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A New Bivariate Exponentiated Family of Distributions: Properties and Applications

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Abstract

The bivariate distributions are useful for the joint modeling of two random variables. In this paper, we have presented a bivariate version of the exponentiated family of distributions. Some desirable properties of the proposed bivariate family of distributions have been explored. These include the conditional distributions, the joint and conditional moments, dependence measures, reliability analysis, and maximum likelihood estimation of the parameters. A specific member of the proposed family has been explored for the power function baseline distribution giving rise to the bivariate exponentiated power function distribution. Some properties of the derived bivariate exponentiated power function distribution have been explored. The derived bivariate exponentiated power function distribution is fitted on some real data sets to see its suitability. It is found that the derived bivariate exponentiated power function distribution performs better than the competing distributions for modeling of the used data.

Key Words: Exponentiated Distributions, Bivariate Distributions, Moments, Dependence Measures, Maximum Likelihood Estimation, Power Function Distribution.

Mathematical Subject Classification: 60E05, 62E15, 62F10, 62H10

1. Introduction

The probability distributions are being used in many areas of life, ranging from engineering to medicine. The probability distributions are also being used in finance and management. The rising complexity of available data has attracted several researchers to propose new probability distributions. In the recent past, the researchers have shifted their attention to generating the families of distributions. The exponentiated family of distributions; proposed by Gupta et al. (1998); is a simple, yet powerful, family of distributions and is obtained by exponentiation of the cumulative distribution function (*cdf*) of any baseline distribution. Since its emergence, the exponentiated family of distributions has been studied by several authors. Gupta and Kundu (2001) have used the exponential baseline distribution in the exponentiated family to propose an exponentiated exponential distribution. Nadarajah (2005) has proposed an exponentiated Pareto distribution by using the Pareto distribution; Pareto (1897); distribution as a baseline distribution in the exponentiated family. Kumar et al. (2017) have proposed an exponentiated Burr–XII distribution by using the Burr–XII distribution as a baseline distribution. An exponentiated power function distribution has been proposed by Arshad et al. (2020) by using the power function distribution as a baseline distribution in the exponentiated family. More details about the exponentiated family of distributions can be found in Nadarajah and Kotz (2006) and Al-Hussaini and Ahsanullah (2015).

Eugene et al. (2002) have proposed the beta family of distributions by using the logit of the beta distribution. The beta family of distributions provides the exponentiated family of distributions as a special case. Cordeiro and Brito (2012) have used the power function distribution in the beta family of distributions to propose a beta power function distribution. Alzaatreh et al. (2013) have proposed a general method of generating new families of distributions by using the combination of any two distributions; one transformer and one transformed. This family of distributions is known in the literature as the *T–X* family of distributions.

In several situations, the joint modeling of two random phenomena is required and in this case, some bivariate distribution is needed. The bivariate distributions are not easy to study and in some cases, even a unique density function is not available. Gumbel (1960) has proposed a method of generating a bivariate distribution by using the univariate marginals. The method proposed by Gumbel is a member of a much larger class, known as *copula*. The copulas and the bivariate distributions are nicely discussed by Nelsen (2006) and Balakrishnan and Lai (2009).

The bivariate families of probability distributions have not been much explored. Sarabia et al. (2014) have proposed a bivariate beta generated family of distributions by using a bivariate beta distribution of Olkin and Liu (2003). This family of distributions has been used by Algarni and Shahbaz (2021) to propose a bivariate beta-inverse Weibull distribution.

Ganji et al. (2018) have extended the *T–X* family of distributions of Alzaatreh et al. (2013) to the bivariate case. Darwish et al. (2021) have proposed a bivariate transmuted family of distributions by using a simplified version of the bivariate *T–X* family of distributions, proposed by Ganji et al. (2018). This proposed bivariate transmuted family of distributions is a simplified version of the Cambanis family of distributions, Cambanis (1977).

In this paper, we have proposed a new bivariate exponentiated family of distributions that provides the univariate exponentiated families of distributions as the marginals. The structure of the paper follows. In Section 2, some desirable materials and methods are discussed. A new bivariate exponentiated family of distributions (*BExFD*) is proposed in Section 3 and some of its desirable properties are discussed in Section 4. In Section 5, a new bivariate exponentiated power function distribution is proposed by using the power function distribution as a baseline distribution in the *BExFD* giving rise to the bivariate exponentiated power function (*BExPF*) distribution. Some properties of the derived *BExPF* distribution are discussed in Section 6. Section 7 contains some numerical studies for the *BExPF* distribution. Conclusions and recommendations are given in Section 8.

2. Materials and Methods

In this section, we have given some desirable materials and methods that are helpful in deriving the new *BExFD*. The exponentiated family of distributions; Gupta et al. (1998); is a simple family of distributions which is obtained by exponentiating the *cdf* of any baseline distribution. The *cdf* of the probability density function (*pdf*) of the exponentiated family of distributions is given as

$$F_{Ex-G}(x)=\left[G_X(x)\right]^{\alpha} \text{ and } f_{Ex-G}(x)=\alpha g_X(x)\left[G_X(x)\right]^{\alpha-1}\ ;\ x\in\mathbb{R},\ \alpha>0\,. \tag{1}$$

The exponentiated family of distributions has been studied for various baseline distributions by various authors. For details, see Al-Hussaini and Ahsanullah (2015). The beta family of distributions is another popular family that is obtained by Eugene et al. (2002) by using the logit of the beta distribution. The *cdf* of the beta family of distributions is given as

$$F_{B-G}(x)=\frac{1}{B(a,b)}\int_0^{G(x)} w^{a-1}(1-w)^{b-1}\,dw\,, \tag{2}$$

where $B(a, b)$ is the complete beta function. The exponentiated family of distributions appears as a special case of the beta generated family of distributions for $b = 1$. The beta family of distributions has been explored by various authors for different baseline distributions $G(x)$. The beta family of distributions has been extended to the bivariate case by Sarabia et al. (2014) by using a bivariate beta distribution of Olkin and Lie (2003). The joint *cdf* of this family of distributions is given as

$$F_{B-G_1G_2}(x_1,x_2)=\frac{1}{B(\alpha_1,\alpha_2,\beta)}\int_0^{G_2(x_2)}\int_0^{G_1(x_1)}\frac{w_1^{\alpha_1-1}w_2^{\alpha_2-1}(1-w_1)^{\alpha_2+\beta-1}(1-w_2)^{\alpha_1+\beta-1}}{(1-w_1w_2)^{\alpha_1+\alpha_2+\beta}}\,dw_1dw_2\,, \tag{3}$$

where $B(\alpha_1, \alpha_2, \beta)$ is the extended beta function defined as

$$B(\alpha,\beta,\gamma)=\int_0^1\int_0^1\frac{w_1^{\alpha-1}w_2^{\beta-1}(1-w_1)^{\beta+\gamma-1}(1-w_2)^{\alpha+\gamma-1}}{(1-w_1w_2)^{\alpha+\beta+\gamma}}\,dw_1dw_2=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)}\,.$$

The marginal families of distributions of (3) are the univariate beta families of distributions, given in (2). The bivariate beta family of distributions, given in (3), has been used by Algarni and Shahbaz (2021) to propose a bivariate beta inverse Weibull distribution.

Alzaatreh et al. (2013) have proposed a general method of proposing the univariate families of distributions. The *cdf* of this family of distributions, named as the *T–X* family of distributions, is given as

$$F_{T-X}(x)=\int_a^{W[G(x)]} r(t)\,dt=R\left[W\left\{G(x)\right\}\right]\ ;\ a<t<b\ ;\ x\in\mathbb{R}\,, \tag{4}$$

where $r(t)$ is the density function of any *transformer* random variable and $G(x)$ is the *cdf* of any transformed distributions. The function $W[G(x)]$ is any real values function such that $W[G(x)] \in [a, b]$, $W(0) \to a$, and $W(1) \to b$.

The *T–X* family of distributions has been extended to the bivariate case by Ganji et al. (2018). The joint *cdf* of the bivariate *T–X* family of distributions is

$$F_{1,2}\left(x_1,x_2\right)=\int_{a_2}^{W_2\left[G_2\left(x_2\right)\right]}\int_{a_1}^{W_1\left[G_1\left(x_1\right)\right]}r\left(u_1,u_2\right)du_1du_2\ ;\ a_1<u_1<b_1\ ;\ a_2<u_2<b_2\ , \quad (5)$$

where $r(u_1,u_2)$ is the joint density function of some random variables with support on $[a_1,b_1]\times[a_2,b_2]$. Also, $W_1[G_1(x_1)]$ and $W_2[G_2(x_2)]$ are some continuous functions of $G_1(x_1)$ and $G_2(x_2)$ such that $W_1(0)\to a_1$, $W_1(1)\to b_1$, $W_2(0)\to a_2$ and $W_2(1)\to b_2$. Darwish et al. (2021) have proposed a simpler version of (5) when the support of $r(u_1,u_2)$ is on $[0,1]\times[0,1]$. The family (5), in this case, is

$$F_{1,2}\left(x_1,x_2\right)=\int_0^{G_2\left(x_2\right)}\int_0^{G_1\left(x_1\right)}r\left(u_1,u_2\right)du_1du_2\ ;\ 0<\left(u_1,u_2\right)<1\ . \quad (6)$$

The bivariate family of distributions can be used to generate new bivariate families of distributions for different choices of $r(u_1, u_2)$. It is to be noted that the Gumbel bivariate family of distributions; Gumbel (1960); can be obtained by using

$$r\left(u_1,u_2\right)=1+\alpha\left(1-2u_1\right)\left(1-2u_2\right),$$

in (6). Also, the bivariate family of distributions by Ali et al. (1978) can be obtained by using

$$r\left(u_1,u_2\right)=\frac{\left(1-\alpha\right)+\alpha\left(\alpha-1\right)\left(1-u_1\right)\left(1-u_2\right)+2\alpha u_1u_2}{\left[1-\alpha\left(1-u_1\right)\left(1-u_2\right)\right]^3}\ ,$$

in (6). Darwish et al. (2021) have used (6) to propose a bivariate transmuted family of distributions. In this paper, we have proposed a new bivariate exponentiated family of distributions by using (6). The new *BExFD* is proposed in the following Section.

3. A New Bivariate Exponentiated Family of Distributions

In this section, we have obtained a new bivariate exponentiated family of distributions (*BExFD*). The new bivariate family has been obtained by using

$$r\left(u_1,u_2\right)=\frac{2ab\,u_1^{2a-1}u_2^{2b-1}}{\left[1-\left(1-u_1^a\right)\left(1-u_2^b\right)\right]^3}\ ;\ \left(u_1,u_2\right)\in\left[0,1\right]\times\left[0,1\right]\ ;\ a,b>0$$

in (6). The joint *cdf* of the new *BExFD* is

$$F_{1,2}\left(x_1,x_2\right)=\int_0^{G_2\left(x_2\right)}\int_0^{G_1\left(x_1\right)}2ab\,u_1^{2a-1}u_2^{2b-1}\left[1-\left(1-u_1^a\right)\left(1-u_2^b\right)\right]^{-3}du_1du_2$$

or

$$F_{1,2}\left(x_1,x_2\right)=\frac{G_1^a\left(x_1\right)G_2^b\left(x_2\right)}{G_1^a\left(x_1\right)+G_2^b\left(x_2\right)-G_1^a\left(x_1\right)G_2^b\left(x_2\right)}=\frac{G_1^a\left(x_1\right)G_2^b\left(x_2\right)}{1-\left[1-G_1^a\left(x_1\right)\right]\left[1-G_2^b\left(x_2\right)\right]}\ . \quad (7)$$

The joint *cdf*, (7), can also be written as

$$F_{1,2}\left(x_1,x_2\right)=\frac{G_1^a\left(x_1\right)G_2^b\left(x_2\right)}{1-\varphi_1\left(x_1\right)\varphi_2\left(x_2\right)}=\frac{G_1^a\left(x_1\right)G_2^b\left(x_2\right)}{\Delta_{1,2}\left(x_1,x_2\right)}\ ;\ \left(x_1,x_2\right)\in\mathbb{R}^2\ ;\ \left(a,b\right)>0,$$

where $\varphi_1\left(x_1\right)=1-G_1^a\left(x_1\right)$, $\varphi_2\left(x_2\right)=1-G_2^b\left(x_2\right)$ and $\Delta_{1,2}\left(x_1,x_2\right)=1-\varphi_1\left(x_1\right)\varphi_2\left(x_2\right)$. The density function corresponding to (7) is

$$f_{1,2}\left(x_1,x_2\right)=\frac{2ab\,g_1\left(x_1\right)g_2\left(x_2\right)G_1^{2a-1}\left(x_1\right)G_2^{2b-1}\left(x_2\right)}{\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)-G_1^a\left(x_1\right)G_2^b\left(x_2\right)\right]^3}=\frac{2ab\,g_1\left(x_1\right)g_2\left(x_2\right)G_1^{2a-1}\left(x_1\right)G_2^{2b-1}\left(x_2\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^3}\ , \quad (8)$$

or

$$f_{1,2}\left(x_1,x_2\right)=\frac{2ab\,g_1\left(x_1\right)g_2\left(x_2\right)G_1^{2a-1}\left(x_1\right)G_2^{2b-1}\left(x_2\right)}{\left[1-\varphi_1\left(x_1\right)\varphi_2\left(x_2\right)\right]^3};\ \left(x_1,x_2\right)\in\mathbb{R}^2\ ;\ \left(a,b\right)>0\ .$$

The marginal *cdf*'s and *pdf*'s for (7) and (8) are

$$F_1\left(x_1\right)=G_1^a\left(x_1\right)\ ;\ f_1\left(x_1\right)=a\,g_1\left(x_1\right)G_1^{a-1}\left(x_1\right)\ ;\ x_1\in\mathbb{R}\ ;\ a>0$$

and

$$F_2\left(x_2\right)=G_2^b\left(x_2\right)\ ;\ f_2\left(x_2\right)=b\,g_2\left(x_2\right)G_2^{b-1}\left(x_2\right)\ ;\ x_2\in\mathbb{R}\ ;\ b>0\ ,$$

which are the univariate exponentiated families of distributions. The conditional distribution of X_1 given $X_2 = x_2$ for the new *BExFD* is

$$f_{1|2}\left(x_1|x_2\right)=\frac{2a\,g_1\left(x_1\right)G_1^{2a-1}\left(x_1\right)G_2^{b}\left(x_2\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^3}\ ;\left(x_1,x_2\right)\in\mathbb{R}^2\ ;\left(a,b\right)>0\,. \tag{9}$$

Also, the conditional distribution of X_2 given $X_1 = x_1$ is

$$f_{2|1}\left(x_2|x_1\right)=\frac{2b\,g_2\left(x_2\right)G_2^{2b-1}\left(x_2\right)G_1^{a}\left(x_1\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^3}\ ;\left(x_1,x_2\right)\in\mathbb{R}^2\ ;\left(a,b\right)>0\,. \tag{10}$$

We will, now, discuss some useful properties for the new *BExFD*.

4. Properties of the New Exponentiated Family of Distributions

In this section, some useful properties of the new *BExFD* have been discussed. These properties are given in the following sub-sections.

4.1. The Joint and Conditional Moments

The moments are useful in studying certain useful properties of a distribution. The (r, s)th order joint moment for a bivariate distribution is defined as

$$\mu_{r,s}^{/}=E\left(X_1^rX_2^s\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1^rx_2^sf_{1,2}\left(x_1,x_2\right)dx_1dx_2\ .$$

The joint moment for the new *BExFD* is

$$\mu_{r,s}^{/}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1^rx_2^s\frac{2ab\,g_1\left(x_1\right)g_2\left(x_2\right)G_1^{2a-1}\left(x_1\right)G_2^{2b-1}\left(x_2\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^3}dx_1dx_2\ .$$

Making the transformation $G_1^a\left(x_1\right)=v_1$ and $G_2^b\left(x_2\right)=v_2$, the joint moment for the new *BExFD* is

$$\mu_{r,s}^{/}=2\int_0^1\int_0^1\left[G_1^{-1}\left(v_1^{1/a}\right)\right]^r\left[G_2^{-1}\left(v_2^{1/b}\right)\right]^s v_1v_2\left[1-\left(1-v_1\right)\left(1-v_2\right)\right]^{-3}dv_1dv_2\ . \tag{11}$$

The moments can be obtained for any baseline distribution. The ratio moment can be easily written from (11) by replacing either r with $-r$ or s with $-s$.

The conditional moments for the new *BExFD* can be obtained from the conditional distribution of X_1 given $X_2 = x_2$ or from the conditional distribution of X_2 given $X_1 = x_1$. Specifically, the rth conditional moment of X_1 given $X_2 = x_2$ is

$$\mu_{r\left(X_1|x_2\right)}^{/}=E\left(X_1^r|x_2\right)=\int_{-\infty}^{\infty}x_1^rf_{1|2}\left(x_1|x_2\right)dx_1=\int_{-\infty}^{\infty}x_1^r\frac{2a\,g_1\left(x_1\right)G_1^{2a-1}\left(x_1\right)G_2^{b}\left(x_2\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^3}dx_1\ .$$

Using the transformation $G_1^a\left(x_1\right)=v_1$, the conditional moment of X_1 given $X_2 = x_2$ is

$$\mu_{r\left(X_1|x_2\right)}^{/}=2G_2^b\left(x_2\right)\int_0^1v_1\left[G_1^{-1}\left(v_1^{1/a}\right)\right]^r\left[1-\left(1-v_1\right)\left\{1-G_2^b\left(x_2\right)\right\}\right]^{-3}dv_1\ . \tag{12}$$

Similarly, the sth conditional moment of X_2 given $X_1 = x_1$ is given as

$$\mu_{s\left(X_2|x_1\right)}^{/}=2G_1^a\left(x_1\right)\int_0^1v_2\left[G_2^{-1}\left(v_2^{1/b}\right)\right]^s\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left(1-v_2\right)\right]^{-3}dv_2\ . \tag{13}$$

The conditional moments can be obtained for specific baseline distribution.

4.2. The Bivariate Reliability and Hazard Rate Functions

The joint reliability function is useful to see the joint survival of two components. The joint reliability function is defined as, see Moore (2016),

$$R\left(x_1,x_2\right)=1-F_1\left(x_1\right)-F_2\left(x_2\right)+F_{1,2}\left(x_1,x_2\right).$$

The bivariate reliability function for the bivariate exponentiated family of distributions is

$$R\left(x_1,x_2\right)=\frac{\left[1-G_1^a\left(x_1\right)\right]\left[1-G_2^b\left(x_2\right)\right]\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]}=\frac{\varphi_1\left(x_1\right)\varphi_2\left(x_2\right)\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]}{\Delta_{1,2}\left(x_1,x_2\right)}\,. \tag{14}$$

The joint reliability function can be obtained for any baseline distribution.

The joint hazard rate function is defined as, see Basu (1971),

$$h_{1,2}\left(x_1,x_2\right)=\frac{f_{1,2}\left(x_1,x_2\right)}{R_{1,2}\left(x_1,x_2\right)}.$$

The joint hazard rate function for the new *BExFD* is

$$h\left(x_1,x_2\right)=\frac{2ab\,g_1\left(x_1\right)g_2\left(x_2\right)G_1^{2a-1}\left(x_1\right)G_2^{2b-1}\left(x_2\right)}{\left[1-G_1^a\left(x_1\right)\right]\left[1-G_2^b\left(x_2\right)\right]\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^2}$$

or

$$h\left(x_1,x_2\right)=\frac{2ab\,g_1\left(x_1\right)g_2\left(x_2\right)G_1^{2a-1}\left(x_1\right)G_2^{2b-1}\left(x_2\right)}{\varphi_1\left(x_1\right)\varphi_1\left(x_1\right)\Delta_{1,2}^2\left(x_1,x_2\right)\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]}. \tag{15}$$

The joint hazard rate function can be obtained for any baseline distribution.

4.3. The Hazard Rate and Mean Residual Life Vectors

The joint hazard rate function, (15), is useful to see the instantaneous failure of both of the components. It is, sometimes, required to see the instantaneous failure of individual components by considering the effect of the other component. This can be easily done by obtaining the hazard rate vector that is defined as

$$\mathbf{h}\left(x_1,x_2\right)=\left[-\frac{\partial}{\partial x_1}\ln R\left(x_1,x_2\right)\quad -\frac{\partial}{\partial x_2}\ln R\left(x_1,x_2\right)\right]^{/}. \tag{16}$$

The entries of the hazard rate vector for the new *BExFD* are

$$\begin{aligned}-\frac{\partial}{\partial x_1}\ln R\left(x_1,x_2\right)&=\frac{a\,g_1\left(x_1\right)G_1^{2a-1}\left(x_1\right)\left[G_1^a\left(x_1\right)+2G_2^b\left(x_2\right)-G_1^a\left(x_1\right)G_2^b\left(x_2\right)\right]}{\left[1-G_1^a\left(x_1\right)\right]\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]}\\&=\frac{a\,g_1\left(x_1\right)G_1^{2a-1}\left(x_1\right)\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\left\{1+\varphi_1\left(x_1\right)\right\}\right]}{\varphi_1\left(x_1\right)\Delta_{1,2}\left(x_1,x_2\right)\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]},\end{aligned} \tag{17}$$

and

$$\begin{aligned}-\frac{\partial}{\partial x_2}\ln R\left(x_1,x_2\right)&=\frac{b\,g_2\left(x_2\right)G_2^{2b-1}\left(x_2\right)\left[2G_1^a\left(x_1\right)+G_2^b\left(x_2\right)-G_1^a\left(x_1\right)G_2^b\left(x_2\right)\right]}{\left[1-G_2^b\left(x_2\right)\right]\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]}\\&=\frac{b\,g_2\left(x_2\right)G_2^{2b-1}\left(x_2\right)\left[G_1^a\left(x_1\right)\left\{1+\varphi_2\left(x_2\right)\right\}+G_2^b\left(x_2\right)\right]}{\varphi_2\left(x_2\right)\Delta_{1,2}\left(x_1,x_2\right)\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]}.\end{aligned} \tag{18}$$

The hazard rate vector can be obtained by using (17) and (18) in (16).

The mean residual life vector is another useful measure in bivariate reliability analysis. The mean residual life vector is defined as

$$\mathbf{m}\left(x_1,x_2\right)=\left[m_1\left(x_1,x_2\right),m_2\left(x_1,x_2\right)\right]^{/}, \tag{19}$$

where

$$m_1\left(x_1,x_2\right)=E\left(X_1-x_1\middle|X_1>x_1,X_2>x_2\right)=\frac{1}{R\left(x_1,x_2\right)}\int_{x_1}^{\infty}R\left(x_1,x_2\right)dx_1=\frac{1}{R\left(x_1,x_2\right)}I_1, \tag{20}$$

and

$$m_2\left(x_1,x_2\right)=E\left(X_2-x_2\middle|X_1>x_1,X_2>x_2\right)=\frac{1}{R\left(x_1,x_2\right)}\int_{x_2}^{\infty}R\left(x_1,x_2\right)dx_2=\frac{1}{R\left(x_1,x_2\right)}I_2. \tag{21}$$

Now, for the new *BExFD*, we have

$$I_1=\int_{x_1}^{\infty}R\left(x_1,x_2\right)dx_1=\left[1-G_2^b\left(x_2\right)\right]\int_{x_1}^{\infty}\frac{\left[1-G_1^a\left(x_1\right)\right]\left[G_1^a\left(x_1\right)+G_2^b\left(x_2\right)\right]}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]}dx_1.$$

Using the transformation, $v_1=G_1^a\left(x_1\right)$, the above integral can be written as

$$I_1 = \frac{\left[1-G_2^b\left(x_2\right)\right]}{a\,g_1\left(x_1\right)}\int_{G_1^a(x_1)}^1 \frac{\left(1-v_1\right)\left[v_1+G_2^b\left(x_2\right)\right]}{\left[1-\left(1-v_1\right)\left\{1-G_2^b\left(x_2\right)\right\}\right]} v_1^{1/a-1} dv_1$$

$$= \frac{\left[1-G_2^b\left(x_2\right)\right]}{a\,g_1\left(x_1\right)}\sum_{j=0}^{\infty}\varphi_2^j\left(x_2\right)\int_{G_1^a(x_1)}^1 v_1^{1/a-1}\left(1-v_1\right)^{j+1}\left[v_1+G_2^b\left(x_2\right)\right]dv_1.$$

Solving the integral, we have

$$I_1 = \frac{\left[1-G_2^b\left(x_2\right)\right]}{a\,g_1\left(x_1\right)}\sum_{j=0}^{\infty}\varphi_2^j\left(x_2\right)B\left(\frac{1}{a}+1, j+2\right)\left[\frac{1}{a\left(j+2\right)+1}\left\{1-I_{G_1^a(x_1)}\left(\frac{1}{a}+1, j+2\right)\right\}\right.$$
$$\left. -G_2^b\left(x_2\right)\left\{1-I_{G_1^a(x_1)}\left(\frac{1}{a}, j+2\right)\right\}\right], \tag{22}$$

where $I_y\left(a,b\right)=B_y\left(a,b\right)/B\left(a,b\right)$ is the incomplete beta function ratio. Similarly, we have

$$I_2 = \frac{\left[1-G_1^a\left(x_1\right)\right]}{a\,g_2\left(x_2\right)}\sum_{j=0}^{\infty}\varphi_1^j\left(x_1\right)B\left(\frac{1}{b}+1, j+2\right)\left[\frac{1}{b\left(j+2\right)+1}\left\{1-I_{G_2^b(x_2)}\left(\frac{1}{b}+1, j+2\right)\right\}\right.$$
$$\left. -G_1^a\left(x_1\right)\left\{1-I_{G_2^b(x_2)}\left(\frac{1}{b}, j+2\right)\right\}\right]. \tag{23}$$

The mean residual life vector is obtained by using (22) and (23) in (20) and (21) and then using the resulting expressions in (19). The hazard rate and mean residual life vectors can be obtained for any baseline distribution.

4.4. Stress–Strength Reliability

The stress and strength reliability is often required in many areas of engineering. If it is assumed that the strength of a component is a positive random variable X_1 and the stress on that component is another positive random variable X_2 then the stress–strength type reliability is obtained as

$$R = P\left(X_1 > X_2\right) = \int_0^{\infty}\int_0^{x_1} f_{1,2}\left(x_1, x_2\right)dx_2 dx_1 .$$

Now, suppose that the strength, X_1, and stress, X_2, have a joint distribution as given in (8) then the stress–strength type reliability is

$$R = P\left(X_2 < X_1\right) = \int_0^{\infty}\int_0^{x_1} f_{1,2}\left(x_1, x_2\right)dx_2 dx_1 = \int_0^{\infty}\int_0^{x_1}\frac{2ab\,g_1\left(x_1\right)g_2\left(x_2\right)G_1^{2a-1}\left(x_1\right)G_2^{2b-1}\left(x_2\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^3}dx_2 dx_1 .$$

Now, using the transformation $v_2 = G_2^b\left(x_2\right)$, we have

$$R = \int_0^{\infty}\int_0^{G_2^b(x_1)}\frac{2a\,g_1\left(x_1\right)G_1^{2a-1}\left(x_1\right)v_2}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left(1-v_2\right)\right]^3}dv_2 dx_1$$

or

$$R = \int_0^{\infty} a\,g_1\left(x_1\right)G_1^{a-1}\left(x_1\right)G_2^{2b}\left(x_1\right)\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_1\right)\right\}\right]^{-2}dx_1 . \tag{24}$$

If X_1 and X_2 have the same distribution, that is if $G_1\left(x_1\right)=G_2\left(x_1\right)=G\left(x_1\right)$, then the reliability is

$$R = \int_0^{\infty}\frac{a\,g\left(x_1\right)G^{a-1}\left(x_1\right)G^{2b}\left(x_1\right)}{\left[1-\left\{1-G^a\left(x_1\right)\right\}\left\{1-G^b\left(x_1\right)\right\}\right]^2}dx_1 .$$

Using the transformation $v_1 = G^a\left(x_1\right)$, the reliability is

$$R = \int_0^1 u_1^{2b/a}\left[1-\left(1-v_1\right)\left(1-v_1^{b/a}\right)\right]^{-2}dv_1 . \tag{25}$$

Further, if $a = b$ then we have

$$R = \int_0^1 v_1^2\left[1-\left(1-v_1\right)^2\right]^{-2}dv_1 = \frac{1}{2}.$$

The reliability coefficient, R, given in (24) is useful when X_1 and X_2 have different parent distributions.

4.5. The Dependence Measures

The dependence measure is a useful measure to see the strength of interdependence between two variables and various dependence measures are available for this purpose. The dependence between two variables can be studied generally or locally. The most popular general measures of dependence for two variables are Kendall's Tau and Spearman's Rho; see Balakrishnan and Lai (2009) and Nelsen (2006). A popular measure of local dependence has been introduced by Holland and Wang (1987). In the following, we have discussed these dependence measures for the new *BExFD*.

4.5.1. Kendall's Tau

The Kendall's Tau coefficient is obtained by using

$$\tau = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{1,2}\left(x_1,x_2\right) f_{1,2}\left(x_1,x_2\right) dx_1 dx_2 - 1 .$$

Now, using the density and distribution functions of the new *BExFD*, we have

$$\begin{aligned}\tau &= 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{G_1^a\left(x_1\right)G_2^b\left(x_2\right)}{1-\left[1-G_1^a\left(x_1\right)\right]\left[1-G_2^b\left(x_2\right)\right]} \frac{2ab\, g_1\left(x_1\right) g_2\left(x_2\right) G_1^{2a-1}\left(x_1\right) G_2^{2b-1}\left(x_2\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^3} dx_1 dx_2 - 1 \\ &= 8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{ab\, g_1\left(x_1\right) g_2\left(x_2\right) G_1^{3a-1}\left(x_1\right) G_2^{3b-1}\left(x_2\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^4} dx_1 dx_2 - 1.\end{aligned}$$

Making the transformation $G_1^a\left(x_1\right)=v_1$ and $G_2^b\left(x_2\right)=v_2$, we have

$$\tau = 8\int_0^1\int_0^1 \frac{v_1^2 v_2^2}{\left[1-\left(1-v_1\right)\left(1-v_2\right)\right]^4} dv_1 dv_2 - 1 = \frac{1}{3} .$$

We can see that Kendall's Tau is fixed for the new *BExFD*.

4.5.2. Spearman's Rho

The Spearman's Rho coefficient is obtained by using

$$\rho = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{1,2}\left(x_1,x_2\right) f_1\left(x_1\right) f_2\left(x_2\right) dx_1 dx_2 - 3$$

Now, using the density and distribution functions of the new *BExFD*, we have

$$\rho = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{ab\, g_1\left(x_1\right) g_2\left(x_2\right) G_1^{2a-1}\left(x_1\right) G_2^{2b-1}\left(x_2\right)}{1-\left[1-G_1^a\left(x_1\right)\right]\left[1-G_2^b\left(x_2\right)\right]} dx_1 dx_2 - 3 .$$

Making the transformation $G_1^a\left(x_1\right)=v_1$ and $G_2^b\left(x_2\right)=v_2$, we have

$$\rho = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{v_1 v_2}{1-\left(1-v_1\right)\left(1-v_2\right)} dv_1 dv_2 - 3 = 4\pi^2 - 39 .$$

We can see that Spearman's Rho is also fixed for the new *BExFD*. We can also see that, for the new *BExFD*, Kendall's Tau is less than Spearman's Rho.

4.5.3. Local Dependence Measure

The local dependence measure is defined by Holland and Wang (1987) as

$$\gamma\left(x,y\right) = \frac{\partial^2}{\partial x \partial y} \ln f_{X,Y}\left(x,y\right),$$

which for the bivariate exponentiated family of distributions is

$$\gamma\left(x_1,x_2\right) = \frac{3ab\, g_1\left(x_1\right) g_2\left(x_2\right) G_1^a\left(x_1\right) G_2^b\left(x_2\right)}{\left[1-\left\{1-G_1^a\left(x_1\right)\right\}\left\{1-G_2^b\left(x_2\right)\right\}\right]^2} . \quad (26)$$

The local dependence measure can be computed for any baseline distribution.

4.6. Random Sampling

A random observation can be drawn from the new *BExFD* by using the conditional distribution approach. For this, we need to solve $F_1(x_1)=u_1$ and $F_{2|1}(x_2|x_1)=u_2$ for x_1 and x_2, where u_1 and u_2 are uniform random numbers. The sampling algorithm is given below:

- Generate a random x_1 observation by solving $F_1(x_1)=u_1$ for x_1. That is, generate x_1 by solving $G_1^a(x_1)=u_1$ for x_1, or x_1 is generated as $x_1=G_1^{-1}\left(u_1^{1/a}\right)$.
- Generate a random x_2 observation by solving $F_{2|1}(x_2|x_1)=u_2$ for x_2.

  Now

$$F_{2|1}(x_2|x_1)=\int_{-\infty}^{x_2} f_{2|1}(w_2|x_1)\,dw_2=\int_{-\infty}^{x_2}\frac{2b\,g_2(w_2)G_2^{2b-1}(w_2)G_1^a(x_1)}{\left[1-\left\{1-G_1^a(x_1)\right\}\left\{1-G_2^b(w_2)\right\}\right]^3}dw_2$$

  Making the transformation $G_2^b(w_2)=v_2$, we have

$$\begin{aligned}F_{2|1}(x_2|x_1)&=2G_1^a(x_1)\int_0^{G_2^b(x_2)}v_2\left[1-\left\{1-G_1^a(x_1)\right\}(1-v_2)\right]^{-3}dv_2\\&=\frac{G_2^{2b}(x_2)}{\left[G_1^a(x_1)+G_2^b(x_2)-G_1^a(x_1)G_2^b(x_2)\right]^2},\end{aligned}$$

  and hence a random x_2 observation is generated by solving

$$\frac{G_2^{2b}(x_2)}{\left[G_1^a(x_1)+G_2^b(x_2)-G_1^a(x_1)G_2^b(x_2)\right]^2}=u_2 .$$

  The solution is given as

$$x_2=G_2^{-1}\left[\left\{\frac{u_2^{1/2}G_1^a(x_1)}{1-u_2^{1/2}\left\{1-G_1^a(x_1)\right\}}\right\}^{1/b}\right]=G_2^{-1}\left(u_2^*\right)\text{ where }u_2^*=\left[\frac{u_2^{1/2}G_1^a(x_1)}{1-u_2^{1/2}\left\{1-G_1^a(x_1)\right\}}\right]^{1/b}.$$

A bivariate observation from the new *BExFD* is therefore obtained by using

$$x_1=G_1^{-1}\left(u_1^{1/a}\right)\text{ and }x_2=G_2^{-1}\left(u_2^*\right)\text{ where }u_2^*=\left[\frac{u_2^{1/2}G_1^a(x_1)}{1-u_2^{1/2}\left\{1-G_1^a(x_1)\right\}}\right]^{1/b}. \tag{27}$$

A random sample of size n, from *BExFD* can, therefore be drawn by using (27) and different values of a and b.

4.7. Maximum Likelihood Estimation of the Parameters

In this section, we have discussed the maximum likelihood estimation for the parameters of *BExFD* with any baseline distributions. For this, we suppose that a bivariate random sample of size n is available from *BExFD* with baseline distributions $g_1(x_1;\xi_1)$ and $g_2(x_2;\xi_2)$, where ξ_1 and ξ_2 are the parameter vectors of the baseline distributions having p_1 and p_2 parameters, respectively. The joint density function for the *BExFD* in this case is

$$f_{1,2}(x_1,x_2)=2ab\,g_1(x_1;\xi_1)g_2(x_2;\xi_2)G_1^{2a-1}(x_1;\xi_1)G_2^{2b-1}(x_2;\xi_2)\left[1-\left\{1-G_1^a(x_1;\xi_1)\right\}\left\{1-G_2^b(x_2;\xi_2)\right\}\right]^3.$$

The log of the density function is

$$\begin{aligned}\ln f_{1,2}(x_1,x_2)&=\ln 2+\ln a+\ln b+\ln g_1(x_1;\xi_1)+\ln g_2(x_2;\xi_2)+(2a-1)\ln G_1(x_1;\xi_1)\\&\quad+(2b-1)\ln G_2(x_2;\xi_2)-3\ln\left[1-\left\{1-G_1^a(x_1;\xi_1)\right\}\left\{1-G_2^b(x_2;\xi_2)\right\}\right].\end{aligned}$$

The log-likelihood function; $\ell=\sum_{i=1}^n \ln f_{1,2}(x_1,x_2)$; is

$$\begin{aligned}\ell=\ln L(a,b;\mathbf{x},\xi_1,\xi_2)&=n\ln 2+n\ln a+n\ln b+\sum_{i=1}^n\ln g_1(x_{1i};\xi_1)+\sum_{i=1}^n\ln g_2(x_{2i};\xi_2)\\&\quad+(2a-1)\sum_{i=1}^n\ln G_1(x_{1i};\xi_1)+(2b-1)\sum_{i=1}^n\ln G_2(x_{2i};\xi_2)\\&\quad-3\sum_{i=1}^n\ln\left[1-\left\{1-G_1^a(x_{1i};\xi_1)\right\}\left\{1-G_2^b(x_{2i};\xi_2)\right\}\right].\end{aligned} \tag{28}$$

The derivatives of the log-likelihood function, (28), are

$$\frac{\partial \ell}{\partial a}=\frac{n}{a}+2\sum_{i=1}^{n}\ln G_1\left(x_{1i};\xi_1\right)-3\sum_{i=1}^{n}\frac{G_1^{a}\left(x_{1i};\xi_1\right)\varphi_2\left(x_{2i};\xi_2\right)\ln G_1\left(x_{1i};\xi_1\right)}{\Delta\left(x_{1i},x_{2i};\xi_1,\xi_2\right)}, \tag{29}$$

$$\frac{\partial \ell}{\partial a}=\frac{n}{b}+2\sum_{i=1}^{n}\ln G_2\left(x_{2i};\xi_2\right)-3\sum_{i=1}^{n}\frac{G_2^{b}\left(x_{2i};\xi_2\right)\varphi_1\left(x_{1i};\xi_1\right)\ln G_2\left(x_{2i};\xi_2\right)}{\Delta\left(x_{1i},x_{2i};\xi_1,\xi_2\right)}, \tag{30}$$

$$\frac{\partial \ell}{\partial \xi_{1k}}=\sum_{i=1}^{n}\frac{g_1'\left(x_{1i};\xi_1\right)}{g_1\left(x_{1i};\xi_1\right)}+\left(2a-1\right)\sum_{i=1}^{n}\frac{G_1'\left(x_{1i};\xi_1\right)}{G_1\left(x_{1i};\xi_1\right)}-3a\sum_{i=1}^{n}\frac{G_1^{a-1}\left(x_{1i};\xi_1\right)\varphi_2\left(x_{2i};\xi_2\right)G_1'\left(x_{1i};\xi_1\right)}{\Delta\left(x_{1i},x_{2i};\xi_1,\xi_2\right)}, \tag{31: 1–p_1}$$

and

$$\frac{\partial \ell}{\partial \xi_{2k}}=\sum_{i=1}^{n}\frac{g_2'\left(x_{2i};\xi_2\right)}{g_2\left(x_{2i};\xi_2\right)}+\left(2b-1\right)\sum_{i=1}^{n}\frac{G_2'\left(x_{2i};\xi_2\right)}{G_2\left(x_{2i};\xi_2\right)}-3b\sum_{i=1}^{n}\frac{G_2^{b-1}\left(x_{2i};\xi_2\right)\varphi_1\left(x_{1i};\xi_1\right)G_2'\left(x_{2i};\xi_2\right)}{\Delta\left(x_{1i},x_{2i};\xi_1,\xi_2\right)}, \tag{32: 1–p_2}$$

where

$$\varphi_1\left(x_{1i};\xi_1\right)=1-G_1^{a}\left(x_{1i};\xi_1\right),\ \varphi_2\left(x_{2i};\xi_2\right)=1-G_2^{b}\left(x_{2i};\xi_2\right),\ \Delta\left(x_{1i},x_{2i};\xi_1,\xi_2\right)=1-\varphi_1\left(x_{1i};\xi_1\right)\varphi_2\left(x_{2i};\xi_2\right),$$

$$g_1'\left(x_{1i};\xi_1\right)=\frac{\partial}{\partial \xi_{1k}}g_1\left(x_{1i};\xi_1\right),\ G_1'\left(x_{1i};\xi_1\right)=\frac{\partial}{\partial \xi_{1k}}G_1\left(x_{1i};\xi_1\right),\ g_2'\left(x_{2i};\xi_2\right)=\frac{\partial}{\partial \xi_{2k}}g_2\left(x_{2i};\xi_2\right),$$

and $$G_2'\left(x_{2i};\xi_2\right)=\frac{\partial}{\partial \xi_{2k}}G_2\left(x_{2i};\xi_2\right).$$

The maximum likelihood estimates of parameters a, b, ξ_1, and ξ_2 is obtained by equating the derivatives in (29), (30), (31: 1–p_1), and (32: 1–p_2) to zero and numerically solving the resulting equations. The entries of the Fisher information matrix for the new *BExFD* are given in Appendix *A*.

The proposed *BExFD* can be used to generate different bivariate exponentiated distributions and in the following, we have discussed one specific member of the family, namely the bivariate exponentiated power function (*BExPF*) distribution.

5. A Bivariate Exponentiated Power Function Distribution

The power function distribution is a simple yet powerful distribution in probability theory. The distribution has been extensively studied by various authors. The moments of lower generalized order statistics for the power function distribution have been studied by Athar and Faizan (2011). Ahsanullah et al. (2013) have given some characterizations of the power function distribution by using the lower record values. Arshad et al. (2020) have proposed an exponentiated power function distribution and have studied some of its useful properties. In the following, we have obtained a bivariate exponentiated power function (*BExPF*) distribution by using the power function distribution as a baseline distribution in the *BExFD*. For this, suppose that the random variables X_1 and X_2 have univariate power function distributions with the marginal density and distribution functions given as

$$g_1\left(x_1\right)=\theta_1 x_1^{\theta_1-1}/c_1^{\theta_1}\ \ \&\ \ G_1\left(x_1\right)=\left(x_1/c_1\right)^{\theta_1}\ ;\ 0<x_1<c_1\ ;\ c_1,\theta_1>0$$

and

$$g_2\left(x_2\right)=\theta_2 x_2^{\theta_2-1}/c_2^{\theta_2}\ \ \&\ \ G_2\left(x_2\right)=\left(x_2/c_2\right)^{\theta_2}\ ;\ 0<x_2<c_2\ ;\ c_2,\theta_2>0.$$

Now, using the marginal distribution functions of the power function distribution in (7) the joint *cdf* of the *BExPF* distribution is obtained as

$$F_{1,2}\left(x_1,x_2\right)=\frac{\left(x_1/c_1\right)^{a\theta_1}\left(x_2/c_2\right)^{b\theta_2}}{\left(x_1/c_1\right)^{a\theta_1}+\left(x_2/c_2\right)^{b\theta_2}-\left(x_1/c_1\right)^{a\theta_1}\left(x_2/c_2\right)^{b\theta_2}}$$

or

$$F_{1,2}\left(x_1,x_2\right)=\frac{\left(x_1/c_1\right)^{a\theta_1}\left(x_2/c_2\right)^{b\theta_2}}{1-\left[1-\left(x_1/c_1\right)^{a\theta_1}\right]\left[1-\left(x_2/c_2\right)^{b\theta_2}\right]}\ ;\ \left(x_1,x_2\right)\in\left[0,c_1\right]\times\left[0,c_2\right]\ ;\ \left(\theta_1,\theta_2,a,b\right)>0. \tag{33}$$

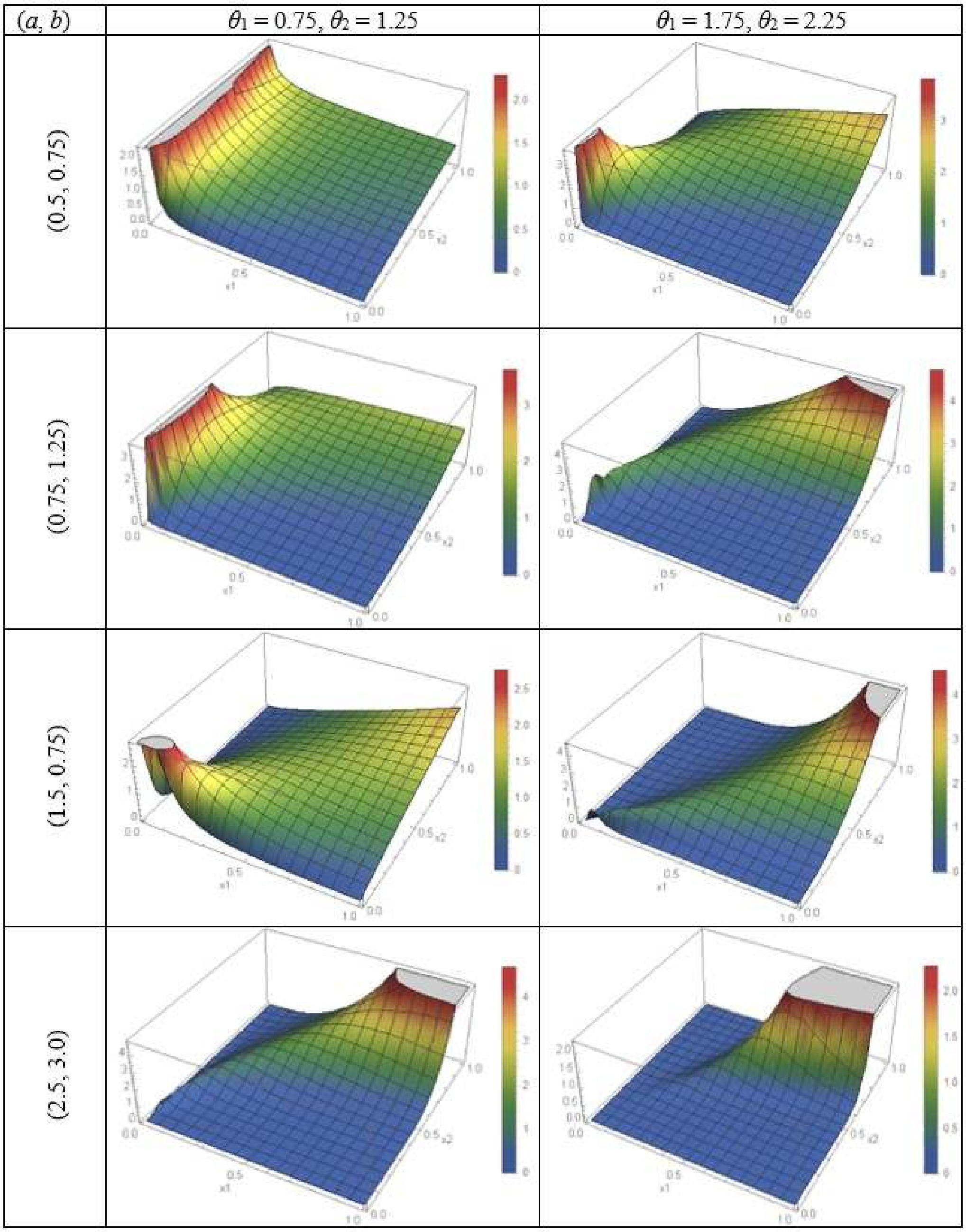
The density function corresponding to (33) is

$$f_{1,2}\left(x_1,x_2\right)=\frac{2ab\left(\theta_1 x_1^{\theta_1-1}/c_1^{\theta_1}\right)\left(\theta_2 x_2^{\theta_2-1}/c_2^{\theta_1}\right)\left(x_1/c_1\right)^{(2a-1)\theta_1}\left(x_2/c_2\right)^{(2b-1)\theta_2}}{\left[1-\left\{1-\left(x_1/c_1\right)^{a\theta_1}\right\}\left\{1-\left(x_2/c_2\right)^{b\theta_2}\right\}\right]^3}, \quad (34)$$

or $$f_{1,2}\left(x_1,x_2\right)=\frac{2ab\theta_1\theta_2 x_1^{2a\theta_1-1}x_2^{2b\theta_2-1}}{c_1^{2a\theta_1}c_2^{2b\theta_2}\Delta_{1,2}^3\left(x_1,x_2\right)};\ \left(x_1,x_2\right)\in\left[0,c_1\right]\times\left[0,c_2\right];\ \left(\theta_1,\theta_2,a,b\right)>0,$$

where $\Delta_{1,2}\left(x_1,x_2\right)=\left[1-\left\{1-\left(x_1/c_1\right)^{a\theta_1}\right\}\left\{1-\left(x_2/c_2\right)^{b\theta_2}\right\}\right]$. The distribution in (34) will be written as $BExPF\left(c_1,c_2,\theta_1,\theta_2,a,b\right)$. The plots of the joint density function for $c_1 = c_2 = 1$ and different values of the other parameters are given in Figure 1, below.

Figure 1: Plots of the Joint Density Function for *BExPF* Distribution



The plots of the joint density function indicate that when both *a* and *b* are less than 1 then the density has a decreasing trend. Also, when both *a* and *b* are greater than 1 then the density function has an increasing trend. We will now discuss some useful properties of the *BExPF* distribution.

6. Properties of Bivariate Exponentiated Power Function Distribution

In this section, we have discussed some useful properties for the *BExPF* distribution. These properties are discussed in the following sub-sections.

6.1. The Marginal and the Conditional Distributions

The marginal density and distribution functions of X_1 and X_2 for the *BExPF* distribution are readily written as

$$f_1\left(x_1\right)=\frac{a\theta_1 x_1^{a\theta_1-1}}{c_1^{a\theta_1}} \ \& \ F_1\left(x_1\right)=\left(\frac{x_1}{c_1}\right)^{a\theta_1} \ ; 0<x_1<c_1 \ , \left(a,\theta_1\right)>0$$

and

$$f_2\left(x_2\right)=\frac{b\theta_2 x_2^{b\theta_2-1}}{c_2^{b\theta_2}} \ \& \ F_2\left(x_2\right)=\left(\frac{x_2}{c_2}\right)^{b\theta_2} \ ; 0<x_2<c_2 \ , \left(b,\theta_2\right)>0 \ .$$

It is easy to see that the marginal distributions are the exponentiated power function distributions. The conditional distribution of X_1 given $X_2 = x_2$ is readily written from (9) as

$$f_{1|2}\left(x_1|x_2\right)=\frac{2a\left(\theta_1 x_1^{\theta_1-1}/c_1^{\theta_1}\right)\left(x_1/c_1\right)^{(2a-1)\theta_1}\left(x_2/c_2\right)^{b\theta_2}}{\left[1-\left\{1-\left(x_1/c_1\right)^{a\theta_1}\right\}\left\{1-\left(x_2/c_2\right)^{b\theta_2}\right\}\right]^3}$$

or

$$f_{1|2}\left(x_1|x_2\right)=\frac{2a\theta_1 x_1^{2a\theta_1-1}x_2^{b\theta_2}}{c_1^{2a\theta_1}c_2^{b\theta_2}\Delta_{1,2}^3\left(x_1,x_2\right)} \ ; \left(x_1,x_2\right)\in\left[0,c_1\right]\times\left[0,c_2\right] ; \left(a,b,\theta_1,\theta_2\right)>0 \ . \tag{35}$$

Again, the conditional distribution of X_2 given $X_1 = x_1$ is written from (10) as

$$f_{2|1}\left(x_2|x_1\right)=\frac{2b\left(\theta_2 x_2^{\theta_2-1}/c_2^{\theta_2}\right)\left(x_1/c_1\right)^{a\theta_1}\left(x_2/c_2\right)^{(2b-1)\theta_2}}{\left[1-\left\{1-\left(x_1/c_1\right)^{a\theta_1}\right\}\left\{1-\left(x_2/c_2\right)^{b\theta_2}\right\}\right]^3}$$

or

$$f_{2|1}\left(x_2|x_1\right)=\frac{2b\theta_2 x_1^{a\theta_1}x_2^{2b\theta_2-1}}{c_1^{a\theta_1}c_2^{2b\theta_2}\Delta_{1,2}^3\left(x_1,x_2\right)} \ ; \left(x_1,x_2\right)\in\left[0,c_1\right]\times\left[0,c_2\right] ; \left(a,b,\theta_1,\theta_2\right)>0 \ . \tag{36}$$

The conditional distributions are useful in obtaining the conditional moments of the distribution that we have obtained in the following.

6.2. The Joint Moments

The (r, s)th joint moment for the *ExFD* is given in (11) as

$$\mu_{r,s}^{/}=2\int_0^1\int_0^1\left[G_1^{-1}\left(v_1^{1/a}\right)\right]^r\left[G_2^{-1}\left(v_2^{1/b}\right)\right]^s v_1 v_2\left[1-\left(1-v_1\right)\left(1-v_2\right)\right]^{-3}dv_1 dv_2 \ ,$$

where $G_1^a\left(x_1\right)=v_1$ and $G_2^b\left(x_2\right)=v_2$. Now, for the power function baseline distribution we have $v_1=\left(x_1/c_1\right)^{a\theta_1}$ or $G_1^{-1}\left(v_1^{1/a}\right)=c_1 v_1^{1/a\theta_1}$. Also, $v_2=\left(x_2/c_2\right)^{b\theta_2}$ or $G_2^{-1}\left(v_2^{1/b}\right)=c_2 v_2^{1/b\theta_2}$ and hence the (r, s)th joint moment for the *BExPF* distribution is

$$\mu_{r,s}^{/}=2\int_0^1\int_0^1\left(c_1^r v_1^{r/a\theta_1}\right)\left(c_2^s v_2^{s/b\theta_2}\right)v_1 v_2\left[1-\left(1-v_1\right)\left(1-v_2\right)\right]^{-3}dv_1 dv_2$$

$$=c_1^r c_2^s\int_0^1\int_0^1 v_1^{r/a\theta_1+1}v_2^{s/b\theta_2+1}\sum_{j=0}^{\infty}\frac{\Gamma\left(j+3\right)}{j!}\left(1-v_1\right)^j\left(1-v_2\right)^j dv_1 dv_2,$$

or

$$\mu_{r,s}^{/}=c_1^r c_2^s\sum_{j=0}^{\infty}\frac{\Gamma\left(j+3\right)}{j!}B\left(\frac{r}{a\theta_1}+2,j+1\right)B\left(\frac{s}{b\theta_2}+2,j+1\right).$$

Now, using the infinite sum of the beta function, see Gradshteyn and Ryzhik (2007), we have

$$\mu_{r,s}^{/} = 2c_1^r c_2^s B\left(\frac{r}{a\theta_1}+2,1\right) B\left(\frac{s}{b\theta_2}+2,1\right) {}_3F_2\left[(1,1,3);\left(\frac{r}{a\theta_1}+3,\frac{s}{b\theta_2}+3\right);1\right], \tag{37}$$

where ${}_3F_2[(\alpha_1,\alpha_2,\alpha_3);(\beta_1,\beta_2);z]$ is the generalized hypergeometric function defined as

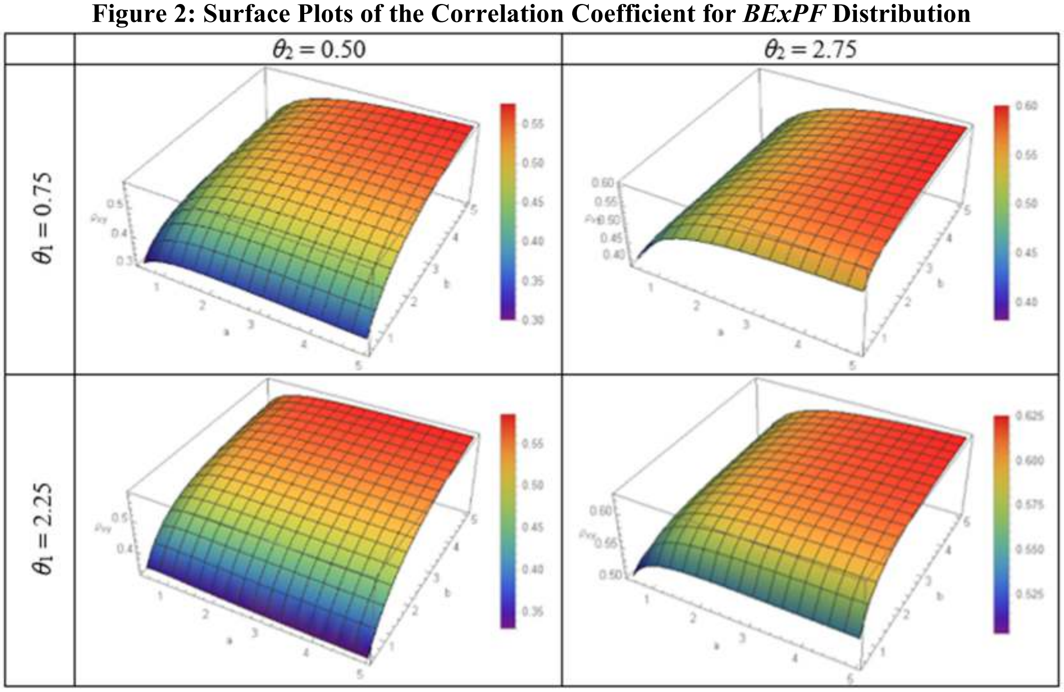
$${}_3F_2\left[(\alpha_1,\alpha_2,\alpha_3);(\beta_1,\beta_2);z\right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{(\beta_1)_k(\beta_2)_k}\frac{z^k}{k!},$$

and $(\alpha)_k$ is the Pochammar's symbol given as

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)},$$

and $\Gamma(\alpha)$ is the complete gamma function. The product moments given in (37) can be used to compute marginal means and variances. Also, the moment expression in (37) can be used to compute the correlation coefficient between the two variables. The values of the correlation coefficient for $c_1 = c_2 = 1$ and for selected values of the other parameters are given in Table *C*.1 in Appendix *C*. Also, the surface plots of the correlation coefficient for various values of θ_1, θ_2, a, and b are given in Figure 2, below.

Figure 2: Surface Plots of the Correlation Coefficient for *BExPF* Distribution



From the above plot, we can see that, for the *BExPF* distribution, the correlation coefficient between the two variables is always positive. We can, also, see that if θ_1 and θ_2 are greater than 1 then the correlation coefficient is much higher as compared with the case when one of these two parameters is less than 1.

6.3. The Conditional Moments

The conditional moments are useful for studying the properties of the conditional distributions. The expressions for the conditional moments for the *BExFD* are given in (12) and (13). Now, from (12), the conditional moment of X_1 given $X_2 = x_2$ is

$$\mu_{r(X_1|x_2)}^{/} = 2G_2^b(x_2)\int_0^1 v_1\left[G_1^{-1}\left(v_1^{1/a}\right)\right]^r\left[1-(1-v_1)\left\{1-G_2^b(x_2)\right\}\right]^{-3} dv_1,$$

where $v_1 = G_1^a(x_1)$. Now, for the power function distribution, we have $v_1 = (x_1/c_1)^{a\theta_1}$ or $G_1^{-1}\left(v_1^{1/a}\right) = c_1 v_1^{1/a\theta_1}$ and hence the rth conditional moment of X_1 given $X_2 = x_2$ is

$$\mu'_{r(X_1|x_2)} = 2c_1^r G_2^b(x_2)\int_0^1 v_1^{r/a\theta_1+1}\left[1-(1-v_1)\left\{1-G_2^b(x_2)\right\}\right]^{-3} dv_1$$
$$= c_1^r \sum_{j=0}^{\infty}\frac{\Gamma(j+3)}{j!}G_2^b(x_2)\left\{1-G_2^b(x_2)\right\}^j \int_0^1 v_1^{r/a\theta_1+1}(1-v_1)^j dv_1.$$

Solving the integral, we have

$$\mu'_{r(X_1|x_2)} = c_1^r \sum_{j=0}^{\infty}\frac{\Gamma(j+3)}{j!}G_2^b(x_2)\left\{1-G_2^b(x_2)\right\}^j B\left(\frac{r}{a\theta_1}+2, j+1\right).$$

Again, using the infinite sum of the beta function, see Gradshteyn and Ryzhik (2007), we have

$$\mu'_{r(X_1|x_2)} = c_1^r G_2^b(x_2) B\left(\frac{r}{a\theta_1}+2,1\right) {}_2F_1\left[1,3;\frac{r}{a\theta_1}+3;1-G_2^b(x_2)\right], \tag{38}$$

where ${}_2F_1[\alpha_1,\alpha_2;\beta;z]$ is the Gauss hypergeometric function defined as

$${}_2F_1\left[\alpha_1,\alpha_2;\beta;z\right] = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k}{(\beta)_k}\frac{z^k}{k!}.$$

Similarly, the sth conditional moment of X_2 given $X_1 = x_1$ is derived as

$$\mu'_{s(X_2|x_1)} = c_2^s G_1^a(x_1) B\left(\frac{s}{b\theta_2}+2,1\right) {}_2F_1\left[1,3;\frac{s}{b\theta_2}+3;1-G_1^a(x_1)\right]. \tag{39}$$

The conditional moments, given in (38) and (39), are useful in obtaining the conditional means and variances.

6.4. The Joint Reliability and Hazard Rate Functions

The joint reliability function for the *BExFD* is given in (14) as

$$R(x_1,x_2) = \frac{\left[1-G_1^a(x_1)\right]\left[1-G_2^b(x_2)\right]\left[G_1^a(x_1)+G_2^b(x_2)\right]}{\left[1-\left\{1-G_1^a(x_1)\right\}\left\{1-G_2^b(x_2)\right\}\right]}.$$

Using the marginal *cdf*'s of the power function distribution in the above equation, the joint reliability function for the *BExPF* distribution is

$$R(x_1,x_2) = \frac{1}{\Delta_{1,2}(x_1,x_2)}\left[\left\{1-(x_1/c_1)^{a\theta_1}\right\}\left\{1-(x_2/c_2)^{b\theta_2}\right\}\left\{(x_1/c_1)^{a\theta_1}+(x_2/c_2)^{b\theta_2}\right\}\right], \tag{40}$$

where $\Delta_{1,2}(x_1,x_2) = \left[1-\left\{1-(x_1/c_1)^{a\theta_1}\right\}\left\{1-(x_2/c_2)^{b\theta_2}\right\}\right]$. Again, the joint hazard rate function for the proposed *BExFD* is given in (15) as

$$h(x_1,x_2) = \frac{2ab\, g_1(x_1) g_2(x_2) G_1^{2a-1}(x_1) G_2^{2b-1}(x_2)}{\varphi_1(x_1)\varphi_1(x_1)\Delta_{1,2}^2(x_1,x_2)\left[G_1^a(x_1)+G_2^b(x_2)\right]}.$$

Now, using the marginal *pdf*'s and *cdf*'s of the power function distribution in the above equation, the joint hazard rate function for the *BExPF* is

$$h(x_1,x_2) = \frac{2ab\theta_1\theta_2 x_1^{2a\theta_1-1}x_2^{2b\theta_2-1}}{c_1^{2a\theta_1}c_2^{2b\theta_2}}\left[\Delta_{1,2}^2(x_1,x_2)\left\{1-\Delta_{1,2}(x_1,x_2)\right\}\left\{(x_1/c_1)^{a\theta_1}+(x_2/\theta_2)^{b\theta_2}\right\}\right]^{-1}. \tag{41}$$

Further, the entries of the hazard rate vector for the *BExPF* distribution can be obtained by using the marginal density and distribution functions of the power function distribution in (17) and (18) and are

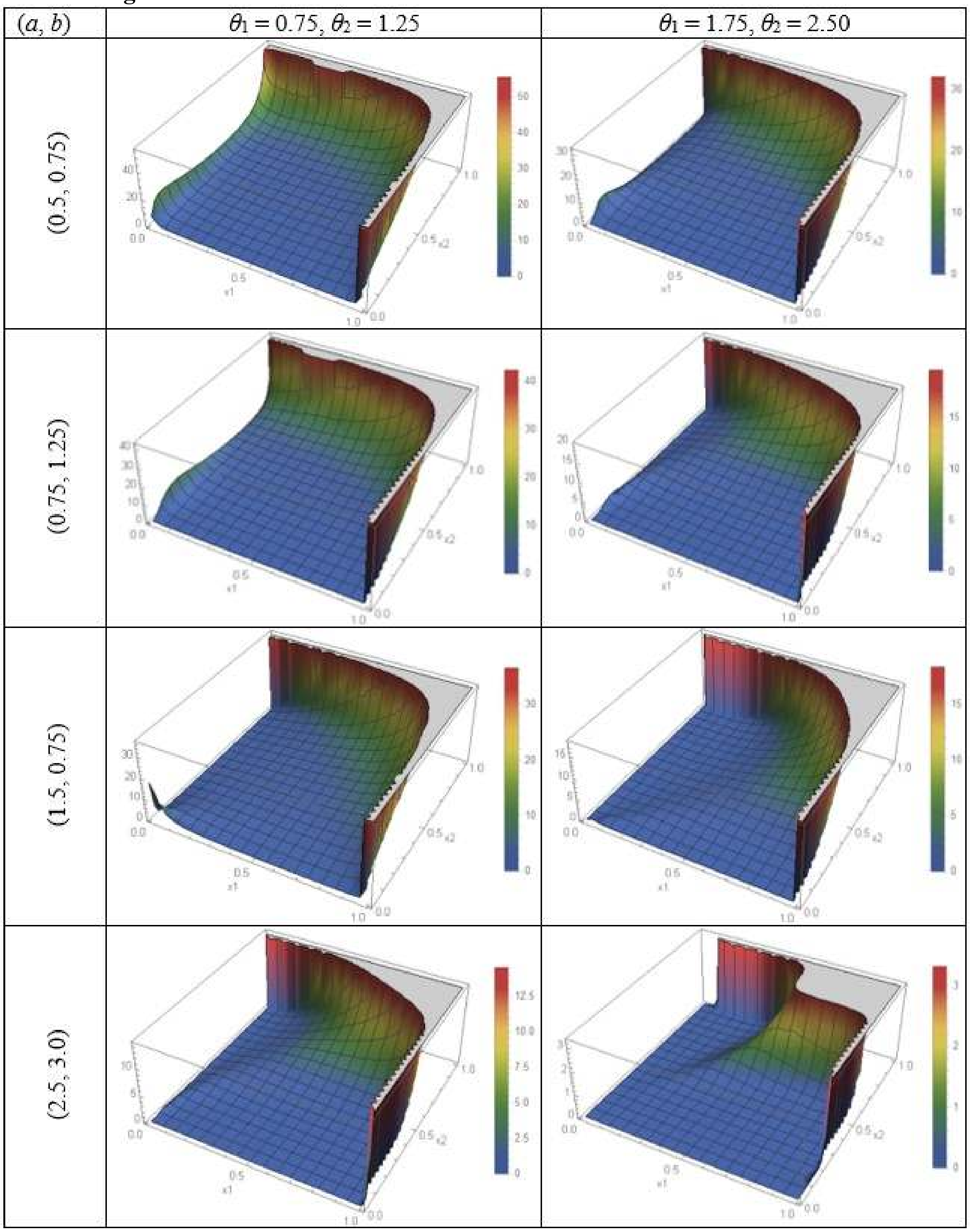
$$-\frac{\partial}{\partial x_1}\ln R(x_1,x_2) = \frac{a\theta_1 x_1^{2a\theta_1-1}\left[(x_1/c_1)^{a\theta_1}+2(x_2/c_2)^{b\theta_2}-(x_1/c_1)^{a\theta_1}(x_2/c_2)^{b\theta_2}\right]}{c_1^{2a\theta_1}\Delta_{1,2}(x_1,x_2)\left[1-(x_1/c_1)^{a\theta_1}\right]\left[(x_1/c_1)^{a\theta_1}+(x_2/c_2)^{b\theta_2}\right]}, \tag{42}$$

and

$$-\frac{\partial}{\partial x_2}\ln R(x_1,x_2) = \frac{b\theta_2 x_2^{2b\theta_2-1}\left[2(x_1/c_1)^{a\theta_1}+(x_2/c_2)^{b\theta_2}-(x_1/c_1)^{a\theta_1}(x_2/c_2)^{b\theta_2}\right]}{c_2^{2b\theta_2}\Delta_{1,2}(x_1,x_2)\left[1-(x_2/c_2)^{b\theta_2}\right]\left[(x_1/c_1)^{a\theta_1}+(x_2/c_2)^{b\theta_2}\right]}. \tag{43}$$

The plots of the joint hazard rate function, given in (41), for $c_1 = c_2 = 1$ and for different values of the other parameters are given in Figure 3 below.

Figure 3: The Joint Hazard Rate Function for the *BExPF* Distribution



The plot of the joint hazard rate function indicates that the *BEXPF* distribution has a monotonically increasing hazard rate.

6.5. The Stress–Strength Reliability Coefficient

The stress-strength reliability coefficient is given as

$$R = P\left(X_2 < X_1\right) = \int_0^{\infty} a\, g_1\left(x_1\right) G_1^{a-1}\left(x_1\right) G_2^{2b}\left(x_1\right) \left[1 - \left\{1 - G_1^{a}\left(x_1\right)\right\}\left\{1 - G_2^{b}\left(x_1\right)\right\}\right]^{-2} dx_1 .$$

Now, for the power function baseline distribution, we have

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$$g_1(x_1) = \theta_1 x_1^{\theta_1 - 1}/c_1^{\theta_1} \;;\; G_1(x_1) = (x_1/c_1)^{\theta_1} \text{ and } G_2(x_1) = (x_1/c_2)^{\theta_2}.$$

The stress-strength reliability coefficient is, therefore,

$$R = \int_0^{c_1} a\left(\frac{\theta_1 x_1^{\theta_1 - 1}}{c_1^{\theta_1}}\right)\left(\frac{x_1}{c_1}\right)^{(a-1)\theta_1}\left(\frac{x_1}{c_2}\right)^{2b\theta_2}\left[1-\left\{1-(x_1/c_1)^{a\theta_1}\right\}\left\{1-(x_1/c_2)^{b\theta_2}\right\}\right]^{-2} dx_1 .$$

Using the transformation $x_1/c_1 = v_1 \Rightarrow dx_1 = c_1 dv_1$, the reliability coefficient is

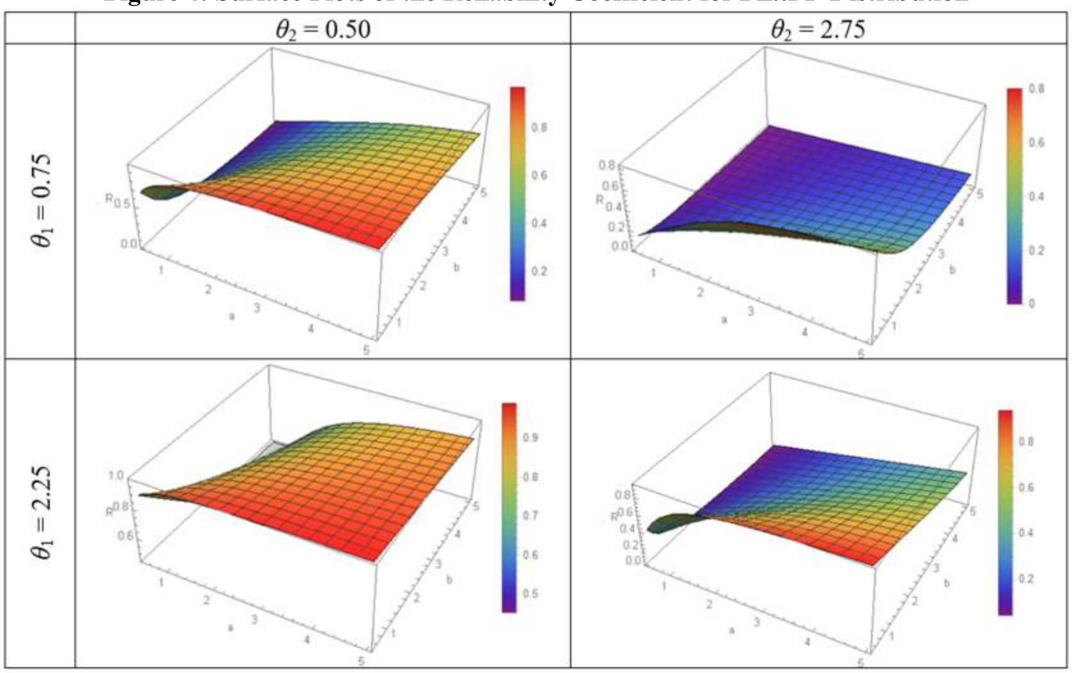
$$R = a\theta_1\left(\frac{c_1}{c_2}\right)^{2b\theta_2}\int_0^1 v_1^{a\theta_1+2b\theta_2-1}\left[1-\left(1-v_1^{a\theta_1}\right)\left\{1-(c_1/c_2)^{b\theta_2}\, v_1^{b\theta_2}\right\}\right]^{-2} dv_1 . \tag{44}$$

If $c_1 = c_2$, then a simplified version of the stress-strength reliability coefficient is

$$R = a\theta_1\int_0^1 v_1^{a\theta_1+2b\theta_2-1}\left[1-\left(1-v_1^{a\theta_1}\right)\left\{1-v_1^{b\theta_2}\right\}\right]^{-2} dv_1 .$$

The values of the reliability coefficient, (44), for various values of θ_1, θ_2, a, and b are given in Table *C*.2 in Appendix *C*. Also, the plots of the reliability coefficient for $c_1 = c_2$ and different combinations of θ_1, θ_2, a, and b are given in Figure 4 below.

Figure 4: Surface Plots of the Reliability Coefficient for *BExPF* Distribution



We can see, from the above plot, that if θ_2 is greater than 1 then the reliability coefficient is on the lower side for almost all values of a and b. Also, if θ_1 is greater than 1 and θ_2 is less than 1 then the reliability is on the higher side. We can also see, from Table *A*.2, that the reliability coefficient is equal to 0.5 if $a\theta_2 = b\theta_1$.

6.6. The Local Dependence Measure

The local dependence measure for the BExFD is given in (26) as

$$\gamma(x_1, x_2) = \frac{3ab\, g_1(x_1) g_2(x_2) G_1^a(x_1) G_2^b(x_2)}{\left[1-\left\{1-G_1^a(x_1)\right\}\left\{1-G_2^b(x_2)\right\}\right]^2} .$$

Using the marginal *pdf*'s and *cdf*'s of the power function distribution, the local dependence measure for the *BExPF* distribution is

$$\gamma\left(x_1,x_2\right)=\frac{3ab\left(\theta_1 x_1^{\theta_1-1}/c_1^{\theta_1}\right)\left(\theta_2 x_2^{\theta_2-1}/c_2^{\theta_2}\right)\left(x_1/c_1\right)^{a\theta_1}\left(x_2/c_2\right)^{b\theta_2}}{\left[1-\left\{1-\left(x_1/c_1\right)^{a\theta_1}\right\}\left\{1-\left(x_2/c_2\right)^{b\theta_2}\right\}\right]^2}=\frac{3ab\theta_1\theta_2 x_1^{(a+1)\theta_1-1}x_2^{(b+1)\theta_2-1}}{c_1^{(a+1)\theta_1}c_2^{(b+1)\theta_2}\Delta_{1,2}^2\left(x_1,x_2\right)}. \tag{45}$$

The local dependence measure can be computed for specific values of the parameters.

6.7. Random Sampling

Random sampling from *BExPF* distribution can be easily done by (27). The steps to draw a random observation from the *BExPF* distribution are given below:

- Generate a random observation x_1 by solving $\left(x_1/c_1\right)^{a\theta_1}=u_1$ or $x_1=c_1u_1^{1/a\theta_1}$
- Generate a random observation x_2 by solving

$\left(x_2/c_2\right)^{\theta_2}=u_2^*$ for x_2 where $u_2^*=\left[\dfrac{u_2^{1/2}\left(x_1/c_1\right)^{a\theta_1}}{1-u_2^{1/2}\left\{1-\left(x_1/c_1\right)^{a\theta_1}\right\}}\right]^{1/b}$ or obtain x_2 by using $x_2=c_2u_2^{*1/\theta_2}$.

A random sample can be generated for a specific sample size and for specific values of the parameters.

6.8. Maximum Likelihood Estimation of the Parameters

In this section, we have discussed the maximum likelihood estimation for the parameters of *BExPF* distribution. For this, suppose that (x_{11}, x_{21}), (x_{12}, x_{22}), ……, (x_{1n}, x_{2n}) be a random sample of n bivariate observations from *BExPF* distribution. The likelihood function is, then,

$$L\left(\theta_1,\theta_2,a,b;\mathbf{x}_1,\mathbf{x}_2\right)=\frac{2^n a^n b^n\theta_1^n\theta_2^n\prod_{i=1}^n x_{1i}^{2a\theta_1-1}\prod_{i=1}^n x_{2i}^{2b\theta_2-1}}{c_1^{2na\theta_1}c_2^{2nb\theta_2}\prod_{i=1}^n\Delta_{1,2}^3\left(x_{1i},x_{2i}\right)},$$

where $\Delta_{1,2}\left(x_{1i},x_{2i}\right)=\left[1-\left\{1-\left(x_{1i}/c_1\right)^{a\theta_1}\right\}\left\{1-\left(x_{2i}/c_2\right)^{a\theta_1}\right\}\right]$.

The log-likelihood function is

$$\begin{aligned}\ell=\ln L\left(\theta_1,\theta_2,a,b;\mathbf{x}_1,\mathbf{x}_2\right)&=n\ln 2+n\ln\theta_1+n\ln\theta_2+n\ln a+n\ln b-2na\theta_1\ln c_1-2nb\theta_2\ln c_2\\&+\left(2a\theta_1-1\right)\sum_{i=1}^n\ln x_{1i}+\left(2b\theta_2-1\right)\sum_{i=1}^n\ln x_{2i}-3\sum_{i=1}^n\ln\Delta_{1,2}\left(x_{1i},x_{2i}\right).\end{aligned} \tag{46}$$

It is easy to see that the maximum likelihood estimators of c_1 and c_2 are $\hat{c}_1=x_{1(n:n)}$ and $\hat{c}_2=x_{2(n:n)}$, where $x_{1(n:n)}$ is the largest observation among x_{1i} and $x_{2(n:n)}$ is the largest observation among x_{2i}. The maximum likelihood estimators of the other parameters are obtained by solving the likelihood equation. For this, we first see that the derivatives of the log-likelihood function, (46), with respect to the unknown parameters are

$$U_a=\frac{\partial\ell}{\partial a}=n\left(\frac{1}{a}-2\theta_1\ln c_1\right)+2\theta_1\sum_{i=1}^n\ln x_{1i}-3\sum_{i=1}^n\frac{\theta_1\left(x_{1i}/c_1\right)^{a\theta_1}\left\{1-\left(x_{2i}/c_2\right)^{b\theta_2}\right\}\ln\left(x_{1i}/c_1\right)}{\Delta_{1,2}\left(x_{1i},x_{2i}\right)}, \tag{47}$$

$$U_b=\frac{\partial\ell}{\partial b}=n\left(\frac{1}{b}-2\theta_2\ln c_2\right)+2\theta_2\sum_{i=1}^n\ln x_{2i}-3\sum_{i=1}^n\frac{\theta_2\left(x_{2i}/c_2\right)^{b\theta_2}\left\{1-\left(x_{1i}/c_1\right)^{a\theta_1}\right\}\ln\left(x_{2i}/c_2\right)}{\Delta_{1,2}\left(x_{1i},x_{2i}\right)}, \tag{48}$$

$$U_{\theta_1}=\frac{\partial\ell}{\partial\theta_1}=n\left(\frac{1}{\theta_1}-2a\ln c_1\right)+2a\sum_{i=1}^n\ln x_{1i}-3\sum_{i=1}^n\frac{a\left(x_{1i}/c_1\right)^{a\theta_1}\left\{1-\left(x_{2i}/c_2\right)^{b\theta_2}\right\}\ln\left(x_{1i}/c_1\right)}{\Delta_{1,2}\left(x_{1i},x_{2i}\right)}, \tag{49}$$

and

$$U_{\theta_2}=\frac{\partial\ell}{\partial\theta_2}=n\left(\frac{1}{\theta_2}-2b\ln c_2\right)+2b\sum_{i=1}^n\ln x_{2i}-3\sum_{i=1}^n\frac{b\left(x_{2i}/c_2\right)^{b\theta_2}\left\{1-\left(x_{1i}/c_1\right)^{a\theta_1}\right\}\ln\left(x_{2i}/c_2\right)}{\Delta_{1,2}\left(x_{1i},x_{2i}\right)}. \tag{50}$$

The maximum likelihood estimators of a, b, θ_1, and θ_2 are obtained by simultaneously solving the equations $U_a=0$, $U_b=0$, $U_{\theta_1}=0$, and $U_{\theta_2}=0$. The entries of the Fisher information matrix for *BExPF* distribution are given in Appendix *B*.

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7. Numerical Studies

In this section, we have given some numerical studies for the *BExPF* distribution. These contain simulation and real data applications.

7.1. Simulation Study

In this section, we have given extensive simulation study to see the consistency of the maximum likelihood estimates. The simulation algorithm is given below:

1. Draw random samples of different sizes from the *BExPF* distribution.
2. Compute the maximum likelihood estimates of the unknown parameters by using a sample of a specific size.
3. Repeat Steps 1 and 2 for a specific number of times, say N.
4. Compute the expected value and the standard error of each parameter by using

   $$\hat{\theta}_h = N^{-1}\sum_{j=1}^{N}\hat{\theta}_{hj} \text{ and } SE\left(\hat{\theta}_h\right) = \sqrt{\left(N-1\right)^{-1}\sum_{h=1}^{N}\left(\hat{\theta}_{hj}-\hat{\theta}_h\right)}$$

   where $\hat{\theta}_h$ is hth element of the parameter vector $\hat{\boldsymbol{\theta}} = \left(\hat{c}_1, \hat{c}_2, \hat{a}, \hat{b}, \hat{\theta}_1, \hat{\theta}_2\right)$ and $\hat{\theta}_{hj}$ is the estimated value of hth parameter at jth simulation.

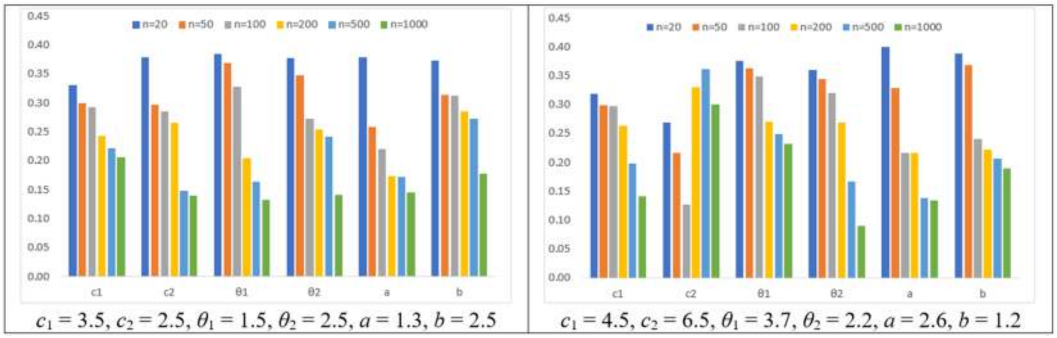
It is to be noted that the maximum likelihood estimates of c_1 and c_2 are the maximum values in the sample at each simulation. The results of the simulation for N = 10000 are given in Table 1 below, where entries in the parenthesis are the standard errors of the estimates.

Table 1: Simulation Results for BExPF Distribution

n	$c_1 = 3.5$	$c_2 = 2.5$	$\theta_1 = 1.5$	$\theta_2 = 3.5$	$a = 1.3$	$b = 2.5$
20	3.50054	2.49937	1.49803	3.49763	1.29763	2.50402
	(0.33022)	(0.37845)	(0.38443)	(0.37714)	(0.37787)	(0.37282)
50	3.49512	2.49649	1.50127	3.4894	1.30121	2.50192
	(0.29931)	(0.29677)	(0.36804)	(0.34797)	(0.25812)	(0.31275)
100	3.5032	2.49625	1.49467	3.5063	1.30137	2.50312
	(0.29152)	(0.28567)	(0.32723)	(0.27247)	(0.22022)	(0.31175)
200	3.49145	2.49924	1.50146	3.50138	1.29233	2.49563
	(0.24297)	(0.26495)	(0.20474)	(0.25371)	(0.17398)	(0.28468)
500	3.50016	2.50124	1.50435	3.50414	1.29968	2.50604
	(0.22084)	(0.14750)	(0.16365)	(0.24172)	(0.17203)	(0.27192)
1000	3.49722	2.49568	1.50467	3.49654	1.29763	2.50281
	(0.20608)	(0.13945)	(0.13161)	(0.14131)	(0.14471)	(0.17811)
n	$c_1 = 4.5$	$c_2 = 6.5$	$\theta_1 = 3.7$	$\theta_2 = 2.2$	$a = 2.6$	$b = 1.2$
20	4.49695	6.50389	3.70073	2.20118	2.60462	1.19935
	(0.31909)	(0.26914)	(0.37643)	(0.35961)	(0.40048)	(0.38921)
50	4.4972	6.49886	3.70654	2.19818	2.59828	1.19403
	(0.29927)	(0.21701)	(0.36357)	(0.34413)	(0.32924)	(0.36928)
100	4.49754	6.50314	3.70424	2.20019	2.59678	1.20341
	(0.29813)	(0.12755)	(0.34911)	(0.31994)	(0.21699)	(0.24023)
200	4.49269	6.5012	3.70479	2.20018	2.59836	1.20657
	(0.26385)	(0.32986)	(0.27022)	(0.26857)	(0.21684)	(0.22271)
500	4.49884	6.49903	3.70408	2.20591	2.59875	1.19895
	(0.19845)	(0.36208)	(0.24866)	(0.16624)	(0.13812)	(0.20683)
1000	4.49287	6.49559	3.7036	2.20334	2.59195	1.19828
	(0.14104)	(0.30078)	(0.23172)	(0.08951)	(0.13448)	(0.18965)

It is evident, from the above table, that the maximum likelihood estimates converge to the true values of the parameters. We can also see, from the above table, that the standard error of each of the parameters decreases with an increase in the sample size. We have also given a comparison of the standard errors by using multiple bar charts. These charts are given in Figure 5 below.

Figure 5: Standard Errors for Different Estimates



The above figure indicates that the standard error for each estimate decreases with an increase in the sample size. This indicates the consistency of the maximum likelihood estimates.

7.2. The Real Data Applications

In this section, we have given two real data applications to see the suitability of the proposed *BExPF* distribution. The first data is about the height and forced expiratory volume of 654 children and is obtained from Rosner (1999). The second data set is about the gross national income of 201 countries of the world for the years 2020 and 2021 and is obtained from the United Nations Development Program (*UNDP*) website *hdr.undp.org/en/data*. The description of the data is given in Table 2, below.

Table 2: Description of the Two Data Sets

Data	*n*	Variables
1	654	X_1: Forced Expiratory Volume X_2: Height, in meters, of the Children
2	201	X_1: GNI; in US$10000; for 2020 X_2: GNI; in US$10000; for 2021

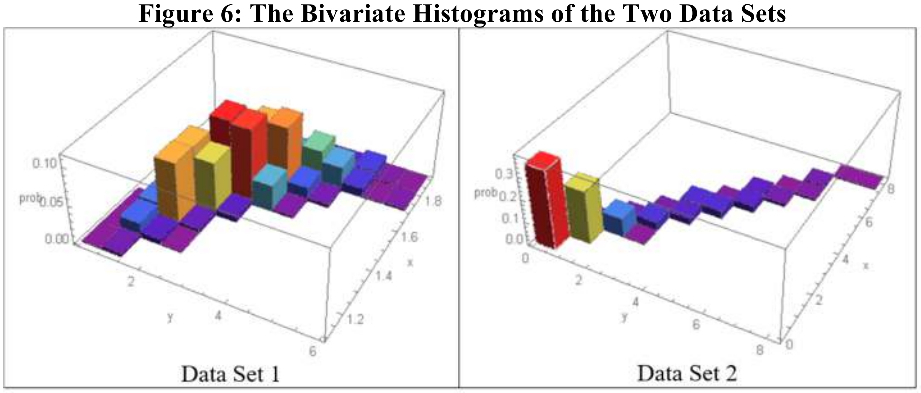
Summary statistics for the two data sets are given in Table 3, below.

Table 3: Summary Measures for the Two Data Sets

Data		Min	Q_1	Median	Mean	Q_3	Max	Var.	Skew.	Corr.
1	X_1	0.791	1.981	2.547	2.637	3.118	5.793	0.752	0.660	0.868
	X_2	1.170	1.450	1.560	1.553	1.660	1.880	0.021	–0.209	
2	X_1	0.072	0.468	1.240	1.783	2.573	7.650	3.045	1.297	0.998
	X_2	0.071	0.467	1.257	1.867	2.748	7.895	3.386	1.270	

From the summary measures we can see that the variables in both data sets are highly correlated. Also, in data set 1, one variable is positively skewed and the other is negatively skewed whereas in data set 2 both variables are positively skewed. The bivariate histograms of the two data sets are given in Figure 6, below.

Figure 6: The Bivariate Histograms of the Two Data Sets



We have fitted four distributions alongside the proposed *BExPF* distributions. The additional fitted distributions are

- Gumbel Bivariate Power Function Distribution (*GBPF*); Gumbel (1960)

$$f_{1,2(G)}\left(x_1,x_2\right)=\left(\theta_1 x_1^{\theta_1-1}/c_1^{\theta_1}\right)\left(\theta_2 x_2^{\theta_2-1}/c_2^{\beta_2}\right)\left[1+\alpha\left\{1-2\left(x_1/c_1\right)^{\theta_1}\right\}\left\{1-2\left(x_2/c_2\right)^{\theta_2}\right\}\right].$$

- Huang and Kotz Bivariate Power Function (*HKBPF*); Hang and Kotz (1999)

$$f_{1,2(HK)}\left(x_1,x_2\right)=\left(\theta_1 x_1^{\theta_1-1}/c_1^{\theta_1}\right)\left(\theta_2 x_2^{\theta_2-1}/c_2^{\beta_2}\right)\left[1+\alpha\left\{1-2\left(x_1/c_1\right)^{(\alpha-1)\theta_1}\right\}\left\{1-2\left(x_2/c_2\right)^{(\alpha-1)\theta_2}\right\}\right].$$

- Ali-Mikhil-Haq Bivariate Power Function (*AMKBPF*); Ali et al. (1978)

$$f_{1,2(AMK)}=\left(\frac{\theta_1 x_1^{\theta_1-1}}{c_1^{\theta_1}}\right)\left(\frac{\theta_2 x_2^{\theta_2-1}}{c_2^{\theta_2}}\right)\times\frac{(\alpha-1)\left[\alpha\varphi_1\left(x_1\right)\varphi_2\left(x_2\right)-1\right]+2\alpha\left(x_1/c_1\right)^{\theta_1}\left(x_2/c_2\right)^{\theta_2}}{\left[1-\alpha\varphi_1\left(x_1\right)\varphi_2\left(x_2\right)\right]^3},$$

where $\varphi_1\left(x_1\right)=1-\left(x_1/c_1\right)^{\theta_1}$ and $\varphi_2\left(x_2\right)=1-\left(x_2/c_2\right)^{\theta_2}$.

- Cambanis Bivariate Power Function (*CBPF*); Cambanis (1997)

$$f_{1,2(C)}\left(x_1,x_2\right)=\left(\theta_1 x_1^{\theta_1-1}/c_1^{\theta_1}\right)\left(\theta_2 x_2^{\theta_2-1}/c_2^{\beta_2}\right)\left[1+\alpha\left\{1-2\left(x_1/c_1\right)^{\theta_1}\right\}+\gamma\left\{1-2\left(x_2/c_2\right)^{\theta_2}\right\}\right.$$
$$\left.+\phi\left\{1-2\left(x_1/c_1\right)^{\theta_1}\right\}\left\{1-2\left(x_2/c_2\right)^{\theta_2}\right\}\right].$$

The maximum likelihood estimates of the parameters, with standard errors, are given in Table 4, below. It is to be noted that the maximum likelihood estimates of c_1 and c_2 in both data sets are the largest observations for each variable.

Table 4: Maximum Likelihood Estimates of the Parameters for Two Data Sets

		Data 1		Data 2	
Distribution	**Parameter**	**Estimate**	**S.E.**	**Estimate**	**S.E.**
Bivariate Exponentiated Power Function Distribution (*BEPF*)	θ_1	2.7408	0.8533	2.0776	0.2718
	θ_2	3.7333	1.3544	2.9154	0.5216
	a	0.5976	0.1895	0.3622	0.0713
	b	1.8296	0.6429	0.2566	0.0353
Gumbel Bivariate Power Function Distribution (*GBPF*)	θ_1	1.5543	0.1986	0.5632	0.0419
	θ_2	6.9018	0.4445	0.5627	0.0420
	α	0.9985	0.3691	0.9873	0.3952
Ali-Mikhil-Haq Bivariate Power Function Distribution (AMKBPF)	θ_1	1.6348	0.1519	0.7523	0.0391
	θ_2	6.9056	0.2204	0.7478	0.0392
	α	0.9782	0.2874	0.9943	0.2718
Huang and Kotz Bivariate Power Function Distribution (*HKBPF*)	θ_1	1.1282	0.0352	0.4830	0.0318
	θ_2	5.4658	1.6359	0.4851	0.0328
	α	0.9748	0.0663	0.9908	0.1390
	γ	10.0191	0.4521	24.8227	3.0126
Cambanis Bivariate Power Function Distribution (*CBPF*)	θ_1	1.7652	0.0527	0.6295	0.0318
	θ_2	2.5925	0.8174	0.6444	0.0339
	α	0.9925	0.1998	0.5352	0.2263
	γ	–0.9973	0.0421	0.5352	0.2385
	ϕ	–0.9897	0.1183	0.9998	0.3023

We have also computed two goodness of fit measures; namely, Akaike Information Criteria (*AIC*) and Bayesian Information Criteria (*BIC*) to see the suitability of various distributions for modeling the two data sets. The results of these measures are given in Table 5, below. From this table, we can see that the proposed *BExPF* distribution fits both data sets reasonably well as compared with the other competing distributions as it has the largest value of the log-likelihood function and the smallest values of *AIC* and *BIC*. The Huang-Kotz bivariate power function distribution is the second best fit for the two data sets. The Gumbel bivariate power function distribution is the worst fitting distribution for the two data sets.

Table 5: Goodness of Fit Measures for Various Distributions

Distribution	Data 1			Data 2		
	Log-Like.	AIC	BIC	Log-Like.	AIC	BIC
BEPFD	–693.2603	1394.5206	1412.4530	–510.5166	1029.0332	1046.9656
GBPFD	–935.2229	1876.4458	1889.8951	–662.0806	1330.1612	1343.6105
AMHBPFD	–793.2604	1592.5208	1605.9701	–610.5167	1227.0334	1240.4827
HKBPFD	–708.1721	1424.3442	1442.2766	–551.5903	1111.1806	1129.1130
CBPFD	–846.9599	1703.9198	1726.3353	–636.8887	1283.7774	1306.1929

8. Conclusions and Further Recommendations

The joint modeling of two variables is often required in many areas of sciences, engineering, biological sciences, etc. The development of bivariate distributions is a tedious task. In this paper, we have proposed a bivariate exponentiated family of distributions (*BExFD*) such that the marginals are the exponentiated families of distributions. Some useful properties of the proposed *BExFD* are studies. The maximum likelihood estimation of the parameters for the proposed family is also done. The proposed *BExFD* has been used to obtain a bivariate exponentiated power function (*BExPF*) distribution. Some useful properties of the derived *BExPF* distribution have been studied. It is found that the variables in the derived *BExPF* distribution are always positively correlated. It is also found that the correlation coefficient increases with an increase in the parameters. The *BExPF* distribution is used to model some real data and it is found that the derived *BExPF* distribution is a better fit as compared with the other competing distributions. It is hoped that the proposed *BExFD* will be useful in deriving new bivariate probability distributions for various choices of baseline distributions. The proposed *BExFD* can also be used for two different baseline distributions to obtain the new bivariate distributions for modeling more complex data.

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Appendix A: Entries of the Fisher Information Matrix for *BExFD*

The entries of the Fisher information matrix for the *BExFD* are given below. In these entries, we have

$$\varphi_1(x_{1i};\xi_1)=1-G_1^a(x_{1i};\xi_1),\ \varphi_2(x_{2i};\xi_2)=1-G_2^b(x_{2i};\xi_2),\ \Delta(x_{1i},x_{2i};\xi_1,\xi_2)=1-\varphi_1(x_{1i};\xi_1)\varphi_2(x_{2i};\xi_2),$$

$$g_1'(x_{1i};\xi_1)=\frac{\partial}{\partial\xi_{1k}}g_1(x_{1i};\xi_1),\ G_1'(x_{1i};\xi_1)=\frac{\partial}{\partial\xi_{1k}}G_1(x_{1i};\xi_1),\ g_2'(x_{2i};\xi_2)=\frac{\partial}{\partial\xi_{2k}}g_2(x_{2i};\xi_2),$$

$$G_2'(x_{2i};\xi_2)=\frac{\partial}{\partial\xi_{2k}}G_2(x_{2i};\xi_2),\ g_1''(x_{1i};\xi_1)=\frac{\partial^2}{\partial\xi_{1k}^2}g_1(x_{1i};\xi_1),\ G_1''(x_{1i};\xi_1)=\frac{\partial^2}{\partial\xi_{1k}^2}G_1(x_{1i};\xi_1),$$

$$g_2''(x_{2i};\xi_2)=\frac{\partial^2}{\partial\xi_{2k}^2}g_2(x_{2i};\xi_2) \text{ and } G_2''(x_{2i};\xi_2)=\frac{\partial^2}{\partial\xi_{2k}^2}G_2(x_{2i};\xi_2).$$

The entries are given below.

$$U_{aa}=\frac{\partial^2\ell}{\partial a^2}=-\frac{n}{a^2}-3\sum_{i=1}^{n}\frac{G_1^a(x_{1i};\xi_1)G_2^b(x_{2i};\xi_2)\varphi_2(x_{2i};\xi_2)\ln^2 G_1(x_{1i};\xi_1)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}$$

$$U_{bb}=\frac{\partial^2\ell}{\partial b^2}=-\frac{n}{b^2}-3\sum_{i=1}^{n}\frac{G_1^a(x_{1i};\xi_1)G_2^b(x_{2i};\xi_2)\varphi_1(x_{1i};\xi_1)\ln^2 G_2(x_{2i};\xi_2)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}$$

$$U_{\xi_{1k}\xi_{1k}}=\frac{\partial^2\ell}{\partial\xi_{1k}^2}=\sum_{i=1}^{n}\left[\frac{g_1''(x_{1i};\xi_1)}{g_1(x_{1i};\xi_1)}-\left\{\frac{g_1'(x_{1i};\xi_1)}{g_1(x_{1i};\xi_1)}\right\}^2\right]+(2a-1)\sum_{i=1}^{n}\left[\frac{G_1''(x_{1i};\xi_1)}{G_1(x_{1i};\xi_1)}-\left\{\frac{G_1'(x_{1i};\xi_1)}{G_1(x_{1i};\xi_1)}\right\}^2\right]$$
$$+3a^2\sum_{i=1}^{n}\frac{G_1^{2(a-1)}(x_{1i};\xi_1)\varphi_2^2(x_{2i};\xi_2)G_1'^2(x_{1i};\xi_1)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}+3\sum_{i=1}^{n}\frac{aG_1^{a-2}(x_{1i};\xi_1)\varphi_2(x_{2i};\xi_2)}{\Delta(x_{1i},x_{2i};\xi_1,\xi_2)}$$
$$\times\left[(a-1)G_1'^2(x_{1i};\xi_1)+G_1(x_{1i};\xi_1)G_1''(x_{1i};\xi_1)\right]$$

$$U_{\xi_{2k}\xi_{2k}}=\frac{\partial^2\ell}{\partial\xi_{2k}^2}=\sum_{i=1}^{n}\left[\frac{g_2''(x_{2i};\xi_2)}{g_2(x_{2i};\xi_2)}-\left\{\frac{g_2'(x_{2i};\xi_2)}{g_2(x_{2i};\xi_2)}\right\}^2\right]+(2a-1)\sum_{i=1}^{n}\left[\frac{G_2''(x_{2i};\xi_2)}{G_2(x_{2i};\xi_2)}-\left\{\frac{G_2'(x_{2i};\xi_2)}{G_2(x_{2i};\xi_2)}\right\}^2\right]$$
$$+3b^2\sum_{i=1}^{n}\frac{G_2^{2(b-1)}(x_{1i};\xi_1)\varphi_1^2(x_{1i};\xi_1)G_2'^2(x_{2i};\xi_2)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}+3\sum_{i=1}^{n}\frac{aG_2^{b-2}(x_{2i};\xi_2)\varphi_1(x_{1i};\xi_1)}{\Delta(x_{1i},x_{2i};\xi_1,\xi_2)}$$
$$\times\left[(b-1)G_2'^2(x_{2i};\xi_2)+G_2(x_{2i};\xi_2)G_2''(x_{2i};\xi_2)\right]$$

$$U_{ab}=\frac{\partial^2\ell}{\partial a\partial b}=3\sum_{i=1}^{n}\frac{G_1^a(x_{1i};\xi_1)G_2^b(x_{2i};\xi_2)\ln G_1(x_{1i};\xi_1)\ln G_2(x_{2i};\xi_2)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}$$

$$U_{a\xi_{1k}}=\frac{\partial^2\ell}{\partial a\partial\xi_{1k}}=2\sum_{i=1}^{n}\frac{G_1'(x_{1i};\xi_1)}{G_1(x_{1i};\xi_1)}+3a\sum_{i=1}^{n}\frac{G_1^{2a-1}(x_{1i};\xi_1)\varphi_2(x_{2i};\xi_2)\ln G_1(x_{1i};\xi_1)G_1'(x_{1i};\xi_1)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}$$
$$+\sum_{i=1}^{n}\frac{G_1(x_{1i};\xi_1)\varphi_2(x_{2i};\xi_2)\left\{a\ln G_1(x_{1i};\xi_1)+1\right\}G_1'(x_{1i};\xi_1)}{\Delta(x_{1i},x_{2i};\xi_1,\xi_2)}$$

$$U_{a\xi_{2k}}=\frac{\partial^2\ell}{\partial a\partial\xi_{2k}}=3b\sum_{i=1}^{n}\frac{G_1^a(x_{1i};\xi_1)G_2^{b-1}(x_{2i};\xi_2)\varphi_1(x_{1i};\xi_1)\varphi_2(x_{2i};\xi_2)\ln G_1(x_{1i};\xi_1)G_2'(x_{2i};\xi_2)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}$$
$$+b\sum_{i=1}^{n}\frac{G_1^a(x_{1i};\xi_1)G_2^{b-1}(x_{2i};\xi_2)\ln G_1(x_{1i};\xi_1)G_2'(x_{2i};\xi_2)}{\Delta(x_{1i},x_{2i};\xi_1,\xi_2)}$$

$$U_{b\xi_{1k}}=\frac{\partial^2\ell}{\partial b\partial\xi_{1k}}=3a\sum_{i=1}^{n}\frac{G_1^{a-1}(x_{1i};\xi_1)G_2^b(x_{2i};\xi_2)\varphi_1(x_{1i};\xi_1)\varphi_2(x_{2i};\xi_2)\ln G_2(x_{2i};\xi_2)G_1'(x_{1i};\xi_1)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}$$
$$+a\sum_{i=1}^{n}\frac{G_1^{a-1}(x_{1i};\xi_1)G_2^b(x_{2i};\xi_2)\ln G_2(x_{2i};\xi_2)G_1'(x_{1i};\xi_1)}{\Delta(x_{1i},x_{2i};\xi_1,\xi_2)}$$

$$U_{b\xi_{2k}}=\frac{\partial^2\ell}{\partial b\partial\xi_{2k}}=2\sum_{i=1}^{n}\frac{G_2'(x_{2i};\xi_2)}{G_2(x_{2i};\xi_2)}+3b\sum_{i=1}^{n}\frac{G_2^{2b-1}(x_{1i};\xi_1)\varphi_1(x_{1i};\xi_1)\ln G_2(x_{2i};\xi_2)G_2'(x_{2i};\xi_2)}{\Delta^2(x_{1i},x_{2i};\xi_1,\xi_2)}$$
$$+\sum_{i=1}^{n}\frac{G_2(x_{2i};\xi_2)\varphi_1(x_{1i};\xi_1)\left\{a\ln G_2(x_{2i};\xi_2)+1\right\}G_2'(x_{2i};\xi_2)}{\Delta(x_{1i},x_{2i};\xi_1,\xi_2)}$$

and

$$U_{\xi_{1k}\xi_{2k}}=\frac{\partial^2\ell}{\partial\xi_{1k}\partial\xi_{2k}}=3ab\sum_{i=1}^{n}\frac{G_1^{a-1}(x_{1i};\xi_1)G_2^{b-1}(x_{2i};\xi_2)\varphi_1(x_{1i};\xi_1)\varphi_2(x_{2i};\xi_2)G_1'(x_{1i};\xi_1)G_2'(x_{2i};\xi_2)}{\Delta(x_{1i},x_{2i};\xi_1,\xi_2)}$$
$$+3ab\sum_{i=1}^{n}\frac{G_1^{a-1}(x_{1i};\xi_1)G_2^{b-1}(x_{2i};\xi_2)G_1'(x_{1i};\xi_1)G_2'(x_{2i};\xi_2)}{\Delta(x_{1i},x_{2i};\xi_1,\xi_2)}.$$

Appendix B: Entries of the Fisher Information Matrix for *BExPF* Distribution

The entries of the Fisher information matrix for the *BExPF* distribution are given below. In these entries we have

$$\varphi_1(x_1)=1-(x_{1i}/c_1)^{a\theta_1},\ \varphi_2(x_2)=\left\{1-(x_{2i}/c_2)^{b\theta_2}\right\} \text{ and } \Delta_{1,2}(x_{1i},x_{2i})=\left[1-\left\{1-(x_{1i}/c_1)^{a\theta_1}\right\}\left\{1-(x_{2i}/c_2)^{a\theta_1}\right\}\right]$$

The entries are given below

$$U_{aa}=\frac{\partial^2\ell}{\partial a^2}=-\frac{n}{a^2}+3\sum_{i=1}^{n}\frac{\theta_1^2(x_{1i}/c_1)^{2a\theta_1}\varphi_2^2(x_2)\ln^2(x_{1i}/c_1)}{\Delta_{1,2}^2(x_{1i},x_{2i})}-3\sum_{i=1}^{n}\frac{\theta_1^2(x_{1i}/c_1)^{a\theta_1}\varphi_2(x_2)\ln^2(x_{1i}/c_1)}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{bb}=\frac{\partial^2\ell}{\partial b^2}=-\frac{n}{b^2}+3\sum_{i=1}^{n}\frac{\theta_2^2(x_{2i}/c_2)^{2a\theta_1}\varphi_1^2(x_1)\ln^2(x_{2i}/c_2)}{\Delta_{1,2}^2(x_{1i},x_{2i})}-3\sum_{i=1}^{n}\frac{\theta_2^2(x_{2i}/c_2)^{b\theta_2}\varphi_1(x_1)\ln^2(x_{2i}/c_2)}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{\theta_1\theta_1}=\frac{\partial^2\ell}{\partial \theta_1^2}=-\frac{n}{\theta_1^2}+3\sum_{i=1}^{n}\frac{a^2(x_{1i}/c_1)^{2a\theta_1}\varphi_2^2(x_2)\ln^2(x_{1i}/c_1)}{\Delta_{1,2}^2(x_{1i},x_{2i})}-3\sum_{i=1}^{n}\frac{a^2(x_{1i}/c_1)^{a\theta_1}\varphi_2(x_2)\ln^2(x_{1i}/c_1)}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{\theta_2\theta_2}=\frac{\partial^2\ell}{\partial \theta_2^2}=-\frac{n}{\theta_2^2}+3\sum_{i=1}^{n}\frac{b^2(x_{2i}/c_2)^{2a\theta_1}\varphi_1^2(x_1)\ln^2(x_{2i}/c_2)}{\Delta_{1,2}^2(x_{1i},x_{2i})}-3\sum_{i=1}^{n}\frac{b^2(x_{2i}/c_2)^{b\theta_2}\varphi_1(x_1)\ln^2(x_{2i}/c_2)}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{ab}=\frac{\partial^2\ell}{\partial a\partial b}=3\sum_{i=1}^{n}\frac{\theta_1\theta_2(x_{1i}/c_1)^{a\theta_1}(x_{2i}/c_2)^{b\theta_2}\varphi_1(x_1)\varphi_2(x_2)\ln(x_{1i}/c_1)\ln(x_{2i}/c_2)}{\Delta_{1,2}^2(x_{1i},x_{2i})}$$
$$+3\sum_{i=1}^{n}\frac{\theta_1\theta_2(x_{1i}/c_1)^{a\theta_1}(x_{2i}/c_2)^{b\theta_2}\ln(x_{1i}/c_1)\ln(x_{2i}/c_2)}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{a\theta_1}=\frac{\partial^2\ell}{\partial a\partial \theta_1}=-2n\ln c_1+2\sum_{i=1}^{n}\ln x_{1i}+3\sum_{i=1}^{n}\frac{a\theta_1(x_{1i}/c_1)^{2a\theta_1}\varphi_2^2(x_2)\ln^2(x_{1i}/c_1)}{\Delta_{1,2}^2(x_{1i},x_{2i})}$$
$$-3\sum_{i=1}^{n}\frac{(x_{1i}/c_1)^{a\theta_1}\varphi_2(x_2)\ln(x_{1i}/c_1)\left\{a\theta_1\ln(x_{1i}/c_1)+1\right\}}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{a\theta_2}=\frac{\partial^2\ell}{\partial a\partial \theta_2}=3\sum_{i=1}^{n}\frac{b\theta_1(x_{1i}/c_1)^{a\theta_1}(x_{2i}/c_2)^{b\theta_2}\varphi_1(x_1)\varphi_2(x_2)\ln(x_{1i}/c_1)\ln(x_{2i}/c_2)}{\Delta_{1,2}^2(x_{1i},x_{2i})}$$
$$+3\sum_{i=1}^{n}\frac{b\theta_1(x_{1i}/c_1)^{a\theta_1}(x_{2i}/c_2)^{b\theta_2}\ln(x_{1i}/c_1)\ln(x_{2i}/c_2)}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{b\theta_1}=\frac{\partial^2\ell}{\partial b\partial \theta_1}=3\sum_{i=1}^{n}\frac{a\theta_2(x_{1i}/c_1)^{a\theta_1}(x_{2i}/c_2)^{b\theta_2}\varphi_1(x_1)\varphi_2(x_2)\ln(x_{1i}/c_1)\ln(x_{2i}/c_2)}{\Delta_{1,2}^2(x_{1i},x_{2i})}$$
$$+3\sum_{i=1}^{n}\frac{a\theta_2(x_{1i}/c_1)^{a\theta_1}(x_{2i}/c_2)^{b\theta_2}\ln(x_{1i}/c_1)\ln(x_{2i}/c_2)}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{b\theta_1}=\frac{\partial^2\ell}{\partial b\partial \theta_2}=-2n\ln c_2+2\sum_{i=1}^{n}\ln x_{2i}+3\sum_{i=1}^{n}\frac{b\theta_2(x_{2i}/c_2)^{2b\theta_2}\varphi_1^2(x_1)\ln^2(x_{2i}/c_2)}{\Delta_{1,2}^2(x_{1i},x_{2i})}$$
$$-3\sum_{i=1}^{n}\frac{(x_{2i}/c_2)^{b\theta_2}\varphi_1(x_1)\ln(x_{2i}/c_2)\left\{b\theta_2\ln(x_{2i}/c_2)+1\right\}}{\Delta_{1,2}(x_{1i},x_{2i})}$$

$$U_{\theta_1\theta_2}=\frac{\partial^2\ell}{\partial \theta_1\partial \theta_2}=3\sum_{i=1}^{n}\frac{ab(x_{1i}/c_1)^{a\theta_1}(x_{2i}/c_2)^{b\theta_2}\varphi_1(x_1)\varphi_2(x_2)\ln(x_{1i}/c_1)\ln(x_{2i}/c_2)}{\Delta_{1,2}^2(x_{1i},x_{2i})}$$
$$+3\sum_{i=1}^{n}\frac{ab(x_{1i}/c_1)^{a\theta_1}(x_{2i}/c_2)^{b\theta_2}\ln(x_{1i}/c_1)\ln(x_{2i}/c_2)}{\Delta_{1,2}(x_{1i},x_{2i})}.$$

Appendix C: Correlation and Stress-Strength Reliability Coefficient for *BExPF* Distribution

Table C.1: Correlation Coefficient for *BExPF* Distribution

		a									
	b	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**	**5.0**
$(\theta_1 = 0.75, \theta_2 = 0.25)$	**0.5**	0.2386	0.3030	0.3375	0.3583	0.3716	0.3805	0.3865	0.3907	0.3935	0.3955
	1.0	0.2603	0.3367	0.3805	0.4086	0.4277	0.4412	0.4510	0.4583	0.4638	0.4680
	1.5	0.2645	0.3455	0.3935	0.4253	0.4476	0.4638	0.4759	0.4851	0.4924	0.4980
	2.0	0.2645	0.3476	0.3978	0.4317	0.4559	0.4737	0.4873	0.4979	0.5063	0.5130
	2.5	0.2633	0.3473	0.3989	0.4341	0.4595	0.4785	0.4931	0.5046	0.5138	0.5212
	3.0	0.2617	0.3463	0.3987	0.4348	0.4610	0.4808	0.4962	0.5083	0.5181	0.5261
	3.5	0.2602	0.3450	0.3979	0.4346	0.4615	0.4819	0.4978	0.5104	0.5206	0.5290
	4.0	0.2587	0.3436	0.3970	0.4341	0.4614	0.4823	0.4986	0.5116	0.5222	0.5309
	4.5	0.2574	0.3423	0.3959	0.4334	0.4611	0.4823	0.4989	0.5122	0.5231	0.5321
	5.0	0.2562	0.3411	0.3949	0.4326	0.4606	0.4821	0.4990	0.5125	0.5236	0.5328
$(\theta_1 = 0.75, \theta_2 = 1.50)$	**0.5**	0.3805	0.3978	0.3987	0.3970	0.3949	0.3930	0.3913	0.3899	0.3886	0.3875
	1.0	0.4412	0.4737	0.4808	0.4823	0.4821	0.4814	0.4805	0.4797	0.4789	0.4782
	1.5	0.4638	0.5063	0.5181	0.5222	0.5236	0.5240	0.5240	0.5238	0.5235	0.5231
	2.0	0.4737	0.5229	0.5382	0.5443	0.5470	0.5484	0.5490	0.5493	0.5494	0.5494
	2.5	0.4785	0.5324	0.5502	0.5578	0.5616	0.5636	0.5648	0.5655	0.5659	0.5662
	3.0	0.4808	0.5382	0.5578	0.5667	0.5713	0.5739	0.5755	0.5765	0.5772	0.5776
	3.5	0.4819	0.5418	0.5630	0.5728	0.5780	0.5811	0.5831	0.5844	0.5853	0.5859
	4.0	0.4823	0.5443	0.5667	0.5772	0.5830	0.5864	0.5887	0.5902	0.5913	0.5920
	4.5	0.4823	0.5459	0.5693	0.5805	0.5867	0.5905	0.5929	0.5946	0.5959	0.5968
	5.0	0.4821	0.5470	0.5713	0.5830	0.5895	0.5936	0.5963	0.5981	0.5995	0.6005
$(\theta_1 = 1.75, \theta_2 = 0.75)$	**0.5**	0.3865	0.4510	0.4759	0.4873	0.4931	0.4962	0.4978	0.4986	0.4989	0.4990
	1.0	0.3988	0.4773	0.5118	0.5298	0.5403	0.5468	0.5510	0.5539	0.5560	0.5574
	1.5	0.3979	0.4819	0.5206	0.5418	0.5547	0.5630	0.5687	0.5728	0.5758	0.5780
	2.0	0.3956	0.4822	0.5233	0.5463	0.5606	0.5700	0.5766	0.5814	0.5850	0.5877
	2.5	0.3933	0.4815	0.5240	0.5482	0.5634	0.5735	0.5807	0.5860	0.5899	0.5930
	3.0	0.3913	0.4805	0.5240	0.5490	0.5648	0.5755	0.5831	0.5887	0.5929	0.5963
	3.5	0.3896	0.4795	0.5237	0.5493	0.5656	0.5766	0.5845	0.5904	0.5949	0.5984
	4.0	0.3882	0.4786	0.5234	0.5494	0.5660	0.5774	0.5855	0.5915	0.5962	0.5998
	4.5	0.3871	0.4778	0.5230	0.5493	0.5662	0.5778	0.5861	0.5923	0.5971	0.6009
	5.0	0.3861	0.4771	0.5226	0.5492	0.5663	0.5781	0.5866	0.5929	0.5978	0.6017
$(\theta_1 = 2.25, \theta_2 = 2.75)$	**0.5**	0.5026	0.5212	0.5239	0.5240	0.5234	0.5228	0.5222	0.5217	0.5212	0.5208
	1.0	0.5326	0.5644	0.5727	0.5759	0.5773	0.5780	0.5783	0.5785	0.5786	0.5786
	1.5	0.5396	0.5775	0.5888	0.5936	0.5960	0.5975	0.5983	0.5989	0.5994	0.5997
	2.0	0.5416	0.5831	0.5961	0.6018	0.6049	0.6068	0.6081	0.6089	0.6095	0.6100
	2.5	0.5421	0.5859	0.6000	0.6064	0.6100	0.6122	0.6136	0.6146	0.6154	0.6159
	3.0	0.5421	0.5874	0.6024	0.6093	0.6131	0.6155	0.6171	0.6183	0.6192	0.6198
	3.5	0.5419	0.5884	0.6039	0.6112	0.6153	0.6178	0.6196	0.6208	0.6217	0.6225
	4.0	0.5416	0.5890	0.6050	0.6125	0.6168	0.6195	0.6213	0.6226	0.6236	0.6244
	4.5	0.5413	0.5894	0.6058	0.6135	0.6179	0.6207	0.6226	0.6240	0.6250	0.6259
	5.0	0.5411	0.5897	0.6063	0.6143	0.6188	0.6217	0.6236	0.6251	0.6261	0.6270

Table C.2: The Reliability Coefficient for *BExPF* Distribution

		a									
	b	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**	**5.0**
$(\theta_1 = 0.75, \theta_2 = 0.25)$	**0.5**	0.8305	0.6586	0.5000	0.3834	0.3073	0.2557	0.2187	0.1910	0.1695	0.1523
	1.0	0.9159	0.8305	0.7443	0.6586	0.5757	0.5000	0.4356	0.3834	0.3414	0.3073
	1.5	0.9441	0.8876	0.8305	0.7731	0.7156	0.6586	0.6028	0.5495	0.5000	0.4557
	2.0	0.9581	0.9159	0.8734	0.8305	0.7875	0.7443	0.7013	0.6586	0.6166	0.5757
	2.5	0.9665	0.9329	0.8989	0.8648	0.8305	0.7961	0.7616	0.7271	0.6927	0.6586
	3.0	0.9721	0.9441	0.9159	0.8876	0.8591	0.8305	0.8018	0.7731	0.7443	0.7156
	3.5	0.9761	0.9521	0.9280	0.9038	0.8795	0.8550	0.8305	0.8059	0.7813	0.7567
	4.0	0.9791	0.9581	0.9371	0.9159	0.8947	0.8734	0.8520	0.8305	0.8090	0.7875
	4.5	0.9814	0.9628	0.9441	0.9253	0.9065	0.8876	0.8686	0.8496	0.8305	0.8114
	5.0	0.9833	0.9665	0.9497	0.9329	0.9159	0.8989	0.8819	0.8648	0.8477	0.8305
$(\theta_1 = 0.75, \theta_2 = 1.50)$	**0.5**	0.2557	0.1266	0.0841	0.0629	0.0503	0.0419	0.0359	0.0314	0.0279	0.0251
	1.0	0.5000	0.2557	0.1695	0.1266	0.1011	0.0841	0.0720	0.0629	0.0559	0.0503
	1.5	0.6586	0.3834	0.2557	0.1910	0.1523	0.1266	0.1084	0.0947	0.0841	0.0756
	2.0	0.7443	0.5000	0.3414	0.2557	0.2039	0.1695	0.1450	0.1266	0.1124	0.1011
	2.5	0.7961	0.5919	0.4243	0.3201	0.2557	0.2125	0.1818	0.1588	0.1409	0.1266
	3.0	0.8305	0.6586	0.5000	0.3834	0.3073	0.2557	0.2187	0.1910	0.1695	0.1523
	3.5	0.8550	0.7074	0.5644	0.4441	0.3583	0.2987	0.2557	0.2233	0.1982	0.1781
	4.0	0.8734	0.7443	0.6166	0.5000	0.4081	0.3414	0.2926	0.2557	0.2269	0.2039
	4.5	0.8876	0.7731	0.6586	0.5495	0.4557	0.3834	0.3293	0.2880	0.2557	0.2298
	5.0	0.8989	0.7961	0.6927	0.5919	0.5000	0.4243	0.3655	0.3201	0.2844	0.2557
$(\theta_1 = 1.75, \theta_2 = 0.75)$	**0.5**	0.7813	0.5644	0.3972	0.2987	0.2384	0.1982	0.1695	0.1480	0.1314	0.1181
	1.0	0.8916	0.7813	0.6707	0.5644	0.4709	0.3972	0.3414	0.2987	0.2653	0.2384
	1.5	0.9280	0.8550	0.7813	0.7074	0.6345	0.5644	0.5000	0.4441	0.3972	0.3583
	2.0	0.9461	0.8916	0.8366	0.7813	0.7259	0.6707	0.6166	0.5644	0.5154	0.4709
	2.5	0.9569	0.9135	0.8697	0.8256	0.7813	0.7369	0.6927	0.6489	0.6059	0.5644
	3.0	0.9641	0.9280	0.8916	0.8550	0.8182	0.7813	0.7443	0.7074	0.6707	0.6345
	3.5	0.9693	0.9384	0.9073	0.8760	0.8445	0.8130	0.7813	0.7496	0.7179	0.6864
	4.0	0.9731	0.9461	0.9189	0.8916	0.8642	0.8366	0.8090	0.7813	0.7536	0.7259
	4.5	0.9761	0.9521	0.9280	0.9038	0.8795	0.8550	0.8305	0.8059	0.7813	0.7567
	5.0	0.9785	0.9569	0.9353	0.9135	0.8916	0.8697	0.8477	0.8256	0.8035	0.7813
$(\theta_1 = 2.25, \theta_2 = 2.75)$	**0.5**	0.4170	0.2086	0.1383	0.1034	0.0825	0.0687	0.0588	0.0514	0.0457	0.0411
	1.0	0.6870	0.4170	0.2792	0.2086	0.1664	0.1383	0.1183	0.1034	0.0918	0.0825
	1.5	0.7923	0.5847	0.4170	0.3143	0.2510	0.2086	0.1784	0.1558	0.1383	0.1243
	2.0	0.8448	0.6870	0.5367	0.4170	0.3352	0.2792	0.2389	0.2086	0.1851	0.1664
	2.5	0.8762	0.7501	0.6249	0.5095	0.4170	0.3491	0.2993	0.2616	0.2321	0.2086
	3.0	0.8970	0.7923	0.6870	0.5847	0.4922	0.4170	0.3590	0.3143	0.2792	0.2510
	3.5	0.9119	0.8223	0.7320	0.6425	0.5569	0.4803	0.4170	0.3664	0.3260	0.2933
	4.0	0.9230	0.8448	0.7659	0.6870	0.6097	0.5367	0.4717	0.4170	0.3721	0.3352
	4.5	0.9316	0.8623	0.7923	0.7220	0.6523	0.5847	0.5214	0.4651	0.4170	0.3766
	5.0	0.9385	0.8762	0.8133	0.7501	0.6870	0.6249	0.5651	0.5095	0.4599	0.4170