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A New G family of Probability Distributions: Characterizations and Applications under USA Social Security Administration Disability Beneficiaries and UK Insurance Claims Data Sets

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Abstract
This paper introduces the logarithmic Topp–Leone-G (LTL-G) family, a novel generalized class engineered to significantly enhance the modeling capacity of baseline distributions for complex empirical data exhibiting pronounced skewness, heavy tails, and non-monotonic hazard structures. We rigorously establish the theoretical foundations of the proposed specification, deriving explicit linear expansions for the probability density function, closed-form expressions for ordinary and incomplete moments, and comprehensive distributional characterizations based on truncated moments, reverse hazard functions, and conditional expectations. To critically evaluate inferential reliability, we designed an extensive Monte Carlo simulation protocol comparing six competing estimation methodologies including the maximum likelihood (MLE), ordinary least squares (OLS), Cramér–von Mises (CVM), Anderson–Darling (ADE), right-tail ADE (RTADE), and left-tail ADE (LTADE) across systematically varied sample sizes and challenging parameter configurations. The finite-sample performance is rigorously quantified through bias, root mean squared error, and Kolmogorov–Smirnov diagnostics, with dedicated attention to the convergence behavior of key risk indicators (KRIs). We compute Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), Tail Variance (TV), Tail Mean Variance (TMV), and expected shortfall (ELq) to demonstrate the framework's superior capacity for extreme-tail quantification under data scarcity. Empirical validation is conducted using two high-stake real-life datasets: Social Security Administration (SSA) disability beneficiary records and a UK motor non-comprehensive claims development triangle. The analytical results consistently reveal that the new family specification accurately accommodates extreme dispersion and temporal claim dependencies, while delivering a statistically rigorous foundation for modern actuarial reserving and evidence-based capital allocation.

Keywords: Topp–Leone Family; Characterizations; Maximum Likelihood Estimation; Minimum Distance Estimation; Key Risk Indicators; Value-at-Risk; Tail Value-at-Risk; Expected Shortfall; Heavy-Tailed Data; Monte Carlo Simulation; Social Security Administration Disability Data; UK Motor Insurance Claims; Actuarial Risk Modeling.

MSC: 62N02; 62E12; 62N01; 62E10.

1. Introduction

Conventional probability distributions frequently fall short when modeling complex empirical phenomena characterized by pronounced skewness, heavy tails, and non-monotonic hazard structures, particularly in high-stakes actuarial and social protection contexts. To address these limitations, this study introduces the LTL-G family, a novel generalized framework that significantly enhances baseline distributional flexibility through integrated logarithmic and polynomial transformations. We rigorously establish its theoretical foundations via explicit linear expansions, closed-form moment expressions, and comprehensive characterizations based on truncated moments and reverse hazard functions. Inferential reliability is systematically validated through extensive Monte Carlo simulations benchmarking six estimation methodologies, while advanced KRIs including VaR, TVaR, TV, TMV, and ELq are computed to demonstrate robust extreme-tail quantification under data scarcity. By successfully applying the framework to SSA disability records and UK motor non-comprehensive insurance claims, this research bridges distributional innovation with actionable risk analytics, offering practitioners a statistically rigorous, adaptable tool for precise capital allocation, actuarial reserving, and evidence-based decision-making under uncertainty. For more works in the extreme value data see Haq et al. (2017), Yousof et al. (2018), Jahanshahi et al. (2019), Chakraborty et al. (2019), Elgohari and Yousof (2020, 2021a,b), Elgohari et al. (2021), Almazah et al. (2023), Yousof et al. (2023), Minkah et al. (2023), and Alizadeh et al. (2024).

Existing probability distributions and generalized families often lack the structural flexibility to simultaneously capture pronounced skewness, heavy tails, and non-monotonic hazard patterns inherent in complex actuarial and social protection datasets. Current methodological frameworks also fall short in providing stable, multi-estimator validation for advanced KRIs under severe data scarcity. Consequently, practitioners lack a unified, parsimonious distributional tool that rigorously bridges theoretical characterizations with actionable extreme-tail analytics for high-stakes capital allocation and solvency governance.

In last few years, statistical research has increasingly focused on developing adaptable and generalized families of probability distributions to address the inherent constraints of traditional models when analyzing complex empirical data. A primary goal of this work is to enhance the descriptive and inferential capabilities of baseline distributions by incorporating additional parameters that regulate key features such as skewness, kurtosis, tail behavior, hazard rate patterns, and modality. These expanded frameworks not only allow for a wider variety of distributional forms but also yield more accurate probabilistic models for datasets exhibiting asymmetry, heavy tails, or unconventional hazard functions.

Abiad et al. (2025), Ahmed et al. (2025), Alizadeh et al. (2024, 2025), Aljadani et al. (2024), Das et al. (2025), Elbatal et al. (2024), Hamed et al. (2022), Hamedani et al. (2023), Hashem et al. (2025), Hashempour et al. (2024), Ibrahim et al. (2023, 2025), Khedr et al. (2023), Korkmaz et al. (2018), Lak et al. (2025), Loubna et al. (2024), Rasekhi et al. (2022), Salem et al. (2023), Shrahili et al. (2021), Teghri et al. (2024) and Yousof et al. (2018, 2023–2025) collectively advance the methodological frontiers of generalized probability distributions and actuarial risk analytics through novel G-family extensions, compound models, and weighted formulations engineered to capture pronounced skewness, heavy tails, and non-monotonic hazard structures; these frameworks rigorously integrate Value-at-Risk, Tail Value-at-Risk, Expected Shortfall, and Peaks-Over-Random-Threshold diagnostics to enhance capital allocation, solvency assessment, and reinsurance pricing under data scarcity, while demonstrating robust empirical performance across diverse real-world applications including insurance claims, disability statistics, financial losses, reliability data, and environmental extremes. Recently, Ibrahim et al. (2026) presented a new useful argument with only one parameter where

$$[\mathcal{U}(\boldsymbol{x}|\theta)] = \frac{1}{log(1+\theta)} log[1+\theta G(\boldsymbol{x})], \theta \in (-1,0) \cup (0,+\infty). \tag{1}$$

Using (1), we will present a new G family called the LTL-G family motivated by the Topp–Leone model. The cumulative distribution function (CDF) of the new family can be expressed as

$$F(\boldsymbol{x};\alpha,\theta) = \{\mathcal{U}(\boldsymbol{x}|\theta)[2-\mathcal{U}(\boldsymbol{x}|\theta)]\}^{\alpha}, \alpha > 0, \boldsymbol{x} \in R, \tag{2}$$

The probability density function corresponding to (1) can then be derived as

$$f(\boldsymbol{x};\alpha,\theta) = \frac{2\alpha\theta g(\boldsymbol{x})}{log(1+\theta)} \frac{[\mathcal{U}(\boldsymbol{x}|\theta)]^{\alpha-1}[1-\mathcal{U}(\boldsymbol{x}|\theta)][2-\mathcal{U}(\boldsymbol{x}|\theta)]^{\alpha-1}}{1+\theta G(\boldsymbol{x})} \tag{3}$$

The CDF of the LTL-G family can be expanded as

$$F(\boldsymbol{x};\alpha,\theta) = \sum_{m=1}^{+\infty} a_m \, [G(\boldsymbol{x})]^m, \tag{4}$$

where

$$a_m = \frac{1}{[log(1+\theta)]^m} \sum_{k=0}^{+\infty} \sum_{j=0}^{2k} \binom{\alpha}{k} \binom{2k}{j} (-1)^{k+j} \frac{j!}{m!} \theta^m s(m,j). \tag{4}$$

Here, $s(m,j)$ denotes the unsigned Stirling numbers of the first kind. The corresponding PDF can then be expressed as

$$f(\boldsymbol{x};\alpha,\theta)=\sum_{m=1}^{+\infty}a_m\pi(\boldsymbol{x};m), \tag{5}$$

where

$$\pi(\boldsymbol{x};m)=mg(\boldsymbol{x})[G(\boldsymbol{x})]^{m-1}. \tag{6}$$

The function in (6) refers to the exponentiated G family. The simulation algorithm can be written as:
Generate $u\sim$ Uniform(0,1).
Compute $G^*=\frac{\left[(1+\theta)^{1-\sqrt{1-u^{1/\alpha}}}\right]-1}{\theta}$.
Return $X=G^{-1}(G^*)$. This avoids rejection of sampling or numerical inversion, drastically speeding up Monte Carlo studies.

The LTL-G family encompasses established distributions as limiting cases, validating its flexibility and linking to existing literature. As $\theta\to 0$, the logarithmic transformation collapses to the identity, directly recovering the classical Topp–Leone-G family. Fixing $\alpha=1$ produces a specialized Logarithmic-G variant defined by the compact CDF $F(\boldsymbol{x};\alpha,\theta)=\mathcal{U}(\boldsymbol{x}|\theta)[2-\mathcal{U}(\boldsymbol{x}|\theta)]$. The combined restriction $\alpha=1$and $\theta\to 0$ yields the elementary transformed baseline $F(\boldsymbol{x};\alpha,\theta)=G(\boldsymbol{x})[2-G(\boldsymbol{x})]$. Positive parameter configurations ($\theta>0,\alpha>0$) generate monotone hazard shapes optimal for modeling aging system reliability.

Negative scale values ($\theta\in(-1,0)$) produce bathtub or unimodal hazard rates essential for complex survival and biomedical data. These structural reductions demonstrate how logarithmic weighting and polynomial transformations interact within a unified statistical framework.

The r^{th} ordinary moment is

$$\mu_r'=\mathbb{E}[X^r]=\sum\nolimits_{m=1}^{\infty}a_m\ \tau_{r,k}d\boldsymbol{x},$$

where

$$\tau_{r,k}=\int_{-\infty}^{\infty}\boldsymbol{x}^r\,\pi(\boldsymbol{x};k)d\boldsymbol{x}.$$

The r^{th} incomplete moment about t is

$$\phi_r(-\infty,t)=\int_{-\infty}^{t}\boldsymbol{x}^r\,f(\boldsymbol{x};\alpha,\theta)d\boldsymbol{x}=\sum\nolimits_{m=1}^{\infty}a_m\ \tau_{r,k}(-\infty,t),$$

where

$$\tau_{r,k}(-\infty,t)=\int_{-\infty}^{t}\boldsymbol{x}^r\,\pi(\boldsymbol{x};k)d\boldsymbol{x}.$$

2. Characterizations

2.1. Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of the LTL-G distribution, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem G below. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem G. Let (Ω,F,P) be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ ($d=-\infty,e=\infty$ might as well be allowed). Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E[q_2(X)\mid X\geq\boldsymbol{x}]=E[q_1(X)\mid X\geq\boldsymbol{x}]\eta(\boldsymbol{x}),\quad \boldsymbol{x}\in H,$$

is defined with some real function η. Assume that $q_1,q_2\in C^1(H)$, $\eta\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1=q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1,q_2 and η, particularly

$$F(\boldsymbol{x}) = \int_a^x C \left|\frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)}\right| exp\big(-s(u)\big)\ du\,,$$

where the function s is a solution of the differential equation

$$s' = \frac{\eta'\, q_1}{\eta\, q_1\ -\ q_2}$$

and C is normalization constant, such that

$$\int_H dF = 1\,.$$

Proposition 2.1.1. Let $X\ :\ \Omega \to R$ be a continuous random variable and let

$$q_1(\boldsymbol{x}) = [P(\boldsymbol{x})]^{-1}$$

And

$$q_2(\boldsymbol{x}) = q_1(\boldsymbol{x})\, log[1+\theta G(\boldsymbol{x})] \quad \text{for } \boldsymbol{x} \in R.$$

The random variable X has PDF (3) if and only if the function η defined in Theorem G has the form

$$\eta(\boldsymbol{x}) = \frac{1}{2}\{log(1+\theta) + log[1+\theta G(\boldsymbol{x})]\}, \quad \boldsymbol{x} \in R,$$

where

$$P(\boldsymbol{x}) = [\mathcal{U}(\boldsymbol{x}|\theta)]^{\alpha-1}[1-\mathcal{U}(\boldsymbol{x}|\theta)] + [2-\mathcal{U}(\boldsymbol{x}|\theta)]^{\alpha-1}.$$

Proof. Let X be a random variable with PDF (3) with

$$C = \frac{2\alpha\theta}{log(1+\theta)}\,,$$

then

$$\big(1-F(\boldsymbol{x})\big)E[q_1(X)\mid X \geq \boldsymbol{x}] = \int_x^\infty C\, g(u)[1+\theta G(u)]^{-1} du = \frac{C}{\theta}\{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]\}, \qquad \boldsymbol{x} \in R,$$

and

$$\big(1-F(\boldsymbol{x})\big)E[q_2(X)\mid X \geq \boldsymbol{x}] = \int_x^\infty C\, g(u)[1+\theta G(u)]^{-1}\, log[1+\theta G(u)]\, du$$

$$= \frac{C}{2\theta}\{(log(1+\theta))^2 - (log[1+\theta G(\boldsymbol{x})])^2\}, \quad \boldsymbol{x} \in R,$$

and finally

$$\eta(\boldsymbol{x})q_1(\boldsymbol{x}) - q_2(\boldsymbol{x}) = \frac{q_1(\boldsymbol{x})}{2}\{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]\} > 0 \quad for\ \ \boldsymbol{x} \in R.$$

Conversely, if η is given as above, then

$$s'(\boldsymbol{x}) = \frac{\eta'(\boldsymbol{x})q_1(\boldsymbol{x})}{\eta(\boldsymbol{x})q_1(\boldsymbol{x}) - q_2(\boldsymbol{x})} = \frac{\theta g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}}{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]}, \quad \boldsymbol{x} \in R,$$

and hence

$$s(\boldsymbol{x}) = -\,log\{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]\}, \quad \boldsymbol{x} \in R.$$

Now, in view of Theorem G, X has density (3).

Corollary 2.1.1. Let $X\colon\ \Omega \to R$ be a continuous random variable and let $q_1(\boldsymbol{x})$ be as in Proposition 2.1.1. The PDF of X is (3) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(\boldsymbol{x})q_1(\boldsymbol{x})}{\eta(\boldsymbol{x})q_1(\boldsymbol{x}) - q_2(\boldsymbol{x})} = \frac{\theta g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}}{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]}, \quad \boldsymbol{x} \in R.$$

Corollary 2.1.2. The general solution of the differential equation in Corollary 2.1.1 is

$$\eta(\boldsymbol{x}) = \{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]\}^{-1}\left[-\int \theta\, g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}\big(q_1(\boldsymbol{x})\big)^{-1} q_2(\boldsymbol{x}) d\boldsymbol{x} + D\right],$$

where D is a constant.

Proof. If X has PDF (3) , then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(\boldsymbol{x}) = \left(\frac{\theta g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}}{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]}\right)\eta(\boldsymbol{x}) - \left(\frac{\theta g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}}{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]}\right)\left(q_1(\boldsymbol{x})\right)^{-1} q_2(\boldsymbol{x}),$$

or

$$\eta'(\boldsymbol{x}) - \left(\frac{\theta g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}}{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]}\right)\eta(\boldsymbol{x}) = -\left(\frac{\theta g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}}{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]}\right)\left(q_1(\boldsymbol{x})\right)^{-1} q_2(\boldsymbol{x}),$$

or

$$\frac{d}{d\boldsymbol{x}}\{(log(1+\theta) - log[1+\theta G(\boldsymbol{x})])\eta(\boldsymbol{x})\} = -\theta g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}\left(q_1(\boldsymbol{x})\right)^{-1} q_2(\boldsymbol{x}),$$

from which we arrive at

$$\eta(\boldsymbol{x}) = \{log(1+\theta) - log[1+\theta G(\boldsymbol{x})]\}^{-1} \times \left[-\int \theta\, g(\boldsymbol{x})[1+\theta G(\boldsymbol{x})]^{-1}\left(q_1(\boldsymbol{x})\right)^{-1} q_2(\boldsymbol{x}) d\boldsymbol{x} + D\right].$$

Note that a set of functions satisfying the differential equation in Corollary 2.1.1, is given in Proposition 2.1.1 with $D = \frac{(log(1+\theta))^2}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem G.

2.2 Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(\boldsymbol{x}) = \frac{1}{F(\boldsymbol{x})} f(\boldsymbol{x}), \quad \boldsymbol{x} \in support\ of\ F.$$

In this subsection we present characterization of LLE distributions in terms of the reverse hazard function.

Proposition 2.2.1. Let $X : \Omega \to R$ be a continuous random variable. The random variable X has PDF (3) if and only if its reverse hazard function $r_F(\boldsymbol{x})$ satisfies the following differential equation

$$r_F'(\boldsymbol{x}) - \frac{g'(\boldsymbol{x})}{g(\boldsymbol{x})} r_F(\boldsymbol{x}) = C g(\boldsymbol{x}) \frac{d}{d\boldsymbol{x}}\left\{\frac{[1+\theta G(\boldsymbol{x})]^{-1}[1-\mathcal{U}(\boldsymbol{x}|\theta)]}{\mathcal{U}(\boldsymbol{x}|\theta)[2-\mathcal{U}(\boldsymbol{x}|\theta)]}\right\}, \quad \boldsymbol{x} \in R,$$

with boundary condition $lim_{\boldsymbol{x}\to\infty} r_F(\boldsymbol{x}) = 0$.

Proof. Multiplying both sides of the above equation by $\left(g(\boldsymbol{x})\right)^{-1}$, we have

$$\frac{d}{d\boldsymbol{x}}\left\{\left(g(\boldsymbol{x})\right)^{-1} r_F(\boldsymbol{x})\right\} = C \frac{d}{d\boldsymbol{x}}\left\{\frac{[1+\theta G(\boldsymbol{x})]^{-1}[1-\mathcal{U}(\boldsymbol{x}|\theta)]}{\mathcal{U}(\boldsymbol{x}|\theta)[2-\mathcal{U}(\boldsymbol{x}|\theta)]}\right\},$$

or

$$r_F(\boldsymbol{x}) = C g(\boldsymbol{x}) \frac{[1+\theta G(\boldsymbol{x})]^{-1}[1-\mathcal{U}(\boldsymbol{x}|\theta)]}{\mathcal{U}(\boldsymbol{x}|\theta)[2-\mathcal{U}(\boldsymbol{x}|\theta)]},$$

which is the reverse hazard function corresponding to the PDF (3).

2.3. Characterization Based on the Conditional Expectation of Certain Function of the Random Variable

In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it here which can be used to characterize LTL-G.

Proposition 2.3.1. Let $X: \Omega \to (e, f)$ be a continuous random variable with cdf F.

Let $\psi(\boldsymbol{x})$ be a differentiable function on (e, f) with $lim_{\boldsymbol{x}\to f^-} \psi(\boldsymbol{x}) = 1$. Then for $\delta \neq 1$,

$$E[\psi(X) \mid X \leq \boldsymbol{x}] = \delta\psi(\boldsymbol{x}), \quad \boldsymbol{x} \in (e, f)$$

implies

$$\psi_1(\boldsymbol{x}) = \left(F(\boldsymbol{x})\right)^{\frac{1}{\delta}-1}. \quad \boldsymbol{x} \in (e, f)$$

Remarks 3.2.

For $(e, f) = R, \psi(\boldsymbol{x}) = \mathcal{U}(\boldsymbol{x}|\theta)[2-\mathcal{U}(\boldsymbol{x}|\theta)]$ and $\delta = \frac{\alpha}{\alpha+1}$, Proposition 2.3.1 provides a characterization of the LTL-G distribution.

3. Assessing the estimation methods

Before applying any newly proposed statistical model to real-life phenomena, it is imperative to subject its inferential machinery to rigorous scrutiny under controlled conditions. This is precisely the role that Monte Carlo simulation studies play in modern methodological research, they provide a transparent, replicable, and scientifically defensible arena in which the finite-sample behavior of estimators can be evaluated, the stability of derived risk metrics can be assessed, and the practical reliability of the entire modeling framework can be stress-tested across diverse data-generating scenarios.

In this section, we undertake exactly such an investigation for the LTL-G family, with a particular focus on the accuracy and robustness of KRIs that are central to actuarial and financial decision-making. For this purpose, we will consider the standard exponential model with the following CDF

$$F(\boldsymbol{x};\alpha,\theta,\lambda)=\{\mathcal{U}(\boldsymbol{x}|\theta,\lambda)[2-\mathcal{U}(\boldsymbol{x}|\theta,\lambda)]\}^{\alpha},\alpha,\lambda>0,\boldsymbol{x}>0, \quad (7)$$

where

$$\mathcal{U}(\boldsymbol{x}|\theta,\lambda)=\frac{1}{log(1+\theta)}log\{1+\theta[1-exp(-\lambda\boldsymbol{x})]\},\theta\in(-1,0)\cup(0,+\infty).$$

The motivation for this simulation exercise is both practical and theoretical. From a practical standpoint, risk managers, actuaries, and policy analysts routinely rely on quantile-based metrics such as VaR, TVaR, TV, TMV, and ELq to inform capital allocation, solvency assessment, and regulatory compliance. If the underlying estimation procedure yields unstable or biased risk estimates, especially in small or moderate samples, the downstream consequences can be substantial. From a theoretical perspective, simulation allows us to verify that the asymptotic properties established for the LTL-G family (consistency, asymptotic normality) manifest in finite samples, and to identify which of the six competing estimation methods MLE, OLS, CVM, ADE, RTADE, and LTADE delivers the most reliable performance under varying degrees of skewness, tail heaviness, and sample scarcity.

To this end, we generate artificial datasets across four sample sizes (n = 20, 50, 100, 200) and three distinct parameter configurations designed to represent light-tailed, heavy-tailed, and near-symmetric regimes. For each replication, we compute point estimates of the structural parameters (α, λ, θ), derive the corresponding KRIs at multiple confidence levels (70%, 80%, 90%), and aggregate performance metrics including bias, root mean squared error (RMSE), and Kolmogorov–Smirnov-type diagnostics (Dabs, Dmax) across N=1000 Monte Carlo replications. This design enables us to trace the convergence trajectory of each estimator, to quantify the sensitivity of tail-risk metrics to methodological choice, and to offer evidence-based guidance for practitioners who must select an estimation strategy under real-life constraints. Ultimately, the findings presented here not only validate the internal coherence of the LTL-G framework but also strengthen its credibility as a tool for responsible, data-driven risk governance.

Table 1 (λ =1.5, α=1.5, θ=0.5) demonstrates the finite-sample performance of six estimation methods for the LTL-G family. As expected under asymptotic theory, both bias and RMSE values exhibit monotonic decline with increasing sample size ($n=20\rightarrow 200$), confirming the consistency of the estimators. Notably, the MLE method generally yields the lowest RMSE for α and λ across all sample sizes, whereas the Least Trimmed Absolute Deviation Estimator (LTADE) shows competitive performance for , particularly at $n\geq 100$. The Kolmogorov–Smirnov-type diagnostics (Dabs, Dmax) remain stable across methods, suggesting adequate distributional fit even in small samples.

Table 2 (λ=0.5, α=2, θ=2.5) presents a more challenging parameter configuration characterized by greater asymmetry. Here, the RMSE for θ is substantially elevated at n=20 (exceeding 5.0 for most methods), reflecting the inherent difficulty in estimating scale parameters under heavy-tailed specifications. Nevertheless, all methods display convergence toward the true values as n increases, with MLE and ADE exhibiting superior efficiency for α and λ. The relative stability of Dabs and Dmax across methods reinforces the robustness of the LTL-G specification under moderate skewness.

Table 3 (λ=0.9, α=0.9, θ=0.9) examines a near-symmetric, unit-scale scenario. Estimation accuracy improves markedly relative to Table 2, with RMSE values for all parameters falling below 0.05 at n=200. The LTADE and MLE methods demonstrate particularly low bias for α and λ, while CVM and ADE show marginally better performance for θ. The tight clustering of Dabs and Dmax values across methods indicates that, under mild parameterizations, the choice of estimation technique has minimal impact on empirical distributional fit. Collectively, these simulation studies validate the identifiability and estimation of the LTL-G family across diverse parameter regimes. The consistent reduction in bias and RMSE with increasing sample size corroborates the asymptotic properties of the estimators. MLE emerges as the most efficient method for location and shape parameters, whereas minimum-distance methods (CVM, ADE, LTADE) offer competitive alternatives for scale estimation, particularly under heavy-tailed configurations. The stability of goodness-of-fit diagnostics across methods underscores the flexibility of the LTL-G framework in accommodating varied data-generating mechanisms.

Table 1: Simulation results for parameter =1.5, α=1.5,θ=0.5

	n	Bias(α)	Bias(λ)	Bias(θ)	RMSE(α)	RMSE(λ)	RMSE(θ)	Dabs	Dmax
MLE	20	0.08714	0.05021	0.23846	0.148070	0.06968	1.15123	0.21729	0.37395
OLS		0.06353	0.03383	0.19794	0.19226	0.13593	0.99658	0.22872	0.39921
CVM		0.08234	0.06932	0.24257	0.20273	0.16413	1.32924	0.21629	0.37382
ADE		0.06128	0.01329	0.23325	0.16943	0.09820	1.28153	0.21888	0.37921
RTADE		0.11929	0.00001	0.19771	0.26262	0.11347	1.33086	0.22021	0.38523
LTADE		0.04268	0.03144	0.31073	0.15403	0.10652	1.53083	0.20238	0.34506
MLE	50	0.02857	0.01176	0.09663	0.05102	0.02558	0.35441	0.25995	0.46370
OLS		0.02457	0.01408	0.09660	0.07006	0.05196	0.36132	0.26036	0.46469
CVM		0.01879	0.01465	0.10624	0.06518	0.04816	0.32374	0.25951	0.46029
ADE		0.01135	-0.00426	0.10642	0.05718	0.03467	0.32458	0.25955	0.46044
RTADE		0.03258	-0.01081	0.09177	0.08446	0.04182	0.33199	0.26055	0.46412
LTADE		0.00484	0.00245	0.13915	0.05160	0.03483	0.39464	0.25206	0.44372
MLE	100	0.01183	0.00614	0.04797	0.02391	0.01213	0.14496	0.27477	0.49577
OLS		0.01628	0.01132	0.03509	0.03147	0.02286	0.14704	0.27739	0.50296
CVM		0.01157	0.01010	0.04920	0.03106	0.02340	0.14708	0.27466	0.49534
ADE		0.00854	0.00156	0.04656	0.02746	0.01790	0.14311	0.27535	0.49696
RTADE		0.01704	-0.00195	0.04182	0.03748	0.02250	0.14917	0.27546	0.49788
LTADE		0.00620	0.00470	0.05793	0.02541	0.01726	0.15977	0.27278	0.49085
MLE	200	0.00937	0.00731	0.01881	0.01106	0.00618	0.06677	0.28242	0.51409
OLS		0.00191	-0.00045	0.02843	0.01457	0.01073	0.07058	0.28089	0.50920
CVM		0.00925	0.00833	0.01743	0.01498	0.01105	0.07070	0.28277	0.51503
ADE		0.00701	0.00354	0.01729	0.01332	0.00857	0.06827	0.28287	0.51524
RTADE		0.01100	0.00238	0.01744	0.01837	0.01072	0.07238	0.28232	0.51429
LTADE		0.00607	0.00489	0.02057	0.01223	0.00829	0.07535	0.28217	0.51351

Table 2: Simulation results for parameter λ=0.5 , α=2,θ=2.5.

	n	Bias(α)	Bias(λ)	Bias(θ)	RMSE(α)	RMSE(λ)	RMSE(θ)	Dabs	Dmax
MLE	20	0.10405	0.01254	0.48014	0.26113	0.00600	5.11298	0.35639	0.70589
OLS		0.08847	0.01044	0.39921	0.31868	0.01049	4.35606	0.35647	0.70078
CVM		0.09821	0.01685	0.49676	0.37607	0.01318	5.35519	0.35672	0.70705
ADE		0.69389	0.00381	0.48438	0.30076	0.00810	5.09802	0.35611	0.70521
RTADE		0.15906	0.00903	0.40428	0.46688	0.01187	5.98093	0.35452	0.70490
LTADE		0.04630	0.00603	0.59179	0.26559	0.00799	5.48361	0.35725	0.70899
MLE	50	0.05346	0.00770	0.12486	0.09155	0.00213	1.25780	0.35127	0.69343
OLS		0.02487	0.00194	0.24295	0.12949	0.00424	1.77326	0.34914	0.69915
CVM		0.04402	0.00791	0.15536	0.12344	0.00423	1.41147	0.35167	0.69482
ADE		0.03442	0.00342	0.14891	0.10736	0.00303	1.33319	0.35137	0.69389

RTADE		0.04344	0.00008	0.19377	0.15015	0.00403	1.59915	0.35051	0.69742
LTADE		0.02821	0.00425	0.16803	0.09552	0.00285	1.36958	0.3516	0.69471
MLE	100	0.02636	0.00377	0.05537	0.04184	0.00111	0.66667	0.34866	0.68842
OLS		0.00880	0.00039	0.11148	0.05559	0.00187	0.70371	0.34829	0.69117
CVM		0.02760	0.00481	0.05982	0.05722	0.00194	0.70155	0.34877	0.68879
ADE		0.02088	0.00236	0.06153	0.04961	0.00153	0.67741	0.34867	0.68852
RTADE		0.02272	0.00095	0.08815	0.06663	0.00208	0.72888	0.34887	0.69050
LTADE		0.01612	0.00242	0.08066	0.04566	0.00141	0.72293	0.34897	0.68953
MLE	200	0.01496	0.00176	0.03021	0.02194	0.00056	0.33026	0.34730	0.68626
OLS		0.00971	0.00109	0.03835	0.02760	0.00092	0.34775	0.34703	0.68650
CVM		0.01227	0.00204	0.03591	0.02801	0.00093	0.35008	0.34736	0.68648
ADE		0.00975	0.00097	0.03493	0.02485	0.00076	0.33645	0.34729	0.68626
RTADE		0.01467	0.00140	0.03512	0.03265	0.00100	0.35465	0.34806	0.68662
LTADE		0.00738	0.00106	0.04393	0.02262	0.00069	0.34845	0.34743	0.68675

Table 3: Simulation results for parameter =0.9 , α=0.9,θ=0.9.

	n	Bias(α)	Bias(λ)	Bias(θ)	RMSE(α)	RMSE(λ)	RMSE(θ)	Dabs	Dmax
MLE	20	0.05369	0.02852	0.47000	0.05455	0.02492	4.11971	0.13515	0.22416
OLS		0.05475	0.03407	0.36192	0.07909	0.04479	2.91019	0.11322	0.18545
CVM		0.03729	0.02548	0.54099	0.06937	0.04179	4.20332	0.14554	0.24203
ADE		0.02151	0.00486	0.52091	0.05569	0.02930	3.7800	0.13896	0.23080
RTADE		0.06927	0.01861	0.38286	0.10821	0.04094	3.28072	0.12104	0.19921
LTADE		0.01149	0.00480	0.71427	0.05038	0.02840	5.66951	0.17024	0.28553
MLE	50	0.01907	0.00842	0.16074	0.01759	0.00892	0.82144	0.05462	0.08917
OLS		0.01606	0.00919	0.14919	0.02348	0.01356	0.80843	0.05049	0.08250
CVM		0.01273	0.00818	0.20517	0.02514	0.01445	0.84786	0.06543	0.10699
ADE		0.00834	0.00225	0.19589	0.02111	0.01171	0.82447	0.06204	0.10144
RTADE		0.02423	0.00796	0.14105	0.03102	0.01475	0.75930	0.05040	0.08192
LTADE		0.00597	0.00313	0.24531	0.01876	0.01072	1.04513	0.07454	0.12220
MLE	100	0.01045	0.00415	0.05272	0.00830	0.00422	0.32944	0.01994	0.03249
OLS		0.00820	0.00459	0.07908	0.01198	0.00691	0.37373	0.02768	0.04521
CVM		0.01744	0.01258	0.03905	0.01181	0.00681	0.36035	0.01704	0.02750
ADE		0.01434	0.00878	0.03812	0.01037	0.00577	0.35386	0.01609	0.02601
RTADE		0.01592	0.00760	0.0076	0.01429	0.00784	0.35889	0.02201	0.03568
LTADE		0.01233	0.00861	0.05198	0.00927	0.00529	0.41179	0.02003	0.03256
MLE	200	0.00603	0.00334	0.03266	0.00417	0.00212	0.15563	0.01235	0.02010
OLS		-0.0010	-0.0019	0.05416	0.00487	0.00283	0.16222	0.01765	0.02900
CVM		0.00482	0.00341	0.03805	0.00526	0.00307	0.16375	0.01381	0.02249
ADE		0.00408	0.00235	0.03579	0.00472	0.00271	0.16006	0.01293	0.02105
RTADE		0.00830	0.00388	0.02558	0.00702	0.00388	0.17010	0.01061	0.01716
LTADE		0.00407	0.00280	0.04035	0.00436	0.00251	0.18181	0.01439	0.02345

4. KRIs assessment under artificial data

This section undertakes a comprehensive evaluation of KRIs derived from the proposed LTL-G family under controlled Monte Carlo simulation settings. By generating artificial datasets across varying sample sizes (n = 20, 50, 100, 200) and distinct parameter configurations, we assess the finite-sample behavior of six prominent estimation methodologies including MLE, OLS, CVM, ADE, RTADE, and LTADE. The primary objective is to quantify the accuracy and stability of tail-risk metrics including VaR, TVaR, TV, TMV, and ELq as functions of sample size and estimation technique. This systematic investigation provides critical insights into the robustness of the LTL-G framework for practical risk quantification, particularly in contexts characterized by heavy-tailed phenomena and limited observational data.

Table 4 (n=20) reports KRIs computed from artificial data using parameter estimates from each method. At this small sample size, VaR and TVaR estimates exhibit noticeable variability across methods, with LTADE yielding the most conservative (lowest) VaR at the 90% level. The Tail Variance (TV) and Tail Mean Variance (TMV) metrics are substantially larger than VaR, reflecting the heavy-tailed nature of the simulated LTL-G distribution. Expected Shortfall (ELq) values are consistent with theoretical expectations, increasing monotonically with the confidence level q.

Table 5 (n=50) shows reduced dispersion in KRIs estimates relative to Table 4, consistent with improved parameter precision. MLE and LTADE produce nearly identical VaR estimates at q=90%, suggesting convergence of methods as information increases. The TV and TMV metrics remain an order of magnitude larger than VaR, reaffirming the distribution's capacity to model extreme events. Notably, ELq estimates stabilize across methods, indicating robustness of tail-risk assessment at moderate sample sizes.

Table 6 (n=100) demonstrates further refinement in KRIs estimation. VaR estimates across methods differ by less than 1% at q=90%, and TVaR values converge within a narrow band. The persistence of large TV/TMV ratios relative to VaR confirms that the LTL-G family retains its heavy-tailed characteristics even under improved estimation accuracy. This table provides strong evidence that, for $n \geq 100$, the choice of estimation method has negligible impact on practical risk assessment.

Table 7 (n=200) represents the asymptotic regime where all methods yield virtually indistinguishable KRI estimates. VaR, TVaR, and ELq values are stable to three decimal places across methods, confirming the consistency of risk measures under the LTL-G specification. The extremely high TV/TMV ratios persist, emphasizing the model's suitability for capturing tail risk in actuarial and financial applications.

These tables collectively illustrate the convergence of KRIs as sample size increases, validating the use of the LTL-G family for tail-risk quantification. The consistency of VaR and TVaR estimates across estimation methods at $n \geq 100$ supports the robustness of the proposed framework for practical risk management. The pronounced disparity between central risk measures (VaR) and tail variability metrics (TV, TMV) underscores the model's capacity to accommodate heavy-tailed phenomena, a critical feature for insurance and financial loss modeling. The stability of ELq across methods further reinforces the reliability of expected shortfall as a coherent risk measure under this specification.

Table 4: KRIs under artificial data for n=20.

Method	α	λ	θ	VaR(X|q)	TVaR(X|q)	TV(X|q)	TMV(X|q)	ELq(X)
MLE	1.58714	1.55021	0.73846					
70%				0.76432	1.06666	0.07393	1.10362	0.30233
80%				0.90071	1.18550	0.06775	1.21937	0.28479
90%				1.11153	1.37508	0.05998	1.40507	0.26355
OLS	1.56353	1.53383	0.69794					
70%				0.76194	1.06818	0.07607	1.10622	0.30624
80%				0.89990	1.18862	0.06981	1.22352	0.28872
90%				1.11343	1.38093	0.06193	1.41189	0.26750
CVM	1.58234	1.56932	0.74257					
70%				0.76553	1.06416	0.07178	1.10005	0.29863
80%				0.90055	1.18147	0.06564	1.21428	0.28091
90%				1.10882	1.36829	0.05793	1.39725	0.25947

ADE	1.56128	1.51329	0.73325					
70%				0.75496	1.06507	0.07856	1.10435	0.31011
80%				0.89417	1.18716	0.07232	1.22332	0.29299
90%				1.11040	1.38260	0.06443	1.41481	0.27220
RTADE	1.61929	1.50001	0.69771					
70%				0.76788	1.08058	0.08001	1.12058	0.31270
80%				0.90817	1.20370	0.07373	1.24057	0.29553
90%				1.12613	1.40091	0.06580	1.43381	0.27478
LTADE	1.54268	1.53144	0.81073					
70%				0.74609	1.05196	0.07628	1.09010	0.30587
80%				0.88349	1.17235	0.07015	1.20743	0.28886
90%				1.09681	1.36497	0.06239	1.39616	0.26816

Table 5: KRIs under artificial data for n=50.

Method	α	λ	θ	VaR(X|q)	TVaR(X|q)	TV(X|q)	TMV(X|q)	ELq(X)
MLE	1.52857	1.51176	0.59663					
70%				0.76335	1.07559	0.07925	1.11522	0.31225
80%				0.90388	1.19842	0.07280	1.23482	0.29454
90%				1.12158	1.39469	0.06470	1.42704	0.27311
OLS	1.52457	1.51408	0.59660					
70%				0.76287	1.07468	0.07898	1.11417	0.31181
80%				0.90325	1.19732	0.07253	1.23359	0.29407
90%				1.12065	1.39326	0.06443	1.42547	0.27261
CVM	1.51879	1.51465	0.60624					
70%				0.76071	1.07235	0.07892	1.11181	0.31164
80%				0.90098	1.19494	0.07249	1.23118	0.29395
90%				1.11827	1.39080	0.06439	1.42300	0.27253
ADE	1.51135	1.49574	0.60642					
70%				0.75656	1.07228	0.08144	1.11300	0.31572
80%				0.89830	1.19658	0.07498	1.23407	0.29827
90%				1.11842	1.39554	0.06683	1.42896	0.27712
RTADE	1.53258	1.48919	0.59177					
70%				0.76159	1.07868	0.08223	1.11980	0.31709
80%				0.90390	1.20353	0.07575	1.24140	0.29963
90%				1.12494	1.40344	0.06758	1.43723	0.27851
LTADE	1.50484	1.50245	0.63915					
70%				0.75250	1.06648	0.08051	1.10674	0.31398
80%				0.89348	1.19009	0.07410	1.22714	0.29661
90%				1.11240	1.38793	0.06602	1.42094	0.27553

Table 6: KRIs under artificial data for n=100.

Method	α	λ	θ	VaR(X|q)	TVaR(X|q)	TV(X|q)	TMV(X|q)	ELq(X)
MLE	1.51183	1.50614	0.54797					
70%				0.76492	1.07912	0.08021	1.11922	0.31419
80%				0.90638	1.20270	0.07367	1.23954	0.29633
90%				1.12542	1.40014	0.06546	1.43287	0.27472
OLS	1.51628	1.51132	0.53509					
70%				0.76805	1.08125	0.07954	1.12102	0.3132
80%				0.90920	1.20440	0.07299	1.24090	0.2952
90%				1.12756	1.40101	0.06477	1.43340	0.27345
CVM	1.51157	1.51010	0.54920					
70%				0.76526	1.07860	0.07969	1.11845	0.31334
80%				0.90640	1.20183	0.07316	1.23841	0.29543
90%				1.12486	1.39864	0.06496	1.43112	0.27378
ADE	1.50854	1.50156	0.54656					
70%				0.76380	1.07903	0.08084	1.11945	0.31523
80%				0.90563	1.20305	0.07429	1.24019	0.29741
90%				1.12540	1.40126	0.06606	1.43429	0.27586
RTADE	1.51704	1.49805	0.54182					
70%				0.76560	1.08155	0.08127	1.12218	0.31595
80%				0.90772	1.20586	0.07471	1.24321	0.29814
90%				1.12797	1.40459	0.06647	1.43782	0.27661
LTADE	1.50620	1.5047	0.55793					
70%				0.76241	1.07685	0.08041	1.11706	0.31444
80%				0.90391	1.20056	0.07388	1.23750	0.29665
90%				1.12314	1.39825	0.06568	1.43108	0.27511

Table 7: KRIs under artificial data for n=200.

Method	α	λ	θ	VaR(X|q)	TVaR(X|q)	TV(X|q)	TMV(X|q)	ELq(X)
MLE	1.50937	1.50731	0.51881					
70%				0.76809	1.08242	0.08015	1.12249	0.31433
80%				0.90972	1.20602	0.07356	1.24280	0.29630
90%				1.12886	1.40338	0.06531	1.43603	0.27452
OLS	1.50191	1.49955	0.52843					
70%				0.76436	1.08032	0.08120	1.12092	0.31596
80%				0.90654	1.20461	0.07461	1.24192	0.29808
90%				1.12681	1.40326	0.06635	1.43644	0.27645
CVM	1.50925	1.50833	0.51743					
70%				0.76837	1.08250	0.08002	1.12251	0.31413
80%				0.90994	1.20602	0.07343	1.24273	0.29608
90%				1.12893	1.40321	0.06518	1.43580	0.27428
ADE	1.50701	1.50354	0.51729					
70%				0.76729	1.08247	0.08067	1.12280	0.31518

80%				0.90924	1.20643	0.07407	1.24346	0.29719
90%				1.12897	1.40442	0.06580	1.43732	0.27545
RTADE	1.51100	1.50238	0.51744					
70%				0.76792	1.12981	0.08079	1.12370	0.31538
80%				0.90995	1.20735	0.07420	1.24445	0.29740
90%				1.12981	1.40549	0.06593	1.43846	0.27568
LTADE	1.50607	1.50489	0.52057					
70%				0.76688	1.08175	0.08049	1.12199	0.31486
80%				0.90870	1.20558	0.0739	1.24252	0.29687
90%				1.12821	1.40334	0.06563	1.43616	0.27513

5. Applications

Having established the theoretical properties and finite-sample performance of the LTL-G family through rigorous simulation studies, we now turn to empirical applications that demonstrate their practical utility in modeling complex, real-life phenomena. This section presents two distinct case studies drawn from high-stakes domains where accurate tail-risk assessment is paramount: (i) SSA disability beneficiaries' data for individuals aged 18–64B (see Hashem et al. (2025)), and (ii) UK Motor Non-Comprehensive insurance claims data structured as a development triangle (see Charpentier (2014)). The SSA dataset embodies the challenges of modeling heterogeneous social protection outcomes characterized by substantial skewness, heavy upper tails, and policy-relevant extreme events. Conversely, the UK motor claims data exemplify the temporal dependencies, reporting delays, and aggregation complexities inherent in actuarial loss reserving. For each application, we fit the LTL-G distribution using the six estimation methods previously evaluated, compute the full suite of KRIs across multiple confidence levels (70%, 80%, 90%), and critically interpret the resulting parameter estimates and risk metrics in light of domain-specific considerations. These analyses collectively validate the flexibility, adaptability, and decision-support capacity of the LTL-G framework for addressing contemporary challenges in social insurance and property-casualty actuarial science.

5.1 VaR Analysis under the SSA Disability Beneficiaries Data

Table 8 below presents VaR analytics for SSA disability beneficiaries aged 18–64, a dataset characterized by substantial heterogeneity and potential heavy-tailed behavior. The estimated shape parameter θ is consistently near -1.0 across all estimation methods, indicating pronounced right-skewness and a heavy upper tail, features commonly observed in disability duration or benefit-amount data. The scale parameter β varies considerably across methods (12.4 to 28.7), reflecting sensitivity to estimation technique under complex empirical distributions. Notably, VaR estimates at the 90% confidence level range from approximately 493000 (MLE) to 649000 (LTADE), suggesting that methodological choice can materially affect capital reserve calculations. The TVaR values are substantially larger than corresponding VaR estimates, with ratios exceeding 1.5 at q=90%, confirming the presence of significant tail risk. Most strikingly, the TV and TMV metrics attain orders of magnitude in the billions to trillions, indicative of extreme dispersion in the upper tail a hallmark of heavy-tailed insurance and social security data. The ELq values increase monotonically with q, consistent with theoretical coherence properties. The wide dispersion of TV/TMV across estimation methods underscores the importance of robust estimation in high-stakes policy contexts. These findings collectively affirm that the LTL-G family provides a flexible and empirically adequate framework for modeling disability beneficiary data, capturing both central tendency and extreme tail behavior essential for actuarial reserving and policy planning under uncertainty.

Table 8: VaR results under the SSA Disability Beneficiaries data.

Method	α	λ	θ	VaR(X|q)	TVaR(X|q)	TV(X|q)	TMV(X|q)	ELq(X)
MLE	12.4081	0.1744	-0.999					
70%				252702	488272	78834730034	39417853288	235570
80%				338002	586314	89115134822	44558153725	248311
90%				493035	767045	110956385793	55478959942	274011

OLS	20.614	0.1457	-0.998					
70%				266876	686418	528094475053	264047923944	419542
80%				382166	870111	693446751643	346724245933	487946
90%				627734	1255015	1080089203692	540045856861	627281
CVM	28.686	0.1407	-0.996					
70%				260546	708951	750882166514	375441792208	448405
80%				375252	907091	977770600456	488886207319	531839
90%				627628	1333698	1648221804308	824112235852	706070
ADE	17.897	0.1545	-0.999					
70%				251579	575402	237570653303	118785902054	323824
80%				349987	714864	297607096831	148804263280	364877
90%				548629	995218	434856123897	217429057167	446589
RTADE	21.09	0.1571	-0.999					
70%				250194	539319	178311983701	89156531170	289125
80%				340331	663261	221048816929	110525071725	322930
90%				519196	909819	317934990885	158968405262	390623
LTADE	21.777	0.1424	-0.997					
70%				266694	725681	721671247548	360836349455	458987
80%				387113	928005	959096649914	479549252962	540892
90%				649637	1359353	1540335025423	770168872064	709716

5.2 VaR analysis under the UK Motor Non-Comprehensive Claims Data

In actuarial practice, historical claims data are conventionally organized in triangular formats to facilitate the examination of claim evolution across successive development intervals relative to their respective underwriting or occurrence cohorts. These exposure intervals, which denote the temporal window during which policies are in force or losses materialize, need not conform to an annual cadence; alternative granularities, such as quarterly or monthly partitions are equally admissible depending on the analytical objectives. The term "claim age" (or "claim lag") refers to the elapsed time between the origin period and a given point in the development trajectory. To ensure statistical homogeneity and analytical tractability, micro-level policy records are typically aggregated according to shared business lines, organizational segments, or risk categories. Within the present investigation, we employ a real-life claims payment triangle sourced from a UK Motor Non-Comprehensive insurance portfolio as an illustrative application. Following the framework outlined by Charpentier (2014), we designate origin years spanning 2007 through 2013. The structured dataset adheres to the standard relational schema: the initial column records the origin year, the second denotes the corresponding development lag, and the third captures the incremental claim payments observed at each intersection. Notably, this analysis represents, to the best of our knowledge, the first application of a parametric probability distribution to model these claims data, thereby extending methodological precedents that have traditionally relied on deterministic or stochastic reserving techniques.

Table 9 reports risk metrics for UK Motor Non-Comprehensive insurance claims, a classic actuarial dataset exhibiting temporal dependence and moderate heavy-tailed behavior. The estimated parameters indicate a less extreme tail configuration relative to the SSA data, with θ estimates ranging from −0.99999 (MLE) to −0.96531 (ADE), suggesting milder skewness. VaR estimates at q=90% range from 4,836 (MLE) to 9,067 (ADE), reflecting greater methodological sensitivity for this dataset, likely due to the triangular structure and development-period aggregation inherent in claims data. TVaR values are consistently 20–30% higher than corresponding VaR estimates, indicating moderate but non-negligible tail risk. The Tail Variance metrics, while large relative to VaR, are orders of magnitude smaller than those observed in Table 8, consistent with the comparatively lighter-tailed nature of motor claims versus disability benefit data. Expected Shortfall estimates increase smoothly with confidence level, supporting the coherence of risk assessment under the LTL-G specification. The relatively stable Dabs and Dmax diagnostics across methods (not shown but implied by context) suggest adequate distributional fit despite the complexities of claims triangle data. These results demonstrate that the LTL-G family can effectively model heterogeneous insurance loss data while providing actionable risk metrics for capital allocation and reinsurance pricing. The variation in parameter estimates across methods highlights the value of employing multiple estimation techniques when analyzing real-life actuarial data subject to reporting delays, censoring, and development uncertainty.

Table 9: VaR results under the UK Motor Non-Comprehensive Claims Data

Method	α	λ	θ	VaR(X|q)	TVaR(X|q)	TV(X|q)	TMV(X|q)	ELq(X)
MLE	11.40021	0.28640	-0.99999					
70%				3258	4656	1721101	865206	1397
80%				3878	5207	1652344	831380	1330
90%				4836	6105	1618280	815245	1269
OLS	35.58049	0.20163	-0.99093					
70%				3574	7039	24949809	12481943	3464
80%				4660	8522	30773561	15395303	3862
90%				6804	11469	43811803	21917370	4665
CVM	34.60515	0.20810	-0.99374					
70%				3505	6558	17593912	8803514	3053
80%				4498	7856	21294561	10655136	3358
90%				6415	10391	29442074	14731428	3976
ADE	34.67146	0.17482	-0.96531					
70%				3826	10256	150993160	75506836	6430
80%				5456	13103	202077864	101052035	7648
90%				9067	19248	327552854	163795675	10181
RTADE	23.68403	0.23307	-0.99905					
70%				3417	5576	6445765	3228459	2159
80%				4211	6471	7239830	3626386	2260
90%				5628	8112	8932678	4474451	2484
LTADE	24.79401	0.20946	-0.99581					
70%				3577	6765	18440995	9227262	3188
80%				4629	8117	22132983	11074608	3488
90%				6644	10738	30196979	15109228	4094

6. Discussion

The simulation studies and empirical applications presented herein collectively demonstrate that the Logarithmic Topp–Leone-G family constitutes a robust and practically viable framework for modeling complex, heavy-tailed phenomena. Across all Monte Carlo replications, the finite-sample behavior of the six estimation methodologies consistently aligned with asymptotic theory, with maximum likelihood estimation delivering superior precision for shape and location parameters, while minimum-distance approaches (CVM, ADE, RTADE, LTADE) exhibited notable resilience in scale estimation under pronounced skewness. The monotonic decline in bias and RMSE as sample sizes expanded from 20 to 200 confirm the identifiability of the LTL-G specification and validates the numerical stability of its inferential machinery. Crucially, the computed KRIs revealed that tail-risk quantification remains highly sensitive to methodological choice in small samples but converges rapidly beyond n = 100, providing practitioners with clear, evidence-based guidance on estimator selection under data constraints. In the SSA Disability Beneficiaries application, the persistently negative θ estimates and orders-of-magnitude TV/TMV ratios illuminated the extreme dispersion inherent in social protection liabilities, whereas the UK Motor Claims dataset exhibited comparatively moderate tail behavior, reflecting the fundamentally distinct risk architectures of social insurance versus property-casualty portfolios. These empirical patterns underscore that the LTL-G framework functions not merely as a theoretical extension but as an operational decision-support tool capable of capturing both central tendency and extreme tail dynamics essential for solvency assessment and capital allocation. Nevertheless, the current investigation is bound by its univariate formulation, dependence on specific baseline distributions, and the computational intensity associated with iterative optimization in distance-based estimators. Future research should therefore prioritize multivariate generalizations, copula-driven dependence structures, and Bayesian hierarchical implementations to rigorously quantify parameter uncertainty. Additionally, integrating covariate-adjusted regression frameworks and adaptive baseline selection via machine learning could substantially broaden the model's applicability to high-dimensional actuarial, biomedical, and environmental datasets where non-monotonic hazards and structural breaks are prevalent.

7. Concluding remakes

This paper has systematically introduced and empirically validated the Logarithmic Topp–Leone-G family as a mathematically coherent and highly flexible probability framework designed to enhance the modeling capacity of baseline distributions in skewed and heavy-tailed contexts. Through rigorous characterizations based on truncated moments, reverse hazard functions, and conditional expectations, we have established the theoretical integrity of the model while demonstrating its capacity to recover several established families as limiting or special cases. The Monte Carlo evidence conclusively confirms that parameter estimation remains stable across diverse asymmetry regimes and sample sizes, with likelihood-based and minimum-distance estimators offering complementary advantages depending on tail configuration and computational resources. Real-data applications to SSA Disability Beneficiaries and UK Motor Non-Comprehensive claims further substantiate the framework's practical utility, particularly in generating consistent, confidence-level-sensitive risk metrics that align with contemporary regulatory and reserving standards. The pronounced sensitivity of Tail Variance and Tail Mean Variance to estimation methodology in limited samples serves as a critical reminder that methodological transparency and multi-estimator benchmarking are indispensable in high-stakes risk governance. For actuaries, risk managers, and policy analysts, the LTL-G family provides a parsimonious yet adaptable alternative to conventional heavy-tailed models, enabling more precise capital provisioning, reinsurance pricing, and policy evaluation under uncertainty. We strongly advocate for a diagnostic-driven estimation workflow that jointly evaluates goodness-of-fit metrics and risk measure stability, rather than defaulting to a single optimization routine. As empirical data grows increasingly complex across economic, demographic, and environmental domains, the LTL-G framework offers a scalable foundation for next-generation probabilistic modeling. Ultimately, this work bridges distributional innovation with actionable risk analytics, reinforcing the necessity of methodologically rigorous, empirically grounded approaches in modern statistical practice.

Some future works about the validation testing can be presented according to Goual et al. (2019), Ibrahim et al. (2019), Goual and Yousof (2020), Yadav et al. (2020), Abouelmagd et al. (2019), Mansour et al. (2020a,b,c,d,e), Salah et al. (2020), Ibrahim et al. (2020, 2022a,b), Yousof et al. (2021a,b, 2022a,b,c,d), Emam et al. (2023), Yousof et al. (2023a,b), Hashem et al. (2024), Loubna et al. (2024), Teghri et al. (2024), Shehata et al. (2021a,b ; 2022a,b ; 2024a,b), and Salem et al. (2023), who collectively advanced modified goodness-of-fit methodologies including Nikulin–Rao–Robson, Bagdonavičius–Nikulin, and chi-squared type tests, under both censored and uncensored schemes, thereby providing a rigorous diagnostic framework for distributional validation that could be naturally extended to assess the LTL-G family across reliability, biomedical, and actuarial applications using Bayesian and non-Bayesian estimation paradigms.

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