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The Kth-Order Equilibrium Rayleigh Distribution: Characterization and Estimation

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Abstract

This manuscript presents a new extension of Rayleigh distribution by employing the concept of Kth order equilibrium method. The introduced model is termed as the Kth-order equilibrium Rayleigh distribution (KERD). Various statistical properties of the new distribution, including its aging behavior and stochastic ordering relations are analyzed. Explicit expressions are derived for moments, conditional moments, incomplete moments, the mean residual function, the mean waiting function, entropy measures and order statistics. Distribution characterization has been examined. Maximum likelihood estimation method is used to estimate the parameters. A simulation study using the Anderson–Darling test statistic is carried out to analyze the asymptotic behavior of maximum likelihood estimators. The behaviors of bias and mean square error are observed with the increase in sample size. The applications of new distribution are demonstrated using two different real life datasets. Ultimately, a comparison is conducted among KERD and its sub-models regarding their fit using information criterion tools.

Key Words: Rayleigh distribution, Equilibrium distribution, Contour plots, Maximum likelihood estimation, Anderson–Darling test statistic.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Lifetime distributions have been extensively used to model and analyze data across diverse fields, including medicine, engineering and business. The accuracy and reliability of statistical analysis largely depend on the selection of an appropriate probability distribution. For example, many commonly used lifetime models such as the exponential, Weibull and Rayleigh distributions only admit monotonic hazard rate behavior. However, real-world datasets often diverge from standard, well-documented probability distributions in the literature, leading to unique challenges. This leads to the growth and expansion of generalized probability models. Designing a new probability model from the formerly constructed models by using different approaches has got enormous scope in the recent years. One such approach used by different researchers is weighted probability technique introduced by Fisher (1934), further Gupta and Kirmani (1990) showed that the weighted distribution is special case of equilibrium distribution of Cox (1962) when we take inverse of the hazard rate as the weight. Equilibrium distributions play a vital role in reliability in

proving the characteristics of stochastic orders. Since 1962 after introduction of equilibrium distributions, researchers from a variety of domains have become interested in it. Some important contributions include, Willmot et al. (2005), Nair and Preeth (2008), Wu (2013) and Anas et al. (2023).
This paper introduces a novel extension of the Rayleigh distribution, called the Kth order equilibrium Rayleigh distribution utilizing weighted equilibrium technique of Anas et al. (2023). For continuous random variable Y the density function of Kth-order weighted equilibrium distribution is given by

$$f_E(y) = \frac{ky^{k-1}R(y)}{E\left[\frac{ky^{k-1}}{h(y)}\right]} \; ; \quad y > 0, k \in R \tag{1}$$

where, $E(Y^k) < \infty$. and $f(y), F(y), R(y)$, and $h(y)$ represents density, distribution, survival and hazard functions, respectively.
Rayleigh distribution was Originally developed by Rayleigh (1880) to describe noise generated by multiple independent sources, it has been extensively applied in life testing, reliability analysis, applied statistics and medical studies. The PDF and CDF of Rayleigh distribution are respectively given by:

$$f(y) = \frac{2y}{\zeta} e^{-\frac{y^2}{\zeta}} \; ; \quad y > 0, \zeta > 0 \tag{2}$$

$$F(y) = 1 - e^{-\frac{y^2}{\zeta}} \; ; \quad y > 0, \zeta > 0 \tag{3}$$

Many researchers developed various generalizations of Rayleigh probability distribution to increase the flexibility in lifetime sample modeling for instance, Ateeq et al. (2019) explored an extension of the Rayleigh distribution, performing parameter estimation and simulations, along with applying the model to real-life datasets. Bhat and Ahmad (2020) studied the power Rayleigh distribution and demonstrated its adaptability to real-life datasets. Kilai et al. (2022) utilized the Rayleigh distribution in the medical field, introducing a versatile modification specifically designed for modeling COVID-19 mortality rates. Mir and Ahmad (2024) studied Sine Power Rayleigh distribution.
Although several generalizations of the Rayleigh distribution have been proposed in the literature to increase flexibility, many of these models rely primarily on algebraic extensions that do not provide a distinct reliability-based interpretation and may exhibit limited hazard rate behavior. In particular, some existing models fail to accommodate a wide range of hazard rate shapes within a unified framework. To address this gap, we introduce the KERD, which is constructed using a reliability-based weighting mechanism. This formulation not only offers a new structural interpretation but also enhances flexibility, allowing the hazard rate function to exhibit increasing, decreasing, decreasing-increasing and constant forms, as illustrated in figure (3). Therefore, the proposed model provides both theoretical novelty and practical applicability in reliability and lifetime data analysis.

2. Definition of KERD.

The PDF of KERD is obtained by substituting the PDF (2) of the Rayleigh distribution into the basic definition of the kth-order weighted equilibrium distribution (1). Thus, the resulting PDF of KERD is given by:

$$f_E(y) = \frac{2y^{k-1} e^{-\frac{y^2}{\zeta}}}{\zeta^{\frac{k}{2}} \Gamma(\frac{k}{2})} \; ; \quad y > 0, \zeta > 0, k > 0 \tag{4}$$

The corresponding CDF of KERD is given by:

$$F_E(y) = 1 - \frac{\Gamma(\frac{k}{2}, \frac{y^2}{\zeta})}{\Gamma(\frac{k}{2})} \; ; \quad y > 0, \zeta > 0, k > 0 \tag{5}$$

The first order derivative of the density $f_E(y)$ is given by

$$f_E'(y) = \frac{2y^{k-2}e^{-\frac{y^2}{\zeta}}}{\zeta^{\frac{k}{2}}\Gamma(\frac{k}{2})}\left(\frac{k-1}{2} - \frac{y^2}{\zeta}\right) \; ; \quad y > 0, \zeta > 0, k > 0 \tag{6}$$

- for $k \leq 1$, $f_E'(y)$ is negative which implies $f_E'(y)$ is decreasing for all $y > 0$. Therefore the mode of KERD is zero for $k \leq 1$.

- for $k > 1$, $f_E'(y)$ is increasing for $y < \sqrt{(\frac{\zeta(k-1)}{2})}$ and is decreasing for $y < \sqrt{(\frac{\zeta(k-1)}{2})}$ and maximum at $y = \sqrt{(\frac{\zeta(k-1)}{2})}$. Therefore mode of KERD is $y = \sqrt{(\frac{\zeta(k-1)}{2})}$.

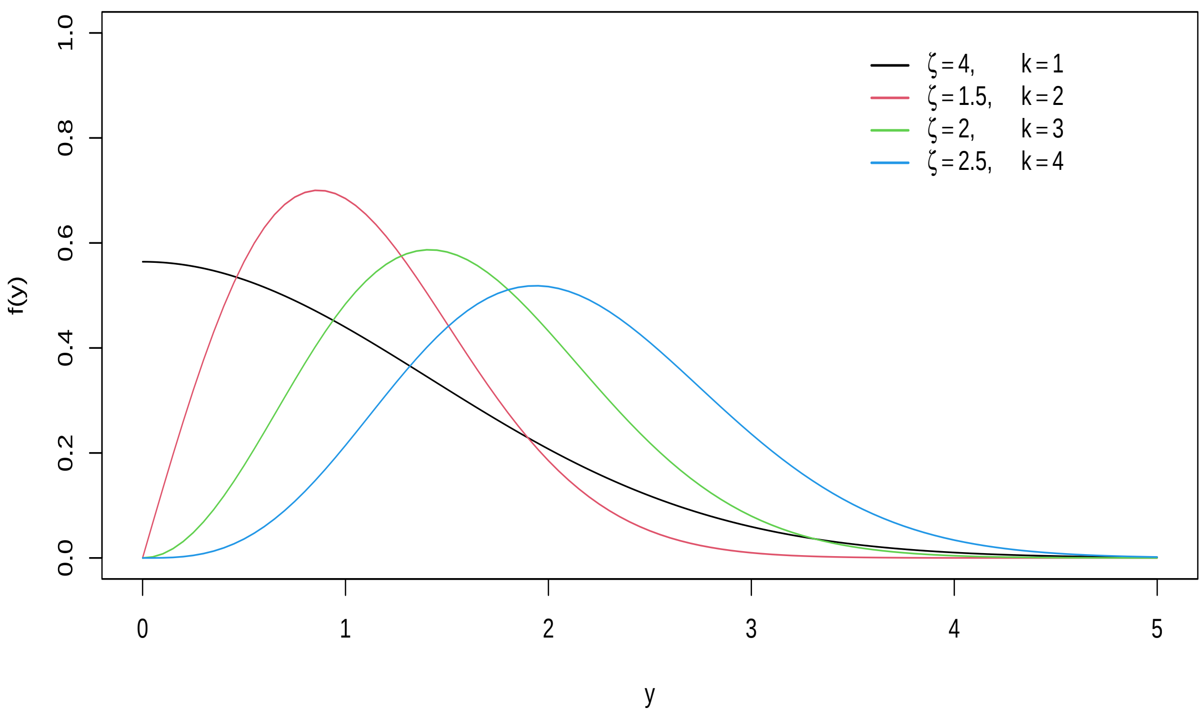

Figure 1: The PDF curves of KERD for different parameter values.

Remark: For $k = 1$, the pdf curve resembles a half-normal distribution. Elsewhere, the graph is positively skewed Gaussian type. As ζ and k increase, the curve becomes more symmetric and less peaked, indicating higher concentration around the center and reduced relative spread.

2.1. Sub models of KER Distribution:

The Kth-order equilibrium Rayleigh distribution reduces to several well-known distributions under specific choices of the parameter k and ζ . The reductions below are obtained by substituting the stated parameter values directly into Equation (4) and simplifying to their standard forms. For algebraic convenience, the scale parameter ζ is expressed in terms of the parameter ζ' and β in the standard definitions of the corresponding sub models.

Table 1: The sub-models of Kth-order equilibrium Rayleigh distribution are:

Sub-Models	Parameter restriction	PDF
Half normal distribution (HND)	$k=1, \zeta=2(\zeta')^2$	$f(y)=\frac{\sqrt{2}e^{-\frac{y^2}{2(\zeta')^2}}}{\sqrt{\pi}\zeta'}$
Rayleigh distribution (RD)	$k=2, \zeta=2(\zeta')^2$	$f(y)=\frac{ye^{-\frac{y^2}{2(\zeta')^2}}}{\zeta'^2}$
Maxwell-Boltzmann distribution (MB)	$k=3, \zeta=2(\zeta')^2$	$f(y)=\sqrt{\frac{2}{\pi}}(\zeta')^{-3}y^2e^{\frac{-y^2}{2(\zeta')^2}}$
Area biased Rayleigh distribution (ABRD)	$k=4, \zeta=2(\zeta')^2$	$f(y)=\frac{y^3e^{\frac{-y^2}{2(\zeta')^2}}}{2(\zeta')^4}$
Length biased weighted rayleigh distribution (LBWRD)	$k=4, \zeta=\frac{\zeta'}{\beta}$	$f(y)=\frac{2y^3\beta^2e^{\frac{-\beta y^2}{\zeta'}}}{(\zeta')^2}$
Area biased Maxwell distribution (ABMD)	$k=5, \zeta=\frac{2}{\zeta'}$	$f(y)=\frac{y^4(\zeta')^{\frac{5}{2}}e^{\frac{-\zeta' y^2}{2}}}{2^{\frac{3}{2}}\Gamma(\frac{5}{2})}$

3. Reliability Analysis

3.1. Reliability function and hazard rate function of KERD

The reliability function $(R_E(Y))$ and hazard rate function $(h_E(Y))$ of KERD are given respectively by:

$$R_E(y)=\frac{\Gamma(\frac{k}{2},\frac{y^2}{\zeta})}{\Gamma(\frac{k}{2})}\ ; \quad y>0, \zeta>0, k>0 \tag{7}$$

$$h_E(y)=\frac{2y^{k-1}e^{\frac{-y^2}{\zeta}}}{\zeta^{\frac{k}{2}}\Gamma(\frac{k}{2},\frac{y^2}{\zeta})}\ ; \quad y>0, \zeta>0, k>0 \tag{8}$$

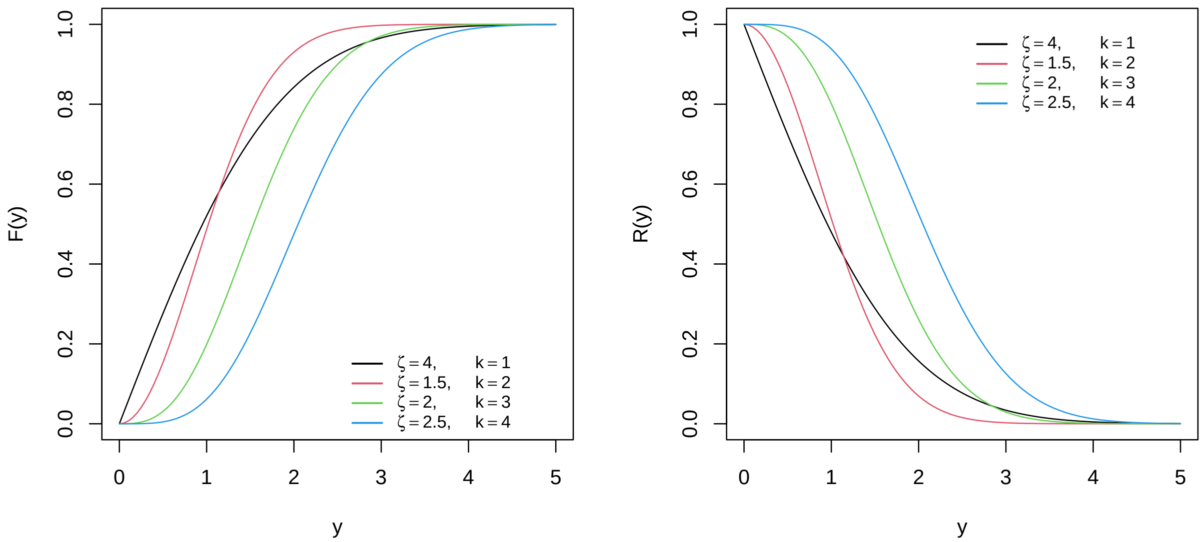

Figure 2: The cumulative distribution function and reliability function of KERD for selected parameter values.

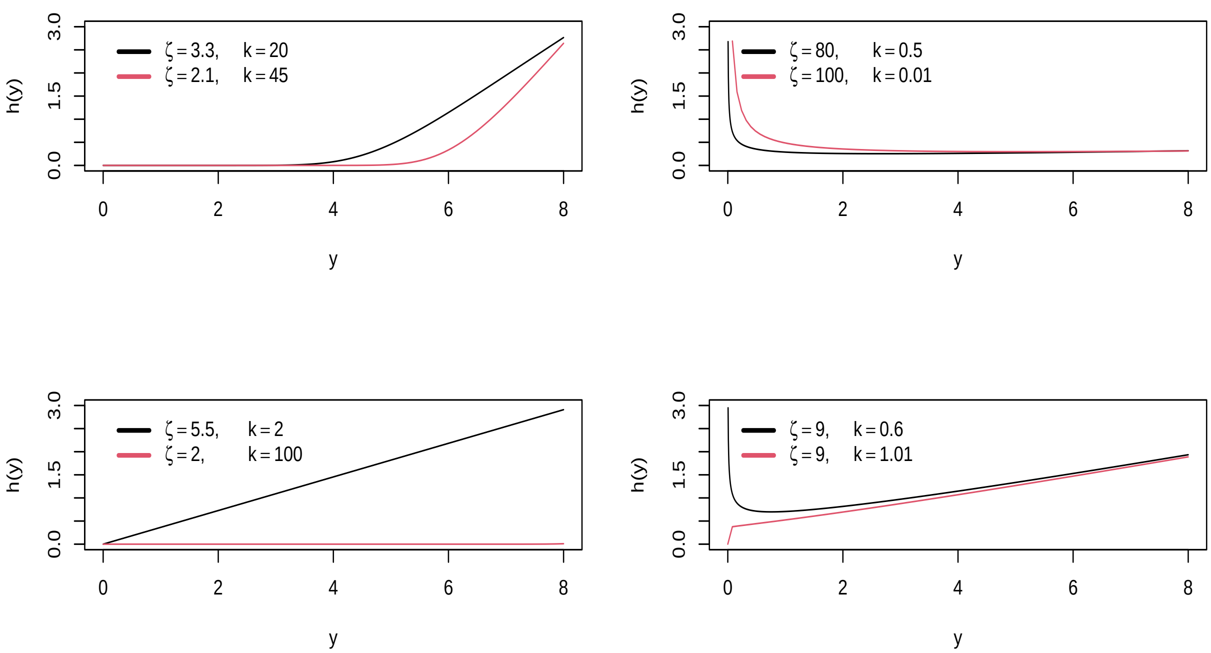

Figure 3: The hazard rate function curves of KERD for selected value of parameters.

Remark: The figure 3 describes the behaviour of hazard rate function of KERD for different choice of parameters and it depicts that the hazard rate function of KERD is increasing, linearly increasing, constant and decreasing-increasing.

3.2. Ordering properties

The ratio of the densities of original random variable (i.e. RD Y_R) and its corresponding Kth-order equilibrium version Y_{KER} is given by

$$r(y) = \frac{y^{2-k}\Gamma(\frac{k}{2})}{\zeta^{(1-\frac{k}{2})}} \tag{9}$$

and the first order derivative of $r(y)$ is given by

$$r'(y) = \frac{y^{1-k}(2-k)\Gamma(\frac{k}{2})}{\zeta^{(1-\frac{k}{2})}} \tag{10}$$

The expression indicates that $r'(y)$ is positive for $k < 2$, negative for $k > 2$, and zero for $k = 2$. Consequently, it follows that,

- for $k < 2$, Y_{KER} exhibits a lower likelihood ratio compared to Y_R ($Y_{KER} \leq_{lr} Y_R$). Based on the chain of implications within various stochastic orders, it can be confidently concluded that $Y_{KER} \leq_{hr} Y_R$ and $Y_{KER} \leq_{mlr} Y_R$. for $k > 2$ converse is true.

The validity of these properties between Y_{KER} and Y_R has been confirmed at predefined values of ζ and k, as detailed in Table 2. This analysis encompasses three cases: $k < 2$, $k > 2$, and $k = 2$. And the results are presented in Table 2 and Figure 4 distinctly support the aforementioned ordering relations.

Table 2: Ordering relations between KERD and Rayleigh distribution.

		$Y = 10, \zeta = 2$			$Y = 10, \zeta = 4$		
Cases	k	$r'(y)$	$h(y)$		$r'(y)$	$h(y)$	
			Y_R	Y_{KER}		Y_R	Y_{KER}
$k < 2$	0.1	152.1048	10	10.1864	78.7344	5	5.1832
	0.5	10.2259	10	10.1472	6.0803	5	5.1446
	1	1.2533	10	10.0981	0.8862	5	5.0964
	1.5	0.1629	10	10.0490	0.1370	5	5.0482
$k = 2$	2	0	10	10	0	5	5
$k > 2$	2.5	-0.0170	10	9.9510	-0.0203	5	4.9519
	3	-0.0125	10	9.9019	-0.0177	5	4.9038
	3.5	-0.0073	10	9.8529	-0.0123	5	4.8557
	4	-0.0040	10	9.8039	-0.0080	5	4.8077

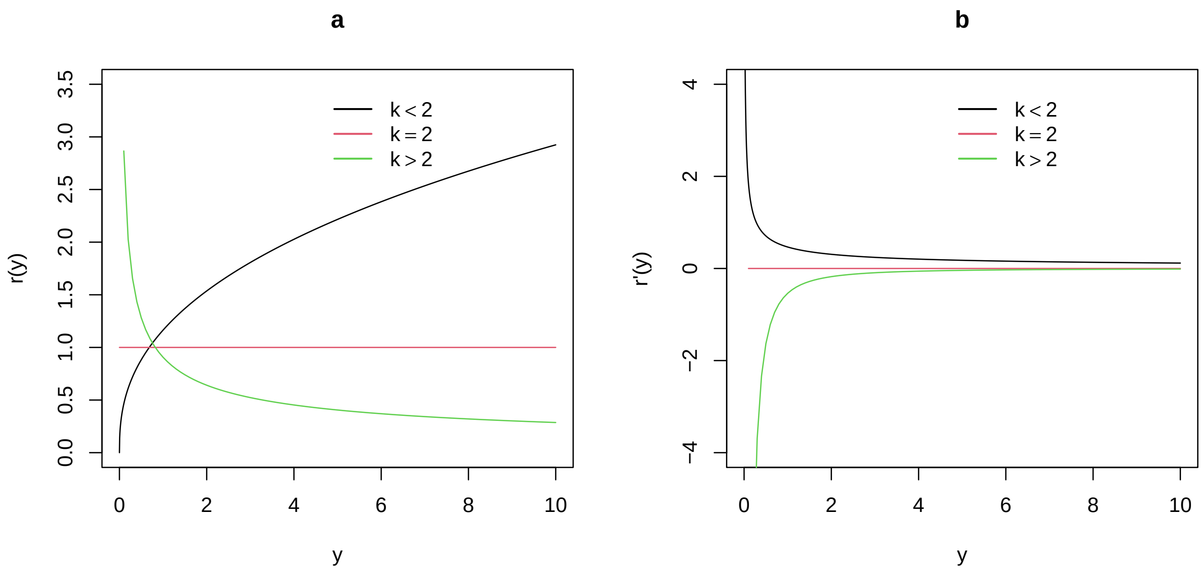

Figure 4: (a) Ratio of density function of RD and KERD, (b) First order derivative of ratio of density function of RD and KERD.

3.3. Reverse hazard rate function

The reverse hazard rate function $((r_E(y))$ of KERD is given by

$$r_E(y) = \frac{f_E(y)}{F_E(y)} = \frac{2y^{k-1}e^{-\frac{y^2}{\zeta}}}{\zeta^{\frac{k}{2}}\left[\Gamma(\frac{k}{2}) - \Gamma(\frac{k}{2}, \frac{y^2}{\zeta})\right]} \;;\quad y > 0, \zeta > 0, k > 0 \tag{11}$$

4. Structural properties of KERD

In this section, some statistical properties of KERD are discussed.

4.1. Moments

In this subsection we compute the r^{th} moment, mean, variance, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of KERD and is given respectively by:

$$E(Y^r) = \zeta^{\frac{r}{2}}\frac{\Gamma(\frac{k+r}{2})}{\Gamma(\frac{k}{2})} \tag{12}$$

$$E(Y) = \zeta^{\frac{1}{2}}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} \tag{13}$$

$$V(Y) = \frac{\zeta}{\left(\Gamma(\frac{k}{2})\right)^2}\left[\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) - \left(\Gamma(\frac{k+1}{2})\right)^2\right] \tag{14}$$

$$CV = \frac{\left[\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) - \left(\Gamma(\frac{k+1}{2})\right)^2\right]^{\frac{1}{2}}}{\Gamma(\frac{k+1}{2})} \tag{15}$$

$$CS = \frac{\Gamma(\frac{k+3}{2})\left(\Gamma(\frac{k}{2})\right)^2 - 3\Gamma(\frac{k}{2})\Gamma(\frac{k+1}{2})\Gamma(\frac{k+2}{2}) - 2(\Gamma(\frac{k+1}{2}))^3}{\left[\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) - \left(\Gamma(\frac{k+1}{2})\right)^2\right]^{\frac{3}{2}}} \tag{16}$$

$$CK = \frac{\Gamma(\frac{k+4}{2})\left(\Gamma(\frac{k}{2})\right)^3 - 4\Gamma(\frac{k+1}{2})\Gamma(\frac{k+3}{2})\left(\Gamma(\frac{k}{2})\right)^2 + 6\left(\Gamma(\frac{k+1}{2})\right)^2\Gamma(\frac{k+2}{2})\Gamma(\frac{k}{2}) - 3\left(\Gamma(\frac{k+1}{2})\right)^4}{\left[\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) - \left(\Gamma(\frac{k+1}{2})\right)^2\right]^2} \tag{17}$$

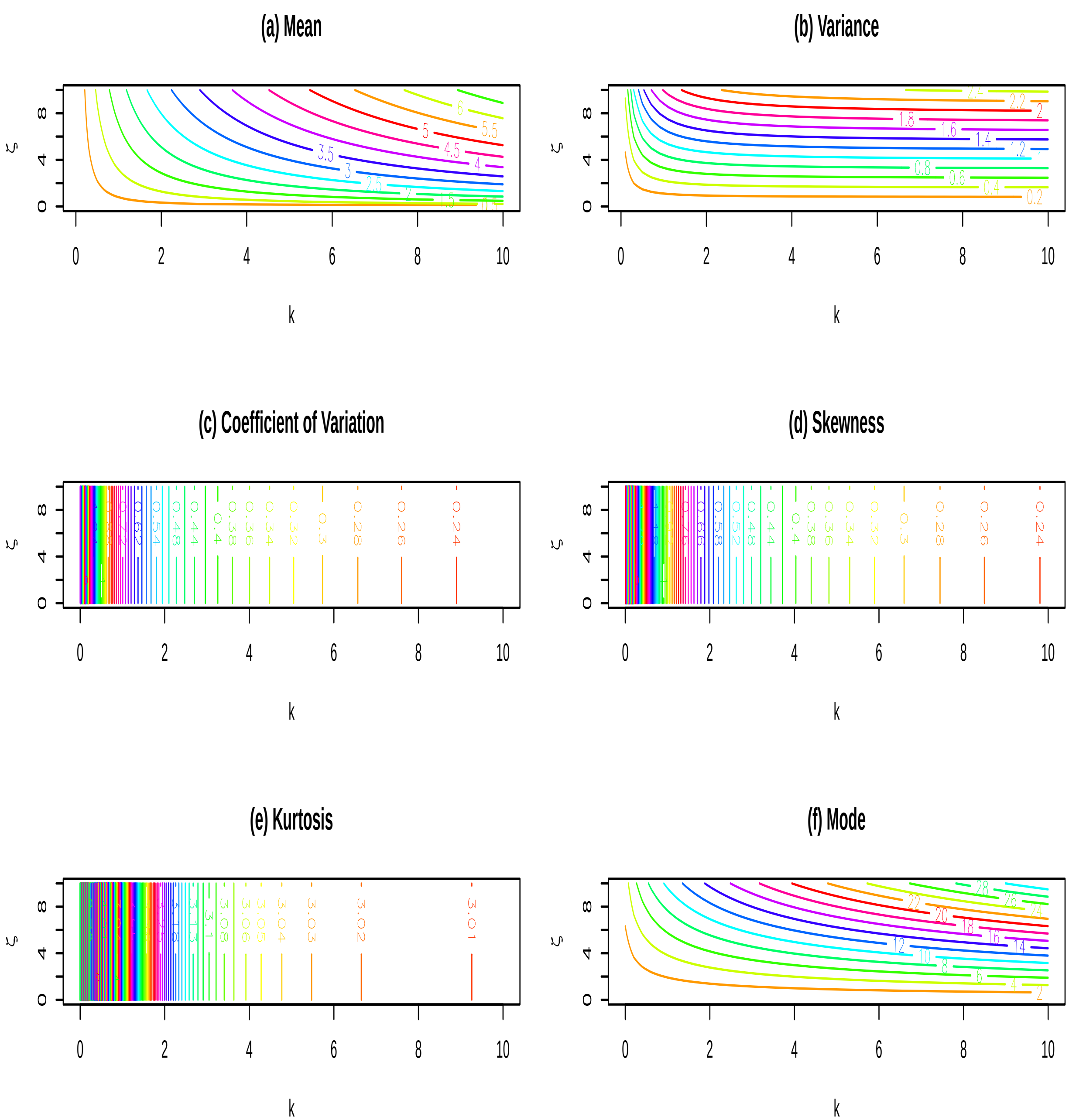

Figure 5: Contour plot for (a) mean, (b) variance, (c) coefficient of variation, (d) skewness, (e) kurtosis and (f) mode.

The contour plots in Figure 5 illustrate the relationships between ζ and k with various statistical properties. They reveal that the mean, variance and mode increase as either ζ or k increases. In contrast, the coefficient of variation (CV), skewness and kurtosis remain unaffected by changes in ζ and decrease as the weight parameter k increases.

4.2. Incomplete moments and Conditional moments

The incomplete moments display the graphical structure of a distribution's moments, which is helpful in various fields such as econometrics, finance and reliability. The n^{th} incomplete moments $(I_E(n,t))$ of KERD is given by:

$$I_E(n,t) = \int_0^t y^n f_E(y)dy$$

$$I_E(n,t) = \frac{\zeta^{\frac{n}{2}}\gamma(\frac{k+n}{2},\frac{t^2}{\zeta})}{\Gamma(\frac{k}{2})} \tag{18}$$

The n^{th} conditional moment of KERD is given by:

$$E(T^n|T>t) = \frac{\int_t^\infty y^n f_E(y)dy}{R(t)}$$

where, $\int_t^\infty y^n f_E(y)dy = \frac{(\zeta)^{\frac{n}{2}}\Gamma(\frac{n+k}{2},\frac{t^2}{\zeta})}{\Gamma(\frac{k}{2})}$

$$E(T^n|T>t) = \frac{(\zeta)^{\frac{n}{2}}\Gamma(\frac{n+k}{2},\frac{t^2}{\zeta})}{\Gamma(\frac{k}{2},\frac{t^2}{\zeta})} \tag{19}$$

4.3. Mean residual life and mean waiting time

Mean residual life $(\mu_E(t))$ of KERD is given by:

$$\mu_E(t) = \frac{1}{R_E(t)}\left[E(t) - \int_0^t y f_E(y)dy\right] - t$$

where, $\int_0^t y f_E(y)dy = \frac{\zeta^{\frac{1}{2}}\gamma(\frac{k+1}{2},\frac{t^2}{\zeta})}{\Gamma(\frac{k}{2})}$

$$\mu_E(t) = \frac{\zeta^{\frac{1}{2}}}{\Gamma(\frac{k}{2},\frac{t^2}{\zeta})}\left[\Gamma(\frac{k+1}{2}) - \gamma(\frac{k+1}{2},\frac{t^2}{\zeta})\right] - t \tag{20}$$

The mean waiting time $\hat{\mu_E}(t)$ of KERD is given by:

$$\hat{\mu_E}(t) = t - \frac{1}{F_E(t)}\int_0^t y f_E(y)dy$$

$$\hat{\mu_E}(t) = t - \frac{\zeta^{\frac{1}{2}}}{\Gamma(\frac{k}{2}) - \Gamma(\frac{k}{2},\frac{t^2}{\zeta})}\gamma(\frac{k+1}{2},\frac{t^2}{\zeta}) \tag{21}$$

4.4. Entropy measures of KERD

Entropy measures provide a quantification of uncertainty or randomness within the system. We have calculated the expressions for Renyi entropy, Arimoto's entropy and Havrda and Charvet entropy. And are respectively given as:

$$H_R(\delta) = \frac{1}{1-\delta}log\left[\frac{(2)^{\delta-1}\zeta^{\frac{1-\delta}{2}}}{\delta^{\frac{(\delta(k-1)+1)}{2}}\left(\Gamma(\frac{k}{2})\right)^{\delta}}\Gamma(\frac{\delta(k-1)+1}{2})\right] \tag{22}$$

$$H_A(\delta) = \frac{1}{2^{\delta-1}-1}\left[\frac{(2)^{1-\delta}\zeta^{\frac{\delta-1}{2}}}{\delta^{\frac{(k-1+\delta)}{2}}\left(\Gamma(\frac{k}{2})\right)^{(\frac{1}{\delta})}}\Gamma(\frac{k-1-\delta}{2\delta}) - 1\right] \tag{23}$$

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$$H_H(\delta) = \frac{1}{1-\delta}\left[\frac{(2)^{\delta-1}\zeta^{\frac{1-\delta}{2}}}{\delta^{\frac{(\delta(k-1)+1)}{2}}\left(\Gamma(\frac{k}{2})\right)^{\delta}}\Gamma(\frac{\delta(k-1)+1}{2})-1\right] \tag{24}$$

4.5. Odds ratio and Mills ratio

The odds ratio $(O_E(y))$ and Mills ratio $(M_E(y))$for KERD are given respectively as:

$$O_E(y) = \frac{\zeta^{\frac{k}{2}}\left[\Gamma(\frac{k}{2})-\Gamma(\frac{k}{2},\frac{y^2}{\zeta})\right]}{2y^{k-1}e^{-\frac{y^2}{\zeta}}} \tag{25}$$

$$M_E(y) = \frac{\zeta^{\frac{k}{2}}\Gamma(\frac{k}{2},\frac{y^2}{\zeta})}{2y^{k-1}e^{-\frac{y^2}{\zeta}}} \tag{26}$$

4.6. Lorenz inequality and Bonferoni inequality

Bonferroni and Lorenz curves measures the inequality of the distribution. The Bonferroni and Lorenz curve for a random variable Y following KERD are respectively given by:

$$L(t) = \frac{\gamma(\frac{k+1}{2},\frac{t^2}{\zeta})}{\Gamma(\frac{k+1}{2})} \tag{27}$$

$$B(t) = \frac{\Gamma(\frac{k}{2})\gamma(\frac{k+1}{2},\frac{t^2}{\zeta})}{\Gamma(\frac{k+1}{2})\left[\Gamma(\frac{k}{2})-\Gamma(\frac{k}{2},\frac{t^2}{\zeta})\right]} \tag{28}$$

4.7. Order statistics

Let us assume that the random sample $Y_1,Y_2,Y_3...,Y_n$ from the KERD, with PDF $(f_E(y))$ and CDF $(F_E(y))$ Let $Y_{(1)},Y_{(2)},Y_{(3)},...,Y_{(n)}$ denote the corresponding order statistics, then the PDF and CDF of the $r^{th}(r=1,2,...n)$ order statistic are respectively, given by:

$$f_r(y) = \frac{n!}{(r-1)!(n-r)!}[F_E(y)]^{r-1}f_E(y)[(1-F_E(y))]^{n-r}.$$

and,

$$F_r(y) = \sum_{j=r}^{n}{}^nC_jF_E^j(y)\left[1-F_E(y)\right]^{n-j}$$

$$f_r(y) = \frac{2n!y^{k-1}\zeta^{-\frac{k}{2}}e^{-\frac{y^2}{\zeta}}}{(r-1)!(n-r)!(\Gamma(\frac{k}{2}))^n}\left[\Gamma(\frac{k}{2})-\Gamma(\frac{k}{2},\frac{y^2}{\zeta})\right]^{r-1}\left[\Gamma(\frac{k}{2},\frac{y^2}{\zeta})\right]^{n-r} \tag{29}$$

and,

$$F_r(y) = \sum_{j=r}^{n}{}^nC_j\frac{1}{(\Gamma(\frac{k}{2}))^n}\left[\Gamma(\frac{k}{2})-\Gamma(\frac{k}{2},\frac{y^2}{\zeta})\right]^{j}\left[\Gamma(\frac{k}{2},\frac{y^2}{\zeta})\right]^{n-j} \tag{30}$$

5. Characterizations of KERD

This section deals with various characterizations of KERD. These characterizations are based on: Coefficient of variation; a simple relationship between two truncated moments; the hazard function; and reverse hazard function.

5.1. Characterization based on coefficient of variation

Let $Y_1, Y_2, Y_3, \ldots, Y_n$ be a random sample of size n drawn from KERD, then the square of sample coefficient of variation is asymptotically unbiased estimator of the square of population coefficient of variation.
Mathematically, $lim_{n\to\infty} E[\frac{S_n{}^2}{\bar{Y_n}^2}] = lim_{n\to\infty} E[\frac{S_n}{\bar{Y_n}}]^2 = [\frac{\sigma}{\mu}]^2$. Where $\bar{Y_n} = \mu$ and $S_n{}^2 = \frac{\sigma^2}{n}$ is mean and variance respectively.
Proof:

$$E(\bar{y_n}^2) = \frac{\zeta}{n\left(\Gamma(\frac{k}{2})\right)^2}\left[\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) + (n-1)\left(\Gamma(\frac{k+1}{2})\right)^2\right] \tag{31}$$

$$E(S_n{}^2) = \frac{\zeta}{\left(\Gamma(\frac{k}{2})\right)^2}\left[\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) - \left(\Gamma(\frac{k+1}{2})\right)^2\right] \tag{32}$$

Now we can write,

$$E(S_n{}^2) = E\left[\frac{S_n{}^2}{\bar{Y_n}^2}\bar{Y_n}^2\right]$$

$$E(S_n{}^2) = E\left[\frac{S_n{}^2}{\bar{Y_n}^2}\right] E\left[\bar{Y_n}^2\right]$$

$$\frac{E\left[S_n{}^2\right]}{E\left[\bar{Y_n}^2\right]} = E\left[\frac{S_n}{\bar{Y_n}}\right]^2$$

Using (31) and (32), we obtain

$$E\left[\frac{S_n}{\bar{Y_n}}\right]^2 = \frac{\left[\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) - \left(\Gamma(\frac{k+1}{2})\right)^2\right]}{\left[\frac{1}{n}\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) + (1-\frac{1}{n})\left(\Gamma(\frac{k+1}{2})\right)^2\right]}$$

taking $\lim_n \to \infty$ on both sides we get

$$\lim_{n\to\infty} E\left[\frac{S_n}{\bar{Y_n}}\right]^2 = \frac{\left(\sqrt(\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2}) - \left(\Gamma(\frac{k+1}{2})\right)^2\right)^2}{\left(\Gamma(\frac{k+1}{2})\right)^2}$$

$$\lim_{n\to\infty} E\left[\frac{S_n}{\bar{Y_n}}\right]^2 = \left(\frac{\sigma}{\mu}\right)^2 \tag{33}$$

5.2. characterization based on truncated moments

This characterization is based on a theorem by Glänzel (1987).(Provided in Appendix as theorem 1)
Let $Y : \Omega \to (0,\infty)$ be a continuous random variable. let $b_1(y) = y^{-(k-2)}$ and $b_2(y) = b_1(y)e^{-\frac{y^2}{\zeta}}$ for $y > 0$ then the random variable Y has PDF (4) iff the function $A(y) = \frac{e^{-\frac{y^2}{\zeta}}}{2}$.
Proof: Suppose the random variable Y has pdf (4).

$$E[b_1(y)|Y \geq y] = q\zeta e^{-\frac{y^2}{\zeta}}$$

$$E[b_2(y)|Y \geq y] = \frac{q}{2}\zeta e^{-\frac{2y^2}{\zeta}} = E[b_1(y)|Y \geq y]A(y)$$

where, $q = \frac{1}{\zeta^{\frac{k}{2}}\Gamma(\frac{k}{2})}$ is constant.
Conversely, if $A(y)$ is of the above form, then it follows that

$$S^{'}(y) = \frac{A^{'}(y)b_(y)}{A(y)b_1(y) - b_2(y)} = \frac{2y}{\zeta}$$

$$S(y) = \frac{y^2}{\zeta}.$$

Thus from theorem Glänzel (1987), F(y) is uniquely determined by the functions $b_1(y), b_1(y)$ and $A(y)$ as follows

$$F(y) = \int_0^y C|\frac{A^{'}(y)}{A(y)b_1(y) - b_2(y)}|e^{-S(y)}dy \tag{34}$$

$$F(y) = \int_0^y C|\frac{2y^{k-1}}{\zeta}|e^{-\frac{y^2}{\zeta}}dy \tag{35}$$

where, C is normalizing constant such that $\int_0^\infty f(y)dy = 1$.

$$c^{-1} = \zeta^{\frac{k}{2}-1}\Gamma(\frac{k}{2}). \tag{36}$$

Thus, using normalizing constant C value in (35) we get the CDF of KERD as follows

$$F(y) = \int_0^y \frac{2y^{k-1}}{\zeta^{\frac{k}{2}}\Gamma(\frac{k}{2})}e^{-\frac{y^2}{\zeta}}dy \tag{37}$$

5.3. Characterization based on HRF

Let $Y : \Omega \to (0, \infty)$ be a continuous random variable. The PDF of Y is (4) iff its hazard rate $h(y)$ satisfies the differential equation,

$$h^{'}(y) - \frac{(k-1)}{y}h(y) = \frac{-y^k}{q\zeta}e^{-\frac{y^2}{2\zeta^2}}\frac{d}{dy}\frac{1}{\Gamma(\frac{k}{2}, \frac{y^2}{\zeta})} \tag{38}$$

where,q is appropriate constant.
Proof: If Y has PDF (4) we get

$$h^{'}(y) - \frac{(k-1)}{y}h(y) = \frac{-y^k}{q\zeta}e^{-\frac{y^2}{2\zeta^2}}\frac{d}{dy}\frac{1}{\Gamma(\frac{k}{2}, \frac{y^2}{\zeta})}$$

Conversely, if HRF satisfies the differential equation above then we have

$$\frac{d}{dy}y^{-k+1}h(y) = \frac{1}{q}\frac{d}{dy}\frac{e^{-\frac{y^2}{\zeta}}}{\Gamma(\frac{k}{2}, \frac{y^2}{\zeta})}$$

$$h(y) = \frac{1}{q}y^{k-1}e^{-\frac{y^2}{\zeta}}\frac{d}{dy}\frac{1}{\Gamma(\frac{k}{2}, \frac{y^2}{\zeta})}$$

where, $q = \frac{2}{\zeta^{\frac{k}{2}}}$,which is HRF of KERD distribution.

5.4. Characterization based on RHRF

Let $Y : \Omega \to (0,\infty)$ be a continuous random variable. The PDF of Y is (4) iff its reverse hazard rate $h(y)$ satisfies the differential equation,

$$r^{'}(y) - \frac{(k-1)}{y}r(y) = \frac{1}{q}y^{k-1}\frac{d}{dy}\frac{e^{-\frac{y^2}{\zeta}}}{\Gamma(\frac{k}{2}) - \Gamma(\frac{k}{2},\frac{y^2}{\zeta})} \tag{39}$$

where,q is appropriate constant.
Proof: If Y has PDF (4) we get

$$r^{'}(y) - \frac{(k-1)}{y}r(y) = \frac{1}{q}y^{k-1}\frac{d}{dy}\frac{e^{-\frac{y^2}{\zeta}}}{\Gamma(\frac{k}{2}) - \Gamma(\frac{k}{2},\frac{y^2}{\zeta})}$$

Conversely, if RHRF satisfies the differential equation above then we have

$$\frac{d}{dy}y^{-k+1}r(y) = \frac{1}{q}\frac{d}{dy}\frac{e^{-\frac{y^2}{\zeta}}}{\Gamma(\frac{k}{2}) - \Gamma(\frac{k}{2},\frac{y^2}{\zeta})}$$

$$r(y) = \frac{1}{q}y^{k-1}\frac{e^{-\frac{y^2}{\zeta}}}{\Gamma(\frac{k}{2}) - \Gamma(\frac{k}{2},\frac{y^2}{\zeta})}$$

where, $q = \frac{2}{\zeta^{\frac{k}{2}}}$, which is RHRF of KERD.

5.5. Parametric estimation of KERD using Maximum likelihood estimation technique

Let $Y_1, Y_2, Y_3, ..., Y_n$ be an observed sample taken from the $KERD(\zeta, k)$ with unknown parameters ζ and k, then the log-likelihood function can be written as

$$logL(y,\zeta,k) = nlog2 + (k-1)\sum_{i=1}^{n} log(y_i) - \frac{\sum_{i=1}^{n} y_i^2}{\zeta} - \frac{nk}{2}log\zeta - nlog\Gamma(\frac{k}{2}) \tag{40}$$

The values of $\hat{\zeta}$ and $\hat{k}$ that maximize the log-likelihood function are called the maximum likelihood estimates of the parameters ζ and k .
The equations obtained on equating the first-order partial derivatives of $logL(y,\zeta,k)$ with respect to ζ and k to zero, are given as

$$\frac{\sum_{i=1}^{n}(y_i)^2}{\zeta^2} - \frac{nk}{2\zeta} = 0 \tag{41}$$

$$\sum_{i=1}^{n} log(y_i) - \frac{n}{2}log\zeta - n\psi(\frac{k}{2}) = 0 \tag{42}$$

$\psi(\frac{k}{2})$ is digamma function, it is not in closed form and cannot be solved algebraically, so we can use Newton–Raphson method.
The associated Fisher information matrix is given by

$$I(\hat{\zeta},\hat{k};y) = -\begin{pmatrix} I_{\zeta\zeta} & I_{\zeta k} \\ I_{k\zeta} & I_{kk} \end{pmatrix}$$

Where,

$$I_{\zeta\zeta} = \frac{\partial^2 \log L}{\partial \zeta^2} = \frac{-2\sum_{i=1}^{n}(y_i)^2}{\zeta^3} + \frac{nk}{2\zeta^2}$$

$$I_{k\zeta} = \frac{\partial^2 \log L}{\partial k \partial \zeta} = \frac{-n}{2\zeta}$$

$$I_{kk} = -n\psi'(\frac{k}{2}).$$

Thus, the approximate variance-covariance matrix is given by

$$\begin{pmatrix} Var(\hat{\zeta}) & Cov(\hat{\zeta}\hat{k}) \\ Cov(\hat{k}\hat{\zeta}) & Var(\hat{k}) \end{pmatrix} = I^{-1}(\hat{\zeta}, \hat{k}; y),$$

and the corresponding 95% confidence intervals of the parameters (ζ, k) are respectively, given by

$$\hat{\zeta} \pm Z_{\alpha/2}\sqrt{\text{var}(\hat{\zeta})}$$

$$\hat{k} \pm Z_{\alpha/2}\sqrt{\text{var}(\hat{k})}$$

where, $Z_{\alpha/2}$ is the upper $\alpha/2$ point of the standard normal distribution and α is the significance level.

6. Simulation study

In this section, a simulation study is conducted with the true parameter values set to $\zeta = 3$ and $k = 2.5$ to assess the behavior of maximum likelihood estimates for ζ and k as the sample size increases. The study also examines the asymptotic normality behavior of these estimators using the Anderson-Darling test statistic. Additionally, the variance-covariance matrix is calculated to explore the relationships between the parameters.

Behavior of Variance, Bias and MSE: To evaluate the behavior of variance, bias and MSE. Random samples are generated using the inverse CDF method for sample sizes $n = 10, 11, \ldots, 200$, with 500 samples for each n. The MLEs of ζ and k are calculated for each sample, along with their varaince, bias and mean squared error (MSE). These measures are plotted against the sample sizes to analyze trends, as illustrated in figure (6).

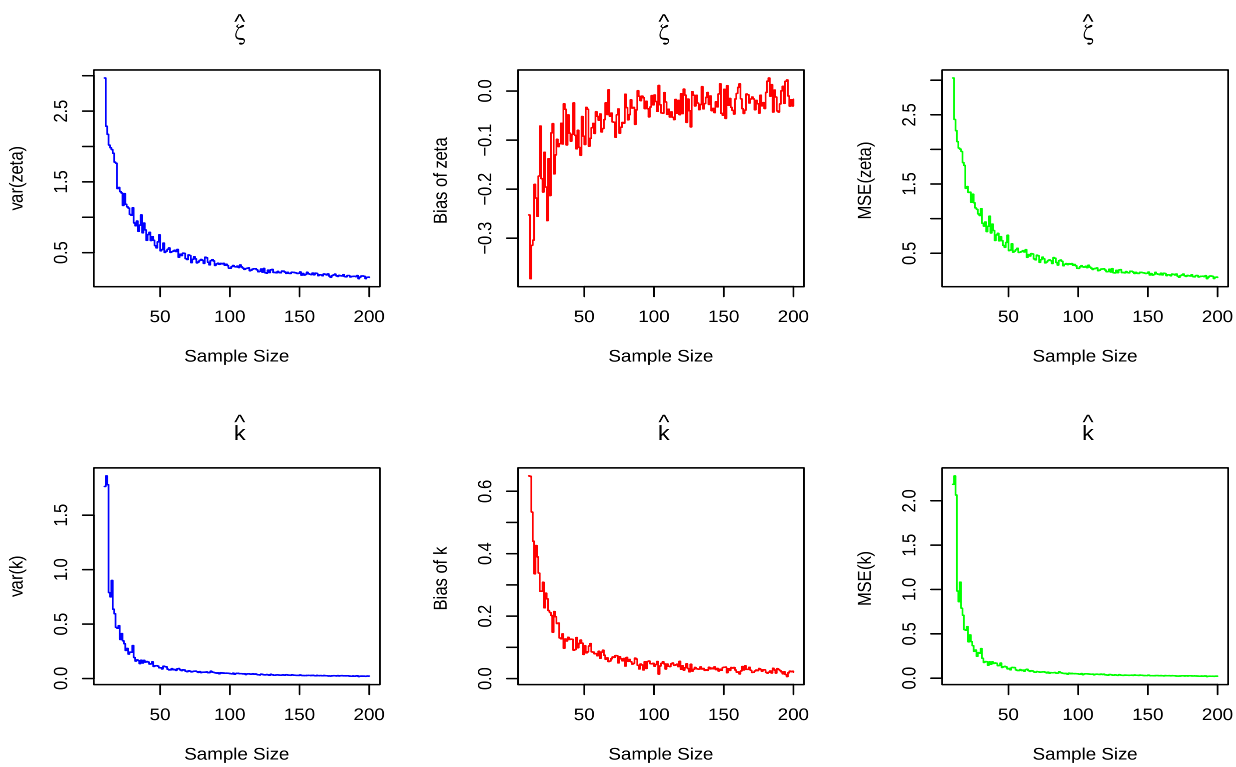

Figure 6: Behavior of Variance ,bias and mean square errors of MLE's.

From Fig. 6, it can be seen that k has a positive bias, whereas ζ has a negative bias. As the sample size increases, the bias for both parameters gradually decreases and approaches zero. Similarly, the variance and mean squared errors (MSE) decrease as the sample size increases. Both trends downward and eventually reaches zero.

Asymptotic normality behavior of MLE's: Asymptotic normality behavior of the MLEs is analyzed by repeatedly generating samples of size $n = 20, 21, \ldots, 300$ for 500 iterations, resulting in 500 sets of parameter estimates for each sample size. The AD test statistic and corresponding P-values are plotted against various sample sizes, with the trace plots shown in Fig. 7.

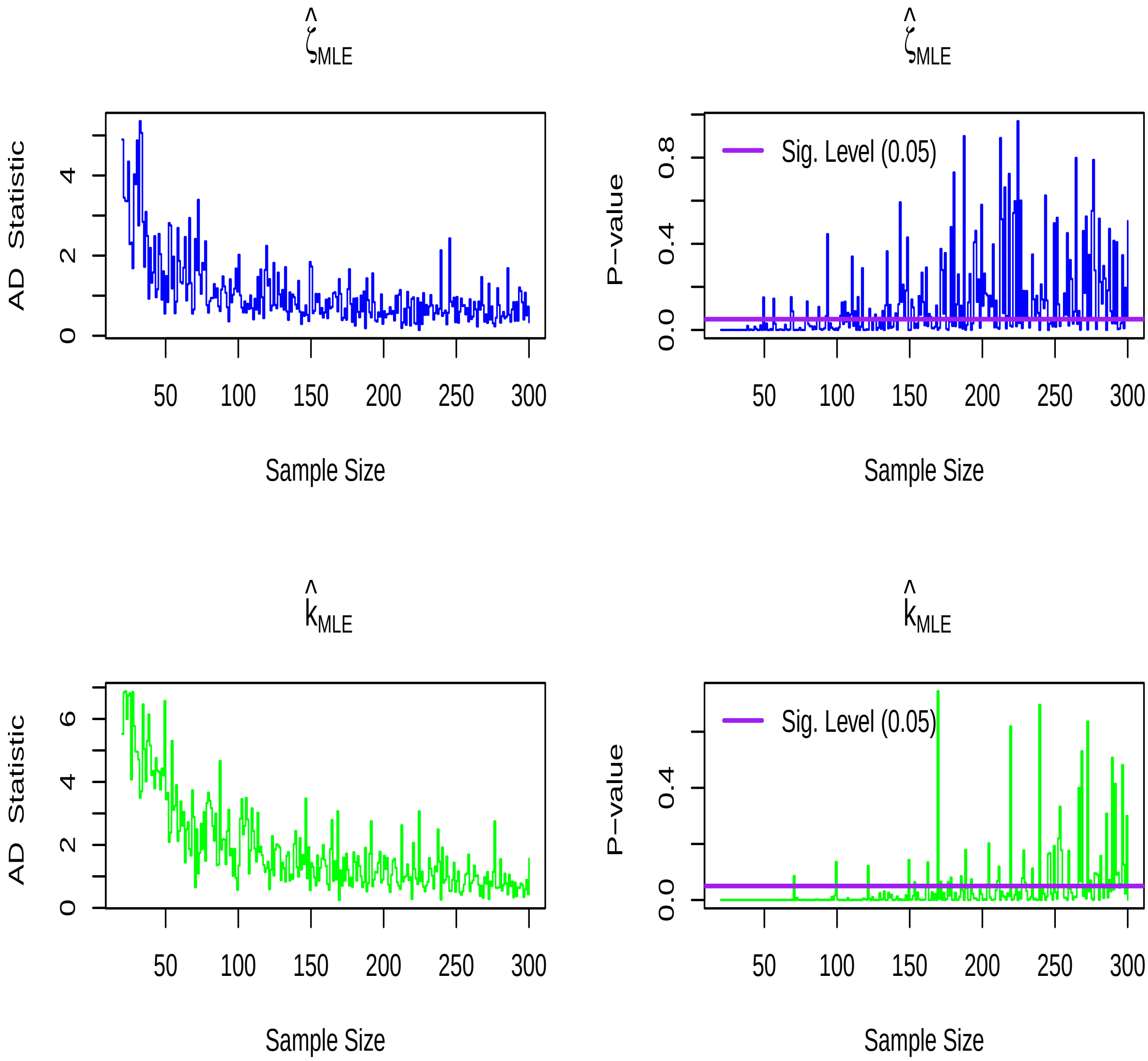

Figure 7: Asymptotic normality behavior of the MLEs for the parameters of KERD with increasing sample size, assessed using the Anderson-Darling test.

From the figure 7, it is evident that as the sample size increases, the AD test statistic decreases and approaches zero for both parameters. Additionally, the P-values frequently cross the significance level of 0.05 as the sample size grows. The MLEs for ζ begin to behave asymptotically normal more quickly, while the MLE for k converges to normality at a slightly slower rate.

Asymptotic variance-covariance and Fisher information matrix: To determine the asymptotic variance-covariance and Fisher information matrix, a sample size of 150 is repeatedly generated 500 times, resulting in an empirical sampling distribution for $\hat{\zeta}$ and $\hat{k}$. The variance-covariance matrix, Fisher information matrix, and 95% asymptotic confidence intervals obtained from the empirical sampling distribution are presented in Table (3). Additionally, the pair plot is shown in Fig. (8).

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Table 3: Fisher Information $I(\mathbf{a}_{\mathbf{mle}})$ and Covariance Matrix $I^{-1}(\mathbf{a}_{\mathbf{mle}})$ of Parameters of KERD

n=150, r=500		$\mathrm{I}(a_{\mathrm{mle}})$		$\mathrm{I}^{-1}(\mathbf{a}_{\mathrm{mle}})$		
Parameters	MLEs	$\hat{\zeta}_{mle}$	$\hat{k}_{mle}$	$\hat{\zeta}_{mle}$	$\hat{k}_{mle}$	95% Asymptotic C.I.
ζ	$\hat{\zeta}_{mle} = (2.444)$	39.7320	22.7905	0.1231	-0.1278	$\hat{\zeta}_{mle}(1.5567, 2.9320)$
k	$\hat{k}_{mle} = (3.7003)$	22.7905	18.0303	-0.1278	0.1819	$\hat{k}_{mle}(2.8645, 4.5361)$

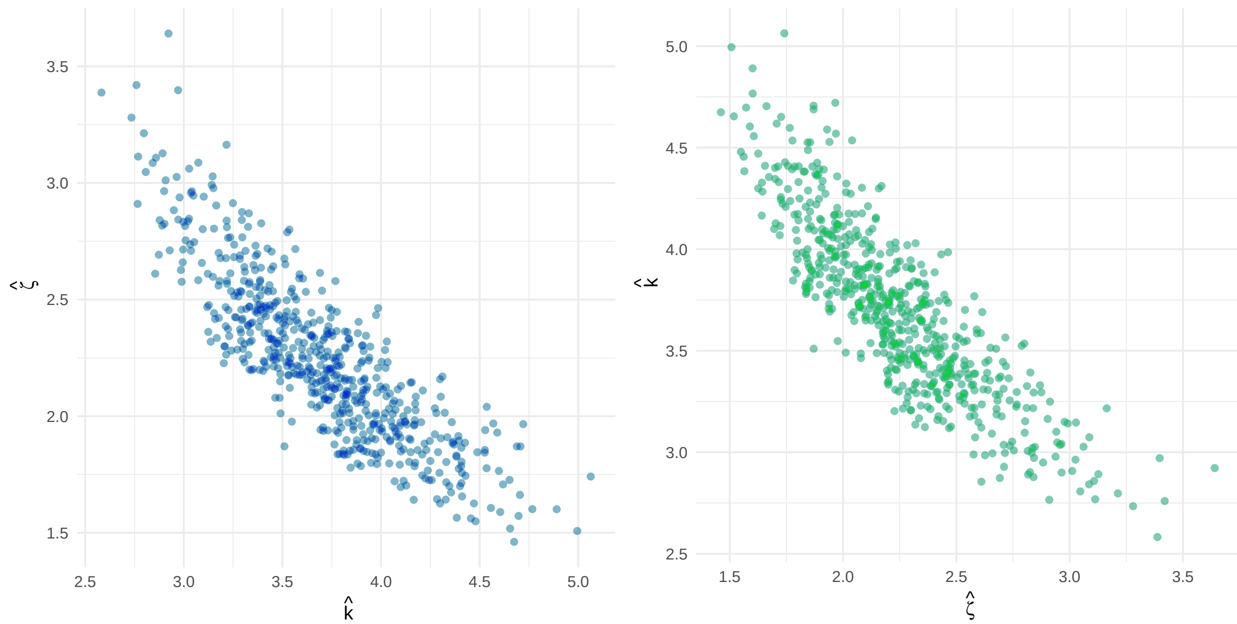

Figure 8: Pair-plot of MLE's of parameters of KERD.

From Table (3), it is observed that higher Fisher information corresponds to lower variance in the parameters. Both Table (3) and Fig. (8) indicate that parameters of KERD $\hat{k}$ and $\hat{\zeta}$ are negatively correlated.

7. Applications

This section demonstrates the practical utility and adaptability of the KERD through analyzing two real-life datasets. The aim is to evaluate the model's versatility and compatibility compared to its sub-models. To achieve this, we employ maximum likelihood estimation for parameter estimation and utilize a range of model selection tools, including AIC, BIC, AICC and HQIC. Typically, a superior distribution is characterized by smaller values of these model selection tools.

Dataset I: This dataset from Lawless 2011 represents represents the strength, measured in GPA, was assessed for single carbon fibers and laden 1000-carbon fiber tows. Single fibers were subjected to tensile testing at a gauge length of 20 mm.

Dataset II: This dataset from Lawless 2011 represents the strength, measured in GPA, was assessed for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were subjected to tensile testing at a gauge length of 10 mm.

Table 4: Discriptive statistics of dataset I and dataset II.

Datasets	Mean	Mode	Q_1	Median	Q_3	Maximum	*Variance*	Skewness	Kurtosis
Dataset I	2.477	2.301	2.150	2.513	2.816	3.585	0.238	-0.154	2.951
Dataset II	3.059	2.937	2.553	2.996	3.422	5.020	0.386	0.633	3.286

Table (4) demonstrates that Dataset I exhibits near symmetry with very mild negative skewness and mesokurtic behavior, indicating moderate dispersion and approximately normal tail weight. In contrast, Dataset II shows moderate positive skewness and slightly heavier tails.

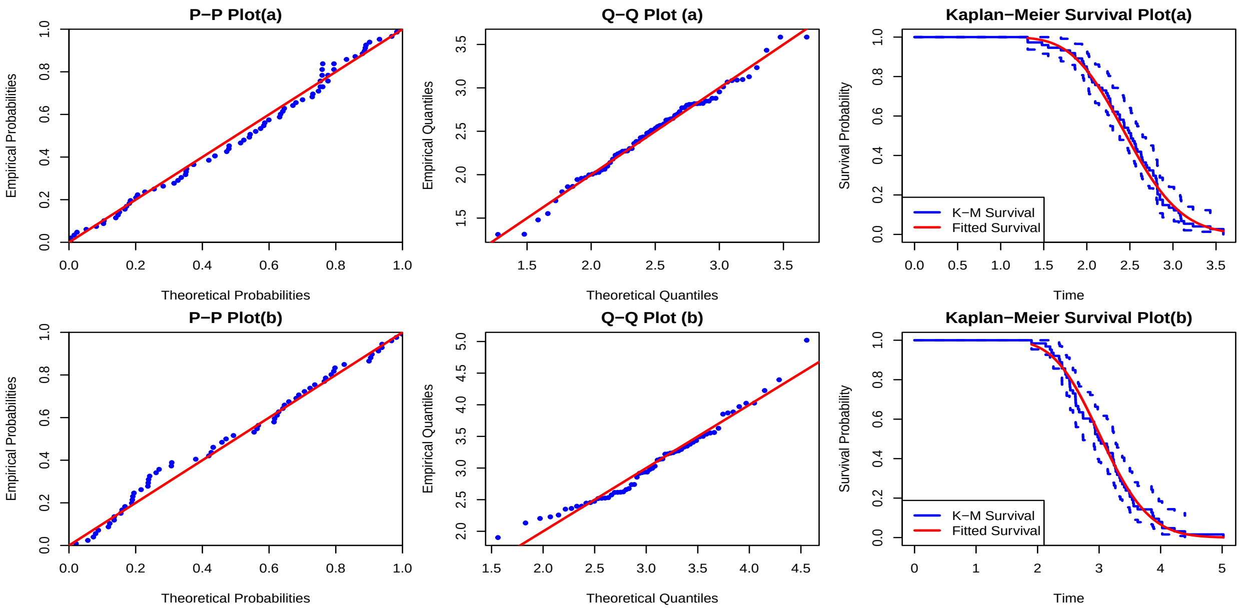

Figure 9: (a) Probability-Probability (P-P) plot, Quantile-Quantile (Q-Q) plot, and Kaplan–Meier survival plot for dataset I and (b) Probability-Probability (P-P) plot, Quantile-Quantile (Q-Q) plot, and Kaplan–Meier survival plot for dataset II.

Figure (9) presents the Probability-Probability (P-P) plot, Quantile-Quantile (Q-Q) plot, and Kaplan–Meier survival plot for both datasets, demonstrating the proposed model's strong compatibility with the data. Table 5, 6 shows the maximum likelihood estimates and different information measure for dataset I and II respectively. and figure 10 shows the plots of estimated density functions of KERD and its submodels.

Table 5: Estimates with corresponding standard error (provided in parenthesis) and Performance measure of the distribution for the dataset I.

Model	Parameters			Information criterion tools				
	ζ	k	β	$-2l$	AIC	BIC	AICC	HQIC
KERD	**1.010**	**12.643**	-	**104.083**	**108.083**	**112.691**	**108.252**	**109.921**
	(0.173)	**(2.079)**						
ABMD	1.01	-	-	143.799	145.799	148.103	145.855	146.718
	(0.074)							
LBMD	1.262	-	-	142.833	144.833	147.137	144.889	145.753
	(0.052)							
LBWRD	2.625	-	0.824	142.833	146.833	151.442	147.002	148.672
	(114.925)		(36.075)					
MD	1.457	-	-	160.264	162.264	164.568	162.320	163.183
	(0.069)							
RD	1.785	-	-	188.302	190.302	192.606	190.357	191.221
	(0.104)							
HND	2.524	-	-	244.453	246.453	248.757	246.508	247.372
	(0.207)							

Table 6: Estimates with corresponding standard error (provided in parenthesis) and Performance measure of the distribution for the dataset II.

Model	Parameters			Information criterion tools				
	ζ	k	β	$-2l$	AIC	BIC	AICC	HQIC
KERD	**1.502** **(0.271)**	**12.971** **(2.254)**	-	**115.311**	**119.311**	**123.598**	**119.511**	**120.997**
ABMD	0.513 (0.041)	-	-	138.222	140.222	142.366	140.288	141.065
LBMD	1.560 (0.070)	-	-	148.330	150.330	152.473	150.395	151.172
LBWRD	2.551 (80.865)	-	0.524 16.607)	148.330	152.330	156.616	152.530	154.015
MD	1.802 (0.093)	-	-	163.170	165.170	167.313	165.235	166.013
RD	2.207 (0.139)	-	-	187.040	189.040	191.183	189.105	189.883
HND	3.121 (0.278)	-	-	234.845	236.845	238.988	236.910	237.688

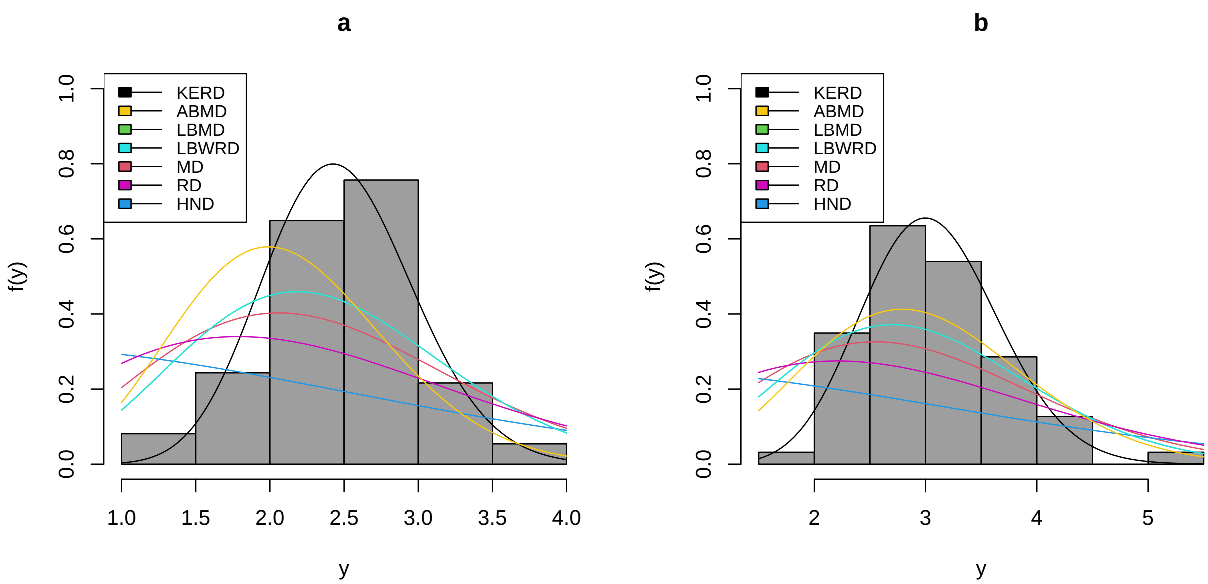

Figure 10: (a) Plots of estimated pdf of KERD and its sub-models for dataset I and (b) plots of estimated pdf of KERD and its sub-models for dataset II.

Table 5 and 6 clearly demonstrate that the proposed model consistently yields the smallest values across the model selection tools. Therefore, we conclude that the KERD distribution proves to be best model than its sub-models. Also the results are validated by figure 10.

8. Conclusion

This manuscript introduces a new generalized model called the Kth order equilibrium Rayleigh distribution (KERD), developed using the concept of weighted probability distributions. We derive explicit formulas for crucial statistical properties such as moments, conditional moments, incomplete moments, mean residual life function, mean waiting time, Renyi entropy and densities of order statistics. The proposed model is verified to be right-skewed and well-suited for modeling various types of hazard rates. Additionally, the behavior of specific structural properties is examined through contour plots, showing that the model becomes increasingly concentrated as parameter values increase. The study explores various statistical characteristics of this model, including aging behavior and stochastic ordering relations. Parameter estimation for the KERD is performed using maximum likelihood estimation techniques.The asymptotic normality of maximum likelihood estimators (MLEs) is examined through simulation studies. Trace plots in figure (6) and (7) illustrate the behavior of the variance, bais, MSE, Anderson-Darling (AD) test statistic and corresponding p-values as the sample size grows, revealing that variance, bias and mean squared error (MSE) of the MLEs decrease with larger sample sizes. The relationship between the parameter estimators k and ζ is analyzed using the variance-covariance matrix, which indicates a negative correlation between the parameters of KERD. We also demonstrate real-life applications of the proposed model, highlighting its superior fit compared to competing models across two different real-life datasets. The results clearly indicate that the KERD provides a significantly better fit than its sub-models and competing distributions, as evidenced by the lowest AIC, BIC, AICC and HQIC values reported in tables (5) and (6). This confirms the advantage of incorporating the kth-order structure in improving model flexibility and goodness of fit. Despite its flexibility and improved goodness of fit, the KERD involves only two parameters, which may increase the risk of overfitting, particularly in small sample situations. This limitation suggests the need for more robust estimation strategies in finite samples. Future research may therefore explore Bayesian estimation frameworks incorporating informative priors, penalized likelihood techniques to control model complexity and extensions to regression, censored and survival data contexts. Such developments would not only strengthen inferential stability but also broaden the practical applicability of the model across diverse statistical modeling scenarios.

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References

1. Anas, M., Dar, A. A., Ahmed, A., and Arshad, M. (2023). K th-order equilibrium weibull distribution: properties, simulation, and its applications. *Communications in Statistics-Simulation and Computation*, pages 1–23.
2. Ateeq, K., Qasim, T. B., and Alvi, A. R. (2019). An extension of rayleigh distribution and applications. *Cogent Mathematics & Statistics*, 6(1):1622191.
3. Bhat, A. and Ahmad, S. P. (2020). A new generalization of rayleigh distribution: Properties and applications. *Pakistan journal of statistics*, 36(3).
4. Cox, D. R. (1962). Renewal theory, methuen and co. *Ltd., London*.
5. Fisher, R. A. (1934). The effect of methods of ascertainment upon the estimation of frequencies. *Annals of eugenics*, 6(1):13–25.
6. Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. In *Mathematical statistics and probability theory: volume B statistical inference and methods proceedings of the 6th Pannonian symposium on mathematical statistics, bad Tatzmannsdorf, Austria, September 14–20, 1986*, pages 75–84. Springer.
7. Gupta, R. C. and Kirmani, S. (1990). The role of weighted distributions in stochastic modeling. *Communications in Statistics-Theory and methods*, 19(9):3147–3162.
8. Kilai, M., Waititu, G. A., Kibira, W. A., Abd El-Raouf, M., and Abushal, T. A. (2022). A new versatile modification of the rayleigh distribution for modeling covid-19 mortality rates. *Results in Physics*, 35:105260.
9. Lawless, J. F. (2011). *Statistical models and methods for lifetime data*. John Wiley & Sons.
10. Mir, A. A. and Ahmad, S. (2024). Modeling and analysis of sine power rayleigh distribution: Properties and applications. *Reliability: Theory & Applications*, 19(1 (77)):703–716.
11. Nair, N. U. and Preeth, M. (2008). Multivariate equilibrium distributions of order n. *Statistics & probability letters*, 78(18):3312–3320.
12. Rayleigh, L. (1880). On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. *Phil. Mag., Series 5*, 43(261):259–272.
13. Willmot, G. E., Drekic*, S., and Cai, J. (2005). Equilibrium compound distributions and stop-loss moments. *Scandinavian Actuarial Journal*, 2005(1):6–24.
14. Wu, X. (2013). Equilibrium distributions of discrete phase type. *Stochastic Models*, 29(2):240–257.

Appendix

Theorem 1:
Let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $H = [a,d]$ be an interval for some $a < d$ (where a and d may be extended to $-\infty$ or ∞). Let Y be a continuous random variable with distribution function F, and let b_1 and b_2 be two real-valued functions defined on H such that:

$$E[b_1(Y)|_{Y>y}] = E[(b_2(y))|_{Y>y}]A(y), y \in H$$

is defined in terms of some real function A. Assume that:

1. $b_1(Y)$ and $b_2(Y)$ are differentiable, and F is twice continuously differentiable and strictly monotone on H.
2. The equation $A(Y)b_1(Y) = b_2(Y)$ has no real solution within the interior of H.

Under these conditions, F is uniquely determined by the functions b_1, b_2, and A. In particular:

$$F(y) = C\int_a^y \left|\frac{A'(u)}{A(u)b_1(u) - b_2(u)}\right| exp(S(u))\,du$$

where S is a solution to the differential equation:

$$s'(y) = \frac{A'(y)b_2(y)}{A(y)b_1(y) - b_2(y)}$$

and C is a normalization constant such that $\int_H F(y)\,dy = 1$.