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A New Alpha Skew Normal Distribution and its Real Life Applications

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Abstract

This paper introduces a novel continuous probability distribution which extends the alpha skew normal distribution of Elal-Olivero (2010). The newy introduced distribution is designed to model data exhibiting tri-modal behavior. A comprehensive overview of the novel distribution is provided including various key statistical properties like moments, moment generating function (mgf), characterization results etc. Besides, to assess the performance of the derived parameters, a simulation study is conducted using Metropolis-Hastings method. Furthermore, an investigation regarding the flexibility and applicability of the distribution is conducted by analyzing two real life datasets. During applications it is found that for the datasets considered, the newly proposed distribution outperforms the existing models in terms of some model selection criterion like AIC and BIC, highlighting its potential in practical applications. Finally, likelihood ratio (LR) test is conducted to differentiate between various nested models.

Key Words: Trimodality, Skew Normal, AIC, Alpha Skew Normal, Metropolis-Hastings

Mathematical Subject Classification: 60E05, 62E15.

1 Introduction

Azzalini (1985) first proposed the skew normal (SN) distribution introducing an asymmetric parameter along with normal distribution. Numerous academics have been continually researching skew normal distribution since Azzalini's findings. Many generalizations, extensions, and developments of Azzalni's SN model were suggested by the researchers in all of these endeavors from specific perspectives. Many skew distributions exist in the literature as a result of these significant efforts and studies, allowing for the fitting of data with a single mode. However, data having a uni-modal, bimodal, or tri-modal character can also be found in real-world situations. Bimodality in the incidence of several cancer types by age was discussed by Anderson et al. (2006). The presence of bimodality during the implications in geosciences was also demonstrated by Rathie and Coutinho (2011) . Contrarily, Funke and Niebuhr (2005) study on the development of West German regions demonstrated the existence of tri-modality in the distribution of relative earnings. Trimodality was also put forth by Downey and Huffman (2001) as a distributional pattern addressing the problem of attitudinal polarization. In this case, tri-modality was put out as a distributional pattern associated with polarization-related conceptual problems. According to their hypothesis, attitudes tend to cluster both toward a central mode and toward distinct tails that appear on Likerttype scales with a midway. Additionally, tri-modality has been noted to regularly manifest in sensation thermometers discovered in the National Election Study. While Baker et al. (1980) and Trunkey (1983) demonstrated that the occurrence of traumatic demise with three distinct modes is 2 observed predominantly in metropolitan areas within the United States. When the mortality rate was plotted depending on the elapsed time since the injury occurred for a sizable sample of deaths, a trimodal distribution was discovered. They classified the initial peak as "Immediate deaths"— signifying that individuals succumb to their conditions right away— the subsequent peak as "Early deaths"—meaning that individuals pass away within the first few hours—and the final peak as "Late deaths"—meaning that individuals pass away days or weeks after the injury. For the Keggin molybdates, Yan et al. (2008) established a tri-modal distribution regarding the measurements of the bonds formed through the bridging of oxygen and metal atoms. Additionally, Nohava et al. (2012) established that data related to grid imprint required tri-modal fitting. Assessing the statistical aspects of the grid indentation posed challenges because of Gaussian distribution with trimodal behavior was necessary which to capture the characteristics of individual stages.

Recently, some authors developed new families of skew distributions which allow the data under consideration for fitting up to two modes. For example Kim (2005) addressed a new asymmetric normal distribution involving linear and non-linear constraints while another class of bimodal skew elliptical distribution was suggested in the work of Elal-Olivero et al. (2009). The alpha skew normal (ASN) distribution was a novel group of skew normal category developed by Elal-Olivero (2010) for fitting both uni-modal and bimodal data with the probability density function (pdf) provided as

$$f(x) = \frac{(1-\alpha x)^2+1}{2+\alpha^2}\phi(x); \quad x \in R, \alpha \in R, \tag{1}$$

where, α is the asymmetric parameter controlling the number of modes in ASN distribution. In order to fit data with up to two modes, Harandi and Alamatsaz (2013) developed another new family with Laplace distribution instead of normal distribution using the same mechanism. Similarly, to fit bimodal data, Hazarika and Chakraborty (2014) suggested another family with logistic distribution. For fitting data with uni-modals as well as bimodal characteristics, Hazarika et al. (2020) also introduced Balakrishnan alpha SN (BASN) distribution. Elal-Olivero et al. (2020) developed another new method and introduced two parameter bimodal skew normal distribution for fitting data with uni-bimodal behavior. Following the same mechanism Das et al. (2024) proposed another new distribution called bimodal $Tanh$ skew normal distribution. Besides, some other notable works on the development of new unimodal or bimodal skew distribution reflected through the works of Alizadeh et al. (2025), Azzalini (1986), Sulewski et al. (2025), Das et al. (2025b), Pathak et al. (2025a), Das et al. (2025c), etc.

However, Shafiei et al. (2016) also developed the alpha beta skew normal (ABSN), a family of distributions that could fit data with up to four modes. Later, Shah et al. (2023) developed a generalization of the ABSN distribution using the generalized alpha skew normal (GASN) distribution (Shafiei et al., 2016), which is more flexible than the alpha skew normal (Elal-Olivero, 2010). Ma and Genton (2004) also developed a novel adaptable class of skew symmetric distributions for fitting multimodal data. Martínez-Flórez et al. (2022) developed some new versatile classes of normal distribution for fitting multimodal data. There were flexible normal distributions that were both symmetric and asymmetric in these classes. Tri-modal Normal (TN) distribution is one of the classes of symmetric distributions proposed by Martínez-Flórez et al. (2022) and its density function was given by

$$f(x) = \frac{(x^2-1)^2+2}{4}\phi(x); \quad x \in R. \tag{2}$$

Trimodal skew logistic distribution (Pathak et al., 2023), Trimodal skew normal distribution (Pathak et al., 2025b), flexible alpha skew normal distribution (Das et al., 2023) (Das et al., 2025a) are four recent distribution which are proposed for fitting data up to three modes.

In this article, a new extension of alpha skew normal distribution of Elal-Olivero (2010) has been proposed that takes into account the concept of tri-modal normal distribution (Martínez-Flórez et al., 2022). Additionally, two real life data sets were used to test the flexibility and utility of the new distribution.

The remaining part of the article is structured as follows: A new extension of the alpha skew normal distribution (Elal-Olivero, 2010) is suggested in Section 2, along with certain specific cases and graphical representations. Some significant statistical properties of the new distribution are presented in Section 3 while characterizations of the new distribution via two truncated moments are taken into consideration in Section 4.On the other hand, Section 5 contains the parameter estimation for the newly proposed distribution.The proposed distribution is presented with two real life applications in Section 6 to demonstrate its adaptability. The likelihood ratio test is then used in Section 7 to distinguish between the proposed models and other nested models, and the article is concluded at Section 8.

2 A Trimodal Extension of the Alpha Skew Normal Distribution

This section proposes a new extension of the alpha skew normal distribution and discusses some of its significant statistical properties.

Definition 2.1. *If X is a continuous random variable, then it is said to follow TASN distribution, denoted as $X \sim TASN(\alpha)$, when its PDF is given as*

$$f(x;\alpha) = \frac{\left[\left(1-\alpha x\right)^2+1\right]}{C(\alpha)}\left[\frac{(x^2-1)^2+2}{4}\right]\phi(x); \quad \alpha \in R, \tag{3}$$

where, $\phi(\cdot)$ is the PDF of standard normal distribution while α is the shape parameter. The normalizing constant is given by $C(\alpha) = 2+3\alpha^2$ and it is calculated as

$$\begin{aligned}
C(\alpha) &= \frac{1}{4}\int_{-\infty}^{-\infty}\Big[(1-\alpha x)^2+1\Big]\Big[(x^2-1)^2+2\Big]\phi(x)dx \\
&= \frac{1}{4}\Bigg[\alpha^2\int_{-\infty}^{-\infty}x^6\phi(x)dx - 2\alpha\int_{-\infty}^{-\infty}x^5\phi(x)dx + (2-2\alpha^2)\int_{-\infty}^{-\infty}x^4\phi(x)dx \\
&\quad + 4\alpha\int_{-\infty}^{-\infty}x^3\phi(x)dx + (3\alpha^2-4)\int_{-\infty}^{-\infty}x^2\phi(x)dx - 6\alpha\int_{-\infty}^{-\infty}x\phi(x)dx + 6\int_{-\infty}^{\infty}\phi(x)dx\Bigg] \\
&= \frac{1}{4}\Bigg[\alpha^2 I_1^N - 2\alpha I_2^N + (2-2\alpha^2)I_3^N + 4\alpha I_4^N + (3\alpha^2-4)I_5^N - 6\alpha I_6^N + 6I_7^N\Bigg]
\end{aligned}$$

Now, I_7^N is the PDF of standard normal distribution. Again, I_1^N, I_3^N, I_5^N are the 6th, 4th and 2nd order moments of normal distribution calculated as 15, 3, and 1, respectively, while I_2^N, I_4^N, I_6^N are the 5th, 3rd and 1st order moments of normal distribution found to be zero. Hence putting the values, final expression of $C(\alpha)$ is calculated as

$$C(\alpha) = \frac{1}{4}\Big[8+12\alpha^2\Big] = 2+3\alpha^2.$$

2.1 Special Cases of TASN Distribution

i. If $\alpha = 0$, then $TASN(\alpha)$ reduces to trimodal normal (TN) distribution of Martínez-Flórez et al. (2022).

Proof. When $\alpha = 0$, then the PDF of $TASN(\alpha)$ becomes,

$$\begin{aligned} f(x;0) &= \frac{\left[(1-0\cdot x)^2+1\right]}{C(0)}\left[\frac{(x^2-1)^2+2}{4}\right]\phi(x) \\ &= \frac{(x^2-1)^2+2}{4}\phi(x). \end{aligned}$$

Hence, $TASN(\alpha)$ reduces to TN distribution of Martínez-Flórez et al. (2022).

ii. If $X \sim TASN(\alpha)$, then $-X \sim TASN(-\alpha)$.

Proof. Let $X \sim TASN(\alpha)$. Define a new random variable $Y = g(X) = -X$. Now using the method of transformations:

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\cdot\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|.$$

where, $f_Y(y)$ is the PDF of Y. First, compute the derivative as

$$\frac{d}{dy}\left[g^{-1}(y)\right] = \frac{d}{dy}(-y) = -1 \quad \Rightarrow \quad \left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = 1.$$

Now,

$$\begin{aligned} f_Y(y) &= f_X(-y;\alpha)\cdot 1 \\ &= \frac{1}{C(\alpha)}\left[(1-\alpha(-y))^2+1\right]\left[\frac{((-y)^2-1)^2+2}{4}\right]\phi(-y). \end{aligned}$$

But,

$$(1+\alpha y)^2+1 = (1-(-\alpha)y)^2+1, C(-\alpha) = C(\alpha)$$

Hence, we get $Y = -X \sim TASN(-\alpha)$.

2.2 Plots of TASN Distribution

This subsection reflects the pictorial view of the $TASN(\alpha)$ distribution for different parameter choices. The respective graphical presentation of the PDF of $TASN(\alpha)$ is depicted in Fig. 1. From Fig. 1 it can be observed that $TASN(\alpha)$ distribution always shows trimodal behavior. More precisely, for the negative values of α, the density function contain higher peak towards left (see Fig. 1(a)) and become symmetric when $\alpha = 0$ (see Fig. 1(c)). Similarly, for positive values of α, the distribution possesses higher peak towards left (see Fig. 1(b) and (d)). So It is worth noting that the shape parameter α plays a crucial role in determining the skewness and mode of the distribution. For smaller values of α, the distribution is more concentrated towards the left, resulting in a left-skewed shape with modes located closer to the origin. As α increases, the peak of the distribution gradually shifts rightward, and the tail becomes heavier on the right-hand side, producing greater right-skewness. This dynamic illustrates how α governs both the asymmetry and the location of the modal region, offering flexibility in modeling datasets with varying degrees of skewness.

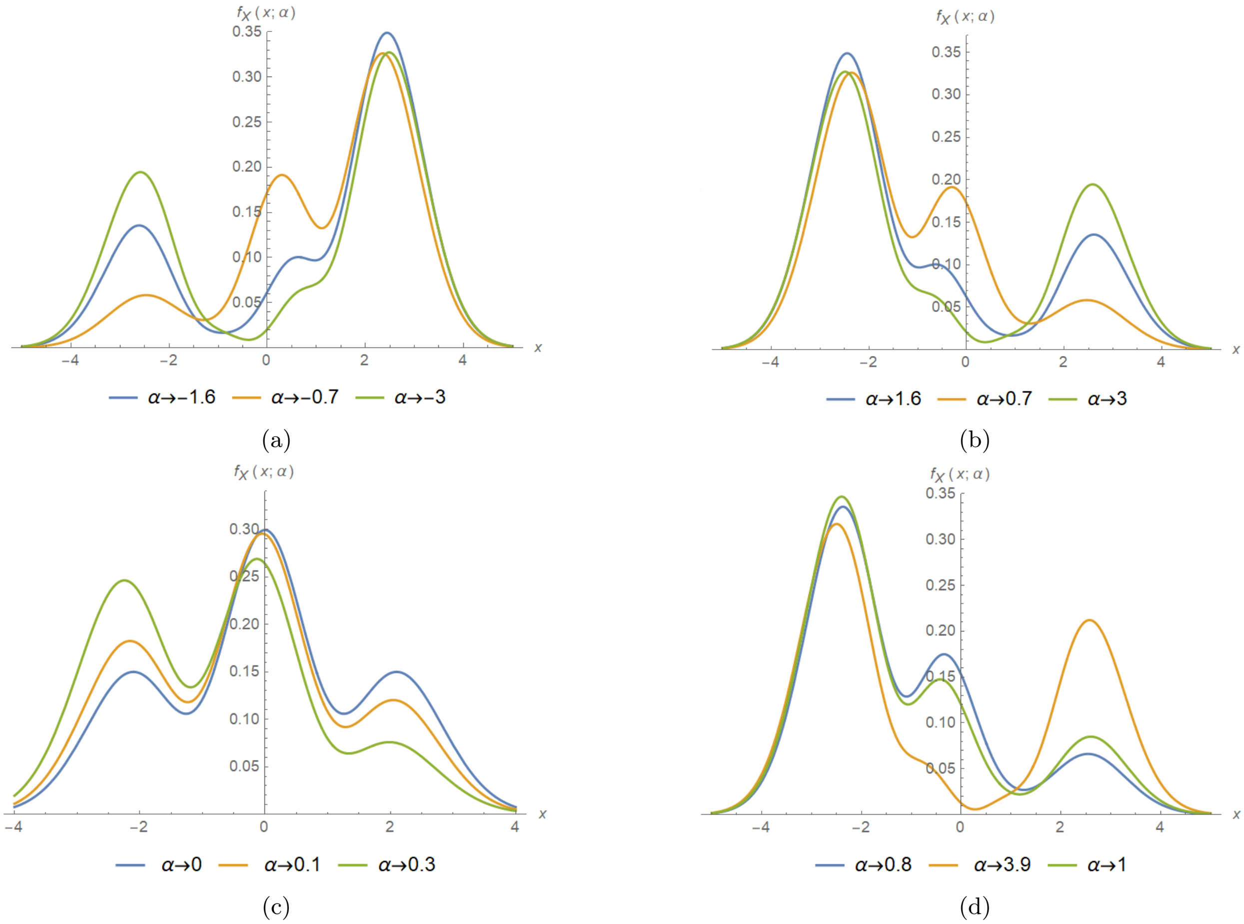


Figure 1: Plots of the PDF of $TASN(\alpha)$

3 Statistical Properties

This section discusses several significant statistical properties of TASN distribution.

3.1 Cumulative Distribution Function (CDF)

Theorem 3.1. *The CDF of $TASN(\alpha)$ distribution is given by*

$$F(x) = \Phi(x) + \frac{\phi(x)}{4C(\alpha)}\Big[14\alpha - x\left(12\alpha^2 + \alpha^2 x^4 - 2\alpha x^3 + \left(3\alpha^2 + 2\right)x^2 - 4\alpha x + 2\right)\Big] \tag{4}$$

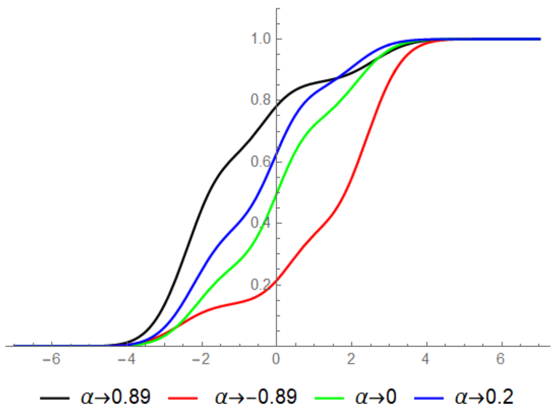
Proof.

$$\begin{aligned} F(X) &= \frac{1}{4C(\alpha)}\int_{-\infty}^{x}\Big[(1-\alpha x)^2+1\Big]\Big[(x^2-1)^2+2\Big]\phi(x)dx \\ &= \frac{1}{4C(\alpha)}\Bigg[\alpha^2\int_{-\infty}^{x}x^6\phi(x)dx - 2\alpha\int_{-\infty}^{x}x^5\phi(x)dx + (2-2\alpha^2)\int_{-\infty}^{x}x^4\phi(x)dx \\ &\quad + 4\alpha\int_{-\infty}^{x}x^3\phi(x)dx + (3\alpha^2-4)\int_{-\infty}^{x}x^2\phi(x)dx - 6\alpha\int_{-\infty}^{x}x\phi(x)dx + 6\int_{-\infty}^{x}\phi(x)dx\Bigg] \\ &= \ = \frac{1}{4C(\alpha)}\Bigg[\alpha^2 I_1 - 2\alpha I_2 + (2-2\alpha^2)I_3 + 4\alpha I_4 + (3\alpha^2-4)I_5 - 6\alpha I_6 + 6I_7\Bigg] \end{aligned}$$

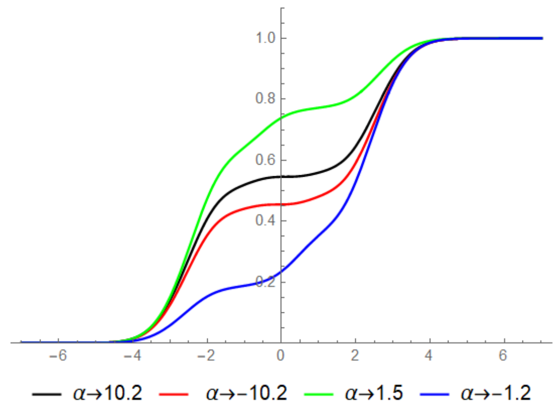
So it can be seen that I_7 is the CDF of the standard normal distribution while I_3 and I_6 are functions of the CDF of the TN distribution of Martínez-Flórez et al. (2022). On the other hand, special functions given by Gradshteyn and Ryzhik (2000) are used to determine the values of the other integrations. Hence, final expression for CDF of $TASN(\alpha)$ distribution is calculated as

$$F(x) = \Phi(x) + \frac{\phi(x)}{4C(\alpha)}\Big[14\alpha - x\left(12\alpha^2 + \alpha^2 x^4 - 2\alpha x^3 + \left(3\alpha^2+2\right)x^2 - 4\alpha x + 2\right)\Big].$$

Proved.
In Fig. 2, CDF of $TASN(\alpha)$ distribution is presented to illustrate how the distribution shapes up concerning the various parameter selections.



(a)



(b)

Figure 2: Plot of the CDF of $TASN(\alpha)$.

3.2 Moments

Theorem 3.2. *Let,* $X \sim TASN(\alpha)$*, then the* r^{th} *order moment of* X *is given by*

$$E(X^r) = \frac{1}{\sqrt{\pi}C(\alpha)}\Bigg[2^{\frac{r}{2}-\frac{5}{2}}\Bigg\{\Big((-1)^r - 1\Big)\alpha r(r(r+4)+7)\Gamma\left(\frac{r}{2}\right) + \frac{1}{\sqrt{2}}\Bigg(\left((-1)^r+1\right)\left(\alpha^2(r+1)(r(r+6)+12)+2r(r+2)+8\right)\Gamma\left(\frac{r+1}{2}\right)\Bigg)\Bigg\}\Bigg] \tag{5}$$

Proof.

$$\begin{aligned}
E(X^r) &= \frac{1}{4C(\alpha)}\int_{-\infty}^{\infty} x^r\Big[(1-\alpha x)^2+1\Big]\Big[(x^2-1)^2+2\Big]\phi(x)dx \\
&= \frac{1}{4C(\alpha)}\Bigg[\alpha^2\int_{-\infty}^{\infty} x^{r+6}\phi(x)dx - 2\alpha\int_{-\infty}^{\infty} x^{r+5}\phi(x)dx + (2-2\alpha^2)\int_{-\infty}^{\infty} x^{r+4}\phi(x)dx \\
&\quad + 4\alpha\int_{-\infty}^{\infty} x^{r+3}\phi(x)dx + (3\alpha^2-4)\int_{-\infty}^{\infty} x^{r+2}\phi(x)dx - 6\alpha\int_{-\infty}^{\infty} x^{r+1}\phi(x)dx \\
&\quad + 6\int_{-\infty}^{\infty} x^{r}\phi(x)dx\Bigg] \\
&= \frac{1}{4C(\alpha)}\Bigg[\alpha^2 I_8 - 2\alpha I_9 + (2-2\alpha^2)I_{10} + 4\alpha I_{11} + (3\alpha^2-4)I_{12} - 6\alpha I_{13} + 6I_{14}\Bigg]
\end{aligned}$$

So it can be seen that I_{14} is the r^{th} order moment of the ordinary normal distribution while I_{10} and I_{13}

are functions of the r^{th} order moment of the tri-modal normal (TN) distribution of Martínez-Flórez et al. (2022). On the other hand, special functions given by Gradshteyn and Ryzhik (2000) are used to determine the values of the other integrations. So, r^{th} order moment of $TASN(\alpha)$ distribution is given by

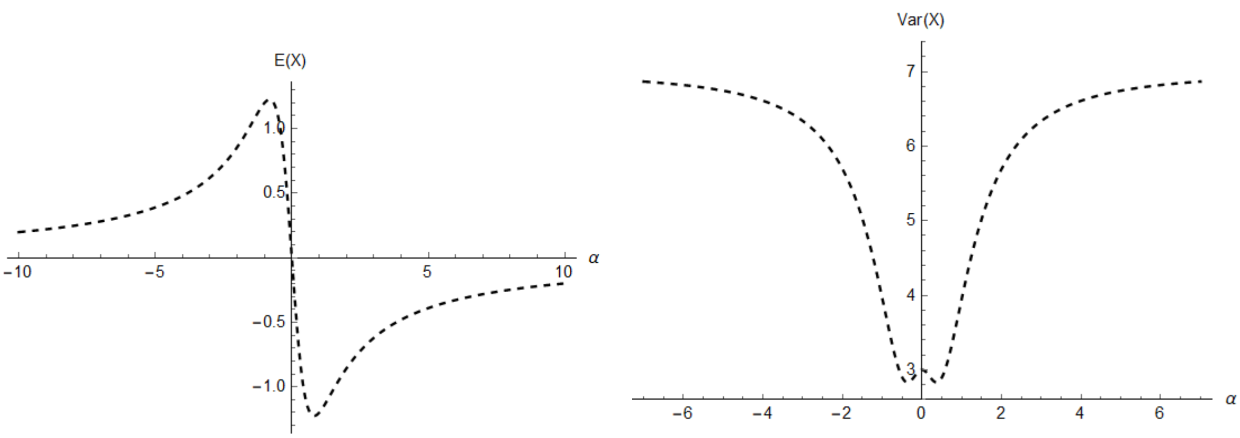
$$E(X^r) = \frac{1}{\sqrt{\pi}C(\alpha)}\Bigg[2^{\frac{r}{2}-\frac{5}{2}}\Bigg\{\Big((-1)^r-1\Big)\alpha r(r(r+4)+7)\Gamma\left(\frac{r}{2}\right) + \frac{1}{\sqrt{2}}\Bigg(((-1)^r+1)\left(\alpha^2(r+1)(r(r+6)+12)+2r(r+2)+8\right)\Gamma\left(\frac{r+1}{2}\right)\Bigg)\Bigg\}\Bigg].$$

Proved.

Remark 3.1. *Putting $r = 1,2,3,4$ in equ. (5), first four order moments of $TASN(\alpha)$ distribution are obtained as*
$E(X = -\dfrac{6\alpha}{C(\alpha)}$, $E(X^2) = 7 - \dfrac{8}{C(\alpha)}$, $E(X^3) = -\dfrac{42\alpha}{C(\alpha)}$, $E(X^4) = 65 - \dfrac{88}{C(\alpha)}$.
Hence the variance of $TASN(\alpha)$ distribution is calculated as

$$Var(X) = \frac{1}{C(\alpha)^2}\Big[3(4+8\alpha^2+21\alpha^4)\Big].$$

Remark 3.2. *Bounds for the mean and variance of $TASN(\alpha)$ distribution are derived by optimizing $E(X)$ and $Var(X)$ with respect to the parameter. The obtained values of the mean and variance are $-1.22474 \leq E(X) \leq 1.22474$ and $2.833 \leq Var(X) \leq 7$ respectively which can be also graphically represented in Figs. 3(a) and 3(b), respectively.*



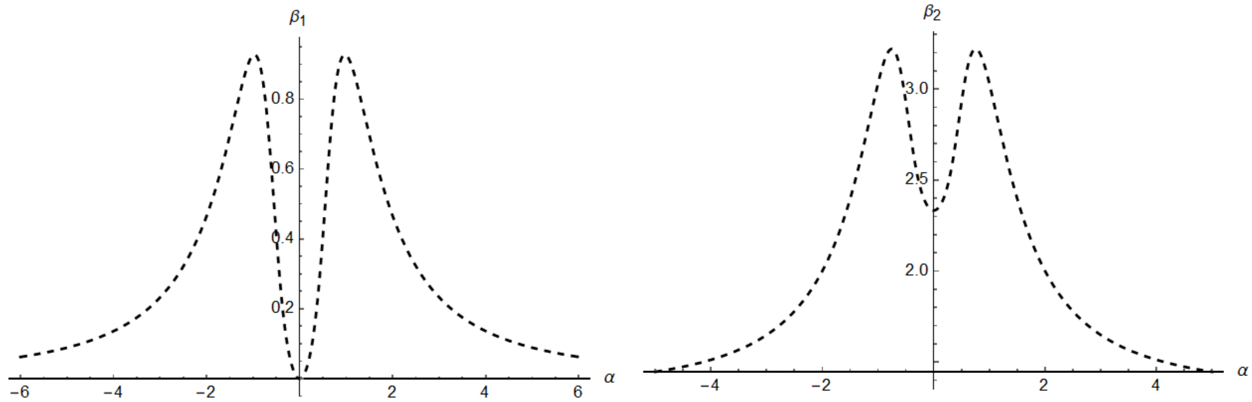
(a) Mean of $TASN(\alpha)$ (b) Variance of $TASN(\alpha)$

Figure 3: Plots of Mean and Variance of $TASN(\alpha)$ distribution

Remark 3.3. *The skewness (β_1) and kurtosis (β_2) of $TASN(\alpha)$ distribution are calculated as*

$$\beta_1 = \frac{16\alpha^2\Big(63\alpha^4+12\alpha^2+4\Big)^2}{3\left(21\alpha^4+8\alpha^2+4\right)^3}, \qquad \beta_2 = \frac{1755\alpha^8+5400\alpha^6+2088\alpha^4+544\alpha^2+112}{3\left(21\alpha^4+8\alpha^2+4\right)^2}$$

Bounds for the skewness (β_1) and kurtosis (β_2) of $TASN(\alpha)$ distribution are derived By optimizing and (β_1) with (β_2) respect to the parameter. The values are obtained as $0 \leq \beta_1 \leq 0.9276$ and $1.3265 \leq \beta_2 \leq 3.2199$ respectively which can be also graphically represented in Fig.s 4 (a) and 4(b), respectively.



(a) Skewness of $TASN(\alpha)$ (b) Kurtosis of $TASN(\alpha)$

Figure 4: Plots of Skewness and Kurtosis of $TASN(\alpha)$ Distribution

3.3 Incomplete Moment

Theorem 3.3. *Let, $X \sim TASN(\alpha)$, then the $r-th$ order incomplete moment of X is given by*

$$m_n(y) = \frac{1}{F(y)4C(\alpha)}\Bigg\{(-1)^n\Big[\sqrt{2}\alpha n P(n)\Gamma_0 + Q(n)\Gamma_1\Big] + y^{-1}\Bigg[-2\sqrt{2}\alpha P(n)y\Gamma_1 + yQ(n)\Gamma_1 - 6\alpha^2 y\Gamma_3^{(y^2)}$$
$$+ 8\alpha^2 y\Gamma_5^{(y^2)} - 8\alpha^2 y\Gamma_7^{(y^2)} - 8\sqrt{2}\alpha y\Gamma_2^{(y^2)} + 6\sqrt{2}\alpha y\Gamma_1^{(y^2)} + 8\sqrt{2}\alpha y\Gamma_3^{(y^2)} - 6y\Gamma_1^{(y^2)} + 8y\Gamma_3^{(y^2)} - 8y\Gamma_5^{(y^2)}\Bigg\} \tag{6}$$

$$\textit{where } P(n) = n^2 + 4n + 7, \quad Q(n) = 12\alpha^2 + \alpha^2 n^3 + (7\alpha^2+2)n^2 + 2(9\alpha^2+2)n + 8,$$
$$\Gamma_k = \Gamma\left(\frac{n+k}{2}\right), \quad \Gamma_k^{(y^2)} = \Gamma\left(\frac{n+k}{2}, \frac{y^2}{2}\right).$$

Proof. The incomplete moment of $TASN(\alpha)$ distribution is given by

$$m_n(y) = E\left[X^n | X < y\right] = \frac{1}{F(y)}\int_{-\infty}^{y} x^n f(x)dx$$

Now,

$$\begin{aligned}
\int_{-\infty}^{y} x^n f(x)dx &= \frac{1}{4C(\alpha)}\int_{-\infty}^{y} x^n\Big[(1-\alpha x)^2+1\Big]\Big[(x^2-1)^2+2\Big]\phi(x)dx \\
&= \frac{1}{4C(\alpha)}\Bigg[\alpha^2\int_{-\infty}^{y} x^{n+6}\phi(x)dx - 2\alpha\int_{-\infty}^{y} x^{n+5}\phi(x)dx + (2-2\alpha^2)\int_{-\infty}^{y} x^{n+4}\phi(x)dx \\
&\quad + 4\alpha\int_{-\infty}^{y} x^{n+3}\phi(x)dx + (3\alpha^2-4)\int_{-\infty}^{y} x^{n+2}\phi(x)dx - 6\alpha\int_{-\infty}^{y} x^{n+1}\phi(x)dx \\
&\quad + 6\int_{-\infty}^{y} x^n\phi(x)dx\Bigg] \\
&= \frac{1}{4C(\alpha)}\Big[\alpha^2 I_8^m - 2\alpha I_9^m + (2-2\alpha^2)I_{10}^m + 4\alpha I_{11}^m + (3\alpha^2-4)I_{12}^m - 6\alpha I_{13}^m + 6I_{14}^m\Big] \qquad (7)
\end{aligned}$$

It is evident that I_{14}^m represents the n^{th} order incomplete moment of the ordinary normal distribution, whereas I_{10}^m and I_{13}^m depend on the n^{th} order incomplete moment of the tri-modal normal (TN) distribution described by Martínez-Flórez et al. (2022). Additionally, the values of the remaining integrations are computed using special functions as outlined by Gradshteyn and Ryzhik (2000). Hence, the final expression can be obtained

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as

$$m_n(y) = \frac{1}{F(y)4C(\alpha)}\Bigg\{(-1)^n\Big[\sqrt{2}\alpha nP(n)\Gamma_0 + Q(n)\Gamma_1\Big] + y^{-1}\Bigg[-2\sqrt{2}\alpha P(n)y\Gamma_1 + yQ(n)\Gamma_1 - 6\alpha^2 y\Gamma_3^{(y^2)}$$
$$+ 8\alpha^2 y\Gamma_5^{(y^2)} - 8\alpha^2 y\Gamma_7^{(y^2)} - 8\sqrt{2}\alpha y\Gamma_2^{(y^2)} + 6\sqrt{2}\alpha y\Gamma_1^{(y^2)} + 8\sqrt{2}\alpha y\Gamma_3^{(y^2)} - 6y\Gamma_1^{(y^2)} + 8y\Gamma_3^{(y^2)} - 8y\Gamma_5^{(y^2)}\Bigg\}$$

where $P(n) = n^2 + 4n + 7, \quad Q(n) = 12\alpha^2 + \alpha^2 n^3 + (7\alpha^2 + 2)n^2 + 2(9\alpha^2 + 2)n + 8,$

$$\Gamma_k = \Gamma\left(\frac{n+k}{2}\right), \quad \Gamma_k^{(y^2)} = \Gamma\left(\frac{n+k}{2}, \frac{y^2}{2}\right).$$

3.4 Mean Deviation

Theorem 3.4. *The mean deviation of $TASN(\alpha)$ distribution about the mean μ is obtained as*

$$\delta_1(x) = 2\mu F(\mu) - 2m_1(\mu)F(\mu). \tag{8}$$

Where, $m_1(\mu)$ is defined using the expression for incomplete moment of $TASN(\alpha)$ in equ. (6) and $F(\mu)$ can be find out using the expression of the cdf given in subsection (3.1).

Proof. The mean deviation about the mean and mean deviation about the median can be difined as follows, respectively:

$$\delta_1(X) = \int_{-\infty}^{\infty} |x-\mu| f(x)dx,$$

$$\delta_2(X) = \int_{-\infty}^{\infty} |x-M| f(x)dx,$$

respectively, where $\mu = E(X)$ and M denotes the median. So, $\delta_1(X)$ can be calculated as

$$\begin{aligned}\delta_1(X) &= \int_{-\infty}^{\infty} |x-\mu| f(x)dx \\ &= 2\mu F(\mu) - 2\int_{-\infty}^{\mu} xf(x)dx \\ &= 2\mu F(\mu) - 2I_1^{\delta}\end{aligned}$$

Now, from the results of incomplete moments for $TASN(\alpha)$ distribution mentioned in (6), I_1^{δ} can be written as

$$I_1^{\delta} = -\Big[\sqrt{2}\alpha(12)\Gamma_0 + (38\alpha^2+14)\Gamma_1\Big] + y^{-1}\Bigg[-2\sqrt{2}\alpha(12)|y|\Gamma_1 + y(38\alpha^2+14)\Gamma_1 - 6\alpha^2 y\Gamma_3^{(y^2)} + 8\alpha^2 y\Gamma_5^{(y^2)}$$
$$- 8\alpha^2 y\Gamma_7^{(y^2)} - 8\sqrt{2}\alpha|y|\Gamma_2^{(y^2)} + 6\sqrt{2}\alpha|y|\Gamma_1^{(y^2)} + 8\sqrt{2}\alpha|y|\Gamma_3^{(y^2)} - 6y\Gamma_1^{(y^2)} + 8y\Gamma_3^{(y^2)} - 8y\Gamma_5^{(y^2)}$$

hence, the final result of mean deviation about mean for $TASN(\alpha)$ distribution is obtained as

$$\delta_1(x) = 2\mu F(\mu) - 2m_1(\mu)F(\mu).$$

Remark 3.6.1. Replacing mean μ by median M in (8), the mean deviation of $TASN(\alpha)$ distribution about median M i.e. $\delta_2(X)$ can be obtained.

3.5 Moment Generating Function (MGF)

Theorem 3.5. *Let, $X \sim TASN(\alpha)$, then the MGF of X is given by*

$$M_X(t) = \frac{1}{4C(\alpha)}\left[e^{\frac{t^2}{2}}\left\{\alpha^2\left(\left(t^2+4\right)\left(t^2+9\right)t^2+12\right)-2\alpha t\left(t^2+6\right)\left(t^2+2\right)\right.\right.$$
$$\left.\left.+2\left(t^2+2\right)^2\right\}\right] \tag{9}$$

Proof.

$$M_X(t) = \frac{1}{4C(\alpha)}\int_{-\infty}^{\infty} e^{xt}\left[(1-\alpha x)^2+1\right]\left[(x^2-1)^2+2\right]\phi(x)dx$$
$$= \frac{1}{4C(\alpha)}\left[\alpha^2\int_{-\infty}^{\infty} e^{xt}x^6\phi(x)dx + (2-2\alpha^2)\int_{-\infty}^{\infty} e^{xt}x^4\phi(x)dx\right.$$

$$-2\alpha\int_{-\infty}^{\infty} e^{xt}x^5\phi(x)dx + 4\alpha\int_{-\infty}^{\infty} e^{xt}x^3\phi(x)dx + (3\alpha^2-4)\int_{-\infty}^{\infty} e^{xt}x^2\phi(x)dx$$
$$\left.-6\alpha\int_{-\infty}^{\infty} e^{xt}x\phi(x)dx + 6\int_{-\infty}^{\infty} e^{xt}\phi(x)dx\right]$$
$$= \frac{1}{4C(\alpha)}\left[\alpha^2 I_{15} - 2\alpha I_{16} + (2-2\alpha^2)I_{17} + 4\alpha I_{18} + (3\alpha^2-4)I_{19} - 6\alpha I_{20} + 6I_{21}\right]$$

It can be observed that I_{21} is the MGF of the ordinary normal distribution while I_{17} and I_{20} are functions of the MGF of the tri-modal normal (TN) distribution of Martínez-Flórez et al. (2022). On the other hand, special functions given by Gradshteyn and Ryzhik (2000) are used to determine the values of the other integrations. Hence, the MGF of $TASN(\alpha)$ distribution is calculated as

$$M_X(t) = \frac{1}{4C(\alpha)}\left[e^{\frac{t^2}{2}}\left\{\alpha^2\left(\left(t^2+4\right)\left(t^2+9\right)t^2+12\right)-2\alpha t\left(t^2+6\right)\left(t^2+2\right)+2\left(t^2+2\right)^2\right\}\right].$$

Proved.

Remark 3.4. *Replacing t by (it) in (9), the characteristic function of $TASN(\alpha)$ distribution is obtained as*

$$\phi_X(t) = \frac{1}{4C(\alpha)}\left[e^{\frac{(it)^2}{2}}\left\{\alpha^2\left(\left((it)^2+4\right)\left((it)^2+9\right)(it)^2+12\right)-2\alpha t\left((it)^2+6\right)\left((it)^2+2\right)+2\left((it)^2+2\right)^2\right\}\right]$$

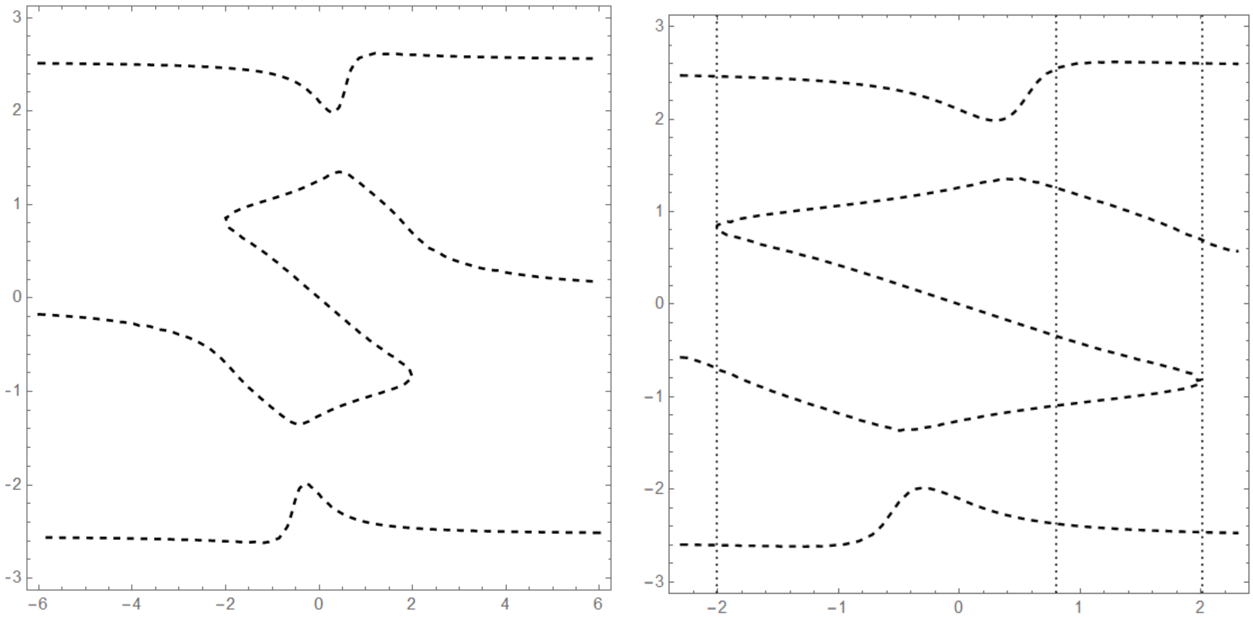
3.6 Mode

Theorem 3.6. *There are a maximum of three modes in the $TASN(\alpha)$ distribution.*

Proof. To show that the $TASN(\alpha)$ distribution no more than three modes, this is similar to show that equ. (10) has five real zeros.

$$\frac{\delta f(x;\alpha)}{\delta x} = Df(x;\alpha) = \frac{\phi(x)}{4C(\alpha)}\left[22(x^6-7x^4+9x^2-3)\alpha - x(x^6-8x^4+11x^2-6)\right.$$
$$\left.-2x(x^4-6x^2-7)\right] \tag{10}$$

Additionally, Fig. 5(a) and Fig. 5(b) depict the contour of the equ. $Df(x;\alpha)=0$, which allows it to be visually understood that the distribution has a maximum of three modes. Besides, from the Fig. 5(b) it can also be visualized clearly that $TASN(\alpha)$ distribution possesses tri-modal shape for $-2 \leq \alpha \leq 2$ (approximately).



(a) For $-6 \leq \alpha \leq 6$ (b) For $-2.3 \leq \alpha \leq 2.3$

Figure 5: The contour diagram of the equ. $Df(x;\alpha) = 0$

4 Characterizations Results

This section considers the characterizations of the $TASN(\alpha)$ distribution via two truncated moments. For these characterizations, the CDF need not have a closed form.

4.1 Characterization based on two truncated moments

The characterizations of the $TASN(\alpha)$ distribution based on the relationship between two truncated moments are discussed in this subsection. The first two characterizations make use of Glänzel (1987) theorem, which is listed below in Theorem 4.1. It is obvious that the result is true when the interval H is not a closed interval. This characterization is stable in the sense of weak convergence; please see reference (Glanzel, 1990).

Theorem 4.1. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$ and $e = \infty$ are allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let k and h be two real functions defined on H such that*

$$E\left[k\left(X\right) \mid X \geq x\right] = E\left[h\left(X\right) \mid X \geq x\right] \eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that k, $h \in C^1\left(H\right)$, $\eta \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that $\eta h = k$ has no real solution in the interior of H. Then F is uniquely determined by the functions k, h and η, particularly

$$F\left(x\right) = \int_a^x C \left| \frac{\eta'\left(u\right)}{\eta\left(u\right) h\left(u\right) - k\left(u\right)} \right| \exp\left(-s\left(u\right)\right) \, du,$$

where the function s is a solution of the differential equ. $s' = \dfrac{\eta' h}{\eta h - k}$ and C is the normalization constant, such that $\int_H dF = 1$.

Proposition 4.1. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable, and let*

$$h(x) = \left\{ \left((1-\alpha x)^2 + 1\right) \left(\frac{(x^2-1)^2+2}{4} \right) \right\}^{-1}$$

and $k(x) = h(x)\Phi(x)$, for $x \in \mathbb{R}$. Then, the density of X is given in equ. (3) if and only if the function η defined in Theorem 4.1 is

$$\eta(x) = \frac{1}{2}\Big\{1+\Phi(x)\Big\}, \quad x \in \mathbb{R}.$$

Proof: If X has PDF given in (3), then

$$(1-F(x))\, E\left[h(X) \mid X \geq x\right] = \frac{1}{C(\alpha)}\Big\{1-\Phi(x)\Big\}, \quad x \in \mathbb{R},$$

and

$$(1-F(x))\, E\left[k(X) \mid X \geq x\right] = \frac{1}{2C(\alpha)}\left\{1-\Phi^2(x)\right\}, \quad x \in \mathbb{R}.$$

Hence

$$\eta(x) = \frac{\dfrac{1}{2C(\alpha)}\left\{1-\Phi^2(x)\right\}}{\dfrac{1}{C(\alpha)}\Big\{1-\Phi(x)\Big\}} = \frac{1}{2}\Big\{1+\Phi(x)\Big\}.$$

Finally

$$\eta(x)\, h(x) - k(x) = \frac{1}{2} h(x) \left\{1-\Phi(x)\right\} > 0, \quad \text{for } x \in \mathbb{R}.$$

Conversely, if η has the above form, then

$$s'(x) = \frac{\eta'(x)\, h(x)}{\eta(x)\, h(x) - k(x)} = \frac{\frac{1}{2}h(x)\phi(x)}{\frac{1}{2}h(x)\left\{1-\Phi(x)\right\}} = \frac{\phi(x)}{1-\Phi(x)},$$

Hence

$$s(x) = -\log\left\{1-\Phi(x)\right\}, \quad x \in \mathbb{R}.$$

Based on Theorem 4.1, X has PDF given in equ. (3).
Proved.

Corollary 4.1. *If $X : \Omega \to \mathbb{R}$ is a continuous random variable and $h(x)$ is as in Proposition 4.1. Then, X has PDF given in equ. (3) if and only if there exist functions k and η defined in Theorem 4.1 satisfying the following first order differential equ.*

$$\frac{\eta'(x)\, h(x)}{\eta(x)\, h(x) - k(x)} = \frac{\phi(x)}{1-\Phi(x)}.$$

Corollary 4.2. *The general solution of the above differential equ. is*

$$\eta(x) = \left[1-\Phi(x)\right]^{-1} \left[-\int \phi(x) \left(h(x)\right)^{-1} k(x) + D \right],$$

where D is a constant. A set of functions satisfying this differential equ. is presented in Proposition 4.1 with $D = 1/2$. Clearly, there are other set (h, k, ξ) satisfying the conditions of Theorem 4.1 of which one is given in the following Proposition.

5 Parameter Estimation

5.1 Location and Scale Extension

The transformation $Y = \mu+\sigma X$ is used in this section to consider the extensions of the $TASN(\alpha)$ distribution using location and scale parameters, where $X \sim TASN(\alpha)$ distribution. Finally, a location scale generalized tri-modal alpha skew normal distribution is obtained with the PDF given by

$$f(y;\mu,\sigma,\alpha) = \frac{1}{C(\alpha)}\left[\left(1-\frac{\alpha(y-\mu)}{\sigma}\right)^2+1\right]\left[\frac{1}{4}\left(\left(\left(\frac{y-\mu}{\sigma}\right)^2-1\right)^2+2\right)\right]\phi\left(\frac{y-\mu}{\sigma}\right) \tag{11}$$

where, $y \in R$, $\alpha \in R$ and $\sigma > 0$. It is denoted as $Y \sim TASN(\mu,\sigma,\alpha)$.

5.2 Maximum Likelihood Estimation

Suppose $y_1, y_2, ..., y_n$ are random variable which are independent and identically distributed and drawn from the tri-modal alpha skew normal distribution, then log-likelihood function for $\theta = (\mu,\sigma,\alpha)$ is calculated as

$$\begin{aligned} l(\theta) = & \sum_{i=1}^{n}\log\left(\left(1-\frac{\alpha\,(y_i-\mu)}{\sigma}\right)^{2}+1\right)+\sum_{i=1}^{n}\log\left(\left(\left(\frac{y_i-\mu}{\sigma}\right)^{2}-1\right)^{2}+2\right)-n\log\sigma \\ & -\sum_{i=1}^{n}\frac{(y_i-\mu)^{2}}{2\sigma^2}-n\log\left(3\alpha^2+2\right)-n\log(4)-\frac{1}{2}n\log(2\pi) \end{aligned} \tag{12}$$

Differentiating equ. (12) with respect to the parameters , the likelihood equ. becomes,

$$\begin{aligned} \frac{\delta l(\theta)}{\delta\mu} &= \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)+\frac{2\alpha}{\beta^2}\sum_{i=1}^{n}\frac{A(y_i;\mu,\sigma,\alpha)}{B(y_i;\mu,\sigma,\alpha)}-\frac{4}{\sigma^3}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)\frac{C(y_i;\mu,\sigma,\alpha)}{D(y_i;\mu,\sigma,\alpha)} \\ \frac{\delta l(\theta)}{\delta\sigma} &= \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)^2+\frac{2\alpha}{\beta^2}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)^2\frac{A(y_i;\mu,\sigma,\alpha)}{B(y_i;\mu,\sigma,\alpha)} \\ &\quad -\frac{4}{\beta^3}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)^2\frac{C(y_i;\mu,\sigma,\alpha)}{D(y_i;\mu,\sigma,\alpha)} \\ \frac{\delta l(\theta)}{\delta\alpha} &= -\frac{6\alpha n}{3\alpha^2+2}-\frac{2}{\sigma}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)A(y_i;\mu,\sigma,\alpha), \end{aligned}$$

where, $A(y_i;\mu,\sigma,\alpha) = \beta+\alpha\mu-\alpha y_i$, $B(y_i;\mu,\sigma,\alpha) = 1+\left(1-\alpha\left(\frac{y_i-\mu}{\beta}\right)\right)^2$,

$C(y_i;\mu,\sigma,\alpha) = y_i^2-2\mu y_i+\mu^2-\beta^2$ and $D(y_i;\mu,\sigma,\alpha) = 2+\left(\left(\frac{y_i-\mu}{\sigma}\right)^{-}1\right)^2$. In order to find the solution to the system of equations, one can adopt optimization routine.

6 Simulation Study

The Metropolis-Hastings (M-H) technique is used to implement a simulation study to assess how the estimated distributional parameters behave. The corresponding results are based on the 1000 samples with three different samples sized $n = 100, 300, 500$.The GenSA packages in R software maximizes the likelihood function for each generated sample. The R code for GenSA package is provided in `https://github.com/jondeep98/Jondeep98.TASN.PJSOR`. Biases and mean square errors of the estimates are used to describe the calculated statistics which are given as $Bias(\hat{\theta}) = E(\hat{\theta})-\theta \quad and \quad MSE(\hat{\theta}) = Var(\hat{\theta})+\left\{Bias(\hat{\theta})\right\}^2$.

For the chosen sample size, the simulation investigation yielded parameter recovery findings that were adequate.It can be also noted that the value of bias as well as MSE of the estimated parameters have decreased with an increasing number of sample size which indicates the excellent behaviour of the MLEs' of the model. The results of the simulation study are presented in Table 1-2.

Table 1: Results of Simulation

		$\mu=0,\quad\sigma=1$					
		$\hat{\mu}$		$\hat{\sigma}$		$\hat{\lambda}$	
α	n	Bias	MSE	Bias	MSE	Bias	MSE
	100	0.4180	1.0425	-0.2630	0.1097	-0.0990	0.4080
-1.5	300	0.2040	0.4669	-0.1410	0.0400	-0.0753	0.3381
	500	0.1728	0.3225	-0.1068	0.0240	-0.0640	0.2820
	100	0.4340	0.9918	-0.269	0.1083	-0.1790	0.3223
-1	300	0.2822	0.5156	-0.1502	0.0446	-0.1058	0.1894
	500	0.1760	0.3484	-0.1120	0.0257	-0.1184	0.1571
	100	0.4205	0.9581	-0.2786	0.1129	-0.1783	0.3354
-0.5	300	0.2266	0.3775	-0.1452	0.0364	-0.1144	0.1328
	500	0.1642	0.2576	-0.0993	0.0228	-0.0807	0.0891
	100	0.0037	0.2560	-0.1523	0.0445	0.0024	0.1586
0	300	0.0029	0.0783	-0.0595	0.0177	-0.0018	0.0338
	500	0.0033	0.0577	-0.0396	0.0138	0.0013	0.0157
	100	-0.3574	0.6834	-0.2486	0.0897	0.2022	0.2886
0.5	300	-0.2038	0.3433	-0.1377	0.0353	0.1215	0.1318
	500	-0.1468	0.2695	-0.0999	0.0213	0.0893	0.0866
	100	-0.4639	0.9914	-0.2652	0.1099	0.1923	0.3270
1	300	-0.1905	0.4603	-0.1444	0.0400	0.0894	0.3406
	500	-0.1837	0.3299	-0.0995	0.0261	0.0942	0.1559
	100	-0.3902	1.0763	-0.2737	0.1124	0.0975	0.4141
1.5	300	-0.1905	0.4603	-0.1444	0.0400	0.0894	0.3406
	500	-0.1602	0.3669	-0.0958	0.0245	0.1152	0.3226

Table 2: Results of Simulation

		$\mu=0,\quad\sigma=1$					
		$\hat{\mu}$		$\hat{\sigma}$		$\hat{\lambda}$	
α	n	Bias	MSE	Bias	MSE	Bias	MSE
	100	0.0384	0.2267	-0.0442	0.0183	-0.0853	0.3419
-1.5	300	-0.0048	0.0589	-0.0080	0.0041	-0.0555	0.1554
	500	-0.0023	0.0180	-0.0067	0.0016	-0.0637	0.0957
	100	0.1294	0.1776	-0.0546	0.0166	-0.0329	0.2726
-1	300	0.0357	0.0331	-0.0115	0.0035	0.0211	0.1303
	500	0.0130	0.0127	-0.0041	0.0018	0.0266	0.0770
	100	0.1537	0.1492	-0.0712	0.0160	-0.1283	0.2331
-0.5	300	0.0589	0.0337	-0.0182	0.0034	-0.0109	0.0722
	500	0.0308	0.0126	-0.0085	0.0016	-0.0042	0.0387
	100	-0.0416	0.0525	-0.0209	0.0044	-0.0533	0.1056
0	300	-0.0049	0.0098	-0.0055	0.0011	-0.0023	0.0161
	500	-0.0015	0.0044	-0.0023	0.0006	-0.0006	0.0081
	100	-0.0744	0.0495	-0.0176	0.0025	0.0037	0.0744
0.5	300	-0.0160	0.0049	-0.0045	0.0006	-0.0010	0.0171
	500	-0.0095	0.0027	-0.0033	0.0004	0.0039	0.0088
	100	-0.0749	0.0493	-0.0224	0.0032	0.0300	0.2177
1	300	-0.0161	0.0085	-0.0065	0.0008	0.0415	0.0829
	500	-0.0133	0.0049	-0.0040	0.0005	0.0137	0.0458
	100	-0.0532	0.0409	-0.0211	0.0044	0.0028	0.4092
1.5	300	-0.0145	0.0101	-0.0059	0.0008	0.0594	0.2446
	500	-0.0111	0.0058	-0.0040	0.0005	0.0498	0.1691

7 Real Life Applications

Two real life data sets are used in this section to assess the suitability of the tri-modal alpha skew normal $TASN(\mu,\sigma,\alpha)$ distribution . Comparisons are made between the fitted model and the normal distribution $N(\mu,\sigma)$, skew normal distribution $SN(\mu,\sigma,\lambda)$, and alpha skew normal $ASN(\mu,\sigma,\alpha)$ distribution . With the help of the GenSA package in R software, the value of the fitted models is calculated using maximum likelihood techniques, and for model comparison, the Akaike and Bayesian information criterion (AIC) are taken into account.

7.1 N Latitude Degrees Dataset

During this illustration, the data set pertaining to N latitude degrees in 69 samples from global lakes is taken into consideration, which was previously analyzed by Shafiei et al. (2016). Table 3 reports the maximum likelihood estimate of the fitted models along with the log-likelihood AIC and BIC values, and Fig. 6 also shows the same behavior of the fitted models.
It becomes apparent from the Table 3 that AIC and BIC values of TASN distribution are lower than those of the other probability distribution being compared. Additionally, Fig. 6 shows that our suggested distribution is well fitted.

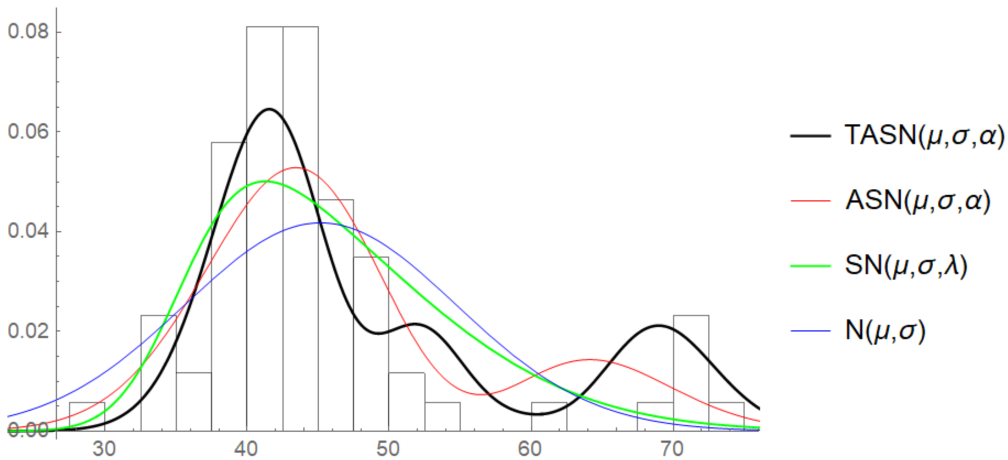


Figure 6: Plots of observed and expected densities for N latitude degrees data.

Table 3: MLEs, log-likelihood, AIC and BIC for N latitude degrees in 69 samples

Distributions	μ	σ	λ	α	$logL$	AIC	BIC
$N(\mu,\sigma)$	45.165	9.548	–	–	-253.599	511.198	515.666
$SN(\mu,\sigma,\lambda)$	35.344	13.698	3.687	–	-243.036	492.072	498.774
$ASN(\mu,\sigma,\alpha)$	52.147	7.7141	–	2.041	-235.370	476.739	483.441
$TASN(\mu,\sigma,\alpha)$	54.755	5.437	–	1.329	-233.564	473.128	479.830

7.2 Lifetime of Certain Dataset

The dataset 30 sample sizes representing the lifetime of a certain device which was reported in Kosznik-Biernacka (2007) is considered for this illustration. The maximum likelihood estimate of the fitted models with the value of log-likelihood AIC and BIC are reported in Table 4 while the behavior of the fitted models are included in Fig. 7.

Table 4: MLEs, log-likelihood, AIC and BIC for lifetime of a certain device

Distributions	μ	σ	λ	α	$logL$	AIC	BIC
$N(\mu,\sigma)$	9.038	3.810	—	—	-82.673	169.346	172.148
$SN(\mu,\sigma,\lambda)$	13.607	5.946	-11.794	—	-76.993	159.986	164.189
$ASN(\mu,\sigma,\alpha)$	6.845	3.064	—	-1.497	-78.609	163.218	167.422
$TASN(\mu,\sigma,\alpha)$	5.907	2.129	—	-0.910	-74.671	155.342	159.546

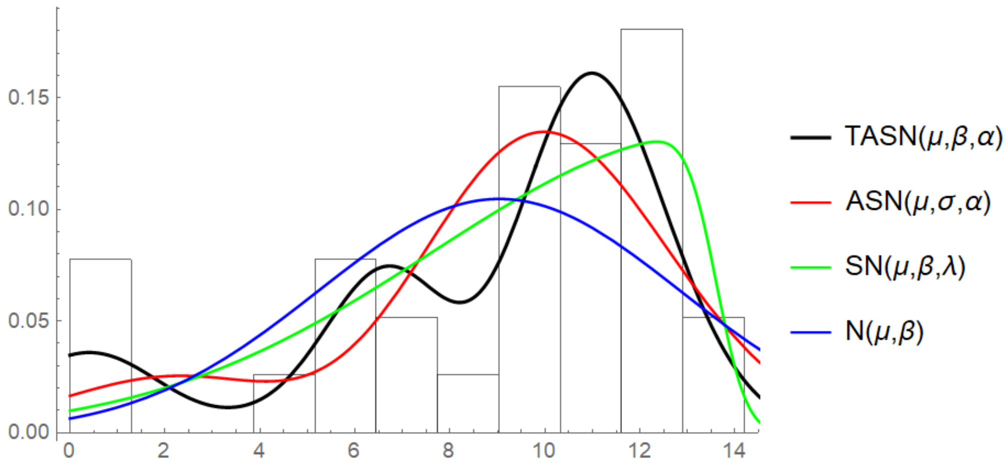


Figure 7: Plots of observed and expected densities for lifetime data of a certain device.

It is also clear from the Table 4 that the AIC and BIC values of TASN distribution are lower than those of the other probability distribution being compared. Fig. 7 also shows that our proposed distribution is good fitted.

8 Hypothesis Testing

To discriminate between $N(\mu, \sigma)$ and $TASN(\alpha)$, the likelihood ratio (LR) test is used. The test statistics as well as the null hypothesis of the test are as follows:

(i) The null hypothesis $H_0 : \alpha = 0$ have to test against the alternative hypothesis $H_1 : \alpha \neq 0$. The test statistic is

$$-2\log(LR) = -\,2[\log L(\hat{\mu}_1, \hat{\sigma}_1, , \alpha = 0|x) - \log L(\hat{\mu}_2, \hat{\sigma}_2, \hat{\alpha}_2,] \sim \chi^2_1,$$

where $(\hat{\mu}_1, \hat{\sigma}_1$ and $(\hat{\mu}_2, \hat{\sigma}_2, \hat{\alpha}_2)$ are the MLEs of $N(\mu, \sigma)$ distribution and $TASN(\mu, \sigma, \alpha)$ distribution respectively; and $r = 1$ (difference between the numbers of parameters).

Table 5: LRT for different hypotheses for the data set I and II.

Hypothesis	LRT statistic		d.f.	Critical Values at 5 %
	Dataset I	Dataset II		
$H_0 : \alpha = 0$ Vs $H_1 : \alpha \neq 0$	39.970	16.004	1	3.841

From Table 5, it can be noted that the value of the LRT statistics is higher than the tabulated critical value at 95% of confidence level for dataset I and II. Consequently, it can be stated that the collected data originates from the $TASN(\mu, \sigma, \alpha)$ distribution.

9 Conclusion

This article introduces a novel family of skew normal distributions called tri-modal alpha skew normal (TASN) distribution considering a new extension of the alpha skew normal distribution. Some essential statistical properties of the new distribution are studied with graphical representations. The characterizations of the TASN distribution using two truncated moments are also presented. In order to explore the issues with estimating the parameters of our new distribution, the MLE method is used. A simulation study using the MH algorithm is carried out to evaluate the performance of maximum likelihood estimators by creating random samples. Besides, for observing the usefulness of the new distribution, two real life applications are included where comparing the novel distribution with the already existed models it may be concluded that the new probability distribution model is better fitted than the others. Finally LR test is performed to examine that the data considered in real life applications comes from the tri-modal alpha skew normal distribution.

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