictly honoring consumer and producer risk boundaries, demonstrates that sequential designs can substantially outperform fixed-sample alternatives without compromising decision accuracy or stakeholder protection. Numerical evaluations confirm that the proposed plans respond predictably to variations in test duration, reliability targets, and confidence specifications, providing quality engineers with a flexible reference for negotiating acceptance criteria and designing economically viable testing protocols. The explicit control table structure ensures that theoretical probabilities translate seamlessly into actionable, step-by-step inspection workflows that require minimal statistical training to implement on the production floor. Looking forward, future research could integrate Bayesian updating into the multi-stage decision process, allowing historical failure records to dynamically adjust truncation times or stage-wise sample allocations in real time. Extending the framework to accommodate progressive censoring, competing failure modes, or environmental stress screening would further broaden its applicability to complex electronic, mechanical, and biomedical reliability contexts. The current methodology also lends itself naturally to software automation, where interactive platforms could instantly generate optimal sample triples and corresponding risk surfaces for user-defined quality thresholds. As industrial systems increasingly demand faster, cheaper, and more precise quality assurance, the X-gamma-based sequential sampling plans presented here offer a statistically coherent pathway toward smarter, more adaptive inspection strategies. By bridging flexible lifetime modeling with controlled sequential sampling logic, this work not only addresses a notable gap in the statistical quality control literature but also equips practitioners with scalable, economically sound tools for next-generation acceptance testing and reliability management.

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