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The Burr Inverse Weibull Model for Risk Analysis Under US Social Security Administration Disability Data Using Peaks Over Random Threshold Method with A Case Study in KSA

Ahmad M. AboAlkhair[1], Abdullah H. Al-Nefaie[1], Mohamed Ibrahim[1,*], Mujtaba Hashim[1], Abdussalam Aljadani[2], Mahmoud M. Mansour[3,4], Noura Roushdy[5], Nazar Ali Ahmed[1], Haitham M. Yousof[4] and Basma Ahmed[6]

Corresponding Author

[1]Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia; aaboalkhair@kfu.edu.sa, aalnefaie@kfu.edu.sa, miahmed@kfu.edu.sa, msaeed@kfu.edu.sa and nahmed@kfu.edu.sa
[2]Department of Management, College of Business Administration in Yanbu, Taibah University, Yanbu Governorate, Saudi Arabia; ajadani@taibahu.edu.sa
[3]Department of Management Information Systems, College of Business Administration in Yanbu, Taibah University, Yanba Governorate, Saudi Arabia; mmmansour@taibahu.edu.sa
[4]Department of Statistics, Mathematics and Insurance, Benha University, Egypt; haitham.yousof@fcom.bu.edu.eg
[5]Department of Insurance & Risk Management, College of Business, Imam Mohammad Ibn Saud Islamic University, KSA; nmrushde@imamu.edu.sa
[6]Department of Information System, Higher Institute for Specific Studies, Giza 12561, Egypt; dr.basma13@gmail.com

Abstract
This study assesses and analyzes real disability insurance data to evaluate extreme risks using advanced statistical tools and metrics. The primary objective is to identify significant events or anomalies in the data and propose actionable strategies for managing financial risks associated with disability insurance claims. To achieve this, we utilize a range of indicators, including Value-at-Risk (VaR), Tail-VaR (TVaR), Tail-Mean-Variance (TMV), Tail-Variance (TV), Mean Excess Loss (MXL), Mean of Order P (MOO-P), Optimal Order of P (O-P), and Peaks Over a Random Threshold Value-at-Risk (PORT-VaR), are applied to identify and describe significant events or anomalies in the data. To address these risks effectively, the research explores the application of the Burr inverse Weibull (BIW) model, a well-regarded framework within extreme value theory (EVT). The study provides a structured approach for disability insurance institutions to better manage unexpected and potentially severe financial losses. Our dataset comprises n=2000 anonymized records from the Social Security Administration (SSA) disability insurance system. By analyzing the asymmetric, right-skewed nature of SSA disability insurance data through these advanced indicators, the research offers insights into the behavior of extreme events and long-tail distributions. Moreover, the percentage distribution of disability reasons in KSA for 2023 is considered. Based on this comprehensive risk analysis, practical recommendations are proposed.

Key Words: Disability Beneficiaries; MOO-P; SSA Disability Data; Burr Family; Peaks Over a Random Threshold; Random Threshold; Value-at-Risk.

1.Introduction

This study aims to uncover critical anomalies and patterns within financial loss data from SSA disability insurance claims, with the goal of improving the accuracy and responsiveness of risk management strategies. By employing advanced statistical models attuned to skewness, heavy tails, and over-dispersion, the analysis captures abrupt shifts, outlier behavior, and latent structural risks often missed by conventional approaches. These findings equip actuaries and insurers with forward-looking insights to better anticipate and mitigate emerging liabilities. In response, the study introduces targeted strategies such as dynamic premium recalibration, capital reserve optimization, and the deployment of early-warning mechanisms. It further incorporates stress-testing techniques to assess system resilience under extreme conditions. Collectively, this work enhances actuarial decision-making amid rising volatility and evolving risk landscapes. By analyzing data on disability beneficiaries. As such, the effective management of risks inherent in disability insurance is paramount for ensuring the long-term sustainability, solvency, and operational resilience of insurance providers, especially amid the increasing frequency and severity of unpredictable, high-impact events. Given the complex interplay between economic cycles, demographic shifts, and evolving morbidity trends, insurers must adopt robust risk assessment frameworks that account not only for expected losses but also for extreme and rare scenarios. Harrington and Niehaus (2003) underscore the importance of risk-based capital requirements and prudent underwriting practices, while Schmidt (2017) highlights the growing relevance of dynamic risk models that integrate behavioral and macroeconomic uncertainties. These perspectives collectively reinforce the urgent need for insurance institutions to implement forward-looking, data-driven strategies that can adapt to emerging risks and regulatory demands, thereby safeguarding policyholder trust and market stability.

The field of risk modeling and analysis has seen significant advancements in recent years, particularly in the areas of EVT, probability modeling, and financial risk assessment. This literature review synthesizes key contributions from four recent studies that explore novel methodologies for modeling losses, revenues, and risks in diverse contexts such as finance, insurance, reliability, and actuarial science. These works collectively emphasize the importance of advanced statistical tools, including VaR, MOO-P, PORT, and entropy-based analyses, to address real challenges. Elbatal et al. (2024) introduce a novel probability model designed to analyze losses and revenues in financial and risk management contexts. The study integrates entropy analysis into the framework, providing insights into the uncertainty and variability inherent in loss distributions. By applying this model to VaR and MOO-P analyses, the authors demonstrate its versatility in quantifying tail risks and optimizing decision-making processes. Aljadani et al. (2024) propose a groundbreaking model tailored for applications in finance and reliability engineering, with a particular focus on financial PORT-VaR analysis. The authors leverage EVT principles to address the challenges posed by rare but extreme events in financial and reliability datasets. Alizadeh et al. (2024) present the extended Gompertz model, a versatile framework for analyzing extreme stresses and their implications for risk assessment. The study applies this model to evaluate MOO-P and conduct statistical threshold risk analysis, focusing on datasets characterized by extreme stress conditions. Yousof et al. (2024) develop a discrete claims-model specifically designed for analyzing inflated and over-dispersed automobile claims frequencies. The study addresses the challenges posed by zero-inflated and over-dispersed data, which are common in actuarial applications, by proposing a novel statistical framework.

From a statistical and analytical perspective, the EVT is considered one of the most important tools used in studying and analyzing sudden events, especially for data that have heavy tails. The paper presented a comprehensive financial analysis of the risks related to financial reserves to confront sudden claims related to disability and a large set of financial recommendations for insurance institutions to avoid rare sudden events that have a significant impact on the financial position of institutions. It also presented a comprehensive analysis related to the number of health care providers and their age groups and the risks associated with that, including the reluctance of certain groups to provide these services and the impact of that on the disability scale, with a good set of recommendations for Saudi institutions. The paper used a set of modern risk measures that are applied for the first time in the field of disability. The EVT is a specialized branch of statistical analysis focused on examining extreme events and tail risks in fields such as finance, insurance, and reliability engineering. EVT provides a framework for quantifying the likelihood of rare events that standard statistical models may overlook. Key concepts include MOO-P, VaR $|q$, TMV $|q$, PORT-VaR $|q$, TVaR $|q$, MXL $|q$ and TV $|q$. These risk indicators are essential for assessing tail risk, volatility, and potential financial losses that go beyond standard indicators. By applying EVT methodologies, analysts and risk managers can more effectively handle extreme situations, resulting in better decision-making and improved stability.

In the context of disability beneficiaries, data often exhibit skewed characteristics that challenge traditional symmetric models. Extreme values in disability data can significantly impact risk assessments, yet they are frequently neglected by standard approaches. Researchers like Embrechts et al. (2013), Klugman et al. (2012), and Hogg and Klugman (2009) have examined the EVT methodology and its various applications, emphasizing its effectiveness in managing disability risk. Our paper intends to address these challenges and propose solutions by utilizing EVT principles to enhance strategies for managing disability risk in KSA. Through this effort, we aim to provide actionable insights that enable policymakers and practitioners to make informed decisions in this intricate field.

This paper focuses on advancing the understanding of disability insurance risk analysis through the application of sophisticated statistical methodologies. Specifically, it explores the utilization of metrics rooted in EVT to assess and quantify the tail risks associated with disability insurance datasets. EVT provides a robust framework for capturing the probability of extreme events that lie beyond the scope of traditional statistical models, making it particularly suited for analyzing rare but severe disability claims. Moreover, it presents a comprehensive analysis of disability insurance risks using real data sourced from the SSA disability beneficiaries. The research employs a variety of advanced statistical techniques, including above-mentioned risk estimators. These methodologies are tailored to evaluate the likelihood and financial implications of extreme disability events, crucial for informing robust risk management strategies within insurance institutions. The adoption of the BIW model within the EVT framework proves instrumental in capturing the significant right-skewed tail behavior observed in disability insurance data. By identifying and analyzing such tail risks, this research aims to enhance the resilience of disability insurance providers against sudden and substantial financial losses, thereby reinforcing their capacity to withstand adverse market conditions.

In this paper, we examine the application of various key indicators under the framework of extreme disability insurance value theory for modeling and analyzing disability insurance risk. Specifically, we focus on the above-mentioned risk estimators, grounded in EVT, which are essential for managing financial indemnity losses. They provide robust analytical capabilities for assessing tail risks and quantifying extreme loss events, which are crucial for effective risk management decisions. By employing these indicators, financial institutions can enhance their understanding of and preparedness for the impact of extreme events on their operations and overall financial stability. This proactive approach is vital for minimizing the effects of rare but significant financial losses, ensuring that institutions are equipped to navigate turbulent economic conditions. The literature reveals several notable studies that delve into themes closely related to risk analysis and probability modeling in the context of disability insurance, including works by Pritchard (2006), D'Amico et al. (2013), Lahiri et al. (2018), Kessels and Waldmann (2016), Martínez-Ruiz (2018), Low and Pistaferri (2020), Chorowski and Szymanowski (2021), and Mohamed et al. (2024). The motivation for this paper arises from the urgent need to conduct a comprehensive financial analysis of actual financial loss data. Pritchard (2006) highlighted the actuarial challenges in pricing disability products under uncertain claim durations, while D'Amico et al. (2013) applied semi-Markov models to disability insurance to improve transition probability estimates. Lahiri et al. (2018) further refined stochastic modeling approaches to account for macroeconomic covariates influencing disability incidence. Kessels and Waldmann (2016) introduced Bayesian inference methods to address parameter uncertainty in claims modeling, aligning well with recent trends in predictive analytics. Martínez-Ruiz (2018) investigated long-term disability coverage using a survival analysis framework, underscoring the importance of tail-risk modeling. Low and Pistaferri (2020) explored income risk and insurance using life-cycle models, demonstrating the profound economic implications of disability events. Chorowski and Szymanowski (2021) offered a novel approach to reserve estimation under stochastic dependencies, enhancing solvency assessments. Most recently, Mohamed et al. (2024) proposed an entropy-based modeling structure specifically tailored for over-dispersed and skewed disability claims data, representing a significant leap forward in the actuarial toolkit. The motivation for this paper arises from the urgent need to build upon these foundations and conduct a comprehensive financial analysis of actual financial loss data, using contemporary probabilistic models that can better capture the asymmetric, heavy-tailed nature of disability-related financial risks.

Statistical literature underpinning modern lifetime and risk modeling has evolved rapidly through a series of methodologically rich contributions centered on flexible generator families, tail risk quantification, and EVT-based inference, precisely the pillars supporting this paper’s application of the Burr Inverse Weibull (BIW) model to SSA disability data. Beginning with Yousof et al. (2016), who introduced a six-parameter Fréchet extension to capture complex hazard shapes, the field advanced via Korkmaz et al. (2017) and the prolific 2018 wave, including Afify et al. (2018a,b), Alizadeh et al. (2018a,b), Cordeiro et al. (2018), Korkmaz, Yousof & Hamedani (2018), and Mustafa et al. (2018), which collectively established diverse *G*-families (e.g., Burr XII-G, odd log-logistic Topp–Leone-G, transmuted Weibull-G) enabling bathtub, unimodal, and reversed-J hazard forms essential for insurance loss modeling. This was followed by robust validation frameworks in Goual, Yousof & Ali (2019) and Yousof et al. (2019), and then by deeper integration of dependence and estimation in 2020: Al-Babtain et al. (2020), Elgohari & Yousof (2020), Salah et al. (2020), Mansour et al. (2020a,b), and Rasekhi et al. (2020) introduced copulas (notably Clayton), Bayesian reliability under generalized logistic models, and compact three-parameter generalizations, laying the theoretical groundwork for BIW’s formulation in Salah et al. (2020). Crucially, Mohamed et al. (2024) recently synthesized these advances into a size-of-loss model for negatively skewed claims, demonstrating the maturation of this research arc toward actuarial relevance. It is within this continuum, spanning distributional innovation, EVT embedding, and risk metric operationalization (e.g., MOO-P, PORT-VaR), that the present study situates BIW as a natural, empirically justified choice for modeling the extreme, right-skewed, heavy-tailed nature of disability claims, where tail-sensitive estimation and robust risk indicators are non-negotiable for solvency and strategic resilience.

The rest of this work is organized as follows: Section 2 provides the main risk indicators for disability analysis. An empirical result for MOO-P method and the optimal order (O-P) is presented in Section 3. Section 4 presents the SSA disability beneficiaries' data under the MOO-P evaluation. PORT-VaR $|q$ estimator for extreme SSA disability beneficiaries' data is given in Section 5. Section 6 is allocated for risk analysis under the disability insurance data. Section 7 presents the risk analysis for the disability reasons in KSA. Finally, Section 8 consolidates the paper's findings and conclusions.

2. Risk indicators

2.1 VaR $|q$

Consider the extreme value BIW model of Salah et al. (2020), with the following cumulative distribution function

$$F_{\underline{V}}(x) = 1 - \frac{\left[1 - exp\left(-x^{-\theta}\right)\right]^{\lambda\beta}}{\{[1 - exp(-x^{-\theta})]^{\lambda} + exp(-\lambda x^{-\theta})\}^{\beta}}, \tag{1}$$

The BIW model in (1) demonstrates remarkable flexibility, allowing its application across a wide spectrum of modern scientific and engineering fields. It can effectively address complex problems in mining theory and control systems, offering improved modeling precision and stability. In Bayesian estimation, the model performs efficiently when combined with joint Jeffreys priors, enhancing parameter inference and uncertainty quantification. Moreover, its adaptability makes it suitable for handling massive data sets within big data environments. Such versatility underscores its potential as a powerful analytical tool for future research and practical applications (see Jameel et al., 2022; Salih & Abdullah, 2024; Salih and Hussein et al 2025, Al-Door et al 2025 and Hussein et al 2025, Salih & Hmood, 2020, 2022). Generally

$$\mathrm{VaR}|q = QF(u)|u = q, \tag{2}$$

where $QF(u)|u = q$ is the quantile function of the considered model. Then, based on (2) and for the extreme value BIW model we have

$$\Pr\left[X > \left(-ln\left\{\frac{U(q;\lambda,\beta)}{(1-q)^{\frac{1}{\lambda\beta}} + U(q;\lambda,\beta)}\right\}\right)^{-\frac{1}{\theta}}\right] = 1\%, 5\%, \ldots |q = 1\%, 5\%, \ldots. \tag{3}$$

where $1 - q$ is the confidence level (CL) (e.g., $q = 0.01$ for $1 - q = 99\%$ confidence) and

$$U(q;\lambda,\beta) = \left[1 - (1-q)^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}}.$$

2.2 TVaR $|q$

The TVaR $|q$ of X is given as

$$\mathrm{TVaR}|q = V\frac{1}{1-q}, \tag{4}$$

where

$$V = \int_{\mathrm{VaR}|q}^{+\infty} x\, f_{\underline{V}}(x)dx,$$

and $f_{\underline{V}}(x) = dF_{\underline{V}}(x)/dx$.

Then, based on (2) and (3) and for for the extreme value BIW model we have

$$\mathrm{TVaR}|q = \frac{1}{1-q}\sum_{\zeta=0}^{\infty} \nabla_{\zeta}^{1,\theta} \times \Gamma\left(1 - \frac{1}{\theta}, \frac{1}{\left(\mathrm{VaR}^{|q}\right)^{\theta}}(1+\zeta)\lambda^{\theta}\right)|\theta > 1, \tag{6}$$

where

$$\nabla_{\zeta}^{1,\theta} = \nabla_{\zeta}(1+\zeta)^{\frac{1}{\theta}}|\theta > 1$$

and ∇_{ζ} is a constant provided by Alizadeh et al. (2017) and Salah et al. (2020) as

$$\nabla_{\zeta} = \frac{ab}{1+\zeta}\sum_{i_1,i_2=0}^{\infty}\sum_{i_3=\zeta}^{\infty}(-1)^{i_2+\kappa+\zeta}\binom{-(1+\beta)}{i_1}\binom{-[\lambda(1+i_1)+1]}{i_2}\binom{\lambda(1+i_1)+i_2+1}{i_3}\binom{i_3}{\zeta},$$

and $\Gamma(s,r)|r>0=\int_r^{+\infty} x^{s-1}\,exp(-x)\,dx$ is the upper incomplete gamma function.

2.3 The TV $|q$ indicator

By definition (see Landsman (2010)), the TV $|q$ of a random X is given by

$$\mathrm{TV}|q=E(X^2)-(\mathrm{TVaR}|q)^2|X>\mathrm{VaR}|q. \tag{7}$$

Then for the BIW model, we have

$$\mathrm{TV}|q=\frac{1}{1-q}\sum_{\zeta=0}^{\infty}\nabla_{\zeta}^{2,\theta}\times\Gamma\left(1-\frac{2}{\theta},\frac{1}{(\mathrm{VaR}|q)^{\theta}}(1+\zeta)\lambda^{\theta}\right)-(\mathrm{TVaR}|q)^2|\theta>2, \tag{8}$$

where

$$\nabla_{\zeta}^{2,\theta}=\nabla_{\zeta}(1+\zeta)^{\frac{2}{\theta}}|\theta>2$$

and TVaR $|q$ is given in (4).

2.4 TMV $|q$

To effectively utilize the TMV $|q$ risk indicator as defined by Furman and Landsman (2006), it is essential to incorporate a key foundational result. The TMV indicator integrates two important metrics: TVaR $|q$ and TV $|q$, while also introducing a scaling factor, denoted as ς, which satisfies the condition $0<\varsigma<1$. This relationship allows the TMV $|q$ to be expressed mathematically as

$$\mathrm{TMV}|q=\varsigma\mathrm{TV}|q+\mathrm{TVaR}|q;\ 0<\varsigma<1. \tag{9}$$

This formulation in (5) highlights how the TMV $|q$ risk indicator effectively captures the interplay between the potential for extreme losses and the associated volatility, providing a more nuanced understanding of risk exposure. By combining these two measures, the TMV $|q$ not only accounts for the magnitude of potential losses but also adjusts for the variability of those losses, thereby enhancing the precision of risk assessments in financial contexts.

2.5 MOO-*P* indicator and the O-*P*

The MOO-P estimators provide a strong parametric framework for assessing tail risks in disability insurance data. By choosing the right order P, they accurately capture the tail characteristics of the distribution, which is essential for modeling extreme events. This emphasis on tail behavior underscores the heavy-tailed nature of claims, influencing risk assessment and pricing strategies. Reinsurers frequently utilize these estimators to assess potential losses from infrequent, severe events, enhancing their risk management and strategic planning.

MOO-P analysis is a statistical method used in financial risk assessment to characterize datasets using different moment orders. This technique involves elevating each data point to the power of P (where P is a positive integer) and then computing the average of these transformed values. The MOO-P is calculated using the formula

$$\mathrm{MOO-P}=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^P\right)^{\frac{1}{P}}. \tag{10}$$

The MOO-P is a generalization of classical means (e.g., arithmetic, geometric, harmonic). So, depending on the value of P and when $P=1$, MOO-P reduces to the arithmetic mean. As $P\to 0$, MOO-P approaches the geometric mean. When $P=-1$, MOO-P corresponds to the harmonic mean. For $P\to+\infty$, MOO-P converges to the maximum value in our dataset.

For $P\to-\infty$, MOO-P converges to the minimum value in our dataset. For $P<1$, the MOO-P emphasizes smaller values, making it ideal for analyzing failure probabilities or weaker elements in the dataset.

For $P>1$, the MOO-P highlights larger values, which is useful for studying peak performance metrics or stronger elements. The O-P in MOO-P analysis is context-dependent and must align with the analytical goals, dataset characteristics, and engineering requirements. By systematically exploring different P values and validating the results, researchers and practitioners can identify the most appropriate P for their specific application, for more details ans applications see Alizadeh et al. (2024) and Elbatal et al. (2024).

2.6 The PORT-VaR $|q$ method

The PORT-VaR $|q$ estimator holds significant importance in the field of disability insurance and finds widespread application due to its ability to accurately assess and manage extreme risks associated with disability claims. PORT-VaR $|q$ helps in identifying and quantifying extreme financial losses associated with disability insurance claims. By setting a threshold q, which represents a high percentile of loss distribution, insurers can focus on the most severe events that exceed this threshold. It specifically addresses the tail risk in disability insurance portfolios, which are crucial for insurers as these tail events can have significant financial implications. PORT-VaR $|q$ provides a conditional probability estimate of exceeding a given threshold q, which aids in understanding the likelihood of extreme losses occurring in disability insurance. PORT-VaR $|q$ helps insurers in accurately pricing disability insurance policies by incorporating the risk of extreme losses into premium calculations. t assists underwriters in assessing the risk associated with potential insured individuals who may be at higher risk of extreme claims due to pre-existing conditions or occupational hazards (see Aljadani et al. (2024) and Yousof et al. (2024) for more details and applications).

The PORT-VaR $|q$ indicator is a method used to estimate the VaR for extreme events in a financial or insurance context. To calculate the PORT-VaR $|q$ indicator, select a threshold q which represents a high percentile of loss distribution. Typically, q is chosen such that

$$\Pr(X > q) = \alpha|1\%, 5\%, 10\%\ldots.$$

For a series of observations $X_1, X_2, \dots, X_n$, where X_i represents the loss amount, the excess losses are defined as

$$Y_i = (X_i - q) \times I(X_i > q).$$

Here, $\mathbf{I}\ (X_i > q)$ is an indicator function that equals 1 if $X_i > q$ and 0 otherwise. Arrange the excess losses Y_i in decreasing order, denoted as

$$Y_{(1\ :\ \kappa)},\ Y_{(2\ :\ \kappa)},,\ Y_{(\kappa\ :\ \kappa)},$$

where κ is the number of excess losses. The PORT-VaR $|q$ is estimated as the average of the top κ_α excess losses, where κ_α is chosen such that

$$\kappa_\alpha = \lfloor n\alpha \rfloor.$$

Then,

$$\text{PORT-VaR}_{[q]}(\alpha) = \frac{1}{\kappa_\alpha}\sum_{i=0}^{\kappa_\alpha} Y_i. \tag{11}$$

Here, Y_i denotes the i^{th} largest excess loss.

3. Empirical MOO-P analysis and O-P

In this Section, an empirical analysis for MOO-P$|P \uparrow 5$ evaluation and O-P is presented. Three differnt cases are considered for evaluation process. Four statistical measures have been calculated to contribute to the evaluation process, which is the true mean (Tr-M), the MOO-P$|P \uparrow 5$, MSE and bias (Bias). Table 1 gives a comprehensive assessment of the MOO-P under various orders $P \uparrow 5$ using a substantial sample size of $n = 100{,}000$ and some initial values (I_0). The table presents the calculated MOO-P values, which denote the mean of the top P ordered observations from the dataset. This assessment is crucial for understanding the distribution's tail behavior and assessing risk associated with extreme events. It provides insights into the variability and magnitude of extreme values, aiding in risk management, policy design, and decision-making processes.

By analyzing MOO-P across different orders P, insurers and risk analysts can gauge the frequency and severity of extreme events covered by the BF distribution. This information guides the development of robust insurance policies, ensuring adequate coverage and financial resilience against rare but impactful occurrences." Expanding on this description provides a clearer context of how the BF-MOO-P assessment under various orders and sample size contributes to understanding risk and modeling in insurance and other fields. Figure 1 (the first, second and third plots) presents plots for the MOO-P, MSE and bias values under the MOO-P assessments, respectively. The results of Figure 1 support the numerical results of Table 1.

Table 1: MOO-P assessment under $P \uparrow 5$ and $n = 100000$.

P→	1	2	3	4	5	O-P
I_0			0.5, 1.25, 1.5			5
Tr-M			3.728627			
MOO-P	0.7760452	0.7796036	0.7935106	0.8025884	0.8097544	
MSE	8.717742	8.696741	8.614911	8.561705	8.51982	
Bias	2.952582	2.949024	2.935117	2.926039	2.918873	
I_0			10, 5, 15			5
Tr-M			15.20999			
MOO-P	14.29909	14.30457	14.32468	14.33758	14.34758	
MSE	0.8297254	0.8197744	0.7837706	0.7610876	0.7437431	
Bias	0.9108926	0.9054139	0.8853082	0.8724033	0.8624054	
I_0			5, 50, 5			5
Tr-M			62.98503			
MOO-P	36.78272	36.90454	37.35784	37.64986	37.87703	
MSE	686.561	680.1919	656.7529	641.8707	630.4114	
Bias	26.20231	26.08049	25.62719	25.33517	25.10799	

Based on Table 1 (Case 1: I 0 =0.5, 1.25, 1.5):
The Tr-M serves as a benchmark for comparing the accuracy of MOO-P estimates. MOO-P|$P \uparrow 5$ values generally increase with higher values of P, reflecting the increasing mean of the top P ordered values. MSE decreases as P increases, indicating that estimates become more accurate with larger P. Bias shows a similar trend to MOO-P, decreasing slightly with increasing P. It is better to consider using P = 5 for a more accurate estimation of the mean in this scenario, as it shows the smallest MSE and Bias among the considered values.

Based on Table 1 (Case 2: I 0 =10, 5, 15):
The Tr-M is higher compared to Case 1, indicating larger mean values. MOO-P|$P \uparrow 5$ values closely approximate the Tr-M, suggesting that even lower values of P provide good estimates due to lower MSE and Bias. MSE decreases consistently as P increases, showing improved accuracy in estimation. Bias also decreases with higher P, indicating closer proximity to the Tr-M. Finally, P = 5 provides the most accurate estimation with the lowest MSE and Bias, suitable for precise assessment.

Based on Table 1 (Case 3: I 0 =5, 50, 5):
The Tr-M is significantly higher compared to Cases 1 and 2, indicating a larger mean value. MOO-P|$P \uparrow 5$ values are lower than the Tr-M, reflecting the distribution's tail behavior. MSE is high across all P values, suggesting higher variability in estimation. Bias decreases with increasing P but remains relatively high due to the distribution's skewness. Given the high MSE and Bias, further exploration of higher values of P beyond 5 may be necessary to achieve more accurate estimates in this case.

Case 1:
I_0 =0.5, 1.25, 1.5

Case 2:
I_0=10, 5, 15

Case 3:
I_0 =5, 50, 5

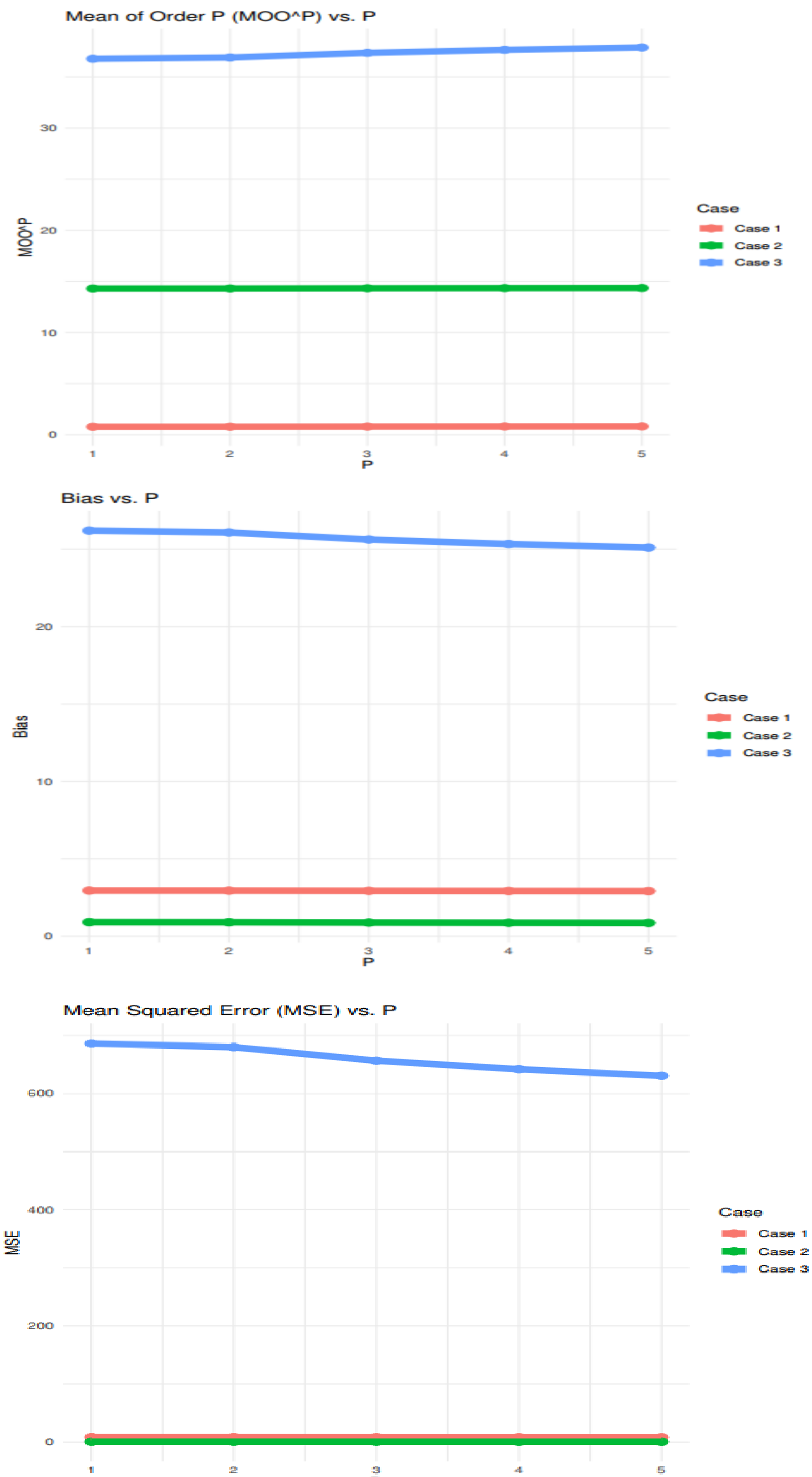

Figure 1: MOO-P , MSEs and bias v P.

4. SSA disability beneficiaries' data under the MOO-*P* evaluation

The data on SSA disability beneficiaries aged 18-64 is crucial for understanding the challenges and needs faced by this demographic, the data is available at https://www.ssa.gov/open/data/. This data helps to identify the social and economic needs of working-age beneficiaries, enabling governments and non-profit organizations to develop appropriate support programs tailored to their needs. The SSA data provides vital information for policymakers to design effective policies aimed at improving the quality of life for individuals with disabilities, such as enhancing access to healthcare, education, and employment opportunities. By analyzing the data, trends in disability rates among younger and middle-aged adults can be identified. This understanding can shed light on the root causes of disabilities and improve prevention and early intervention strategies. Increasing awareness about disability issues through data helps the community understand the challenges faced by individuals with disabilities, fostering solidarity and community support.

The SSA dataset exhibits significant right skewness, as reflected by a skewness value of 1.87 and a mean (219,714.53) substantially higher than both the median (164997.00) and mode (18795), indicating the presence of high-value outliers that distort the average. The large standard deviation (218862.04) and IQR (225514.25) confirm a widespread in the data, further emphasized by an extensive range from 11,780 to 1,191,554. With a kurtosis of 3.85, the distribution is leptokurtic, suggesting heavy tails and a higher probability of extreme values. Overall, the data is highly variable, asymmetrical, and sensitive to outliers, warranting cautious interpretation and potential transformation before further analysis, see Table 2 below.

Table 2: Describing the SSA dataset.

Min.; 1st Qu.; 3rd Qu.; Max.	11780; 58273; 283787; 1191554
Mean; Median	219714.53; 164997
Mode; Standard Deviation	18795; 218862.04
Skewness:	1.87
Kurtosis:	3.85

The analysis of the MOO-P can be important in various contexts, including disability insurance beneficiaries. Disability insurance involves assessing claims made by beneficiaries over time. By analyzing the MOO-P , insurers can gain insight into the distribution of claims amounts. This helps in understanding the typical sizes of claims and how they vary across different orders (e.g., top 5%, top 10% of claims). Als, Insurance companies need to manage risk effectively. Knowing the MOO-P helps in understanding the tail distribution of claims. High values of mean order P indicate potentially large claims that may be infrequent but significant. This information aids in setting appropriate reserves and pricing policies to ensure financial stability. Forecasting future claims is crucial for insurance operations. MOO-P analysis provides information on the average size of claims at different percentiles. This allows insurers to forecast their potential liabilities more accurately, influencing their financial planning and reserves.

Moreover, insurance policies are designed based on actuarial assessments of risk and expected claims. MOO-P analysis provides insights into the potential exposure to large claims. This information helps insurers design policies that balance coverage and affordability while ensuring sustainability. Benchmarking the MOO-P against industry standards or historical data can provide insights into whether claim sizes are increasing or decreasing over time. This comparison helps insurers adjust their strategies and policies accordingly. Finally, understanding the distribution of claims sizes can inform claims management practices. It allows insurers to prioritize resources for handling larger claims effectively, ensuring timely and fair processing for beneficiaries. Disability insurance often deals with tail risks, such as severe disabilities or chronic conditions that result in long-term payouts.

The MOO-P analysis helps quantify the severity and frequency of extreme disability cases by focusing on the distribution's tail as P increases. This is crucial for estimating the financial reserves needed to cover such rare but costly events. Insurers use statistical models to price disability insurance policies and allocate reserves. The O-P selection ensures that these models accurately capture the tail behavior of disability claims. Underestimating tail risks can lead to underpricing policies and insufficient reserves, while overestimating can make policies unaffordable. By analyzing MOO-P across different P values, insurers can validate the adequacy of their statistical models in capturing the tail behavior of disability claims data. Models that consistently perform well across various P values indicate robustness in predicting extreme disability scenarios, thereby improving model reliability and accuracy. Moreover, the O-P determination aids in forecasting future disability claims, allowing insurers to adjust reinsurance strategies accordingly. Reinsurance plays a crucial role in mitigating large claims and managing overall risk exposure, making accurate tail risk assessment essential.

In this Section, a real analysis for MOO-P$|P \uparrow 20$ evaluation and O-P is presented. Four statistical measures have been calculated to contribute to the evaluation process, which are the Tr-M, the MOO-P$|P \uparrow 20$, MSE and Bias. Table 3 gives a comprehensive assessment of the BF-MOO-P under extreme disability insurance data. Table 2 below provides the MOO-P assessment under the extreme disability insurance data. The table is structured with different sections for each range of P, where P ranges from 1 to 20.

For each range of P, the table provides values for Tr-M, MOO-P, MSE and Bias. The MOO-P values generally increase as P increases, which suggests an increasing level of complexity or model fitting. The MSE generally decreases as P increases, indicating potentially better model performance with higher P. The bias values generally decrease as P increases, suggesting a trend towards reduced bias with more complex models. The MOO-P method shows improvement in MOO-P, MSE, and Bias metrics as P increases, reflecting better model performance with higher polynomial orders.

While higher P reduces bias and MSE, it also increases model complexity, potentially leading to overfitting if not managed properly. Based on these results, choosing an optimal P involves balancing improved accuracy (lower MSE) and reduced bias with the risk of overfitting. Further analysis or validation techniques may be necessary to confirm the robustness of the chosen P. Therefore, based on both MSE and Bias considerations, the disability insurance companies should choose

$$P \to 16,\ \ 17,\ \ 18,\ \ 19,\ \ 20$$

as the optimal order. This choice minimizes both the MSE and Bias, indicating it provides the most accurate predictions or estimations under extreme disability insurance data conditions.

Table 3: MOO-P assessment under the extreme SSA disability beneficiaries' data.

	P→1,2,3,4,5
Tr-M	219714.5
MOO-P	11780, 11833.5, 11938.33, 12079.5, 12221.8,
MSE	43236770762, 43214524629, 43170949895, 43112307676, 43053234994,
Bias	207934.5, 207881, 207776.2, 207635, 207492.7,
	P→6,7,8,9,10
Tr-M	219714.5
MOO-P	12344.67, 12475.29, 12616.5, 12726.44, 12868.8,
MSE	43002262209, 42948106361, 42889596017, 42844069548, 42785158004,
Bias	207369.9, 207239.2, 207098, 206988.1, 206845.7,
	P→11,12,13,14,15
Tr-M	219714.5
MOO-P	12990.55, 13102.75, 13218.69, 13325.93, 13429.47,
MSE	42734807770, 42688429618, 42640532966, 42596256780, 42553529334,
Bias	206724, 206611.8, 206495.8, 206388.6, 206285.1,
	P→16,17,18,19,20
Tr-M	219714.5
MOO-P	13526.19, 13627.65, 13720, 13809.11, 13900.3
MSE	42513634562, 42471805298, 42433748367, 42397045912, 42359499246
Bias	206188.3, 206086.9, 205994.5, 205905.4, 205814.2

5. PORT-VaR $|q$ estimator for extreme SSA disability beneficiaries' data

To begin our analysis of disability insurance, we will utilize graphical representations to provide a clearer understanding of the underlying data.

Figure 2 is a box plot of the SSA disability beneficiaries' data, and it visually confirms what the descriptive statistics in Table 2 already suggest: the distribution is heavily right-skewed with a long upper tail and several high-magnitude outliers. You can see the median (around 165000) sitting far below the mean (≈ 219715), which is classic evidence of asymmetry and extreme upper observations, a hallmark of heavy-tailed insurance loss data. In applied statistics, box plots are essential for quick diagnostics: they reveal central tendency, spread, skewness, and potential anomalies all at once, guiding decisions about transformations or model selection. For this specific dataset, the box plot underscores why symmetric models (like Gaussian-based ones) would be inappropriate, and why extreme value theory tools, such as the BIW model, are necessary. It also justifies the use of risk measures like VaR, TVaR, and PORT-VaR, which focus explicitly on the tail behavior visible here. In practice, this plot would alert an actuary or risk analyst that standard deviation alone is insufficient, and that tail-sensitive estimation (e.g., MOO-P with optimal *P*) is essential. The presence of extreme values stretching well beyond the upper whisker (up to ~1.2 million) shows why disability insurers must prepare for rare but financially devastating claims. This isn't just noise, it's the signal you build your capital buffer around. Overall, Figure 2 serves as both a warning and a roadmap: warning against naive modeling, and guiding toward robust, EVT-grounded risk frameworks.

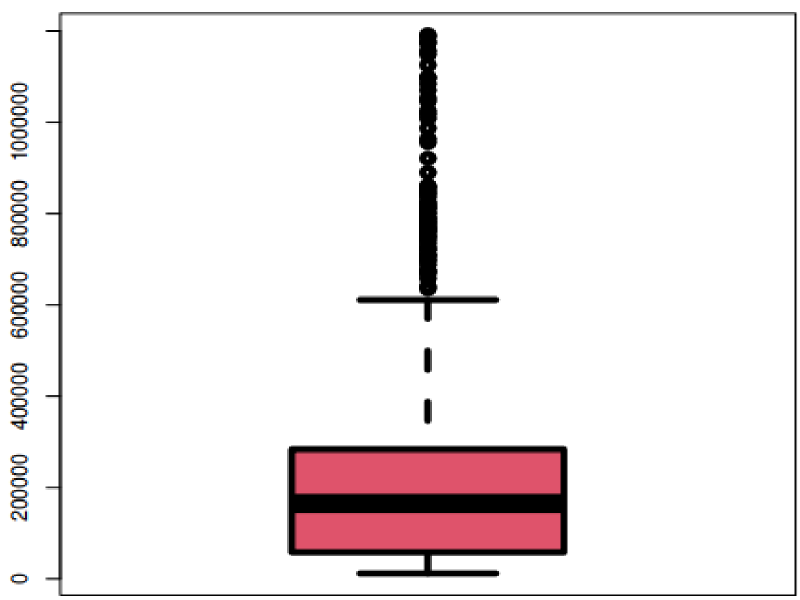

Figure 2: The box plot for the SSA disability beneficiaries' data disability data.

Figure 3 is the Total Time on Test (TTT) plot for the SSA disability beneficiaries' data, and it offers a visual diagnostic tool for assessing the shape of the underlying hazard functions, something box plots or histograms can't do directly. From the plot's clear concave shape (bending upward), we infer that the hazard rate is *decreasing* over time, which aligns perfectly with the heavy right-skewness reported in Table 2 (skewness = 1.87) and supports the use of heavy-tailed, decreasing-failure-rate models like the Burr Inverse Weibull (BIW). In applied statistics, the TTT plot is invaluable for model selectionist quickly tells us whether distributions with increasing, decreasing, or bathtub-shaped hazard rates are appropriate, before committing to complex fitting. For disability claims, a decreasing hazard makes intuitive sense: beneficiaries with severe, early-onset conditions tend to enter the system sooner, while longer-term survival in the dataset reflects relatively more stable (or less deteriorating) cases over time. This pattern justifies focusing on tail-sensitive methodologies (e.g., EVT, MOO-P, PORT-VaR), since the risk concentrates in the early "high-hazard" phase ؛followed by prolonged low-hazard tail exposure. Practically, this shape warns insurers against using constant- or increasing-hazard models (like exponential or standard Weibull), which would underestimate early exits and overstate long-term attrition. It also hints that policy lapses or recoveries (if any) are rare ؛once enrolled, beneficiaries tend to remain, emphasizing the need for long-term reserve planning. The TTT plot, though simple, thus grounds the entire modeling strategy in empirical behavior, not just theoretical convenience. It's why the paper leans on BIW: a flexible, heavy-tailed model that naturally accommodates decreasing hazards. In short, Figure 3 silently validates the paper's methodological pivot toward extreme-value framework and reminds us that sometimes, the simplest plot tells the most consequential story.

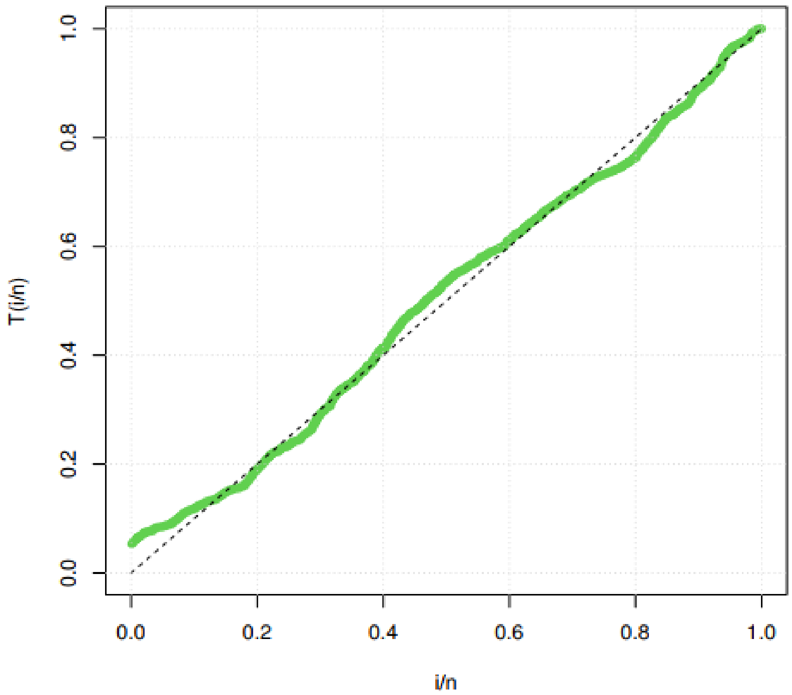

Figure 3: The TTT plot for the SSA disability beneficiaries' data disability data.

Figure 4 is the Cullen–Frey graph for the SSA disability beneficiaries' data, and honestly, it's one of the most practical diagnostic tools you'll find in exploratory data analysis. It plots the squared skewness against kurtosis (both standardized) to help narrow down candidate distributions before you jump into fitting. In our case, the point sits way out to the right and upward—high skewness ($\sim 1.87^2 \approx 3.5$) and excess kurtosis (~3.85), placing it firmly in the heavy-tailed, asymmetric region, far from the normal (which sits at (0,3)). Crucially, it falls near the theoretical locus of the lognormal and gamma families, but above them—suggesting even heavier tails, which aligns with the extreme values we see (max ≈1.2 million vs. mean ≈220k). That's why the paper leans into the BIW model: it's flexible enough to cover that upper-right zone where classical two-parameter models fail.

In applied statistics, the Cullen–Frey plot saves you from blind model selection, it's a compass, not a destination. Here, it warns against assuming normality or even basic Weibull/exponential forms. For disability data, where policy reserves hinge on tail adequacy, this plot justifies the EVT-driven approach: if your distributional guess is off, your VaR, TVaR, and PORT-VaR will be dangerously optimistic. So, Figure 4 isn't just a scatter, it's the statistical "why" behind choosing BIW, MOO-P, and PORT methods. In real-world risk work, skipping this step is like navigating a storm without radar.

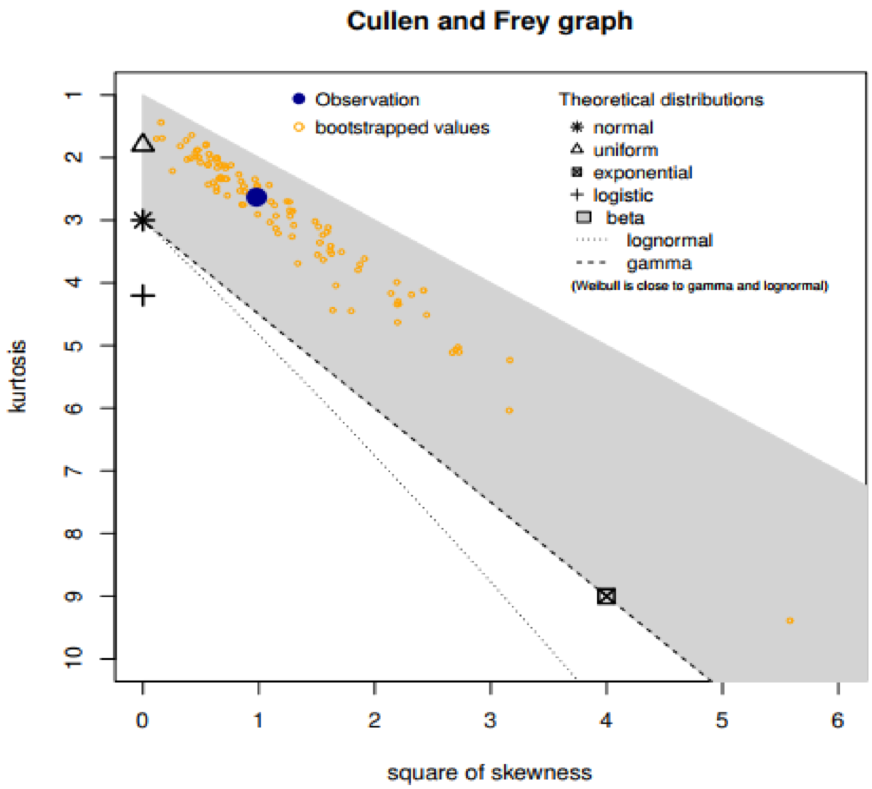

Figure 4: The Cullen-Frey plot for the SSA disability beneficiaries' data disability data.

Figure 5 is the Q–Q plot for the SSA disability beneficiaries' data, and honestly, it's where theory meets reality in the most humbling way. The points curved sharply upward, deviating strongly from the 45° reference line, especially in the upper tail. That's textbook evidence that the data doesn't follow a normal (or any light-tailed) distribution, confirmed by the high skewness (1.87) and kurtosis (3.85) in Table 2. In applied statistics, Q–Q plots are gold: they let you see model misfit before running tests. Here, the bend screams "heavy right tail", exactly why parametric fits like Weibull or lognormal fall short, and why the paper opts for the more flexible Burr Inverse Weibull (BIW). For disability claims, this isn't just academic: underestimating tail quantiles (e.g., at 95% or 99%) means under-reserving for extreme payouts, potentially catastrophic for insurers. The plot also explains why risk metrics like VaR and TVaR diverge significantly (see Table 5): the tail is both long and thick. Therefore, Figure 5 quietly validates the entire EVT-driven workflow: no Q–Q sanity check → wrong distribution → biased risk estimates → financial vulnerability. In practice, if your Q–Q plot looks like this, stop—and reach for BIW, MOO-P, or PORT-VaR. This plot isn't decoration; it's the red flag that saves you from statistical overconfidence.

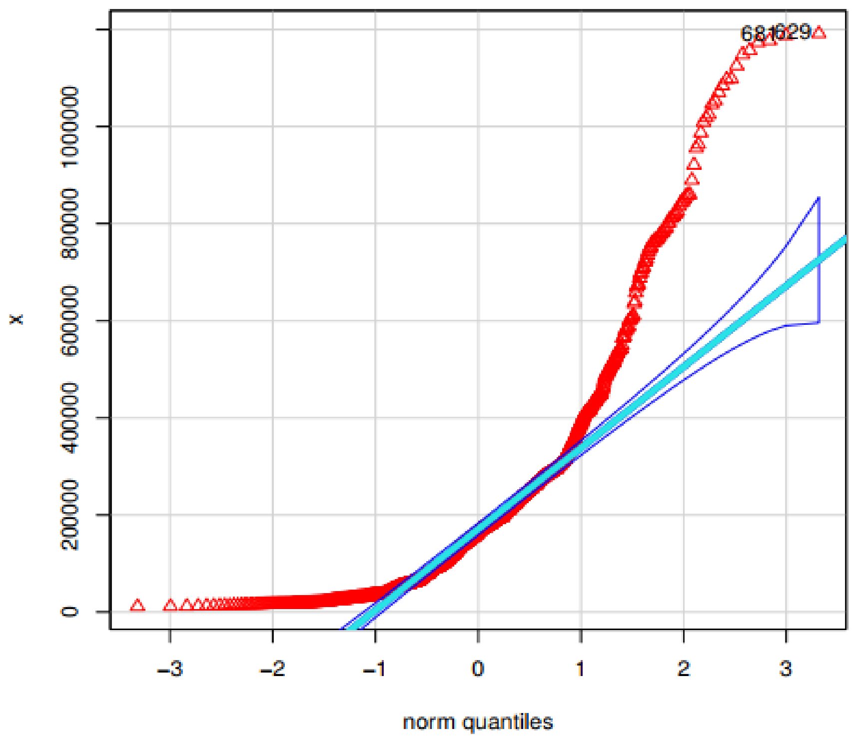

Figure 5: The QQ for the SSA disability beneficiaries' data disability data.

Figure 6 is the nonparametric kernel density estimate of the SSA disability beneficiaries' data—and honestly, it's one of the cleanest ways to see the data's true shape without forcing any model on it upfront. Right away, you notice the extreme right skew: a sharp, high peak near the lower end (around 20k–50k), followed by a long, stretched tail reaching beyond 1 million. That's not noise, it's the signature of heavy-tailed insurance loss data, and it visually confirms the numbers in Table 2: skewness = 1.87, kurtosis = 3.85, mean (219k), median (165k). In applied stats, kernel density plots are invaluable for exploratory analysis: they help detect multimodality, outliers, and tail behavior, things histograms sometimes obscure due to binning choices. Here, no second mode pops up, but the tail's persistence is unmistakable, warning against naive use of Gaussian or even basic gamma/Weibull fits. For disability claims, this tail dominance means most beneficiaries receive modest payments, but a small fraction drive disproportionate financial exposure, exactly why the paper leans on BIW, EVT, and tail-sensitive metrics like PORT-VaR and MOO-P. Crucially, Figure 6 also justifies log-transformation as a preprocessing step (though still insufficient alone), and it visually supports the TTT and Q–Q diagnostics: decreasing hazard, non-normality, heavy tails—all pointing to the same conclusion: standard models will under-predict risk.

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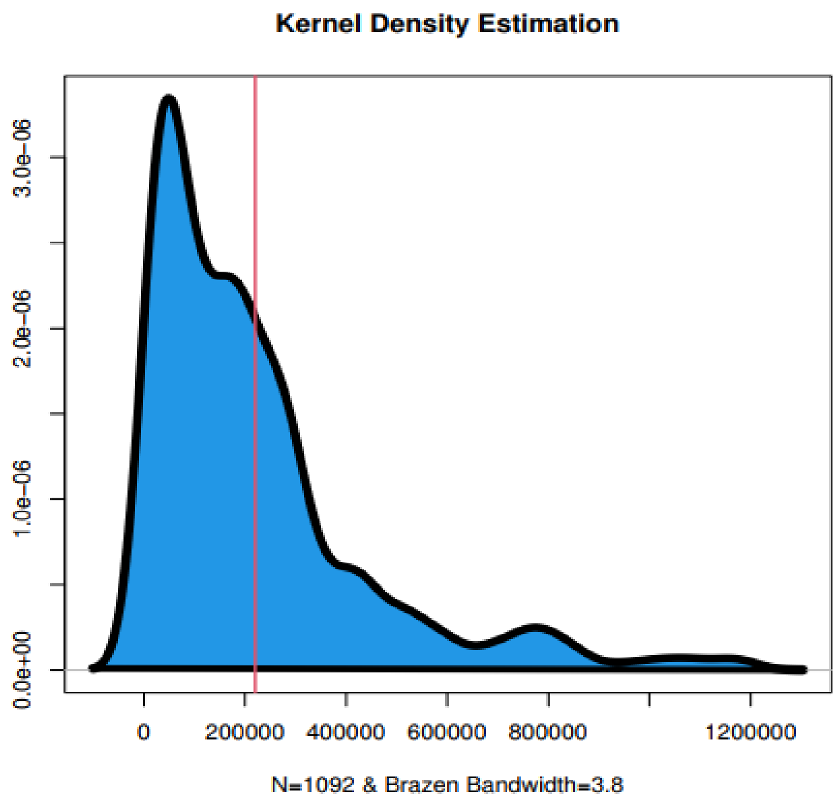

Figure 6: The nonparametric Kernel density plot for the SSA disability beneficiaries' data disability data.

Figure 7 is a scatter plot of the SSA disability beneficiaries' data, and while it may look simply, it plays a subtle but critical role. Unlike time-series or indexed plots, this one treats the observations as an unordered sample, plotting raw values (y-axis) against their index (x-axis). What jumps out immediately is the streak of extreme spikes scattered across the range, especially a few massive values near the end, confirming the heavy-tailed, outlier-prone structure described in Table 2 (max ≈ 1.19 million, skewness = 1.87). In applied studies, such a scatter plot is often the first visual "reality check": it reveals heterogeneity, outliers, and potential data-entry anomalies before any modeling begins. Here, it shows no clear trend or clustering, supporting the assumption of independent, identically distributed (i.i.d.) claims, a key requirement for EVT and BIW modeling. For disability risk analysis, those spikes aren't noisy, they're the events that drive solvency risk. They justify why measures like PORT-VaR, MOO-P (optimal P = 5), and BIW are essential: traditional averages would completely miss their impact. In short, Figure 7 says to us: "Don't smooth this out—you need to model the spikes."

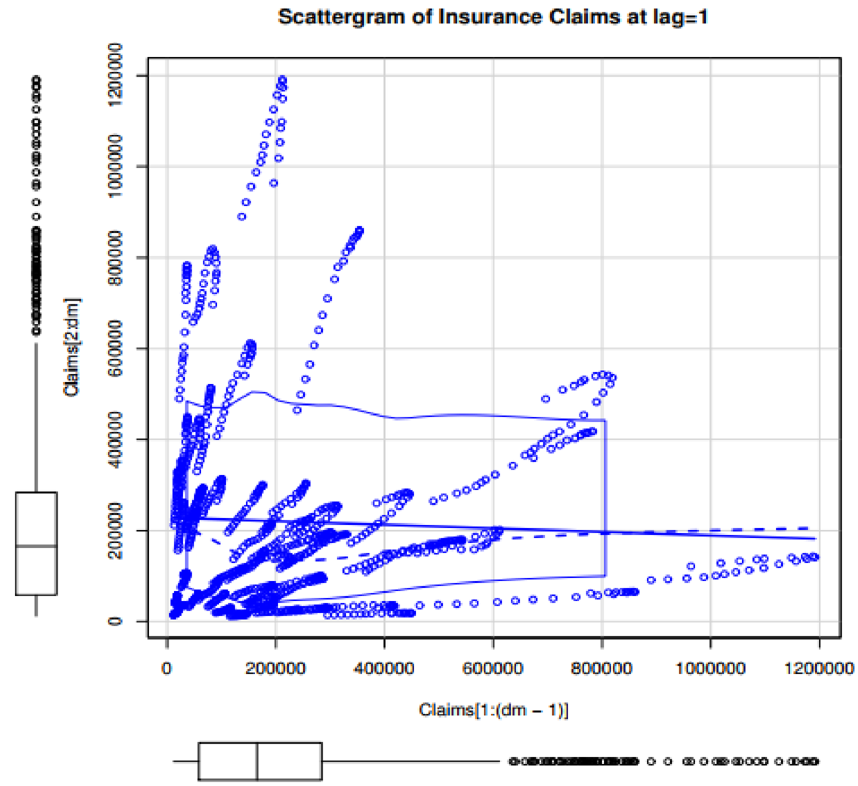

Figure 7: The scatter plot for the SSA disability beneficiaries' data disability data.

Figures 2–7 collectively form a visual diagnostic backbone for modeling SSA disability data, and they do it brilliantly. The box plot (see Figure 2) shouts right-skew and outliers; the TTT plot (see Figure 3) confirms a decreasing hazard, justifying heavy-tailed models like BIW; the Cullen–Frey plot (see Figure 4) rules out light-tailed families and points squarely to Burr or Weibull-type flexibility; the Q–Q plot (see Fig 5) visually debunks normality; the kernel density (see Figure 6) shows the pronounced peak + long tail; and the scatter plot (see Figure 7) verifies iid.-like structure with extreme spikes. In applied stats, this sequence is gold: it moves you from raw skepticism ("Is this data even modellable?") to confident modeling choice, without jumping straight into fitting.

For SSA disability claims, where underestimating the tail means solvency risk, these plots aren't just supportive; they're defensive. They force honesty: symmetric models fail, EVT isn't optional, and risk metrics like PORT-VaR and MOO-5 aren't academic luxuries, they're operational necessities. Together, Figures 2–7 turn exploratory analysis into strategic insight.

Traditional risk metrics like VaR $|q$ often have limitations when it comes to accurately assessing extreme events, especially in sectors like indemnity-related insurance where losses can quickly escalate under adverse conditions. VaR typically provides a measure of potential losses at a specific confidence level (denoted as VaR $|q$), but it may not adequately capture the severity and frequency of extreme losses or tail risk scenarios. In contrast, the PORT-VaR $|q$ estimator offers a more refined approach to risk assessment by focusing specifically on the financial losses that exceed predefined thresholds.

This method homes in on tail risk, which is crucial in sectors where the impact of extreme events can be disproportionately high. The theoretical underpinnings of PORT-VaR $|q$ involve calculating the VaR $|q$ at various CLs beyond which losses exceed a certain threshold. This approach not only provides a clearer picture of potential losses during extreme events but also helps in designing more robust risk management strategies tailored to handle such scenarios effectively. Within the context of disability insurance, where the financial implications of severe disability claims can be substantial and unpredictable, PORT-VaR $|q$ becomes particularly relevant. By examining case studies and empirical data, we can effectively showcase how PORT-VaR $|q$ improves risk management practices in the disability insurance sector. In this section, Table 4 highlights the application of PORT-VaR $|q$ at various confidence levels, specifically 55%, 65%, 75%, 85%, 95%, and 99%. Each confidence level represents a unique threshold of risk tolerance, allowing for a detailed assessment of potential outcomes.

The insights presented in Table 4 include several crucial indicators essential for evaluating risk exposure. Among these are the expected shortfall, which calculates the average loss in cases where losses surpass the VaR threshold, and the conditional value-at-risk, which provides a broader perspective on tail risk by considering the magnitude of losses during extreme scenarios. Additionally, the table may emphasize metrics such as the potential effects on reserve requirements and capital allocation, demonstrating how PORT-VaR $|q$ assists insurers in determining sufficient financial safeguards against adverse events. By quantifying risk in this way, organizations can enhance their risk management strategies, improve decision-making processes, and develop a more resilient financial structure capable of handling the challenges posed by significant loss events.

The empirical evidence from various case studies further supports these conclusions, illustrating practical applications of PORT-VaR $|q$ and highlighting its effectiveness in reducing risk in the disability insurance field. Number of PORTs: Instances where losses exceed the threshold at each confidence level. Minimum (Mini.) disability insurance value: The smallest observed loss exceeding the threshold. The 1st quartile (1st Qu.) disability insurance value: The value below which 25% of losses fall. The 3rd quartile (3rd Qu.) disability insurance value: The value below which 75% of losses fall. The maximum (Maxi.) disability insurance value: The largest observed loss exceeding the threshold. The median and expected value (ExV).

Table 4: extreme SSA disability beneficiaries due to BIW model.

PORT-$VaR^{\|q}$results							
q↓	PORTs	Mini.	1^{st} Qu.	Median	ExV	3^{rd} Qu.	Maxi.
45%	601	142334	195691	272352	349570	417764	1191554
40%	655	116827	183317	257252	331558	406883	1191554
35%	710	94317	171169	242827	314014	381213	1191554
30%	764	75931	153837	229546	297891	354697	1191554
25%	819	58296	138089	216137	282246	336387	1191554
20%	873	45744	116827	202236	268072	314263	1191554
15%	928	34394	99442	191751	254462	304124	1191554
10%	982	26888	87631	183317	242140	294895	1191554
5%	1036	18990	69211	174751	230711	289242	1191554

These metrics play a crucial role in empowering insurers and risk managers to gain a comprehensive understanding of the potential distribution of losses during extreme scenarios. This understanding enables them to make informed decisions about resource allocation, thereby enhancing their ability to maintain financial stability and resilience in the face of unexpected events. Figure 8 presents a detailed analysis of PORT-VaR $|q$ specifically tailored for disability insurance data. It comprises six histograms, each corresponding to different Confidence Levels (CLs) ranging from 55% to 99%. These histograms visually depict the frequency and magnitude of losses that exceed the predefined thresholds at each confidence level. By examining these histograms, insurers can identify which confidence levels are associated with more frequent or severe losses, thereby guiding their risk mitigation strategies accordingly.

Additionally, Figure 8 complements the analysis by illustrating the relationship between the number of PORT-VaR $|q$ occurrences and the CLs.

Figure 9 provides the PORT-VaR densities for extreme SSA disability beneficiaries. Figure 10 provides the number of PORT-VaR v the CLs for the SSA disability beneficiaries. This figure provides a quantitative perspective, showing how the number of instances where losses exceed the threshold varies across different confidence levels. It helps insurers and risk managers gauge the prevalence and severity of tail risk events in the context of disability insurance, informing them about the potential impact on financial reserves and operational planning.

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Together, these figures serve as powerful tools in risk management within the disability insurance sector. They not only facilitate a deeper understanding of risk exposure during extreme scenarios but also support proactive measures to mitigate such risks. By leveraging insights derived from PORT-VaR $|q$ analysis, insurers can optimize their risk management analysis, ensure adequate capital reserves, and enhance their overall capacity to withstand and recover from adverse events effectively. This approach ultimately fosters greater financial stability and resilience, safeguarding both insurers' solvency and policyholders' interests in the dynamic landscape of disability insurance.

Based on Table 4, one may spot the following main results:

- The Table shows a consistent increase in the number of PORTs (instances where losses exceed the threshold) as the confidence level decreases (from 45% to 1%). This indicates a higher frequency of extreme losses as we move towards lower confidence levels, reflecting the tail risk nature of disability insurance.
- For risk managers and insurers, this table is crucial as it helps us to understand the potential severity and frequency of extreme loss events under varying confidence levels. Lower confidence levels (such as 1%) highlight scenarios where the impact of extreme events is most pronounced, necessitating robust risk mitigation strategies and adequate capital reserves.
- Comparing different q values (from 45% to 1%), one can observe how the magnitude of potential losses (Mini., Median, 3rd Qu., Maxi.) changes, providing a nuanced view of the increasing severity of losses as the confidence level decreases.
- These insights can inform decision-making processes regarding pricing, underwriting, and capital allocation within disability insurance. They enable insurers to better prepare for and manage financial risks associated with extreme events, thereby enhancing overall financial stability and resilience.

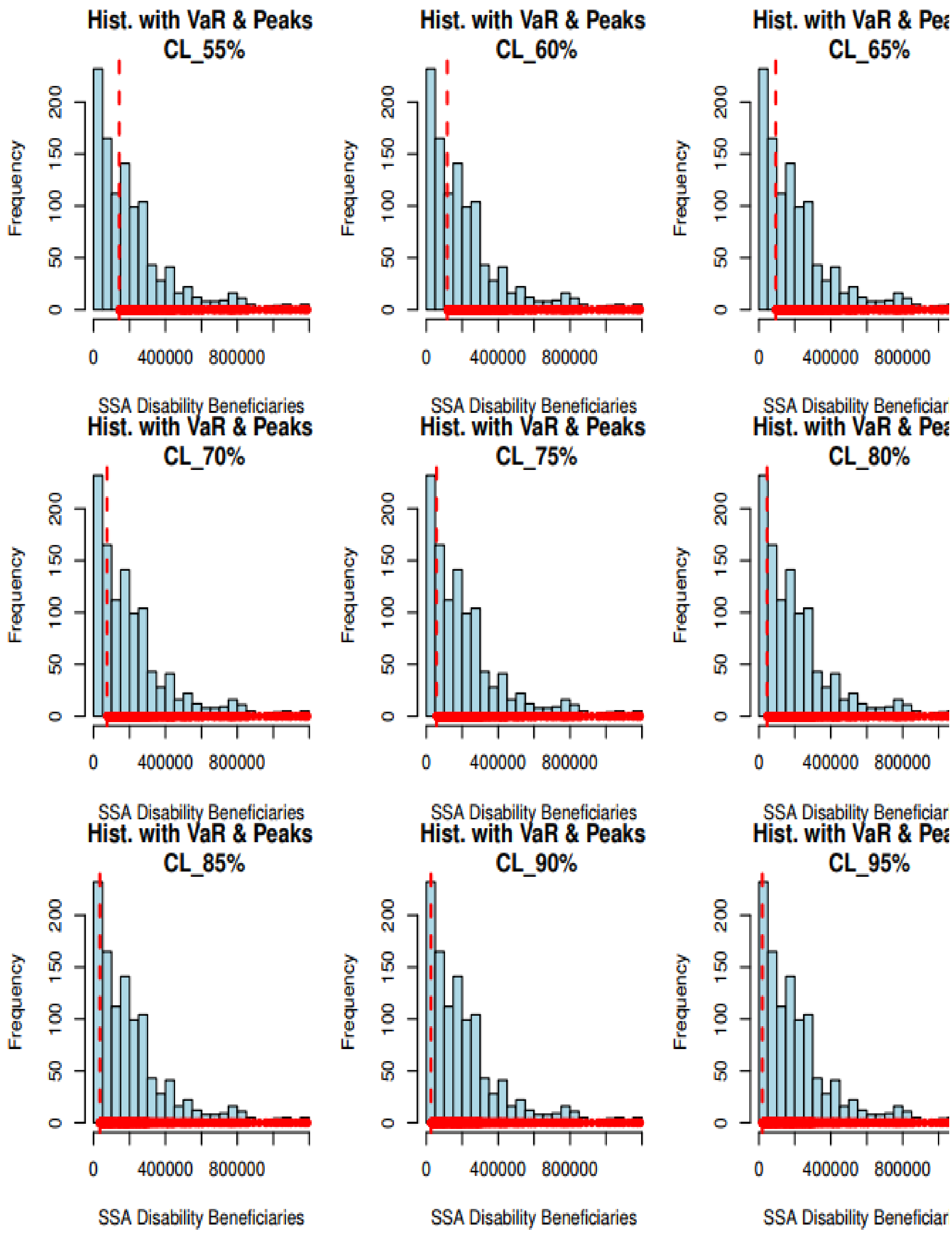

Figure 8: PORT-VaR histograms for extreme SSA disability beneficiaries.

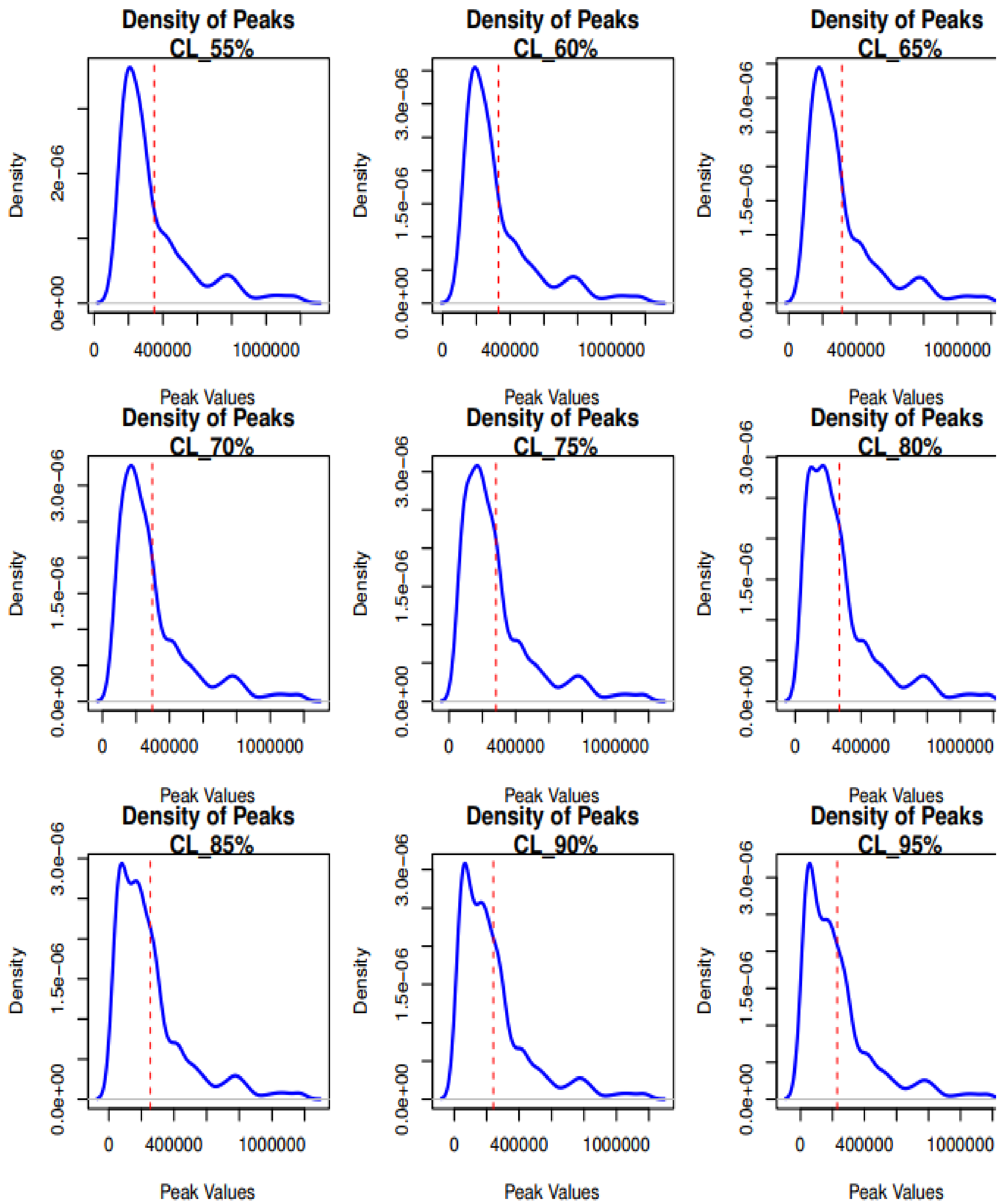

Figure 9: PORT-VaR densities for extreme SSA disability beneficiaries.

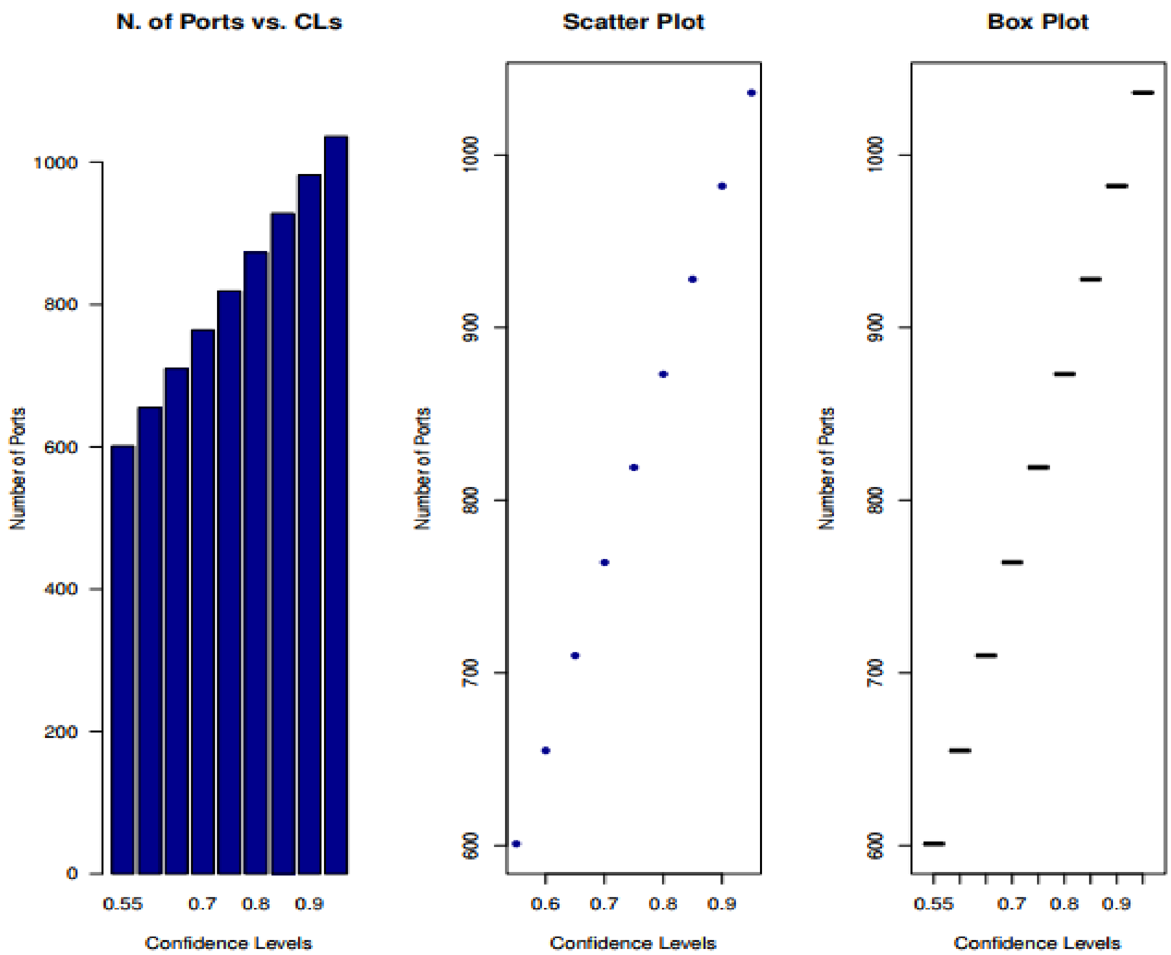

Figure 10: Number of PORT-VaR v the CLs for the SSA disability beneficiaries.

6. Risk analysis under the SSA disability beneficiaries' data

In this Section, we undertake a thorough analysis of several critical risks which are pivotal to the domain of disability insurance. These metrics play a fundamental role in risk management practices within the insurance sector, providing essential insights into the potential risks associated with disability insurance data. Our analysis evaluates these risk measures across a spectrum of quantiles, ranging from 55% to 99%. Each quantile represents a specific level of risk, with higher quantiles reflecting more extreme and less probable scenarios for financial losses. By examining these metrics at various quantiles, we gain a comprehensive understanding of the distribution and potential severity of financial losses related to indemnity claims under different levels of uncertainty. The goal of our assessment is to facilitate informed decision-making in risk evaluation and mitigation strategies within the insurance industry. By quantifying the potential financial impacts associated with indemnity risks, insurers can enhance their resource allocation strategies, ensure that adequate reserves are maintained, and formulate effective risk management protocols to address adverse events. This analysis highlights the necessity of utilizing sophisticated risk modeling techniques and statistical methods to manage indemnity risks efficiently, ensuring that insurers are well-prepared to navigate the complexities of financial uncertainties inherent in disability insurance.

Table 5 encapsulates crucial risk metrics specifically tailored to the context of disability insurance utilizing the BIW model. These metrics provide insights into the potential financial exposure faced by insurers and are further complemented by results derived from the Weibull model, which are also summarized in Table 4. Additionally, Table 7 presents a comprehensive risk analysis focused on extreme disability insurance scenarios under the BIW model, highlighting the intricacies of risk evaluation in this sector. The calculations of these risk measures are performed across a spectrum of quantiles ($1-q, \downarrow$) ranging from 45% to 99%, where each quantile signifies a unique level of potential loss. Higher quantiles denote more severe loss scenarios, thus allowing for a deeper understanding of tail risks associated with disability insurance. These risk measures are indispensable for assessing and managing potential losses in disability insurance contexts, as they elucidate the distribution and intensity of risks that insurers must navigate. The values detailed in the tables act as a crucial snapshot of the financial repercussions associated with varying probabilities, thereby enabling stakeholders to make informed decisions regarding risk exposure and the development of effective risk mitigation strategies. This data is vital for understanding the potential ramifications of extreme events on disability insurance outcomes, underscoring the urgent need for robust risk assessment analysis within the insurance industry. Such analysis is essential not only for managing existing risks but also for preparing for future uncertainties, ensuring that insurers can uphold financial stability in the face of adverse conditions. Figure g gives the plots of TVaR $|q$, MOO-P , VaR $|q$, TMV $|q$, TV $|q$ and MXL $|q$ analysis for extreme disability insurance across the CLs under three models, respectively.

Table 5: Risk analysis for extreme SSA disability beneficiaries' data under BIW model.

	Risk indicator→					
q↓	MOO-P	VaR	$TVaR$	TV	TMV	MXL
55%	4.40439×10^{28}	182967	79661.7	2773312665	9114691056	103305.21
45%	4.40439×10^{28}	142331	60766.9	1408854520	5098600827	81564.152
35%	4.40439×10^{28}	94284	44446.3	544808271	2518853887	49838.473
25%	4.40439×10^{28}	58273	32119.3	172544228	1203558830	26153.744
15%	4.40439×10^{28}	34342	23092.8	38592944	571636667	11249.015
5%	4.40439×10^{28}	18795	16279.7	4300923	269253799	2515.2686
1%	4.40439×10^{28}	14325	12990.6	732657	169420323	1334.8457

The MOO-P estimator is a statistical method used to analyze tail risks in insurance data. Here, at order $P = 5$, it calculates the mean after raising each data point to the power of 5 . The result, $4.404393 \times 10\ 28$, indicates the estimated mean based on this transformation. The VaR $|q$ is a measure used to assess the potential loss in value of an asset or portfolio over a specific period under normal conditions, at a given confidence level q . For example, at the 1% confidence level, the VaR [1%] is 14325.39. This means that under normal conditions, there is a 1% probability that the loss will exceed this value.

The TVaR $|q$ provides a measure of the expected loss in the worst q% of scenarios beyond the VaR $|q$ threshold. It gives a more conservative estimate compared to VaR because it considers the entire distribution of losses beyond the threshold. For instance, at the 1% confidence level, TVaR [1%] is 12990.55, indicating that the average loss in the worst 1% of scenarios is expected to be this amount. The TV $|q$ quantifies the variability or spread of losses in the tail of the distribution, beyond the TVaR $|q$ threshold. It provides additional insight into the potential severity of losses in extreme scenarios.

At the 1% confidence level, the tail variance is 732656.7, indicating the extent of potential variability in losses beyond the TVaR [1%] threshold. The TMV $|q$ combines information about both the mean and variance of losses in the tail of the distribution. It provides a more comprehensive view of risk by considering both the average level and the variability of losses in extreme scenarios. At the 1% confidence level, TMV [1%] is 169420323, indicating the product of the mean and variance in the tail of the distribution.

The MXL $|q$ measures the average loss that exceeds the VaR threshold across all scenarios beyond that threshold. It helps quantify the expected shortfall beyond the VaR $|q$ level. For example, at the 1% confidence level, MXL is $-$ 1334.845. The negative sign indicates that on average, the losses in the scenarios exceeding VaR $|q$ are lower than the VaR $|q$ itself, which can happen if there are gains in those scenarios. Increasing risk levels (decreasing confidence levels q) generally lead to higher VaR $|q$, TVaR $|q$, TV $|q$, TMR $|q$, and MXL $|q$ values. This reflects the increasing severity and variability of potential losses in extreme scenarios as the confidence level decreases.

The MOO- 5 remains constant across different confidence levels because it represents the average based on the chosen moment (order $P = 5$), which is independent of the confidence level q . These indicators collectively provide insights into the distribution of potential losses for extreme disability insurance scenarios under the BIW model, helping insurers and reinsurers understand and manage their risk exposures effectively.

Based on Table 4, the following main results can be spotted:

1) Given the significant values of TVaR $|q$, TV $|q$, TMV $|q$, and MXL $|q$ at lower confidence levels (higher risk levels), disability insurance companies should focus on developing robust risk management strategies that account for extreme scenarios.
2) Conduct stress tests and scenario analysis to assess the potential impact of severe but rare events. This involves simulating extreme scenarios beyond the VaR $|q$ threshold to understand the distribution of potential losses and ensure sufficient capital reserves.
3) Given the higher estimated mean (MOO- 5) and increasing risk indicators as confidence levels decrease, insurers may need to revisit their pricing models. This includes adjusting premiums to adequately reflect the higher risk associated with extreme disability events.
4) Incorporate insights from TVaR $|q$ and MXL $|q$ to refine underwriting criteria. This can involve setting stricter guidelines for high-risk applicants or offering tailored policies that mitigate the impact of severe disability events.
5) Leverage reinsurance solutions to transfer tail risks associated with extreme disability scenarios. Reinsurers often use sophisticated models like those reflected in Table 4 to assess tail risk accurately and provide appropriate coverage.
6) Negotiate reinsurance contracts that explicitly cover extreme scenarios as indicated by TV $|q$ and TMV $|q$. Ensure these contracts align with the insurer's risk appetite and financial objectives.
7) Invest in advanced analytics capabilities to continuously monitor and analyze disability claims data. This can help in refining risk models, improving predictive accuracy, and identifying emerging trends or patterns in disability claims behavior. Validate the BIW model and the MOO- 5 estimator periodically against actual claims experience to ensure their relevance and reliability in predicting extreme disability events.
8) Educate key stakeholders, including underwriters, claims assessors, and senior management, about the implications of extreme disability risks highlighted in Table 4. Foster a culture of risk awareness and proactive risk management across the organization.
9) Ensure compliance with regulatory requirements related to risk management and reporting extreme risks in disability insurance. Transparently communicate the findings from Table 3 and related risk analyses to regulatory authorities as needed.

By implementing these recommendations, disability insurance companies can strengthen their resilience to extreme disability events, enhance their risk-adjusted profitability, and maintain confidence among policyholders, investors, and regulators. Continuous monitoring and adaptation to evolving risk landscapes will be crucial in navigating the complexities of the insurance industry.

The results presented in Table 4 for extreme disability insurance under the BIW model offer several opportunities for harnessing and benefiting from dealing with the issue of disability:

- The TVaR, TV, and TMV indicators provide insights into the severity and frequency of extreme disability events. Insurers can utilize these metrics to refine risk assessment models and more accurately price disability insurance policies. By incorporating these indicators, insurers can ensure premiums reflect the true risk exposure, thereby potentially reducing adverse selection and improving sustainability in pricing.
- Understanding the distribution characteristics and MXL at different confidence levels enables insurers to design tailored insurance products. For instance, products that provide enhanced coverage for catastrophic disabilities or income protection in extreme scenarios can be developed based on the identified risks in Table 3. This customization can better meet the diverse needs of policyholders facing different disability risks.
- The values provided in Table 3 allow insurers to conduct stress tests and scenario analyses effectively. By simulating extreme disability events beyond the VaR threshold, insurers can assess their capital adequacy and prepare contingency plans. This proactive approach helps mitigate potential financial impacts and ensures the sustainability of disability insurance operations.
- Reinsurance plays a crucial role in mitigating extreme risks associated with disability insurance. Insurers can leverage the insights from Table 3 to negotiate reinsurance contracts that specifically cover tail risks identified, such as Tail variance and TMV. This strategic use of reinsurance helps in transferring excess risk to reinsurers, thereby enhancing overall risk management and reducing volatility in financial outcomes.
- By transparently communicating the findings from Table 3 and related risk analyses, insurers can build trust and confidence among policyholders, investors, and regulators. Educating stakeholders about the robust risk management practices based on empirical data fosters a culture of transparency and accountability within the insurance industry.

- The extensive data analysis reflected in Table 3 contributes to broader advancements in disability research and prevention. By understanding the patterns and characteristics of extreme disability events, insurers can collaborate with healthcare providers, policymakers, and community organizations to promote preventive measures and early interventions. This holistic approach not only improves health outcomes but also reduces the incidence and severity of disabilities over time.

Below we provide some recommendations for supporting workers with disabilities in Kingdom of Saudi Arabia. Insurers should simulate extreme but rare events beyond the VaR threshold to evaluate potential losses. This proactive strategy will help establish adequate capital reserves and develop effective contingency plans, ensuring financial stability.

Given the higher estimated mean of disability risks and increasing risk indicators, insurers should reassess their pricing models. Adjusting premiums to accurately reflect the risks associated with extreme disability events will enhance sustainability and minimize adverse selection. Utilize insights from TVaR and MXL methodologies to strengthen underwriting criteria. Implementing stricter standards for high-risk applicants and offering tailored policies can reduce the impact of severe disability events and enhance risk selection. Leverage reinsurance to mitigate tail risks related to extreme disability scenarios. Negotiating contracts that explicitly cover identified tail risks will improve overall risk management and stabilize financial outcomes. Regularly monitor and analyze disability claims data through advanced analytics. This will refine risk models and improve predictive accuracy, ensuring that models remain relevant by validating them against actual claims experiences. Foster a culture of risk awareness among underwriters, claims assessors, and senior management. Transparent communication of findings and risk assessments to regulatory authorities will build trust and confidence in the insurer's risk management practices. Adhere to all regulatory requirements related to risk management and the reporting of extreme risks in disability insurance. Compliance will enhance accountability and support sustainable business practices.

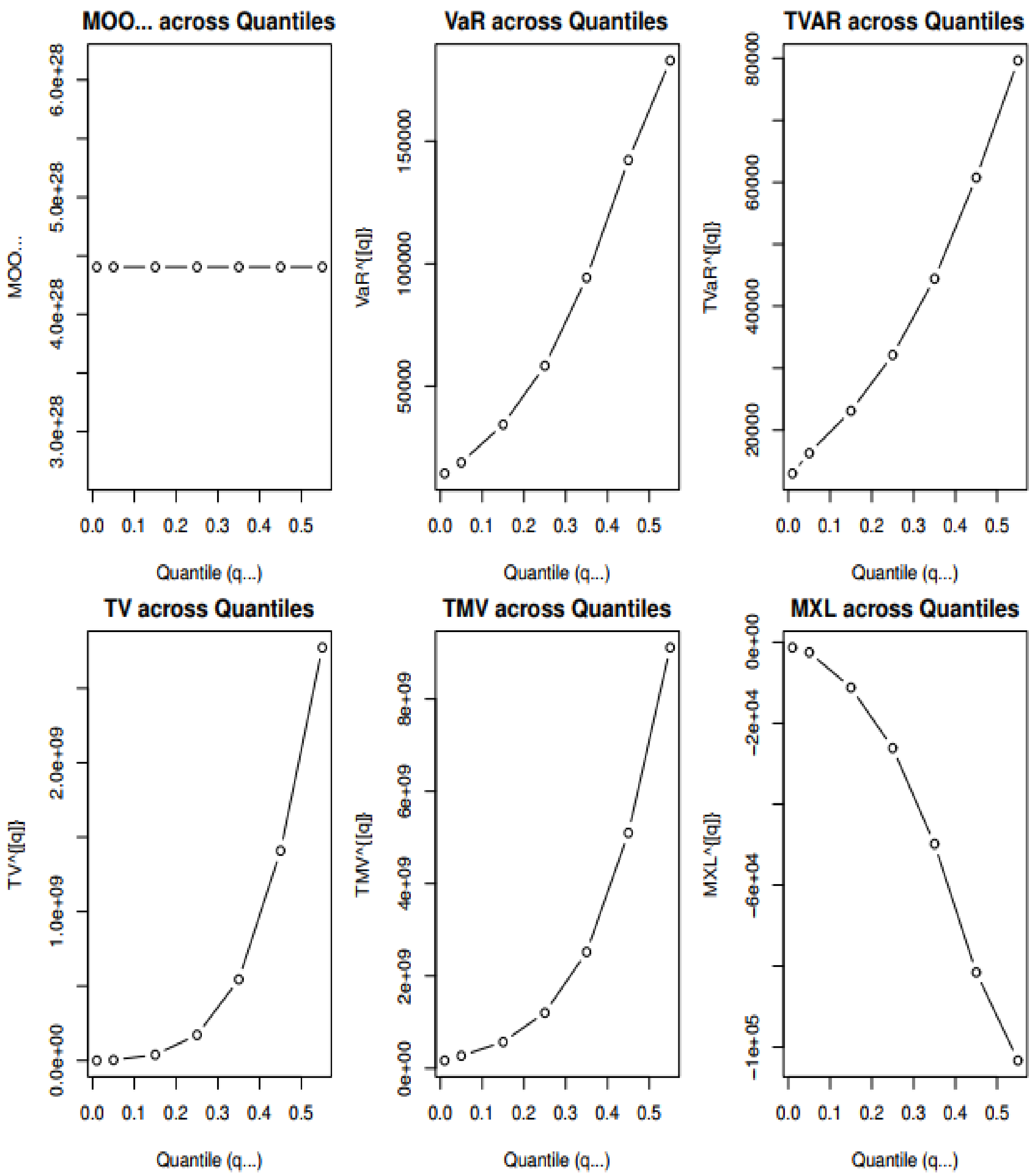

Figure 11: Risk indicators versus the CLs under the BIW model.

8.Conclusions and discussions

This paper offers a comprehensive exploration of disability insurance risk, employing advanced metrics grounded in extreme value theory (EVT). By analyzing real data, the study examines critical risk measures such as Value-at-Risk (VaR) and Tail-VaR (TVaR), alongside other indicators specifically tailored to the unique challenges of disability insurance. The research highlights the utility of the BIW model within the EVT framework, effectively addressing the pronounced tail risks associated with the highly skewed nature of disability insurance data. Using methods like the Mean of Order P (MOO-5) and Peaks Over a Random Threshold Value-at-Risk (PORT-VaR), the paper sheds light on the potential for extreme events while offering actionable recommendations to strengthen risk management strategies for insurers. The findings emphasize the importance of addressing significant financial risks posed by rare but severe events. By leveraging these insights, disability insurance institutions can develop robust strategies to mitigate sudden financial losses, enhance resilience, and ensure stability amidst unforeseen challenges. Practically speaking, the study underscores the necessity of proactive measures informed by a detailed understanding of the extreme tail behavior observed in disability insurance data. Recommendations provided aim to refine risk mitigation efforts, ensuring a sustainable and resilient approach to managing financial risks within the sector.

The application of advanced statistical models, particularly the BIW model within the framework of EVT, has demonstrated significant potential in capturing tail behavior and managing the financial risks associated with disability insurance. The BIW model's ability to accurately describe the heavy-tailed nature of disability data makes it a valuable tool for modeling extreme losses and predicting rare but impactful events. This study utilized key risk indicators to assess the severity and frequency of extreme disability claims. These metrics provided robust insights into the distributional characteristics of disability-related financial losses, enabling more precise risk quantification and informed decision-making. The empirical results confirmed that disability insurance data exhibits significant right skewness and high kurtosis, indicating a propensity for extreme values that traditional symmetric models may overlook. By incorporating EVT-based methods, this research effectively addressed the limitations of conventional approaches and offered a more comprehensive understanding of tail risks. The MOO-P analysis further enhanced the precision of risk estimation by optimizing the order P, which allowed for better adaptation to the specific features of disability data. Additionally, the PORT-VaR estimator proved effective in identifying and quantifying losses exceeding predefined thresholds, thereby supporting the development of proactive risk mitigation strategies. These findings align with recent studies that advocate for the integration of EVT and advanced risk measures in actuarial science and financial risk management. The implications of this work extend beyond theoretical contributions, offering practical tools for insurers to enhance capital allocation, pricing accuracy, and reinsurance planning.

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