Pak.j.stat.oper.res. Vol.22 No. 1 2026 pp 71-101 DOI: https://doi.org/10.18187/pjsor.v22i1.4813

Pakistan Journal of Statistics and Operation Research

An Alternative Exponential Model for Skewed Real Data: Characterizations, Bayesian, Non- Bayesian Estimation and Distributional Validation Testing

Mohamed Ibrahim[1,*] , Abdullah H. Al-Nefaie[1], Hafida Goual[2], Talhi Hamida[2], Aiachi Hiba[2], G.G. Hamedani[3], Ahmad M. AboAlkhair[1], Mujtaba Hashim[1], Nazar Ali Ahmed[1], and Haitham M. Yousof[5]

* Corresponding Author

[1]Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia; (M.I.): miahmed@kfu.edu.sa; (A.H.A.): aalnefaie@kfu.edu.sa; (A.M.A.): aaboalkhair@kfu.edu.sa; (M.H.): msaeed@kfu.edu.sa and (N.A.A.): nahmed@kfu.edu.sa
[2]Laboratory of probabilities and statistics LaPS, Badji Mokhtar – Annaba University, 12, P.O. Box, 23000 Annaba Algeria.; (H.G.): hafida.goual@univ-annaba.dz, (T.H.): talhihamida@yahoo.fr & (A.H.): aiachihiba@yahoo.com
[3]Department of Mathematical and Statistical Sciences, Marquette University, USA; gholamhoss.hamedani@marquette.edu
[4]Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Benha University, Benha, Egypt; haitham.yousof@fcom.bu.edu.eg

Abstract

This paper presents a novel exponential model with two parameters, placing particular attention on its practical applications to skewed data as the central area of investigation. The mathematical characteristics of this atypical distribution are established, in a lucid and succinct manner, by the discoveries made in this investigation. Furthermore, it is worth noting that there exist three distinct approaches to describing the distribution. The process of estimating the parameters of the novel model involves employing a range of established methodologies, including the Bayesian technique. When confronted with censored data, the maximum likelihood technique is commonly considered as a viable approach. Pitman's closeness criteria are employed as the comparative tool when assessing the probability estimate in relation to Bayesian estimation approaches. During the computation of Bayesian estimations, three distinct loss functions, namely generalized quadratic, Linex, and entropy, are employed. A multitude of simulated experiments are conducted to assess the efficacy of various estimation methodologies. The BB algorithm is employed to facilitate the comparison and contrast between the Bayesian technique and the censored maximum likelihood strategy. The Nikulin-Rao-Robson (NKRR) statistic was derived by conducting two empirical studies using real-world data sets characterized by skewed distributions, along with simulation research conducted in an unfiltered environment. Furthermore, this paper delineates two other uses within the same context. The study's findings illustrate the efficacy of the approaches presented for the purposes of distribution and estimation.

Keywords: Bayesian Validation; Censored Applications; Censored Validation; Characterizations; Exponential Model; Nikulin-Rao-Robson; Pitman's Proximity; Censored Validation.

Mathematical Subject Classification: 62N02; 62E12; 62N01; 62E10.

1. Introduction

The NKRR statistic test is a statistical process that is used for the purpose of validating hypotheses or hypotheses-based statements. The fundamental reason for its significance is that it can identify whether or not a certain statistical model is a good fit for a particular dataset. This ability is what gives it its significance. The NKRR test is a useful tool for determining whether or not the model satisfactorily explains the observed data and whether or not the assumptions that underpin the model are reasonable (for additional details, see Nikulin (1973a), Nikulin (1973b), Nikulin (1973c) and Rao and Robson (1974)). To accomplish this, we must first determine how well the model fits the data. Below are some key reasons why the NKRR statistic test is important for statistical validation:

I. The NKRR test provides a quantitative measure of how well a model fits the data. It allows researchers to assess the adequacy of a model in capturing the observed patterns and variability in the data. This is crucial for evaluating

the reliability and usefulness of the model for making predictions or drawing inferences (see Goual et al. (2019) and Ibrahim et al. (2023)).

II. The NKRR test allows for hypothesis testing by comparing the observed data against the model's predictions. It helps determine whether there is evidence to support the hypothesis that the model accurately represents the underlying data-generating process. This is particularly valuable when researchers want to test specific assumptions or hypotheses about the relationship between variables.

III. Statistical models often rely on certain assumptions, such as normality or independence of the data. The NKRR test helps validate these assumptions by assessing whether they hold true for the observed data. If the test indicates a poor fit, it suggests that the underlying assumptions might not be appropriate, prompting researchers to consider alternative models or data transformations.

IV. The NKRR test is helpful for facilitating model selection because it examines the degree to which multiple models are consistent with the same dataset. As a result, this test is beneficial for facilitating model selection. Researchers have the capacity to perform the test in order to evaluate the goodness-of-fit statistics of a variety of models and choose the model that provides the best accurate representation of the data. This helps to avoid overfitting, which is when a model is extremely sophisticated and ends up fitting the noise in the data, as well as underfitting, which is when a model is overly simplistic and fails to capture crucial patterns. Overfitting happens when a model is overly intricate and ends up fitting the noise in the data.

V. Statistical models are often used to make decisions or draw inferences about a population based on the observed data. The NKRR test provides a means to assess the validity of such decisions or inferences. If the model fails the NKRR test, it indicates that the model might not be appropriate for making reliable predictions or drawing valid conclusions.

To conduct out an NKRR type goodness of fit test for a parametric model that is often used in survival analysis, social science research, engineering research, and dependability research, the goal of this study is to explore a two-parameter Poisson-exponential (TPE) model. This will allow us to determine whether or not the model is a good fit for the data. To obtain an improvement in the maximum likelihood estimation of the parameters and the mean time between failure of the TPE distribution, the Bayesian technique is utilized in this study. This was done in order to accomplish the aforementioned goals. To demonstrate how successful this adjustment has been, a suitable loss function has been used here. The TPE distribution is well-known for the ease with which it may be altered mathematically and scientifically in the process of modelling. This ease of manipulation has earned the TPE distribution its well-deserved reputation. This work, on the other hand, focuses on the more practical parts of mathematics and statistical modelling, which is a divergence from the approach that the majority of academics who investigate probability distributions normally adopt in their investigations. When confirming the distribution using censored data, it is important to omit a wide variety of theoretical mathematical properties, algebraic derivations, and other associated theories. This is because these aspects are associated with the distribution. This absence was deliberate and done so by purpose in order to emphasize the significance of the new distribution, as well as its adaptability and versatility, as well as its substantial utility in statistical and mathematical modelling, in particular when dealing with controlled data. In this context, we need to quickly investigate the origin of the new distribution, as well as how it was built and constructed. This is necessary so that academics working in the field can provide more distributions that are comparable to the one that is being discussed here, and maybe distributions that are more flexible. According to Tahir et al. (2020), the following is a description of how the cumulative distribution function (CDF) of the TPE distribution might be expressed

$$\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}) = \frac{1}{1-exp(-\mathcal{V}_1)}\left\{1-exp\left[-\mathcal{V}_1\overline{\mathfrak{z}}_{\mathcal{V}_2}(\boldsymbol{x})\right]\right\}|_{\mathcal{V}_1\in R-\{0\},\mathcal{V}_2>0 \text{ and } \boldsymbol{x}>0}, \tag{1}$$

where $\underline{\mathbb{V}} = (\mathcal{V}_1, \mathcal{V}_2)$ and $\overline{\mathfrak{z}}_{\mathcal{V}_2}(\boldsymbol{x}) = 1 - \mathfrak{z}_{\mathcal{V}_2}(\boldsymbol{x})$ and

$$\mathfrak{z}_{\mathcal{V}_2}(\boldsymbol{x}) = [exp(-\mathcal{V}_2\boldsymbol{x})]^{[1-exp(-\mathcal{V}_2\boldsymbol{x})]}.$$

Then, the probability density function (PDF) corresponding to (1) can then be derived as

$$f_{\underline{\mathbb{V}}}(\boldsymbol{x}) = \frac{\mathcal{V}_1\mathcal{V}_2}{1-exp(-\mathcal{V}_1)}\mathfrak{z}_{\mathcal{V}_2}(\boldsymbol{x})\frac{1}{exp\left\{\mathcal{V}_1\left[\boldsymbol{x}+\overline{\mathfrak{z}}_{\mathcal{V}_2}(\boldsymbol{x})\right]\right\}}[\mathcal{V}_2\boldsymbol{x}-1+exp(\mathcal{V}_2\boldsymbol{x})], \tag{2}$$

for all $\mathcal{V}_1 \in R-\{0\}$, $\boldsymbol{x}>0$ and $\mathcal{V}_2>0$. The exponential distribution and its unique flexible extensions have been the subject of a significant amount of research in academic circles. These scholars have a particular interest in the potential applications of these specific extensions across a variety of scientific fields, such as engineering, actuarial science, medical science, reliability, and others. An example of distributional validation of an exponential extension using a modified goodness of fit test that is applicable to both censored and complete data is given by Goual et al. (2019) and Ibrahim et al. (2023). Both of these studies were conducted by Ibrahim et al. When none of the TPE model's

parameters have been identified and all of the data have been collected, it is recommended that a modified chi-square test of fit be carried out. The NKRR statistic was independently developed by Nikulin (1973a), Nikulin (1973b), Nikulin (1973c) and Rao and Robson (1974), and it was used as the foundation for this test. In comparison to the traditional chi-square test, the NKRR statistic provides an evaluation of the appropriateness of the chosen model of fit that is more objective. To accomplish this goal, the Fisher information that is based on the estimate with the highest likelihood is completely incorporated. We develop a one-of-a-kind goodness-of-fit test by making use of Bagdonavičius et al. (2013), Bagdonavičius and Nikulin (2011a) and Bagdonavičius and Nikulin (2011b) method for data that is right-censored and has unknown parameters. The next step would be to determine how well the model fits the data. Modifications are made to the NKRR statistic so that it can consider filtering as well as additional influences the precise nature of which is unknown. Using a numerical simulation exercise, we establish the invariance of the distribution of the test statistic and analyze the null hypothesis that a sample originates from a TPE model. Both are done to test the alternative hypothesis. This is done to verify that the validity of the null hypothesis has not been compromised. Both censored and uncensored TPE distribution data can be utilized to find the best fit with the help of the provided tests. To demonstrate how useful the TPE model is, we finally put the approach that we developed to use by applying it to data that was collected from the actual world and was gathered from a variety of scientific fields. This article aims to shed light on the practical uses of our modified chi-square test as well as the TPE distribution in a variety of different contexts.

One of the key benefits of utilizing the NKRR test statistic is its resilience to the impact of outliers. Outliers are frequently encountered in real-world datasets and can significantly affect the outcomes of statistical analysis. The NKRR test is specifically designed to withstand the influence of outliers, making it a valuable tool for identifying and examining datasets containing extreme values. This is especially crucial in finance, where events like market crashes and major price fluctuations must be detected and analyzed before any action is taken. Moreover, the NKRR test statistic's ability to handle non-normal distributions adds another compelling reason to use it. Real-world datasets often exhibit non-normal distributions, including those with heavy tails and skewed distributions. Since various distributions can deviate from normality, the NKRR test is an effective approach for analyzing a wide range of datasets by detecting such deviations. This is particularly important in fields like engineering and economics, which extensively utilize non-normal distributions. Furthermore, in addition to its resistance to outliers and capability to handle non-normal distributions, the NKRR test statistic is a versatile tool suitable for analyzing datasets of any size, whether they are small samples or extensive datasets. Its adaptability in accommodating non-normal distributions enables its use in various applications, from scrutinizing individual data points to exploring overarching patterns and trends.

2. Characterization results

Within the field of statistical analysis, the practice of use truncated moments to describe probability distributions has been widely recognized and firmly established. Truncated moments are widely recognized as a valuable tool in multiple disciplines, such as finance, engineering, and physics, owing to their ability to provide significant insights on the characteristics of a distribution, including its location, shape, and scale. In recent times, there has been an increased application of truncated moments in addressing truncated moment problems. These challenges entail the inference of a probability distribution by utilizing its truncated moments. A commonly utilized strategy for addressing such issues entails the utilization of the maximum entropy method. The objective of this technique is to determine the most likely probability distribution that corresponds to the given collection of truncated moments, so aiding in the resolution of the problem.

The characterizations that have been generated from two shortened moments have attracted considerable attention in this particular context due to their wide-ranging application. By employing a limited number of truncated moments, it becomes possible to make inferences about many probability distributions, such as the normal, exponential, and gamma distributions. For more comprehensive information, see Glanzel (1987). The characterizations obtained from two abbreviated moments present notable benefits compared to other approaches, as explained by Glanzel (1990). Truncated moment selection methods are computationally simple and serve as a fundamental approach for determining the most probable distribution that fulfills a given set of truncated moments. Essentially, truncated moments can be regarded as a subset of truncated moments. Moreover, they exhibit resilience in the presence of measurement imperfections, providing a dependable estimation of the examined distribution.

In conclusion, the use of characterizations derived from two truncated moments serves as a powerful methodology for comprehending probability distributions, yielding substantial ramifications inside diverse fields of research. The key

factor that contributes to their significance is their ability to generate accurate and reliable estimations of probability distributions. As a result, they are highly beneficial to both researchers and practitioners. This section focuses on characterizations of the TPE distribution using two truncated moments. The CDF does not need to be closed for this categorization. This categorization makes use of Glanzel (1987) theorem. Obviously, the result holds when the interval $H_{(d,e)}$ is not closed. Glanzel (1990) defines this characterisation as stable in the sense of weak convergence.

Proposition 2.1. Let the random variable $X : \Omega \to (0,\infty)$ be continuous, and assume

$$\boldsymbol{h}_{\underline{\mathbb{V}}}(\boldsymbol{x}) = \frac{[exp(-\mathcal{V}_2\boldsymbol{x})]^{exp(-\mathcal{V}_2\boldsymbol{x})}[\mathcal{V}_2\boldsymbol{x} - 1 + exp(\mathcal{V}_2\boldsymbol{x})]^{-1}}{exp(-\mathcal{V}_1\{1 - [exp(-\mathcal{V}_2\boldsymbol{x})]^{[1-exp(-\mathcal{V}_2\boldsymbol{x})]}\})}$$

and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x})\, exp(-\mathcal{V}_2\boldsymbol{x})$ for $\boldsymbol{x} > 0$. Then, the PDF of X is given in (2) if and only if the function $\boldsymbol{\varkappa}$ defined in Theorem 2.1 is of the form

$$\boldsymbol{\varkappa}(\boldsymbol{x}) = \frac{1}{2} exp(-\mathcal{V}_2\boldsymbol{x})\,|_{\boldsymbol{x}>0}.$$

Proof. If X has PDF (2), then

$$\left(1 - \mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x})\right) E\left[\boldsymbol{h}_{\underline{\mathbb{V}}}(X) \mid_{X\geq\boldsymbol{x}}\right] = \frac{\mathcal{V}_1}{1 - e^{-\mathcal{V}_2}} exp(-\mathcal{V}_2\boldsymbol{x})\,|_{\boldsymbol{x}>0},$$

and

$$\left(1 - \mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x})\right) E[\boldsymbol{g}(X) \mid_{X\geq\boldsymbol{x}}] = \frac{\mathcal{V}_1}{2(1 - e^{-\mathcal{V}_2})} exp(-2\zeta_2\boldsymbol{x})\,|_{\boldsymbol{x}>0},$$

and finally

$$\boldsymbol{\varkappa}(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{h}(\boldsymbol{x})\, exp(-\mathcal{V}_2\boldsymbol{x}) < 0 \text{x>0}.$$

Conversely, if $\boldsymbol{\varkappa}$ has the above form, then

$$s'(\boldsymbol{x}) = \frac{\boldsymbol{\varkappa}'(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x})}{\boldsymbol{\varkappa}(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x})} = \mathcal{V}_2,$$

and hence $s(\boldsymbol{x}) = \mathcal{V}_2\boldsymbol{x}, \quad \boldsymbol{x} > 0$. In view of Theorem 2.1, X has PDF (2).

Corollary 2.1. Let $X{:}\,\Omega \to (0,\infty)$ be a continuous random variable and $\boldsymbol{h}(\boldsymbol{x})$ be as in Proposition 2.1. Then, X has PDF (2) if and only if there exist functions $\boldsymbol{g}$ and $\boldsymbol{\varkappa}$ defined in Theorem of Glanzel (1987) satisfying the following first order differential equation

$$\frac{\boldsymbol{\varkappa}'(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x})}{\boldsymbol{\varkappa}(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x})} = \mathcal{V}_2.$$

Corollary 2.2. The general solution of the above differential equation is

$$\boldsymbol{\varkappa}(\boldsymbol{x}) = exp(\mathcal{V}_2\boldsymbol{x})\left[-\int \mathcal{V}_2\, \boldsymbol{g}(\boldsymbol{x})\big(\boldsymbol{h}(\boldsymbol{x})\big)^{-1} exp(-\mathcal{V}_2\boldsymbol{x}) + D\right],$$

where D is a constant. A set of functions satisfying this differential equation is presented in Proposition 2.1. with $D = 0$. Clearly, there are other triplets $(\boldsymbol{h}, \boldsymbol{g}, \boldsymbol{\varkappa})$ satisfying the conditions of theorem of Glanzel (1987).

3. The NKRR for the TPE model

The NKRR is a valuable tool for validating distortions and testing hypotheses in various fields, including engineering, economics, and finance. It is particularly useful for analyzing data sets that have skewed distributions, heavy tails, and outliers, which are commonly encountered in real-world applications. One of the key benefits of the NKRR test statistic is its ability to detect deviations from normality that other statistical tests may miss. It is also robust to outliers, making it a reliable method for identifying and analyzing data sets with extreme values, especially in finance, where it is crucial to identify and examine significant market events. Another advantage of the NKRR test statistic is its flexibility in handling different types of distributions. Unlike some other statistical tests, the NKRR test can be applied to both continuous and discrete distributions, making it a versatile tool for a wide range of applications. Moreover, the NKRR test can be used with data sets of any size, from small samples to large data sets.

In point of fact, the NKRR statistic is derived from the disparities that exist between two statistical estimators of the probability of falling into specific grouping intervals. Additionally, the NKRR statistic is a modified form of the usual chi-squared tests that are used to evaluate the full data set. Both the empirical distribution function and maximum likelihood are used in the other estimating method in order to determine the values of the tested model's parameters that are unknown to the researcher. Nikulin (1973a), Nikulin (1973b), Nikulin (1973c) and Rao and Robson (1974) offered additional information on this statistic, but Goual et al. (2019) and Ibrahim et al. (2023) demonstrated its

application in a range of different situations. A natural progression from the Pearson statistic is the NKRR test, which makes use of the chi-square distribution in its calculations. However, when random right censoring is involved in addition to the unknown parameters, the conventional test might not be adequate to support the null hypothesis. This is because random right censoring affects the proportion of correct answers. In this particular circumstance, it is necessary to adjust the NKRR statistic in accordance with the recommendations made by Bagdonavičius et al. (2013), Bagdonavičius, V. and Nikulin (2011a) and Bagdonavičius, V. and Nikulin (2011b) and recently adjusted by Yousof et al. (2023a), Yousof et al. (2023b) and Emam et al. (2023). On the other hand, Nikulin (1973a), Nikulin (1973b), Nikulin (1973c) and Rao and Robson (1974) came up with the NKRR statistic that is described below in order to evaluate the hypothesis:

$$H_0: Pr\{\boldsymbol{x}_i \leq \boldsymbol{x}\} = \mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x})|_{\boldsymbol{x}\in R}.$$

The NKRR statistic is utilized as a means of assessing the concordance between two probability distributions, and it holds significant utility across various disciplines such as statistics, hydrology, and meteorology.

4. Estimation and inference

In brief, Bayesian and non-Bayesian approaches to estimation and inference in statistics represent two separate paradigms. Non-Bayesian methodologies mostly focus on estimates derived from likelihood-based approaches, utilizing confidence intervals or hypothesis tests for inference. In contrast, Bayesian methodologies integrate prior beliefs and provide a posterior distribution. Both techniques have their own set of pros and limitations, and the selection of one over the other depends on the unique context and the data that is available. This part is comprised of two subsections, which delve into the examination of Bayesian and non-Bayesian estimating procedures. The first component of this study examines eight estimate strategies that are not based on Bayesian principles. These techniques include Maximum Likelihood estimate (MXLE), Cramer-von-Mises Estimation (CVME), Weighted-Least Square Estimation (WLSQ), and Kolmogorov Estimation (KE). The following section (SELF) examines both the squared error loss function and Bayesian estimating methodologies.

4.1 Classical estimation methods

Maximum likelihood method

Given a set of observed data, the maximum likelihood approach is a useful statistical tool for estimating the parameters of a probability distribution. The primary idea behind this strategy is to discover the parameter values that maximize the probability of detecting the given data. The maximum likelihood technique produces efficient parameter estimators, which are asymptotically unbiased and have the minimum possible variance. This makes the procedure particularly beneficial when we have a limited amount of data and want to acquire the most precise parameter estimates. The greatest likelihood method is frequently resistant to changes in the assumed distributional form. This means that even if the data is not completely normally distributed, the approach can still yield respectable estimates. The maximum likelihood approach can also be used to assess how well a given distribution fits a set of data. We can establish whether the chosen distribution is a good fit for the data by comparing the observed data to the model predictions, or whether a different distribution would be more appropriate. The likelihood function of this sample is

$$L_{\underline{\mathbb{V}}}(\boldsymbol{x}) = \prod_{i=1}^{m} f_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)\left[1 - \mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)\right]^{n-m},$$

where $N = \frac{n!}{(n-m)!}$, then we have

$$L_{\underline{\mathbb{V}}}(\boldsymbol{x}) = N\zeta_1^m \mathcal{V}_2^m (1 - exp(-\mathcal{V}_1))^{-n} C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i) \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)$$

in which

$$M_{\mathcal{V}_2}(\boldsymbol{x}_i) = \mathfrak{Z}_{\mathcal{V}_i}(\boldsymbol{x}_i)[\mathcal{V}_2\boldsymbol{x}_i - 1 + exp(\mathcal{V}_2\boldsymbol{x}_i)], \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i) = exp(-\mathcal{V}_2\boldsymbol{x}_i - \mathcal{V}_1(1 - \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_i))),$$

and $C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m) = 1 - exp(-\mathcal{V}_1(1 - \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_m)))$. For obtaining the MXLE of $\mathcal{V}_1$ and $\mathcal{V}_2$, we have

$$l_{\underline{\mathbb{V}}}(\boldsymbol{x}) = ln\, N + m\, ln\, \mathcal{V}_1 + m\, ln\, \mathcal{V}_2 - n\, ln(1 - exp(-\mathcal{V}_1)) + (n-m)\, ln\, C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m) + \sum_{i=1}^{m} ln\, M_{\mathcal{V}_2}(\boldsymbol{x}_i) + \sum_{i=1}^{m} ln\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i).$$

Then,

$$\frac{\partial l_{\underline{\mathbb{V}}}(\boldsymbol{x})}{\partial \mathcal{V}_1} = \frac{m}{\mathcal{V}_1} - \frac{n\, exp(\mathcal{V}_1)}{1 - exp(-\mathcal{V}_1)} + (n-m)\frac{\frac{\partial C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)}{\partial \mathcal{V}_1}}{C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)} + \sum_{i=1}^{m} \frac{\frac{\partial \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)}{\partial \mathcal{V}_1}}{\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)} = 0$$

$$\frac{\partial l_{\underline{\mathbb{V}}}(\boldsymbol{x})}{\partial \mathcal{V}_2} = \frac{m}{\mathcal{V}_2} + (n-m)\frac{\frac{\partial C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)}{\partial \mathcal{V}_2}}{C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)} + \sum_{i=1}^{m} \frac{\partial M_{\mathcal{V}_2}(\boldsymbol{x}_i)}{\partial \mathcal{V}_2}\frac{1}{M_{\mathcal{V}_2}(\boldsymbol{x}_i)} + \sum_{i=1}^{m} \frac{\partial \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)}{\partial \mathcal{V}_2}\frac{1}{\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)} = 0$$

where

$$\frac{\partial \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_i)}{\partial \mathcal{V}_2} = -[\boldsymbol{x}_i - \boldsymbol{x}_i\, exp(-\mathcal{V}_2\boldsymbol{x}_i)(1-\mathcal{V}_2\boldsymbol{x}_i)]\, exp(-\mathcal{V}_2\boldsymbol{x}_i(1-exp(-\mathcal{V}_2\boldsymbol{x}_i)),$$

$$\frac{\partial M_{\mathcal{V}_2}(\boldsymbol{x}_i)}{\partial \mathcal{V}_2} = \left(\frac{\partial \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_i)}{\partial \mathcal{V}_2}\right)(\mathcal{V}_2\boldsymbol{x}_i - 1 + exp(\mathcal{V}_2\boldsymbol{x}_i)) + \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_i)(\boldsymbol{x}_i + \boldsymbol{x}_i\, exp(\mathcal{V}_2\boldsymbol{x}_i)),$$

$$\frac{\partial \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)}{\partial \mathcal{V}_1} = -(1-\mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_i))\, exp(-\mathcal{V}_2\boldsymbol{x}_i - \mathcal{V}_1(1-\mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_i)),$$

$$\frac{\partial \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)}{\partial \mathcal{V}_2} = \left[\mathcal{V}_1\left(\frac{\partial \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_i)}{\partial \mathcal{V}_2}\right) - \boldsymbol{x}_i\right] exp(-\mathcal{V}_2\boldsymbol{x}_i - \mathcal{V}_1(1-\mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_i)),$$

$$\frac{\partial C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)}{\partial \mathcal{V}_1} = (1-\mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_m))\, exp(-\mathcal{V}_1(1-\mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_m)),$$

and

$$\frac{\partial C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)}{\partial \mathcal{V}_2} = -\mathcal{V}_1\left(\frac{\partial \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_m)}{\partial \mathcal{V}_2}\right) exp(-\mathcal{V}_1(1-\mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}_m)).$$

For the purpose of verifying various TPE distribution sub-models, we can compute the maximum values of the unconstrained and restricted log-likelihoods. he Newton-Raphson method is applied iteratively to solve a set of simultaneous equations, facilitating the computation of maximum likelihood estimators (MXLEs) for the proposed model. In the realm of maximum likelihood estimation (MXLE), these equations often arise from the first-order conditions, where we set the gradient (first derivative) of the log-likelihood function to zero. In simpler terms, we equate the initial derivative of the log-likelihood function to zero. To implement the Newton-Raphson method, we are required to solve a linear system that stems from the second derivative (Hessian) of the log-likelihood function. This process enables us to iteratively refine our estimates of the model parameters.

The method of CVME

The CVME method is typically resilient in the face of shifts in the distributional form that is expected. This indicates that the technique can still produce good estimates, even in the event that the data do not perfectly follow a normal distribution. The CVME method can also be used to evaluate how well a certain distribution fits a collection of data by comparing the two. By comparing the data that has been observed to the data that the model has predicted, we will be able to determine whether or not the selected distribution is a good fit for the data, or whether or not another distribution would be more appropriate. The CVME of the parameters $\mathcal{V}_1$ and $\mathcal{V}_2$ are obtained via minimizing the following CVME $(\underline{\mathbb{V}})$ expression with respect to (w.r.t.) to $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively, where

$$\text{CVME}_{(\underline{\mathbb{V}})} = \frac{1}{12}n^{-1} + \sum_{i=1}^{n}\left[\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n}) - l_{(i,n)}^{[1]}\right]^2 |_{(\mathcal{V}_1 \in I \text{ and } \boldsymbol{x}_{i,n} \in N_{(0)})},$$

where $l_{(i,n)}^{[1]} = \frac{1}{2n}(2i-1)$ and

$$\text{CVME}_{(\underline{\mathbb{V}})} = \sum_{i=1}^{n}\left[\frac{1 - exp\left[-\mathcal{V}_1\overline{\mathfrak{Z}}_{\mathcal{V}_2}(\boldsymbol{x}_{i,n})\right]}{1 - exp(-\mathcal{V}_1)} - l_{(i,n)}^{[1]}\right]^2.$$

Then, CVME of the parameters $\mathcal{V}_1$ and $\mathcal{V}_2$ are obtained by solving the two following non-linear equations

$$0 = \sum_{i=1}^{n}\left(\frac{1 - exp\left[-\mathcal{V}_1\overline{\mathfrak{Z}}_{\mathcal{V}_2}(\boldsymbol{x}_{i,n})\right]}{1 - exp(-\mathcal{V}_1)} - l_{(i,n)}^{[1]}\right)\varsigma_{(\mathcal{V}_1)}(\boldsymbol{x}_{i,n}, \underline{\mathbb{V}}),$$

and

$$0=\sum_{i=1}^{n}\left(\frac{1-exp\left[-\mathcal{V}_1\overline{\overline{z}}_{\mathcal{V}_2}(\boldsymbol{x}_{i,n})\right]}{1-exp(-\mathcal{V}_1)}-l_{(i,n)}^{[1]}\right)\varsigma_{(\mathcal{V}_2)}(\boldsymbol{x}_{i,n},\underline{\mathbb{V}}),$$

where $\varsigma_{(\mathcal{V}_1)}(\boldsymbol{x}_{i,n},\underline{\mathbb{V}})=\partial\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n})/\partial\mathcal{V}_1$ and $\varsigma_{(\mathcal{V}_2)}(\boldsymbol{x}_{i,n},\underline{\mathbb{V}})=\partial\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n})/\partial\mathcal{V}_2$ are the first derivatives of the CDF of TPE distribution w.r.t. $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively.

The OrLS method

Estimating the parameters of a linear regression model using the OrLS method is a common statistical approach that has gained popularity in recent years. Using this method, the parameters are estimated by minimizing the sum of squared residuals that exists between the values that have been observed and the values that have been predicted by the model. The OrLS is an uncomplicated and easy-to-understand method that is also straightforward and quick to put into action. It is something that is frequently covered in introductory level statistics courses and may be accomplished with relatively straightforward software programmes like as Excel or R. Estimates of model parameters are provided by the OrLS, and these estimates are both asymptotically unbiased and have the smallest feasible variance. When the assumptions of the linear regression model are met, this makes OrLS a highly effective method. The OrLS offers regression coefficient estimates that may be easily comprehended in terms of the slope and intercept of the linear connection between the predictor and response variables. This enables us to draw significant conclusions about the variables' relationship. The OrLS may be expanded to handle more sophisticated regression models including multiple regression, polynomial regression, and predictor interactions. As a result, OrLS is a versatile method that may be used to answer a wide range of research issues. The OrLS provides a framework for statistical inference on estimated regression coefficients, such as hypothesis testing and confidence interval estimation. This helps us to determine whether the link between the predictor and response variables is statistically significant and to estimate the range of plausible regression coefficient values. Let $\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n})$ denotes the CDF of TPE model and let $z_1<z_2<\cdots<z_n$ be the n ordered RS. The OLSEs ($O_{(\underline{\mathbb{V}})}$) are obtained upon minimizing

$$O_{(\underline{\mathbb{V}})}=\sum_{i=1}^{n}\left[\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n})-l_{(i,n)}^{[2]}\right]^2,$$

where $l_{(i,n)}^{[2]}=\frac{i}{n+1}$. Then, we have

$$O_{(\underline{\mathbb{V}})}=\sum_{i=1}^{n}\left[\frac{1-exp\left[-\mathcal{V}_1\overline{\overline{z}}_{\mathcal{V}_2}(\boldsymbol{x}_{i,n})\right]}{1-exp(-\mathcal{V}_1)}-l_{(i,n)}^{[2]}\right]^2,$$

The LSEs can then be obtained after solving:

$$0=\sum_{i=1}^{n}\left[\frac{1-exp\left[-\mathcal{V}_1\overline{\overline{z}}_{\mathcal{V}_2}(\boldsymbol{x}_{i,n})\right]}{1-exp(-\mathcal{V}_1)}-l_{(i,n)}^{[2]}\right]\varsigma_{(\mathcal{V}_1)}(\boldsymbol{x}_{i,n},\underline{\mathbb{V}}),$$

and

$$0=\sum_{i=1}^{n}\left[\frac{1-exp\left[-\mathcal{V}_1\overline{\overline{z}}_{\mathcal{V}_2}(\boldsymbol{x}_{i,n})\right]}{1-exp(-\mathcal{V}_1)}-l_{(i,n)}^{[2]}\right]\varsigma_{(\mathcal{V}_2)}(\boldsymbol{x}_{i,n},\underline{\mathbb{V}}),$$

where $\varsigma_{(\mathcal{V}_1)}(\boldsymbol{x}_{i,n},\underline{\mathbb{V}})$ and $\varsigma_{(\mathcal{V}_2)}(\boldsymbol{x}_{i,n},\underline{\mathbb{V}})$ defined above.

The WLSQ method

The WLSQ is a statistical method used to estimate the parameters of a regression model when the errors are heteroscedastic (i.e., when the variance of the errors is not constant across the range of the predictor variable). The basic idea behind WLSQ is to assign weights to each observation based on their variance and then use these weights to estimate the parameters of the model. When the errors are heteroscedastic, the (OrLS estimates can be inefficient and biased. WLSQ, on the other hand, provides more efficient and unbiased estimates of the regression coefficients. By accounting for the heteroscedasticity of the errors, WLSQ can provide a better fit to the data compared to OrLS. This can result in better predictions and more accurate estimates of the parameters. The WLSQ is a flexible method that can be applied to a wide range of regression models, including linear, nonlinear, and generalized linear models. The WLSQ method assigns weights to the observations based on their variability. This means that observations with higher variance (i.e., more variability) are given less weight in the estimation process, which can lead to more accurate estimates of the parameters. The WLSQ can be used to conduct statistical inference on the estimated regression

coefficients, such as hypothesis testing and confidence interval estimation. The WLSQ are obtained by minimizing the function $W_{(\underline{\mathbb{V}})}$ w.r.t. $\mathcal{V}_1$ and $\mathcal{V}_{2j}$

$$W_{(\underline{\mathbb{V}})} = \sum_{i=1}^{n} d_{(i,n)}^{[3]} \left[\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n}) - l_{(i,n)}^{[2]}\right]^2,$$

where $l_{(i,n)}^{[3]} = [(1+n)^2(n+2)]/[i(-i+1+n)]$. The WLSEs are obtained by solving

$$0 = \sum_{i=1}^{n} l_{(i,n)}^{[3]} \left[\frac{1 - exp\left[-\mathcal{V}_1 \overline{\overline{z}}_{\mathcal{V}_2}(\boldsymbol{x}_{i,n})\right]}{1 - exp(-\mathcal{V}_1)} - l_{(i,n)}^{[2]}\right] \varsigma_{(\mathcal{V}_1)}(\boldsymbol{x}_{i,n}, \underline{\mathbb{V}}),$$

and

$$0 = \sum_{i=1}^{n} l_{(i,n)}^{[3]} \left[\frac{1 - exp\left[-\mathcal{V}_1 \overline{\overline{z}}_{\mathcal{V}_2}(\boldsymbol{x}_{i,n})\right]}{1 - exp(-\mathcal{V}_1)} - l_{(i,n)}^{[2]}\right] \varsigma_{(\mathcal{V}_2)}(\boldsymbol{x}_{i,n}, \underline{\mathbb{V}}),$$

where $\varsigma_{(\mathcal{V}_1)}(\boldsymbol{x}_{i,n}, \underline{\mathbb{V}})$ and $\varsigma_{(\mathcal{V}_2)}(\boldsymbol{x}_{i,n}, \underline{\mathbb{V}})$ defined above.

Method of L-moments

The method of L-moments estimation is a statistical technique used to estimate the parameters of a probability distribution. It is an alternative to traditional maximum likelihood estimation and is particularly useful when the sample size is small, or the data is non-normal. The L-moments are robust statistics that are less sensitive to outliers and non-normality than traditional moments. This makes them particularly useful in situations where the data may be skewed or have heavy tails. The L-moments provide efficient estimators of the parameters, which means that they have a lower mean squared error than traditional moment estimators, especially for small sample sizes. The L-moments for the population can be obtained from

$$L_{(r)} = \frac{1}{r}\sum_{d=0}^{r-1} \binom{r-1}{d} (-1)^d E(X_{r-d\ :\ d})|_{\ (r\geq 1)}.$$

The first four L-moments are generated by the following:

$$L_{(1)}(\underline{\mathbb{V}}) = E(\boldsymbol{x}_{1\ :\ 1}) = \mu_1' = L_{(1)}, L_{(2)}(\underline{\mathbb{V}}) = \frac{1}{2}E(\boldsymbol{x}_{2\ :\ 2} - \boldsymbol{x}_{1\ :\ 2}) = \frac{1}{2}(\mu_{2\ :\ 2}' - \mu_{1\ :\ 2}') = L_{(2)},$$

where $L_i|_{(i=1.2)}$ is the L-moments for the sample. Then, the L-moments of the parameters $\mathcal{V}_1$ and $\mathcal{V}_2$ can be obtained by solving the two equations numerically.

KE method

The KS approach gives efficient parameter estimators, which are asymptotically unbiased and have the minimum possible variance. This makes the procedure particularly beneficial when we have a limited amount of data and want to acquire the most precise parameter estimates. The KS approach is frequently resistant to changes in the assumed distributional form. This means that even if the data is not completely normally distributed, the approach can still yield respectable estimates. The KS technique can also be used to determine how well a given distribution fits a piece of data. We can establish whether the chosen distribution is a good fit for the data by comparing the observed data to the model predictions, or whether a different distribution would be more appropriate. The KS technique is a non-parametric approach, which means it makes no assumptions about the data's underlying distribution. This makes the approach especially effective in circumstances when the data distribution is unknown or difficult to model. The Kolmogorov estimates (KEs) of $\mathcal{V}_1$ and $\mathcal{V}_2$ are obtained by minimizing the function

$$K = K(\underline{\mathbb{V}}) = \underset{1\leq i\leq n}{max}\left\{-\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n}) + \frac{1}{n}i, -(i-1)\frac{1}{n} + \mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n})\right\}.$$

For estimating each parameter, the KEs of $\mathcal{V}_1$ and $\mathcal{V}_{2j}$ are obtained by comparing $\left[-\mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n}) + \frac{1}{n}i\right]|_{1\leq i\leq n}$ and $\left[-\frac{1}{n}(i-1) + \mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}_{i,n})\right]|_{1\leq i\leq n}$ and selecting the max one.

4.2 Bayesian estimation

Bayesian estimation is a statistical technique that is widely used in various fields, including engineering, economics, and social sciences. The primary importance and motivations of Bayesian estimation lie in its ability to provide a flexible framework for modeling complex data sets and estimating the distributions of unknown parameters. One of the primary motivations for using Bayesian estimation is its ability to incorporate prior knowledge or beliefs about the

parameters into the analysis. Unlike classical frequentist methods, which only use the data to estimate the parameters, Bayesian estimation allows the researcher to incorporate prior knowledge or beliefs about the parameters into the analysis. This prior information can come from previous studies, expert opinions, or subjective assessments. By incorporating prior knowledge, Bayesian estimation can provide more accurate and reliable estimates of the parameter distributions, particularly in situations where data are limited.

Another motivation for using Bayesian estimation is its ability to handle complex data sets. Bayesian estimation is particularly useful in situations where the data are noisy or incomplete, or where the underlying model is complex and difficult to specify. Bayesian estimation can handle these types of situations by allowing the researcher to use a flexible framework for modeling the data and estimating the parameters. This flexibility allows the researcher to incorporate different types of data, such as categorical or ordinal data, and to handle missing data. A third motivation for using Bayesian estimation is its ability to provide uncertainty estimates for the parameter estimates. Bayesian estimation provides a posterior distribution for the parameters, which describes the probability distribution of the parameters given the data and prior knowledge. This posterior distribution provides a measure of uncertainty around the parameter estimates, which can be used to assess the reliability and precision of the estimates. This is particularly important in decision-making contexts, where the uncertainty around the parameter estimates can have significant implications. Assume the gamma priors for the parameters $\mathcal{V}_1$ and $\mathcal{V}_{2j}$ were

$$\mathcal{V}_{1;(\varsigma_1,\varpi_1)}(\mathcal{V}_1) \sim \text{Gamma}(\varsigma_1,\varpi_1) \text{ and } \mathcal{V}_{2;(\varsigma_2,\varpi_2)}(\mathcal{V}_2) \sim \text{Gamma}(\varsigma_2,\varpi_2).$$

Because gamma priors are able to reflect such a diverse range of preconceived conceptions, they are utilized rather frequently in research. Because of their adaptability, Gamma distributions can be utilized to model a diverse selection of hypotheses regarding the parameters. Because of its versatility, the research method allows the researcher to take into account a diverse set of preconceived notions and assumptions. It is possible to utilize gamma priors to represent positive continuous variables, which is one reason why they are appealing. The terms "rates," "concentrations," and "time periods" are only a few examples of the numerous parameters that are utilized in practical contexts and are considered to be positive continuous variables. Modeling such variables works very well using gamma distributions because they are continuous and cannot produce negative results. As a result of this, gamma priors are an alternative worth considering for a wide variety of applications in which only positive continuous variables are relevant. Gamma priors are appealing for a number of reasons, including the fact that they can be conjugated with a broad range of likelihood functions. When performing Bayesian estimation, it is advantageous to have a closed-form equation for the posterior distribution because this simplifies both the computation and the interpretation of the results. In conclusion, the information that is currently available, the nature of the problem that is currently available, and the outcomes that are wanted from the model should all have an impact on the selection of priors when using Bayesian estimation. It is essential to provide an explanation for and defend the selection of priors, and sensitivity analyses should be conducted whenever possible to evaluate how well results hold up under varying prior specifications. Since different priors can result in vastly different posterior distributions and interpretations, it is essential to provide an explanation for and defend the selection of priors. The openness with which priors are chosen and utilized is a critical factor in determining the validity and reproducibility of Bayesian analysis.

Assume that the parameters are independently distributed. Then, the joint prior distribution is denoted by the following:

$$\pi_{(\varsigma_i,\varpi_i)}(\underline{\mathbb{V}}) = \frac{1}{\Gamma(\varsigma_1)\Gamma(\varsigma_2)}\varpi_1^{\varsigma_1}\varpi_2^{\varsigma_2}\mathcal{V}_1^{\varsigma_1-1}\mathcal{V}_2^{\varsigma_2-1}\, exp[-(\mathcal{V}_1\varpi_1 + \mathcal{V}_2\varpi_2)].$$

The posterior distribution of the parameters, denoted by the notation $p(\underline{\mathbb{V}}|\underline{X})$, is defined as

$$p(\underline{\mathbb{V}}|\underline{X}) \propto L(\mathcal{V}_1,\mathcal{V}_{2j}|\underline{X}) \times \pi_{(\varsigma_i,\varpi_i)}(\underline{\mathbb{V}}),$$

where the L $(\mathcal{V}_1,\mathcal{V}_2;|\underline{X}) = \prod_{n}^{i=1} f_{\underline{\mathbb{V}}}(z_i)$ and $\pi_{(\varsigma_i,\varpi_i)}(\underline{\mathbb{V}})$ refer to the joint prior model. Then, the Bayesian estimators are the means of their respective marginal posteriors. We can't use the above formulas to figure out the Bayesian predictions. So, the estimate with numbers is needed. Markov Chain Monte Carlo (MCMC) methods are a group of algorithms used in statistics to sample from probability distributions. MCMC is useful because it can deal with complex probability distributions, especially those that can't be sampled directly. In many real-world situations, the posterior distribution of interest is not available in closed form, which makes it hard or impossible to get samples using standard methods. MCMC gives an answer by making a Markov chain that converges to the desired posterior distribution, even if the distribution is complicated or has a lot of dimensions. This lets researchers get data from the posterior distribution, which they can use to draw conclusions and make decisions. The Metropolis-Hastings algorithm, the Gibbs sampler, and the Hamiltonian Monte Carlo are all examples of MCMC algorithms. The

Metropolis-Hastings algorithm is a general-purpose algorithm that works by making proposals from a candidate distribution and accepting or rejecting them based on a chance of acceptance. The Gibbs sampler is a special case of the Metropolis-Hastings algorithm that generates proposals by updating each parameter conditional on the others. Hamiltonian Monte Carlo is a more advanced algorithm that uses Hamiltonian dynamics to generate proposals that move efficiently through the parameter space.

5. Simulations and assessment

In statistics, simulation studies are a typical way for evaluating the performance of estimate methods. Simulation studies have grown in popularity in recent years due to their capacity to give a controlled and thorough evaluation of various estimation approaches under a variety of scenarios. This post tries to emphasize the statistical significance and rationale behind simulation studies for evaluating estimation methods in this context. One of the key reasons for carrying out simulation studies is to assess the performance of estimating methods under various circumstances. Simulation studies enable researchers to produce data with known properties and compare the accuracy, precision, bias, and other relevant criteria of alternative estimating approaches. This allows researchers to identify the strengths and weaknesses of different estimation methods and to choose the most appropriate method for a given problem. Another motivation for conducting simulation studies is to assess the robustness of estimation methods to different assumptions and sources of variability. Simulation studies can be used to evaluate the impact of different types of errors, such as measurement errors or missing data, on the performance of estimation methods. This allows researchers to determine the conditions under which different estimation methods are most appropriate and to identify potential sources of bias or error. Simulation studies can also be used to compare different estimation methods in terms of computational efficiency and scalability. For example, simulation studies can be used to compare the performance of different optimization algorithms or Bayesian inference methods under different sample sizes or parameter spaces. This allows researchers to identify the most computationally efficient method for a given problem and to optimize the use of computing resources.

We conducted MCMC simulation experiments and provided the results in Tables 1, 2, and 3 using mean squared errors (MSEs) as the numerical assessment to compare the classical and Bayesian techniques. We created N=1000 samples of the TPE model and found that as n rose, the performance of all estimate methods improved. Despite the abundance of traditional ways, the MXLE method was found to be the most effective and consistent, as seen in Tables 1, 2, and 3. We advocate the Bayesian method and the MXLE method for statistical modelling and applications based on a complete simulation analysis. Although this section focuses on simulation studies to assess rather than contrast various estimation procedures, simulation can also be used to contrast various estimation methodologies. Real data, on the other hand, is frequently used to evaluate various estimating strategies, which is why we present four examples for this reason. In addition, we provide two further applications for comparing competing models.

Table 1: MSEs under $v_{1,[0]}=0.5$ and $v_{2,[0]}=0.5$.

		MSE						
n		MXLE	OrLS	WLSQ	CrVM	Bayes	Moment	KE
50	$v_{1,[0]}$	0.25196	0.26686	0.30517	0.25692	0.19364	0.62123	0.27959
	$v_{2,[0]}$	0.00293	0.00360	0.00393	0.00325	0.00275	0.00447	0.00373
100	$v_{1,[0]}$	0.12212	0.12778	0.15529	0.12492	0.17317	0.30932	0.13272
	$v_{2,[0]}$	0.00143	0.00168	0.00192	0.00159	0.00139	0.00232	0.00171
200	$v_{1,[0]}$	0.06064	0.06296	0.07617	0.06174	0.01575	0.14376	0.06576
	$v_{2,[0]}$	0.00070	0.00082	0.00092	0.00079	0.00079	0.00108	0.00085
300	$v_{1,[0]}$	0.04028	0.04182	0.04668	0.04166	0.01197	0.08907	0.04259
	$v_{2,[0]}$	0.00045	0.00053	0.00054	0.00053	0.00042	0.00066	0.00054

Table 2: MSEs under $v_{1,[0]}=1.5$, $v_{2,[0]}=0.3$.

		MSE						
n		MXLE	OrLS	WLSQ	CrVM	Bayes	Moment	KE
50	$v_{1,[0]}$	0.29753	0.32128	0.37973	0.30464	0.05022	0.61511	0.33906
	$v_{2,[0]}$	0.00118	0.00138	0.00159	0.00126	0.01353	0.00203	0.00144
100	$v_{1,[0]}$	0.14264	0.15380	0.17884	0.14950	0.02645	0.32494	0.16061

	$v_{2,[0]}$	0.00058	0.00065	0.00072	0.00062	0.00078	0.00103	0.00067
200	$v_{1,[0]}$	0.06737	0.07013	0.08584	0.06881	0.00421	0.16012	0.07397
	$v_{2,[0]}$	0.00027	0.00030	0.00035	0.00029	0.00067	0.00052	0.00031
300	$v_{1,[0]}$	0.04745	0.04958	0.05968	0.04889	0.00353	0.11265	0.05025
	$v_{2,[0]}$	0.00019	0.00021	0.00024	0.00020	0.00016	0.00036	0.00021

Table 2: MSEs under $v_{1,[0]}$= 0.7, $v_{2,[0]}$= 1.5.

		MSE						
n		MXLE	OrLS	WLSQ	CrVM	Bayes	Moment	KE
50	$v_{1,[0]}$	0.23600	0.25867	0.32186	0.24319	0.05962	0.34264	0.27247
	$v_{2,[0]}$	0.02517	0.03134	0.03779	0.02794	0.00562	0.02381	0.03305
100	$v_{1,[0]}$	0.13654	0.14252	0.16598	0.14057	0.05311	0.20697	0.14862
	$v_{2,[0]}$	0.01420	0.01681	0.01813	0.01608	0.00364	0.01415	0.01746
200	$v_{1,[0]}$	0.06207	0.06545	0.07682	0.06371	0.01707	0.11545	0.06786
	$v_{2,[0]}$	0.00657	0.00765	0.00826	0.00733	0.00188	0.00779	0.00782
300	$v_{1,[0]}$	0.03818	0.03816	0.04552	0.03911	0.00490	0.08773	0.04169
	$v_{2,[0]}$	0.00405	0.00442	0.00487	0.00450	0.00041	0.00588	0.00476

6. Applications

6.1 Applications for comparing methods

In this section, we are going to look at and investigate two different kinds of real data, and then we are going to utilize those data to make comparisons between the various estimating approaches that have already been presented and evaluated. The first category of data, which is known as the failure time data, includes information about the total number of minutes that it takes for a patient to begin to feel better after receiving an analgesic (see Gross and Clark (1975)). This data is collected over the course of a patient's lifetime. As the second set of data, Bjerkedal (1960) measured the lengths of time (in days) that 72 guinea pigs that had been infected with virulent tubercle bacilli survived after being infected with the bacteria. Table 4 displays the application results (p-value) for several methods that make use of relief data. Table 5 displays the application results (p-value) for several methods that make use of survival data. When compared to the OrLS method, which has a p-value of 0.99167, it can be demonstrated that the CrVM approach has superior performance by referring to Table 4. It can be demonstrated, using Table 5, that the CrVM technique is superior to the MXLE method, which has a p-value of 0.75720, whilst the CrVM approach has a p-value of 0.7747.

Table 4: P-values, $\widehat{v_1}$ and $\widehat{v_2}$ under relief data.

Method	$\widehat{v_1}$	$\widehat{v_2}$	p-value
ML	-27.897582	2.132678	0.893201
OrLS [2]	-27.022477	2.166062	0.991672
WLSQ	-19.355131	1.951620	0.922483
CrVM [1]	-39.500664	2.377092	0.996619
Bayes	-27.074214	2.076913	0.719456
Moment	-9.277645	1.575578	0.917356
KE	-7.768978	1.528459	0.698564

Table 5: P-values, $\widehat{v_1}$ and $\widehat{v_2}$ under relief data.

Method	$\widehat{v_1}$	$\widehat{v_2}$	p-value
ML	-2.695623	1.085237	0.757201
OrLS [2]	-2.969704	1.180724	0.703422
WLSQ	-2.838867	1.117713	0.609303
CrVM [1]	-3.181756	1.213274	0.774721
Bayes	-2.015189	0.966565	0.628113
Moment	-2.586704	1.098555	0.675484
KE	0.173866	0.635453	0.159873

6.2 Applications for comparing models

The process of modeling real data involves fitting different probability distributions to real datasets and evaluating the degree to which they are a good fit for the datasets. Researchers are able to calculate the optimal distribution for a particular dataset and draw inferences about the demographic characteristics of the population by following this process. The modeling of data from the real world is important for a number of different reasons. To begin, real data modeling enables researchers to evaluate the applicability of multiple probability distributions for a particular dataset. This may be done for any dataset. In actuality, a single probability distribution is unable to perfectly characterize a large number of datasets. Instead, the various patterns found in the data may need a distribution that is either more flexible or more complex. Researchers have the capacity to use real data modeling to analyze several probability distributions and determine which one best matches the process that lies behind the data that is being generated. Second, researchers are able to infer population parameters based on the fitted distribution when actual data modeling is used. After discovering the most optimal and appropriate distribution, the researchers can proceed to estimate the parameters of the distribution and arrive at conclusions on the population's properties. These findings can be put to use to validate theories and make projections about forthcoming observations. Third, modeling real data gives researchers the ability to evaluate how well the fitted distribution matches the data. Goodness-of-fit tests are used to determine the extent to which a fitted distribution corresponds to the data that have been collected. A distribution that fits the data well should have a low goodness-of-fit test statistic, which indicates that the distribution is a good fit for the data that was observed.

A number of distinct models, including the standard exponential (E), the Odd-Lindley-exponential (ODLE), the Marshall-Olkin-exponential (MROE), the moment exponential (ME), the Burr-Hatke-exponential (BRHE), the generalized Marshall-Olkin-exponential (GZMOE), the beta-exponential (BE), the Marshall-Olkin-Kumaraswamy-exponential (MOKE), and the Kumaraswamy Marshall-Olkin-exponential(KRMOE), will be contrasted with the fit of the TPE distribution. Researchers such as Merovci et al. (2017), Merovci et al. (2020), Karamikabir et al. (2020), Korkmaz et al. (2020), and Hamedani et al. (2018) are capable of developing models that are more competitive than others. To do a comparison between the models, we will be using the Cramer-Von Mises (CVMC), Anderson-Darling (ANDC), and Kolmogorov-Smirnov (KS) statistics. In addition, to achieve a higher degree of precision, we will make use of the following five goodness-of-fit metrics: the Akaike Information Criterion (AK-INC), the Bayesian Information Criterion (BS-INC), the Consistent Akaike Information Criterion (CAK-INC), and the Hannan-Quinn Information Criterion (HQ-INC). In these applications, we will evaluate the first fits of theoretical distributions such as the normal, uniform, exponential, logistic, beta, lognormal, and Weibull using the skewness-kurtosis plot (also known as the Cullen and Frey plot). The bootstrapping method will be utilized and plotted in order to achieve a greater level of precision. We will also show scattergram plots, the "nonparametric Kernel density estimation (NKDE)" method for investigating the initial shape of the insurance claims density, the "Quantile-Quartile (Q-Q)" plot for investigating the "normality" of the current data, the "total time in test (TTT)" plot for investigating the initial shape of the empirical HRF, and the "box plot" for identifying extreme data. All of these plots will investigate the "normality" The concept of "normality" will be looked into by each of these plots.

Figure 1 illustrates various graphical representations of the relief times data, including a Q-Q plot (top row, right panel), a box plot (top row, left panel), a Kernel nonparametric density estimation plot (middle row, right panel), a TTT plot (middle row, left panel), a Cullen and Frey plot (bottom row, left panel), and scattergrams (bottom row, right panel). Meanwhile, Figure 2 displays the ES-CDF (right panel) and ES-PDF (left panel) for the relief periods data. Figure 3 presents the P-P plot (right panel) and Kaplan-Meier survival (KM-S) plot (left panel) for the relief periods data. According to the information in Figure 1 (second row, left panel), the Hazard Rate Function (HRF) of the relief times exhibits a "monotonically increasing HRF." Additionally, as depicted in Figure 1 (first row, right panel), there is only one extreme observation in the relief data. Notably, Figure 1 portrays a bimodal and right-skewed density, displaying an asymmetric shape in the nonparametric Kernel density estimate plot (second row, right panel). Furthermore, Figure 1 (third row, left panel) illustrates that the relief times data do not conform to any of the theoretical distributions, including the normal, uniform, exponential, logistic, beta, lognormal, and Weibull distributions.

Figure 2, comprising the ES-PDF (left panel) and ES-CDF (right panel), along with Figure 3, showcasing the KM-S plot (left panel) and the P-P plot (right panel), indicate a strong fit of the new model to the data. Figures 1q and 3q

demonstrate that the empirical findings align with the practical observations. In Table 6, you can find the Maximum Likelihood Estimators (MXLEs) and Standard Errors (SD-ER) calculated using the maximum likelihood technique for the relief data. Meanwhile, Table 7 presents the maximum likelihood-based statistics, including AK-INC, BS-INC, CAK-INC, HQ-INC, ANDC, CVMC, KG-SM, and p-value for the relief data. With values of AK-INC=36.198, BS-INC=38.19, CAK-INC=36.904, HQ-INC=36.587, ANDC=0.2891, CVMC=0.0494, KG-SM=0.12900, and p-value=0.8933, it becomes evident that the proposed lifetime TPE model surpasses all previously considered models. Consequently, when modeling the relief times dataset, the new lifespan model proves to be a compelling alternative to these existing models. These results unequivocally establish the superiority of the novel distribution in capturing the characteristics of the bimodal and right-skewed relief times data with an asymmetric shape. Figures 1q, 2q, and 3q are included in the Appendix for reference.

Figure 4, Figure 5, and Figure 6 present the outcomes of various statistical techniques and approaches employed to analyze the survival times data. These figures include visual representations such as box plots, Q-Q plots, TTT plots, scattergrams, and nonparametric Kernel density estimation plots to illustrate the distribution and patterns within the data. The results from these analyses consistently indicate that the data do not adhere to any of the theoretical distributions; instead, they exhibit characteristics of being bimodal, right skewed, and asymmetric in shape. Furthermore, the KM-S plot, ES-PDF, ES-CDF, P-P plot, and KM-S plot all provide evidence that the new model is a good fit for the data. This alignment with both experimental and real-world observations is noteworthy. The proposed lifetime TPE model's parameters were estimated using the maximum likelihood approach, and Tables 6 and 8 contain the corresponding Maximum Likelihood Estimators (MXLEs) and Standard Errors (SEs) for the two datasets, respectively. The model selection criteria, including AK-INC, BS-INC, CAK-INC, HQ-INC, ANDC, CVMC, KG-SM, and p-value for the two datasets, are listed in Tables 7 and 9. With values of AK-INC=202.08, BS-INC=206.6, CAK-INC=202.258, HQ-INC=203.897, ANDC=0.562, CVMC=0.0948, KG-SM=0.1290, and p-value=0.8933, it is evident that the proposed lifetime TPE model outperforms all other models based on these criteria. These findings underscore the superior performance of the new model in accurately representing the survival times data, implying that it serves as an excellent replacement for existing models when modeling relief times datasets. Figures 4q, 5q, and 6q are included in the Appendix for reference.

Table 6: MXLEs and SD-ER for the relief times.

Models		The MXLEs (SD-ER)
$E_{(\tau)}$	MXLE	0.52554
	SD-ER	(0.11741)
$ODLE_{(\tau)}$	MXLE	0.60443
	SD-ER	(0.05345)
$ME_{(\tau)}$	MXLE	0.950324
	SD-ER	(0.15032)
$BRHE_{(\tau)}$	MXLE	0.526543
	SD-ER	(0.118322)
$MROE_{(\alpha,\tau)}$	MXLE	54.4743, 2.3167
	SD-ER	(35.5845), (0.3725)
$GZMOE_{(\lambda,\alpha,\tau)}$	MXLE	0.51941, 89.46419, 3.16492
	SD-ER	(0.2615), (66.2812), (0.7717)
$KE_{(\vartheta,\beta,\tau)}$	MXLE	83.7548, 0.56844, 3.33016
	SD-ER	(42.646), (0.32871), (1.18772)
$BE_{(\vartheta,\beta,\tau)}$	MXLE	81.63743, 0.54433, 3.514433
	SD-ER	(120.4132), (0.3332), (1.41022)
$MOKE_{(\alpha,\beta,\lambda,\tau)}$	MXLE	0.133, 33.24, 0.571, 1.67
	SD-ER	(0.332), (57.85), (0.7), (1.8)
$KRMOE_{(\alpha,\beta,\lambda,\tau)}$	MXLE	8.8618, 34.8262, 0.299, 4.8929
	SD-ER	(9.1462), (22.3112), (0.2393), (3.1716)
$BRXE_{(\vartheta,\tau)}$	MXLE	1.163544, 0.320661
	SD-ER	(0.33312), (0.034249)
$TPE_{(v_1,v_2)}$	MXLE	-27.89843, 2.13353
	SD-ER	(20.2522), (0.43441)

Table 7: AK-INC, BS-INC, CAK-INC, HQ-INC, ANDC, CVMC, KG-SM and (p-value) for the relief times.

Models	AK-INC, BS-INC, CAK-INC, HQ-INC	ANDC	CVMC	KG-SM and (p-value)
ODLE	49.162, 50.664, 49.373, 49.347	1.3532	0.2245	0.8533(<0.001)
ME	54.323, 55.347, 54.574, 54.538	2.7616	0.5344	0.3204(0.1)
BRHE	67.764, 68.703, 67.839, 67.955	0.6235	0.1047	0.4432(<0.001)
MROE	43.521, 45.513, 44.264, 43.905	0.8414	0.1433	0.1862(0.55)
GZMOE	42.755, 45.755, 44.235, 43.344	0.5154	0.0832	0.1533(0.78)
KE	41.784, 44.754, 43.283, 42.326	0.4565	0.0708	0.1434(0.86)
BE	43.481, 46.452, 44.928, 44.426	0.7476	0.1241	0.1615(0.80)
MOKE	41.583, 45.543, 44.245, 42.315	0.6580	0.1344	0.1435(0.87)
KRMOE	42.812, 46.836, 45.535, 43.651	1.1821	0.1945	0.1553(0.86)
BRXE	48.143, 50.165, 48.873, 48.534	1.3913	0.2555	0.2484(0.171)
TPE	36.1979, 38.191, 36.9041, 36.5866	0.2892	0.0493	0.12903(0.8934)

Table 8: MXLEs and SD-ER for the survival times.

Models		The MXLEs (SD-ER)
$E_{(\tau)}$	MXLE	0.54013
	SD-ER	(0.06315)
$ODLE_{(\tau)}$	MXLE	0.38156
	SD-ER	(0.02126)
$ME_{(\tau)}$	MXLE	0.92553
	SD-ER	(0.07732)
$BRHE_{(\tau)}$	MXLE	0.544242
	SD-ER	(0.0634)
$MROE_{(\alpha,\tau)}$	MXLE	8.7835, 1.3815
	SD-ER	(3.5643), (0.1943)
$GZMOE_{(\lambda,\alpha,\tau)}$	MXLE	0.18354, 47.6465, 4.4742
	SD-ER	(0.0733), (44.921), (1.3334)
$KE_{(\vartheta,\beta,\tau)}$	MXLE	3.30425, 1.10015, 1.03734
	SD-ER	(1.10635), (0.7645), (0.614)
$BE_{(\vartheta,\beta,\tau)}$	MXLE	0.80734, 3.46155, 1.33146
	SD-ER	(0.69619), (1.0036), (0.8662)
$MOKE_{(\alpha,\beta,\lambda,\tau)}$	MXLE	0.0084, 2.7165, 1.9866, 0.0997
	SD-ER	(0.0033), (1.31653), (0.7844), (0.055)
$KRMOE_{(\alpha,\beta,\lambda,\tau)}$	MXLE	0.37331, 3.47821, 3.3063, 0.29925
	SD-ER	(0.1365), (0.8615), (0.7788), (1.1124)
$BRXE_{(\vartheta,\tau)}$	MXLE	0.48447, 0.21355
	SD-ER	(0.06154), (0.01235)
$TPE_{(\nu_1,\nu_2)}$	MXLE	-2.69563, 1.08548
	SD-ER	(1.01171), (0.15244)

Table 9: AK-INC, BS-INC, CAK-INC, HQ-INC, ANDC, CVMC, KG-SM and (p-value) for survival times data.

Models	AK-INC, BS-INC, CAK-INC, HQ-INC	ANDC	CVMC	KG-SM and (p-value)
E	234.645, 236.955, 234.618, 235.525	6.534	1.2544	0.3454(0.066)
ODLE	229.444, 231.483, 229.245, 230.145	1.943	0.3358	0.55(<0.001)
ME	210.444, 212.677, 210.456, 211.306	1.526	0.2519	0.1576(0.135)
BRHE	234.634, 236.965, 234.796, 235.516	0.715	0.1155	0.2834(<0.001)
MROE	210.366, 214.935, 210.524, 212.174	1.177	0.1753	0.1045(0.444)

GZMOE	210.544, 217.387, 210.894, 213.265	1.029	0.1635	0.0935(0.510)
KE	209.444, 216.665, 209.772, 212.174	0.747	0.1553	0.09521(0.532)
BE	207.374, 214.551, 207.73, 210.496	0.988	0.1546	0.1144(0.344)
MOKE	209.445, 218.566, 210.344, 213.743	0.796	0.1369	0.1055(0.443)
KRMOE	207.854, 216.942, 208.419, 211.449	0.615	0.1566	0.0954(0.501)
BRXE	235.361, 239.956, 235.529, 237.544	2.964	0.5265	0.2255(0.0024)
TPE	202.088, 206.61 202.257, 203.8976	0.5622	0.0948	0.0793(0.7576)

7. Bayesian estimation and loss functions

7.1 Priors and posteriors

In Bayesian estimation, prior and posterior distributions play a crucial role in the estimation process. Prior distributions represent the researcher's prior beliefs about the distribution of the population parameters, while the posterior distribution represents the updated beliefs based on the observed data. The choice of prior distribution can significantly impact the posterior distribution and, consequently, the estimation results. The loss function used in Bayesian estimation also plays a critical role in determining the posterior distribution and estimation results. In this essay, we will discuss the role of prior and posterior distributions in Bayesian estimation under different loss functions. Under squared error loss function, the posterior distribution can be obtained by multiplying the likelihood function with the prior distribution and normalizing the resulting distribution. The resulting posterior distribution is also a normal distribution with mean and variance determined by the prior and likelihood functions. This approach is commonly used in Bayesian estimation for continuous data. Under absolute error loss function, the posterior distribution can be obtained by minimizing the expected absolute error. The posterior distribution obtained under this loss function is known as the posterior median and represents the median value of the posterior distribution. Under asymmetric loss function, the posterior distribution can be obtained by minimizing the expected loss. The posterior distribution obtained under this loss function is known as the posterior mode and represents the mode value of the posterior distribution. This approach is commonly used in Bayesian estimation when the costs associated with underestimation and overestimation are not equal. As prior distributions, we assume the parameters $\mathcal{V}_1$ follows the Gamma distribution, while $\mathcal{V}_2$ has a non informative distribution as a prior

$$\pi(\mathcal{V}_1) = \frac{\tau b}{\Gamma(\tau)} \mathcal{V}_1^{\tau-1}\, exp(-\vartheta\zeta_1)|_{\vartheta,\tau>0}\ , \pi(\mathcal{V}_2) = \frac{1}{\mathcal{V}_2}, \tag{3}$$

where the constants ϑ, τ are called hyper-parameters. Thus, the joint prior distribution of $(\underline{\mathbb{V}})$ is,

$$\pi(\underline{\mathbb{V}}) = \frac{\tau b}{\mathcal{V}_2 \Gamma(\tau)} \mathcal{V}_1^{\tau-1}\, exp(-\vartheta\zeta_1). \tag{4}$$

The joint posterior distribution of $(\underline{\mathbb{V}})$ is

$$\pi\left(\underline{\mathbb{V}}|\boldsymbol{x}\right) = \frac{\mathcal{V}_1^{m+\tau-1}\mathcal{V}_2^{m-1}\, exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n}}{\int_0^{+\infty}\int_0^{+\infty} \mathcal{V}_1^{m+\tau-1}\mathcal{V}_2^{m-1}\, exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n}} \frac{C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m}\prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\, \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)}{C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m}\prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\, \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i) d\zeta_1 d\zeta_2}, \tag{5}$$

then

$$\pi(\underline{\mathbb{V}}|\boldsymbol{x}) = Q\zeta_1^{m+\tau-1}\mathcal{V}_2^{m-1}\, exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n} C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\, \boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i), \tag{6}$$

where Q refers to the constant of normalizing.

7.2 Posterior risks

In Bayesian statistics, Bayesian estimators are used to estimate population parameters based on observed data and prior beliefs. Bayesian estimators are typically obtained by finding the posterior distribution of the parameter of interest, given the observed data and the prior distribution. Once the posterior distribution is obtained, Bayesian estimators can be computed based on different risk functions. In this essay, we will discuss the importance and applications of Bayesian estimators and their posterior risks. One of the main advantages of Bayesian estimators is that they allow researchers to incorporate prior information into the estimation process. This prior information can be in the form of expert knowledge, historical data, or previous research findings. By incorporating this prior information, Bayesian estimators can provide more accurate and reliable estimates of the population parameters. Bayesian estimators and their posterior risks have important applications in many fields, including engineering, finance, and medicine. For example, in engineering, Bayesian estimators can be used to estimate the lifetime of a component based

on observed failure times and prior knowledge about the reliability of the component. In finance, Bayesian estimators can be used to estimate the volatility of financial assets based on historical data and prior knowledge about the behavior of financial markets. In medicine, Bayesian estimators can be used to estimate the effectiveness of medical treatments based on clinical trial data and prior knowledge about the efficacy of similar treatments. Under the assumption of a generalized quadratic (GQ) loss function, the following are some Bayesian estimators

$$\mathcal{V}_{1,GQ} = \frac{\int_0^{+\infty} \int_0^{+\infty} \mathcal{V}_1^{m+\tau+\delta-1} \mathcal{V}_2^{m-1} exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n}}{\int_0^{+\infty} \int_0^{+\infty} \mathcal{V}_1^{m+\tau+\delta-2} \mathcal{V}_2^{m-1} exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n}} \times \frac{C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)d\zeta_1 d\zeta_2}{C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)d\zeta_1 d\zeta_2},$$

and

$$\mathcal{V}_{2,GQ} = \frac{\int_0^{+\infty} \int_0^{+\infty} \mathcal{V}_1^{m+\tau-1}\mathcal{V}_2^{m+\delta-1} exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n}}{\int_0^{+\infty} \int_0^{+\infty} \mathcal{V}_1^{m+\tau-1}\mathcal{V}_2^{m+\delta-2} exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n}} \times \frac{C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)d\zeta_1 d\zeta_2}{C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)d\zeta_1 d\zeta_2}.$$

The corresponding posterior risks are then

$$PR(\mathcal{V}_{1,GQ}) = E_\pi(\mathcal{V}_1^{\delta+1}) - 2\hat{\zeta}_{1,GQ}E_\pi(\mathcal{V}_1^{-\delta}) + \mathcal{V}_{1,GQ}^2\, E_\pi(\mathcal{V}_1^{\delta-1}), \tag{7}$$

$$PR(\mathcal{V}_{2,GQ}) = E_\pi(\mathcal{V}_2^{\delta+1}) - 2\hat{\zeta}_{2,GQ}E_\pi(\beta^{-\delta}) + \mathcal{V}_{2,GQ}^2\, E_\pi(\mathcal{V}_2^{\delta-1}). \tag{8}$$

In Bayesian analysis, the entropy loss function is used as a measure of the expected information loss associated with decision-making under uncertainty. It quantifies the discrepancy between the true but unknown parameter value and the decision or estimate made based on observed data. The entropy loss function is particularly relevant when making decisions that involve predicting or estimating uncertain quantities. In Bayesian analysis, the true parameter value is often treated as a random variable with a probability distribution called the prior distribution. The goal is to update this prior distribution based on observed data using Bayes' theorem, resulting in a posterior distribution. The entropy loss function comes into play when making decisions or choosing estimates based on this posterior distribution. The entropy loss function is defined in terms of the posterior distribution and the decision rule or estimate. It measures the expected information or uncertainty associated with the decision. The function is derived by taking the average (expectation) of the logarithm of the posterior distribution. Under the entropy loss function, we obtain the following estimators

$$\mathcal{V}_{1,E} = \left[K \int_0^{+\infty} \int_0^{+\infty} \mathcal{V}_1^{m+\tau-p-1}\mathcal{V}_2^{m-1} exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n} \times C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)d\zeta_1 d\zeta_2\right]^{-\frac{1}{p}}, \tag{9}$$

and

$$\mathcal{V}_{2,E} = \left[K \int_0^{+\infty} \int_0^{+\infty} \mathcal{V}_1^{m+\tau-1}\mathcal{V}_2^{m-p-1} exp(-\vartheta\zeta_1)(1-exp(-\mathcal{V}_1))^{-n} \times C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)d\zeta_1 d\zeta_2\right]^{-\frac{1}{p}}. \tag{10}$$

The corresponding posterior risks are then

$$PR(\mathcal{V}_{1,E}) = PE_\pi(ln(\mathcal{V}_1) - ln(\mathcal{V}_{1,E})), \tag{11}$$

and

$$PR(\mathcal{V}_{2,E}) = PE_\pi(ln(\mathcal{V}_2) - ln(\mathcal{V}_{2,E})). \tag{12}$$

finally, under the Linex loss function, the Bayesian estimators

$$\mathcal{V}_{1,L} = \frac{-K}{r} ln\left[\int_0^{+\infty} \int_0^{+\infty} \mathcal{V}_1^{m+\tau-1}\mathcal{V}_2^{m-1} exp(-\vartheta\zeta_1 - r\zeta_1)(1-exp(-\mathcal{V}_1))^{-n} \times C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m} \prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)d\zeta_1 d\zeta_2\right], \tag{13}$$

$$\mathcal{V}_{2,L} = \frac{-K}{r} ln\left[\int_0^{+\infty}\int_0^{+\infty} \mathcal{V}_1^{m+\tau-1}\mathcal{V}_2^{m-1}\, exp(-\vartheta\zeta_1 - r\zeta_2)(1-exp(-\mathcal{V}_1))^{-n} \times C_{\underline{\mathbb{V}}}(\boldsymbol{x}_m)^{n-m}\prod_{i=1}^{m} M_{\mathcal{V}_2}(\boldsymbol{x}_i)\boldsymbol{\varkappa}_{\underline{\mathbb{V}}}(\boldsymbol{x}_i)d\zeta_1 d\zeta_2\right], \quad (14)$$

and the corresponding posterior risks are then

$$PR(\mathcal{V}_{1,L}) = r(\mathcal{V}_{1,GQ} - \mathcal{V}_{1,L}), \quad (15)$$

and

$$PR(\mathcal{V}_{2,L}) = r(\mathcal{V}_{2,GQ} - \mathcal{V}_{2,L}). \quad (16)$$

7.3 Likelihood and Bayesian estimation

Estimating model parameters is accomplished with a variety of techniques, two of the most used of which being likelihood estimation and Bayesian estimation. In order to obtain the parameter estimates using the likelihood technique, one must first maximize the likelihood function. However, when using Bayesian estimation, one must first provide a prior distribution for the parameters, and then one must update this prior distribution using the observed data in order to achieve the posterior distribution. In this essay, we will compare the likelihood estimation method with the Bayesian estimation method by utilizing Pitman's proximity criteria as a measuring stick. Pitman's closeness criterion is a method for comparing two estimators that is based on the difference in the predicted values of the estimators when the parameter is set to all of its conceivable values. The criterion indicates that one estimate is closer to the actual value of the parameter than another estimator if its expected value is closer to the true value under all conceivable values of the parameter. In other words, if an estimator's expected value is closer to the true value, then that estimator is closer to the true value. This criterion can be used to the context of likelihood and Bayesian estimation, and it can be used to compare the performance of the two methodologies. The performance of likelihood and Bayesian estimates may be compared using Pitman's proximity criterion since it takes into consideration the risks that each technique poses, which are the anticipated values of each method's loss functions. This enables us to evaluate the effectiveness of the two estimation methods. The loss function for Bayesian estimation is the anticipated value of the posterior distribution, however the loss function for likelihood estimation is typically the squared error loss. It is possible for us to use Pitman's proximity criterion to evaluate and contrast the hazards that are linked with the two different approaches. Numerous research has been conducted to compare Bayesian estimate to likelihood utilizing the Pitman's closeness criterion; however, the results of these studies differ depending on the model and loss function that is being utilized. On the other hand, it has been proved that, in general, Bayesian estimation can do better than likelihood estimation in terms of Pitman's proximity criterion. This is especially true in situations in where the sample size is small, and the prior distribution is informative. This is due to the fact that Bayesian estimation takes into account the uncertainty that is present in the estimations of the parameter, which ultimately leads to a more reliable estimate of the parameter being produced.

We are going to compare and contrast the statistical performance of Bayesian estimators with that of MXLEs that are comparable. For this main aim, we perform a MCMC simulation method, we assume that $\mathcal{V}_1 = 1$, $\mathcal{V}_2 = 1.5$ and $\vartheta = \tau = 1$, we will generate N type-II censored samples following the new model, we use different sample sizes $n = 30,100,200$ while $m = 10,40,160$ respectively, we obtain the following results. The values of the estimators that were generated by applying the function BB method are presented in Table 7. Here, we notice that, particularly as the sample size n is increased, the estimated values of v1 and v2 are quite close to the parameter's true values. This is especially the case when the sample size n is increased. Table 8 presents the Bayesian estimators and the PR for the GQ loss function. The PR is displayed in brackets. In table 9, the Bayesian estimators for the entropy loss function are presented, along with the PR for the function. In Table 10, the Bayesian estimators for the Linex loss function are listed, along with the PR (in brackets). The Bayesian estimators and the PR (in brackets) for each of the three loss functions are presented in Table 11. The Bayesian estimator that uses the GQ loss function and has a value of $\lambda = -0.5$ provides the lowest posterior risk, as shown in Table 8. In addition, when the value of n is large, we obtain the posterior risk that is the least suitable. As can be seen in Table 10 of the estimation performed under the entropy loss function, the value $p = 0.5$ yields the best results in terms of the posterior risk when $n = 200$ is used. It is abundantly evident that the value $r = -1$ offers the most favourable PR. A cursory examination of the relationships between the three loss functions reveals that the quadratic loss function produces the best results. Table 11, 12, 13 and 14 provides additional information regarding these discoveries. Comparing the most accurate maximum likelihood

estimators with the most accurate Bayesian estimators is something that we strongly recommend doing. In order to accomplish this, we make use of the Pitman closeness criterion (you can find additional information regarding this in Pitman et al. (1937), Fuller (1982) and Jozani (2012).

Table 10: The MXLE and quadratic error (in brackets).

	$n=30,m=10$	$n=100,m=40$	$n=200,m=160$
N=5000|$\mathcal{V}_1$	1.0501 (0.0143)	0.9234 (0.0217)	1.2394 (0.0065)
N=5000|$\mathcal{V}_2$	1.1134 (0.0076)	1.2872 (0.0078)	1.6572 (0.0094)

Table 11: Bayesian estimators with PR (in brackets) under GQ loss function.

$N=5000,\lambda$		$n=30,m=10$	$n=100,m=40$	$n=200,m=160$
-2	$\mathcal{V}_{1,[0]}$	0.6490 (0.0089)	0.6825(0.0041)	0.6432(0.0016)
	$\mathcal{V}_{2,[0]}$	1.2657(0.1491)	1.5033(0.0611)	1.7113(0.0008)
-1.5	$\mathcal{V}_{1,[0]}$	0.7990(0.0087)	0.0825(0.0061)	1.2127(0.0016)
	$\mathcal{V}_{2,[0]}$	1.8657(0.7091)	1.7039(0.0633)	1.7120(0.0008)
-1	$\mathcal{V}_{1,[0]}$	0.9181(0.0005)	0.9739(0.0001)	2,0018(0.0001)
	$\mathcal{V}_{2,[0]}$	1.9195(0.0002)	1.7870(0.0012)	1.9898(0.0001)
-0.5	$\mathcal{V}_{1,[0]}$	1.0994(0.0081)	1.0888(0.0070)	1.0138(0.0018)
	$\mathcal{V}_{2,[0]}$	1.3999(0.0825)	1.4701(0.711)	1.61310,0012)
0.5	$\mathcal{V}_{1,[0]}$	0.7510(0.0085)	0.7926(0.0077)	1.1839(0.0020)
	$\mathcal{V}_{2,[0]}$	0.6891(0.0909)	1.3591(0.995)	1.6132(0.0019)
1	$\mathcal{V}_{1,[0]}$	0.7575(0.0087)	0.0977(0.0078)	1.1841(0.0031)
	$\mathcal{V}_{2,[0]}$	1.4228(0.1094)	1.3803(0.1071)	1.6549(0.0025)
1.5	$\mathcal{V}_{1,[0]}$	0.6743(0.0073)	0.5632(0.0081)	1.1232(0.0042)
	$\mathcal{V}_{2,[0]}$	0.4768(0.1241)	1.6754(0.1181)	1.6403(0.0033)
2	$\mathcal{V}_{1,[0]}$	0.1099(0.0088)	0.0990(0.0081)	1.1841(0.0042)
	$\mathcal{V}_{2,[0]}$	1.4768(0.1241)	1.4191(0.1181)	1.7158(0.0033)

Table 12: Bayesian estimators under the entropy loss function.

$N=5000,P$		$n=30,m=10$	$n=100,m=40$	$n=200,m=160$
-2	$\mathcal{V}_{1,[0]}$	1.0942(0.008)	1.3990(0.1644	1.2144(0.0019)
	$\mathcal{V}_{2,[0]}$	1.8181(0.0699)	1.2839(0.009)	1.7034(0.011)
-1.5	$\mathcal{V}_{1,[0]}$	1.1067(0.0091)	1.2188(0.1443)	1.2179(0.0017)
	$\mathcal{V}_{2,[0]}$	0.9407(0.0611)	0.4077(0.0661)	1.7060(0.0012)
-1	$\mathcal{V}_{1,[0]}$	1.1041(0.0009)	0.6205(0.0171)	1.2167(0.0001)
	$\mathcal{V}_{2,[0]}$	1.9177(0.0072)	1.1633(0.0073)	1.7051(0.0003)
-0.5	$\mathcal{V}_{1,[0]}$	17981(0.0038)	0.7830(0.0733)	1.2148(0.0009)
	$\mathcal{V}_{2,[0]}$	1.8493(0.0308)	0.7757(0.319)	1.7037(0.0009)
0.5	$\mathcal{V}_{1,[0]}$	1.6981(0.0038	0.4830(0.0733)	1.2148(0.0009)
	$\mathcal{V}_{2,[0]}$	1.8491(0.0308)	1.6155(0.319)	1.6037(0.0009)
1	$\mathcal{V}_{1,[0]}$	0.8998(0.0008)	0.8895(0.0729)	0.9814(0.0001)
	$\mathcal{V}_{2,[0]}$	1.5638(0.0071)	1.6001(0.0065)	1.502(0.0002)
1.5	$\mathcal{V}_{1,[0]}$	1.7053(0.0035)	0.6701(0.0667)	1.0969(0.0009)
	$\mathcal{V}_{2,[0]}$	1.4239(0.0199)	14881(0.0303)	1.5759(0.0003)
2	$\mathcal{V}_{1,[0]}$	1.7697(0.0099)	0.7654(0.1173)	1.0886(0.0031)
	$\mathcal{V}_{2,[0]}$	1.4579(0.0997)	1.4354(0.0944)	1.4571(0.0014)

Table 13: Bayesian estimators under Linex loss function.

$N = 5000, r$		$n = 30, m = 10$	$n = 100, m = 40$	$n = 200, m = 160$
-2	$\mathcal{V}_{1,[0]}$	0.6043(0.0039)	0.5861(0.0009)	0.7174(0.0003)
	$\mathcal{V}_{2,[0]}$	0.9547(0.1041)	1.4193(0.0131)	1.4045(0.0004)
-1.5	$\mathcal{V}_{1,[0]}$	0.5211(0.0039	0.6815(0.0038)	0.5179(0.0012)
	$\mathcal{V}_{2,[0]}$	0.9806(0.0411)	1.4455(0.0519)	1.4054(0.0013)
-1	$\mathcal{V}_{1,[0]}$	0.5455(0.0519	0.5815(0.0183)	1.0070(0.0057)
	$\mathcal{V}_{2,[0]}$	1.1191(0.0049)	1.1251(0.0195)	1.4094(0.0057)
-0.5	$\mathcal{V}_{1,[0]}$	1.2041(0.0013)	0.7813(0.0007)	0.8153(0.0003)
	$\mathcal{V}_{2,[0]}$	1.2080(0.0014)	1.2609(0.0199)	1.4011(0.0004)
0.5	$\mathcal{V}_{1,[0]}$	1.2041(0.0013)	0.7813(0.0007)	0.8153(0.0003)
	$\mathcal{V}_{2,[0]}$	1.2080(0.0014)	1.2609(0.0199)	1.4011(0.0004)
1	$\mathcal{V}_{1,[0]}$	0.7228(0.0105)	0.7919(0.0081)	1.0019(0.0004)
	$\mathcal{V}_{2,[0]}$	1.2117(0.1033)	1.1139(0.0581)	1.4059(0.0013)
1.5	$\mathcal{V}_{1,[0]}$	1.1082(0.0107)	1.0700(0.0081)	0.9634(0.0025)
	$\mathcal{V}_{2,[0]}$	1.5495(0.01213)	0.9160(0.1155)	1.5939(0.0027)
2	$\mathcal{V}_{1,[0]}$	0.6991(0.0007)	0.8058(0.0147)	1.2061(0.0015)
	$\mathcal{V}_{2,[0]}$	1.3815(0.0183)	1.3251(0.0195)	1.4091(0.0032)

Table 14: Bayesian estimators under the three loss functions.

N=5000		$n = 10, m = 10$	$n = 50, m = 40$	$n = 200, m = 160$
GQ$\|\gamma = -1$	$\mathcal{V}_{1,[0]}$	1.0994(0.0081)	1.0888(0.0070)	1.0138(0.0018)
	$\mathcal{V}_{2,[0]}$	1.3999(0.0825)	1.4701(0.711)	1.61310,0012)
Entropy$\|p = 0.5$	$\mathcal{V}_{1,[0]}$	0.8998(0.0008)	0.8895(0.0729)	0.9814(0.0001)
	$\mathcal{V}_{2,[0]}$	1.5638(0.0071)	1.6001(0.0065)	1.502(0.0002)
Linex$\|r = 1.5$	$\mathcal{V}_{1,[0]}$	1.1082(0.0107)	1.0700(0.0081)	0.9634(0.0025)
	$\mathcal{V}_{2,[0]}$	1.5495(0.01213)	0.9160(0.1155)	1.5939(0.0027)

Pitman estimators are a class of statistical estimators that have a strong foundation in Bayesian analysis and decision theory. Here, we'll explore Pitman estimators and their applications in Bayesian analysis:

- Pitman estimators are commonly used in Bayesian statistics for estimating parameters of probability distributions. In Bayesian analysis, we start with a prior distribution representing our beliefs about the parameters and update these beliefs based on observed data to obtain a posterior distribution. Pitman estimators can be employed to find estimators that have desirable properties within this Bayesian framework.
- Pitman estimators are often minimax estimators, which means they minimize the maximum possible estimation error (the worst-case scenario). In Bayesian analysis, minimizing the maximum posterior risk is a crucial consideration. Pitman estimators offer robustness against large errors, making them suitable for applications where avoiding extreme estimation errors is essential.
- Pitman estimators are admissible, meaning they cannot be uniformly dominated by any other estimator within a certain class. Admissibility ensures that Pitman estimators are competitive and do not have alternatives that consistently perform better.
- Pitman estimators are particularly well-suited for location-scale families of distributions. These are distributions that can be transformed to fit different locations and scales. For example, the normal distribution is a location-scale family, and Pitman estimators work effectively for estimating its parameters.
- In Bayesian linear regression, Pitman estimators can be used to estimate regression coefficients and their uncertainties. They provide robustness against outliers and leverage points.
- In survival analysis, Pitman estimators can be employed to estimate the parameters of survival distributions,

which are commonly used to model time-to-event data.

- Pitman estimators can play a role in Bayesian hypothesis testing by providing estimators of test statistics and decision boundaries.

Bayesian Model Selection: In model selection, Pitman estimators can assist in choosing the most appropriate model from a set of candidate models based on their posterior probabilities.

Pitman estimators are known for their robustness, making them suitable for applications where the data may not precisely follow the assumed model. They are less sensitive to deviations from model assumptions compared to some other estimators.

Pitman estimators can be adapted for nonparametric Bayesian analysis, such as estimating unknown probability distributions without assuming a specific parametric form. This flexibility is valuable in a wide range of applications.

Definition. An estimator θ_1 of a parameter θ dominates another estimator θ_2 in the sense of Pitman's closeness criterion if, for all $\theta \in \Theta$,

$$P_\theta[|\theta_1 - \theta| < |\theta_2 - \theta|] > 0.5.$$

In summary, Pitman estimators are valuable tools in Bayesian analysis, especially when robustness and minimaxity are important considerations. They find applications in various fields, including linear regression, survival analysis, hypothesis testing, model selection, and nonparametric Bayesian analysis. Their properties make them particularly useful for situations where reliable parameter estimation is critical. In Table 15, we show the values of the Pitman probabilities, which allows us to compare the Bayesian estimators with the MXLE estimators when $\gamma = -0.5, p = 0.5$ and $r = -1.5$. Bayesian estimators are said to perform better than MXLE estimators whenever the probability is greater than 0.5, according to definition 1 of the term. When assessed against this criterion, the performance of the Bayesian estimators of the two parameters is better than that of the MXLE; as a result, this conclusion can be formed. Also the GQ loss function has the best values in comparison with the other two loss functions with $\mathcal{V}_1 = 0.779|_{n=100,m=40}$ and $\mathcal{V}_2 = 0.761|_{n=100,m=40}$.

Table 15: Pitman estimators.

$N = 5000$		$n = 10, m = 10$	$n = 50, m = 40$	$n = 200, m = 160$
GQ$\|\gamma = -1$	$\mathcal{V}_{1,[0]}$	0.779	0.779	0.674
	$\mathcal{V}_{2,[0]}$	0.734	0.579	0.634
Entropy$\|p = 0.5$	$\mathcal{V}_{1,[0]}$	0.589	0.667	0.634
	$\mathcal{V}_{2,[0]}$	0.544	0.523	0.589
Linex$\|r = 1.5$	$\mathcal{V}_{1,[0]}$	0.699	0.634	0.5789
	$\mathcal{V}_{2,[0]}$	0.612	0.581	0.5523

8. Uncensored validity

8.1 Simulating the NKRR

There are several different motivations for why researchers could choose to carry out an unfiltered simulation study utilizing the NKRR statistics. Assessing the statistical strength of the NKRR tests in a variety of different contexts is one of the most important motivations. It is possible for aspects such as sample size and effect size to have an impact on a test's statistical power, which is a measurement of the ability of a test to identify an actual effect or difference. Researchers can establish the minimal sample size necessary to attain the desired degree of statistical power and analyze how the performance of the test may be impacted by other factors by performing a study that utilizes simulation. Performing unrestricted simulation research while using the NKRR statistics is necessary for a number of reasons, one of which is to evaluate the correctness and precision of calculated distribution parameter values. In many instances, the purpose of a goodness-of-fit test is to estimate the values of the distribution's parameters in addition to determining whether a specific distribution is suitable for the data. Simulation studies have the potential to give researchers with useful insights into the accuracy and precision of parameter estimations under a variety of different scenarios, which in turn assists the researchers in deciding which distribution to employ for further analyses. We conducted extensive study using computer modeling to validate the assertions made in this work. Our findings are shown here. We particularly constructed N statistics for 15,000 simulated samples with sample sizes of 25, 50, 150, 350, and 600 respectively so that we could test the null hypothesis that the sample is an accurate representation of the TPE model. After determining the average number of times, a null hypothesis is rejected for each of the several theoretical degrees, the relevant empirical and theoretical levels are shown in Table 16. We discovered that the level

value estimated empirically was quite like the level value theoretically determined, which suggests that the suggested test is suitable for the TPE distribution.
Table 16: Uncensored validity under $\varepsilon = 0.01,0.02,0.05,0.1$ and $N = 15000$.

Table 16: Uncensored validity under ε=0.01,0.02,0.05,0.1 and N=15000.

n↓&ε→	ε=0.01	ε=0.02	ε=0.05	ε=0.1
n_1=25	0.9938	0.9835	0.9522	0.9032
n_2=50	0.9933	0.9826	0.9515	0.9020
n_3=150	0.9917	0.9815	0.9509	0.9011
n_4=350	0.9910	0.9809	0.9504	0.9005
n_5=600	0.9904	0.9803	0.9501	0.9002

8.2 Uncensored applications

Example 1: Uncensored strengths of glass fibers

The strengths of glass fibers are important because they determine the overall strength and durability of the materials in which they are used. For example, glass fibers are commonly used in the construction of reinforced concrete, where their strength and durability are crucial for ensuring the structural integrity of the building. Uncensored strength data on glass fibers can provide valuable information on the variability of fiber strength and can help engineers and designers make informed decisions about material selection and structural design. One common statistical model used for analyzing uncensored strength data is the normal distribution. The normal distribution assumes that the data is normally distributed, meaning that it has a symmetric, bell-shaped curve. This model is appropriate when the data is continuous, and the distribution is well-behaved. In the case of the strengths of glass fibers, the normal distribution may be appropriate if the data is approximately symmetric and does not contain outliers. However, if the data is not well-modeled by the normal distribution, alternative statistical models may be necessary. For example, the lognormal distribution is often used to model data that is positively skewed, as it has a natural interpretation in terms of ratios or percentages. Another alternative is the Weibull distribution, which is commonly used in reliability analysis and could describe both increasing and decreasing failure rates. Consider the Strengths of glass fibers of Necholas (2006). Using the BB algorithm, we can get the MXLE value of the parameter vector $\underline{\mathbb{V}}$ assuming that our TPE model can fit the strength data of $1.5cm$ glass fiber is 1.86403, 3.57192. Then, the NKRR value Y^2 =11.89066, while the critical value is 12.59159. As a result, we can say that the new distribution can efficiently conduct statistical modelling operations on unfiltered glass fibre strengths data. We can't say it's the best, but it can be utilised in data modelling or that the unfiltered strengths of glass fibres data match this distribution.

Example 2: Uncensored heat exchanger tube crack

Heat exchanger tubes are critical components of many industrial systems, including power plants, chemical processing plants, and refrigeration systems. The occurrence of cracks in these tubes can lead to leakage and potential safety hazards, making accurate modeling of crack sizes and growth rates essential for effective maintenance and repair. One common statistical model used for analyzing uncensored crack data is the Weibull distribution. The Weibull distribution is a commonly used model in reliability analysis, as it has the ability to describe both increasing and decreasing failure rates. This makes it a useful tool for modeling the growth and propagation of cracks in heat exchanger tubes over time. The Weibull distribution can also be used to estimate the probability of failure at a given time, which is important for maintenance scheduling and risk assessment. Consider the uncensored heat exchanger tube crack data (see Meeker (1998)) where the sum of inspection are 902, 606, 1077, 186, 1209, 1932, 1592 and 1377) and number of fans found to have cracks are 12, 16, 18, 5, 18, 17, 6 and 2. Then if the estimates are 13.41602 and 1.00614, then, the NKRR value $Y^2 = 20.137906$, on the other hand, crucial value is 21.02607 . Accordingly, we can say that the new distribution can efficiently perform statistical modeling operations for the uncensored heat exchanger tube crack data. We cannot say that it is absolutely the best, but we can say that it can be used in data modeling or that the uncensored heat exchanger tube crack data follows this distribution.

9. Censored distributional validation

One key reason for carrying out censored simulation research using NKRR statistics is to evaluate the statistical power of the tests under various types and levels of censoring. Because censoring can result in information loss and decreased statistical power, it is critical to calculate the minimum sample size required to attain a given level of statistical power

under various types and levels of censoring. Another reason to conduct censored simulation research using NKRR statistics is to assess the correctness and precision of the predicted distribution parameters, especially when working with right-censored data. In many circumstances, the purpose of a goodness-of-fit test is to estimate the values of a distribution's parameters as well as to determine whether a certain distribution fits the data. Simulation studies can provide insights into the accuracy and precision of parameter estimates under various types and levels of censoring, and they can help inform decisions about which distribution to utilize in subsequent analyses. Now, if we were going to test the null hypothesis that the data are distributed, then for the TPE distribution the survival function (SF) and cumulative hazard function (CHRF) are

$$S_{\underline{\mathbb{V}}}(\boldsymbol{x}) = 1 - \mathcal{F}_{\underline{\mathbb{V}}}(\boldsymbol{x}) = 1 - \left(\frac{1 - exp\{-\mathcal{V}_1[1 - \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x})]\}}{1 - exp(-\mathcal{V}_1)}\right),$$

and

$$A_{TPE}(\boldsymbol{x}, \underline{\mathbb{V}}) = -ln[S_{\underline{\mathbb{V}}}(\boldsymbol{x})] = -ln\left(1 - \frac{1 - exp\{-\mathcal{V}_1[1 - \mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x})]\}}{1 - exp(-\mathcal{V}_1)}\right),$$

where $\mathfrak{Z}_{\mathcal{V}_2}(\boldsymbol{x}) = [exp(-\mathcal{V}_2\boldsymbol{x})]^{[1-exp(-\mathcal{V}_2\boldsymbol{x})]}$. For all ω, we have $e_{\omega,X} = E_k/k$. Let's divide a finite time period $[0,\tau]$ into $k > s$ smaller periods $I_\omega = (\vartheta_{\omega-1}, \vartheta_{\omega,\boldsymbol{x}}]$, where is the study's maximum duration and

$$0 =< \vartheta_{0,\boldsymbol{x}} < \vartheta_{1,\boldsymbol{x}} \ldots < \vartheta_{k-1,\boldsymbol{x}} < \vartheta_{k,\boldsymbol{x}} = +\infty.$$

Bagdonavicius and Nikulin's (2011) and Bagdonavicius et al. ($2011a$)'s test statistic is expressed as:

$$Y_n^2 = \sum_{\omega=1}^{k} (U_{\omega,X} - e_{\omega,X})^2 \frac{1}{U_{\omega,X}} + Q_{W,G}.$$

9.1 Censored simulation study under the NKRR statistics Y^2

A censored simulation study under the NKRR statistics involves generating simulated data sets with censoring and testing the goodness-of-fit of the censored data to a particular distribution using the NKRR test statistic. In a censored simulation study, observations beyond a certain value (known as the censoring threshold) are not fully observed but instead are recorded as being above this threshold. This type of simulation study can be particularly useful when dealing with real-world data sets in which observations may be censored or truncated due to various factors such as measurement limitations or data collection methods. By conducting a censored simulation study under the NKRR statistics, researchers can assess the statistical power of the NKRR test under different censoring scenarios, evaluate the accuracy and precision of estimated distribution parameters, and make informed decisions about which distribution to use for subsequent analyses. Overall, censored simulation studies using the NKRR statistics are a useful tool for analyzing the goodness-of-fit of distributions to censored data and can help influence decision-making in a variety of sectors, including engineering, economics, and finance. In this study, it is assumed that a sample of N=14000 is subject to 25% right censoring. The null hypothesis H0 is related to the TPE model, and a goodness-of-fit test is performed with df=5 grouping intervals. The study's goal is to calculate the average number of non-rejection occurrences of the null hypothesis for different theoretical levels ($\varepsilon = 0.01; 0.02; 0.05; 0.1$). Table 17 summarises the findings and displays the degree of consistency between the theoretical and empirical levels. Based on the findings, it is possible to infer that the custom test is well matched to the TPE model.

Table 17: Censored validation under $\varepsilon = 0.01; 0.02; 0.05; 0.1$ and $N = 14000$.

n↓&ε→	ε=0.01	ε=0.02	ε=0.05	ε=0.1
n_1=25	0.9928	0.9763	0.9533	0.9027
n_2=50	0.9919	0.9781	0.9523	0.9014
n_3=150	0.9911	0.9796	0.9511	0.9007
n_4=350	0.9906	0.9799	0.9504	0.9002
n_5=600	0.9903	0.9801	0.9501	0.9001

Based on our analysis, we have arrived at the inference that the theoretical level of the chi-square distribution on degrees of freedom aligns with the empirical significance level of the NKRR statistics at which it attains statistical significance. This observation leads us to the conclusion that the proposed test can aptly fit the censored data originating from the TPE distribution.

9.2 Censored applications under the NKRR statistics Y^2

Censored lung cancer data set

The statistical modelling of censored data is an important part of survival analysis because it allows for the precise calculation of the underlying survival function as well as the identification of factors that impact survival time. The

censored lung cancer dataset exemplifies the significance of adequate statistical modelling in survival data analysis. The survival period of patients is censored in the censored lung cancer dataset, which means that the actual time of death for certain patients is unclear. When a patient withdraws from the study, is lost to follow-up, or is still alive at the end of the trial, this occurs. If this censoring is not accounted for in statistical modelling, the estimates of the survival function and the impact of covariates on survival time may be skewed. The Cox proportional hazards model is a popular statistical method for analyzing censored data because it allows for the calculation of the hazard function, or the instantaneous risk of an event occurring, as a function of covariates. In many circumstances, this model implies that the hazard function is proportionate across multiple levels of variables, which is a legitimate assumption. However, it is critical to evaluate the validity of this assumption in each individual dataset. The selection of censoring distribution is another critical part of statistical modelling for censored data. The censoring distribution in the censored lung cancer dataset is unknown and must be calculated. One popular strategy is to assume that censoring is non-informative, which means that the censoring distribution is unrelated to the survival time and variables. This assumption, however, should be carefully assessed in each given dataset, as informative filtering can occur in some situations, resulting in biassed results. Then, for the lung cancer data (see Loprinizi (1998)) where $n = 228$ and censored items= 63 , the estimates are 4.06698,1.102493 and

$U_{\omega,X}$	29	30	35	31	32	25	28	18
$e_{\omega,X}$	7.19224	7.19224	7.19224	7.19224	7.19224	7.19224	7.19224	7.19224

Based on our analysis, we have obtained a critical value of 15.50731 for the chi-squared test. Upon calculation of the estimated statistic for the suggested test using the previous findings, we have arrived at a value of 15.413172. Our hypothesis may therefore be accepted, as the computed value is lower than the tabulated value of the NKRR statistic. This suggests that there is a 5% likelihood of deviation of the censored lung cancer data from the TPE distribution.

Censored capacitor reliability data set

In the censored capacitor data reliability dataset, the lifetime of the capacitors is censored, meaning that the exact lifetime is not known for some capacitors. This occurs when the capacitors are still functioning at the end of the study or when they are withdrawn from the study before failure. If this censoring is not properly accounted for in the statistical modeling, the estimates of the lifetime distribution and the reliability measures can be biased. One common statistical method for analyzing censored data in reliability analysis is the Weibull distribution. The Weibull distribution is a flexible and widely used model for lifetime data, as it can describe both increasing and decreasing hazard rates. The Weibull distribution also allows for the estimation of several reliability measures, such as the mean time to failure and the failure rate. Another important aspect of statistical modeling for censored data is the choice of censoring distribution. In the censored capacitor data reliability dataset, the censoring distribution is not known and must be estimated. One approach is to assume that the censoring is non-informative, meaning that the censoring distribution is independent of the lifetime of the capacitors. However, it is important to evaluate the validity of this assumption in each specific dataset, as informative censoring can occur in some cases and can lead to biased estimates. By considering the reliability data (see Meeker (1998)) where $n = 64$ and censored observations= 32 . Then, the estimates as 1.99674, 1.20933 and

$U_{\omega,X}$	11	15	6	10	6	5	6	5
$e_{\omega,X}$	5.91775	5.91775	5.91775	5.91775	5.91775	5.91775	5.91775	5.91775

Based on our analysis, we have obtained a critical value of 15.50731 for the chi-squared test. Upon calculation of the estimated statistic for the suggested test using the previous findings, we have arrived at a value of 14.899051. Our hypothesis may therefore be accepted, as the computed value is lower than the tabulated value of the NKRR statistic. This suggests that there is a 5% likelihood of deviation of the censored capacitor reliability data from the TPE distribution.

10. Conclusions

The Nikulin-Rao-Robson (NKRR) test statistic has gained widespread acceptance as a practical instrument for the validation of distortional assumptions and the examination of hypotheses in a wide range of fields, including economics, engineering, and finance, among others. The NKRR test is well-known for its resistance against outliers and capacity to deal with non-normal distributions. As a result, it is a dependable and flexible instrument that may be utilized for the analysis of data sets that contain large tails, skewed distributions, and outliers. In this paper, the Poisson-exponential model with two parameters, also known as the TPE model, is analyzed and presented. The exponentiated exponential model can be reformulated into a novel and more adaptable version called the Poisson-

exponential model with two parameters. In this study, we analyze, describe, and put into practice six traditional methods of estimating, namely the Cramer-von-Mises technique, the ordinary least squares method, the L-moments method, the maximum likelihood method, the Kolmogorov method, and the weighted-least squares method. These methods are listed in order from simplest to most complex. In addition to this, we construct Bayesian estimation and investigate its application in conjunction with the squared error loss function. To establish a full comparison of the techniques, we conduct an in-depth analysis of two real data sets in addition to two generated data sets. In order to demonstrate the adaptability of the TPE distribution as well as its usefulness, we will be using not one but two separate real-world data sets. When it comes to modeling relief times and survival times data sets, the new model outperforms a large number of competing models. This is demonstrated by the Akaike Information Criterion, the Consistent Akaike Information Criterion, the Hannan-Quinn Information Criterion, the Bayesian Information Criterion, the Cramer-Von Mises statistics, and the Anderson-Darling statistics. Even though the study and its applications contain a plethora of additional findings, the following discoveries are of particular interest:

1) The maximum probability method remains the classical approach with the highest level of accuracy and efficiency, despite the plethora of other classical methods available. In the realm of statistical modeling and application work, our team highly recommends the Bayesian approach and the Maximum Likelihood method. When it comes to modeling asymmetric bimodal right-skewed relief data, the proposed lifespan TPE model outperforms all other models. This model is characterized by the following parameters: AK-INC=36.198, BS-INC=38.19, CAK-INC=36.904, HQ-INC=36.587, ANDC=0.2891, CVMC=0.0494, K. S=0.12900, and p-value=0.8933. Consequently, the new lifespan model serves as an excellent alternative to previously employed models for modeling relief time datasets. These results underscore the superiority of the newly developed distribution over existing ones.
2) When dealing with the modeling of asymmetric bimodal right-skewed survival data, the proposed lifetime TPE model similarly outperforms all other models. This model is associated with the following statistics: AK-INC=202.08, BS-INC=206.6, CAK-INC=202.258, HQ-INC=203.897, ANDC=0.562, CVMC=0.0948, K. S=0.1290, and p-value=0.8933. In the context of modeling survival time datasets, this new lifetime model serves as a valuable alternative to previously employed models. These results reinforce the superior performance of the newly developed distribution in comparison to others.
3) We provide Bayesian estimation under various loss functions, utilizing the generalized quadratic, Linex, and entropy loss functions, respectively. We offer a wealth of valuable information for obtaining Bayesian estimators. Each estimating approach described here undergoes a series of simulation tests, each with its specific controls and settings. The results of these simulation studies are discussed in their respective sections of the study.
4) To process estimation using censored samples, we compare the Bayesian method with the censored maximum likelihood method employing the BB algorithm. In this section, we comprehensively outline the NKRR statistic for the TPE model in the uncensored scenario. We conduct a simulation analysis, the results of which demonstrate the appropriateness of the NKRR test for the TPE model. To assert that the data fits the TPE model correctly, we provide two real data applications under the uncensored scenario: the strengths of glass fibers data and the heat exchanger tube crack data. Both datasets were collected under the uncensored situation.
5) Under the censored scenario, we present a thorough development of the Bagdonavicius and Nikulin statistic for the new model. Additionally, we offer a simulation study to evaluate the statistic. We provide an analysis of two real data applications in the context of the censored scenario: one concerning capacitors (reliability data) and the other concerning lung cancer (medical data). The results from these applications conclude that the proposed test effectively fits censored data from the TPE distribution.

Looking ahead, here are some future directions:

1) Explore variations and extensions of the alternative exponential model to account for different degrees and types of skewness in real-world data. This could involve developing a family of skewed exponential distributions or hybrid models that combine exponential and other distributional components.
2) Investigate the robustness of the proposed model to outliers and anomalies in real data. Develop methods for outlier detection and handling within the context of the alternative exponential model, which can enhance the model's applicability in practical settings.
3) Develop rigorous model validation techniques specific to the alternative exponential model. This could include goodness-of-fit tests, graphical diagnostics, and cross-validation procedures tailored to assess the model's performance in capturing skewness and other characteristics of real data.
4) Explore the incorporation of covariates or explanatory variables into the alternative exponential model. This would allow for modeling how skewness and other parameters change as a function of other factors, providing a more comprehensive understanding of the data generation process.

5) Extend the Bayesian estimation approaches for the alternative exponential model. Investigate the development of informative prior distributions, Bayesian model selection criteria, and Bayesian hierarchical modeling techniques to improve parameter estimation and uncertainty quantification.
6) Conduct comparative studies between the alternative exponential model and other commonly used skewed distributions, such as the gamma or log-normal distributions. Assess the model's performance in terms of goodness-of-fit and predictive accuracy across a wide range of real datasets.
7) Apply the alternative exponential model to various fields where skewed data are prevalent, such as finance, economics, environmental science, and epidemiology. Explore how the model performs in capturing skewness patterns unique to these domains.
8) Develop scalable algorithms and computational techniques for fitting the alternative exponential model to large datasets. Consider parallel computing, optimization algorithms, and distributed computing frameworks to handle big data scenarios effectively.
9) Enhance the interpretability of the model's parameters by investigating how they relate to meaningful characteristics of the data-generating process. This could involve developing parameter interpretations that are more intuitive and actionable for practitioners.
10) Create user-friendly software packages or libraries for estimating and analyzing the alternative exponential model. Open-source tools can facilitate broader adoption and application of the model by researchers and practitioners.
11) Investigate methods for quantifying and propagating uncertainty in model parameters to downstream analyses and decision-making processes. This is particularly important in cases where parameter estimates have a substantial impact on decision outcomes.
12) Conduct in-depth case studies using real datasets from various domains to showcase the practical utility of the alternative exponential model. Illustrate how the model can provide valuable insights and improve predictive accuracy compared to existing methods.

Acknowledgments: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU260874].

Data Availability: Links to access the data used in this paper are given.

Contributions: All authors contributed equally to the preparation of the paper, commentary on the results and review of the paper.

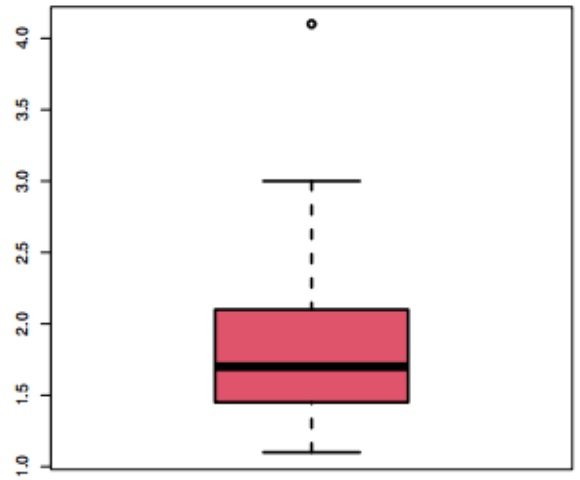
Conflict of Interest: There are no conflicts of interest.

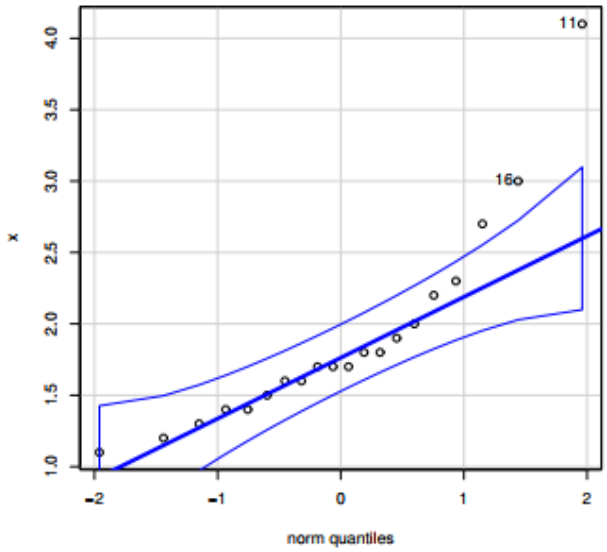
References

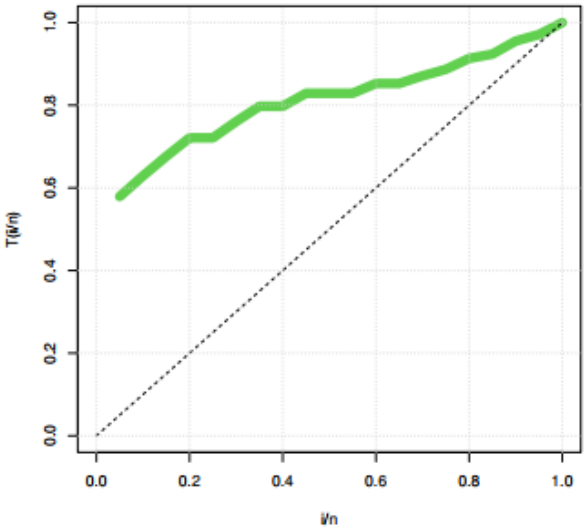
1. Nikulin. M.S, (1973a). Chi-squared test for normality. in proceedings of the International Vilnius Conference on Probability Theory and Mathematical Statistics, 2, 119-122.
2. Nikulin. M.S, (1973b). Chi-squared test for continuous distributions with shift and scale parameters, Theory of Probability and its Aplications. 18, 559-568.
3. Nikulin. M.S, (1973c). On a Chi-squared test for continuous distributions, Theory of Probability and its Aplications. 19, 638-639.
4. Rao, K. C., Robson, D. S. (1974). A Chi-square statistic for goodness-of-fit tests within the exponential family. Communication in Statistics, 3, 1139-1153.
5. Goual, H., Yousof, H. M., and Ali, M. M. (2019). Validation of the odd Lindley exponentiated exponential by a modified goodness of fit test with applications to censored and complete data. Pakistan Journal of Statistics and Operation Research, 745-771.
6. Ibrahim, M., Aidi, K., Ali, M. M. and Yousof, H. M. (2023). A Novel Test Statistic for Right Censored Validity under a new Chen extension with Applications in Reliability and Medicine. Annals of Data Science, 10,1285--1299.
7. Tahir, M. H., Hussain, M. A., Cordeiro, G. M., El-Morshedy, M. and Eliwa, M. S. (2020). A new Kumaraswamy generalized family of distributions with properties, applications, and bivariate extension. Mathematics, 8(11), 1989.
8. Bagdonavičius, V., Levuliene, R., J., and Nikulin, M. (2013). Chi-squared goodness-of-fit tests for parametric accelerated failure time models. Communications in Statistics-Theory and Methods, 42, 15, 2768-2785.
9. Bagdonavičius, V. and Nikulin, M. (2011a). Chi-squared goodness-of-fit test for right censored data. International

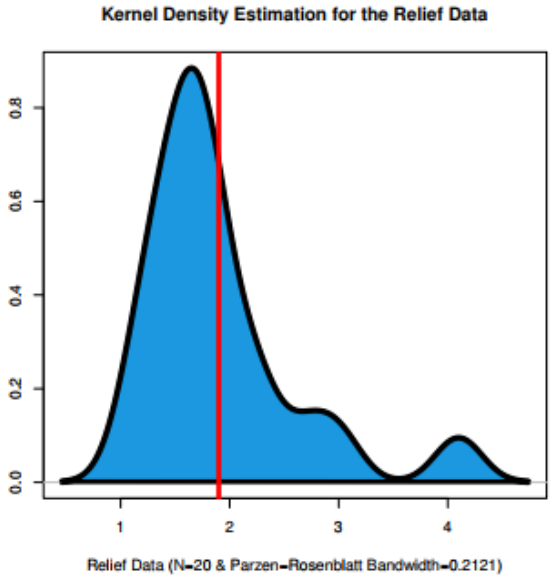
Journal of Applied Mathematics and Statistics, 24, 30-50.
10. Bagdonavičius, V. and Nikulin, M. (2011b). Chi-squared tests for general composite hypotheses from censored samples. Comptes Rendus Mathematique, 349(3-4), 219-223.
11. Glanzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 1987, 75-84.
12. Glanzel, W. (1990). Some consequences of a characterization theorem based on truncated moments, Statistics: A Journal of Theoretical and Applied Statistics, 21 (4), 613-618.
13. Yousof, H. M., Aidi, K., Hamedani, G.G. and Ibrahim, M. (2023a). Chi-squared type test for distributional censored and uncensored validity with numerical assessments and real data applications. Japanese Journal of Statistics and Data Science, forthcoming.
14. Yousof, H. M., Ali, M. M., Aidi, K., Ibrahim, M. (2023b). The modified Bagdonavičius-Nikulin goodness-of-fit test statistic for the right censored distributional validation with applications in medicine and reliability. Statistics in Transition New Series, 24(4), forthcoming.
15. Emam, W., Tashkandy, Y., Goual, H., Hamida, T., Hiba, A., Ali, M.M., Yousof, H.M., Ibrahim, M. (2023). A New One-Parameter Distribution for Right Censored Bayesian and Non-Bayesian Distributional Validation under Various Estimation Methods. Mathematics 2023, 11, 897. https://doi.org/10.3390/math11040897
16. Gross, J. and Clark, V. A. (1975). Survival Distributions: Reliability Applications in the Biometrical Sciences, John Wiley, New York, USA.
17. Bjerkedal, T. (1960). Acquisition of resistance in Guinea pigs infected with different doses of virulent tubercle bacilli. American Journal of Hygiene, 72, 130--148.
18. Merovci, F., Alizadeh, M., Yousof, H. M. and Hamedani, G. G. (2017). The exponentiated transmuted-G family of distributions: theory and applications. Communications in Statistics-Theory and Methods, 46(21), 10800-10822.
19. Merovci, F., Yousof, H. M., & Hamedani, G. G. (2020). The Poisson Topp Leone Generator of Distributions for Lifetime Data: Theory, Characterizations and Applications. Pakistan Journal of Statistics and Operation Research, 16(2), 343-355.
20. Karamikabir, H., Afshari, M., Yousof, H. M., Alizadeh, M., & Hamedani, G. (2020). The Weibull Topp-Leone Generated Family of Distributions: Statistical Properties and Applications. Journal of The Iranian Statistical Society, 19(1), 121-161.
21. Korkmaz, M. C., Altun, E., Yousof, H. M. and Hamedani, G. G. (2020). The Hjorth's IDB Generator of Distributions: Properties, Characterizations, Regression Modeling and Applications. Journal of Statistical Theory and Applications, 19(1), 59-74.
22. Hamedani, G. G., Yousof, H. M., Rasekhi, M., Alizadeh, M. and Najibi, S. M., (2018). Type I general exponential class of distributions. Pakistan Journal of Statistics and Operation Research, 14(1), 39-55.
23. Pitman, E. (1937). The closest estimates of statistical parameters. Mathematical proceeding of the Cambridge philosophical society, 33(2).
24. Fuller, W.A. (1982). Closeness of estimators. Encyclopedia of statistical sciences, Vol 2, Wiley
25. Jozani, M. J., Davies, K. F. and Balakrishnan, N. (2012). Pitman closeness results concerning ranked set sampling. Statistics & Probability Letters, 82(12), 2260-2269.
26. Nicholase, M.D.; Padgett, W.J. A bootstrap control chart for Weibull percentiles. Qual. Reliab. Eng. Int. 2006, 22, 141--151.
27. Meeker, W.Q.; Escobar, L.A.; Lu, C.J. Accelerated degradation tests: Modeling and analysis. Technometrics 1998, 40, 89--99.
28. Loprinzi, C. L., Laurie, J. A., Wieand, H. S., Krook, J. E., Novotny, P. J., Kugler, J. W. and Klatt, N. E. (1994). Prospective evaluation of prognostic variables from patient-completed questionnaires. North Central Cancer Treatment Group. Journal of Clinical Oncology, 12(3), 601-607.

Appendix:


The Relief Times Data


11
16
x
norm quantiles


T(i/n)
i/n


Kernel Density Estimation for the Relief Data
Relief Data (N=20 & Parzen−Rosenblatt Bandwidth=0.2121)

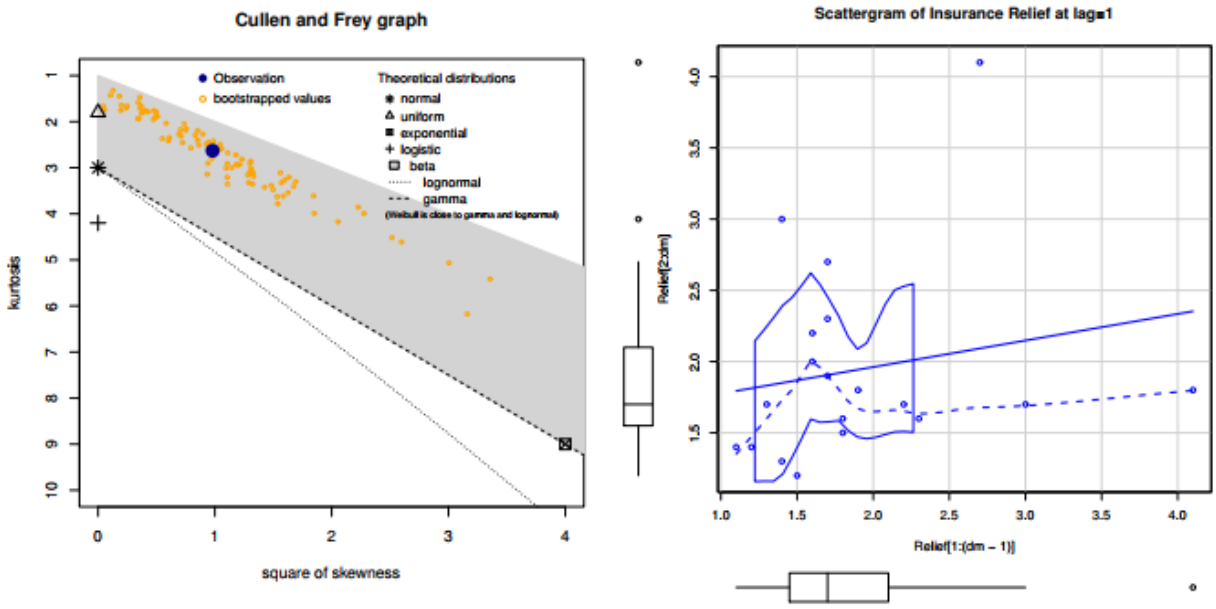


Figure 1: Describing the relief times data.

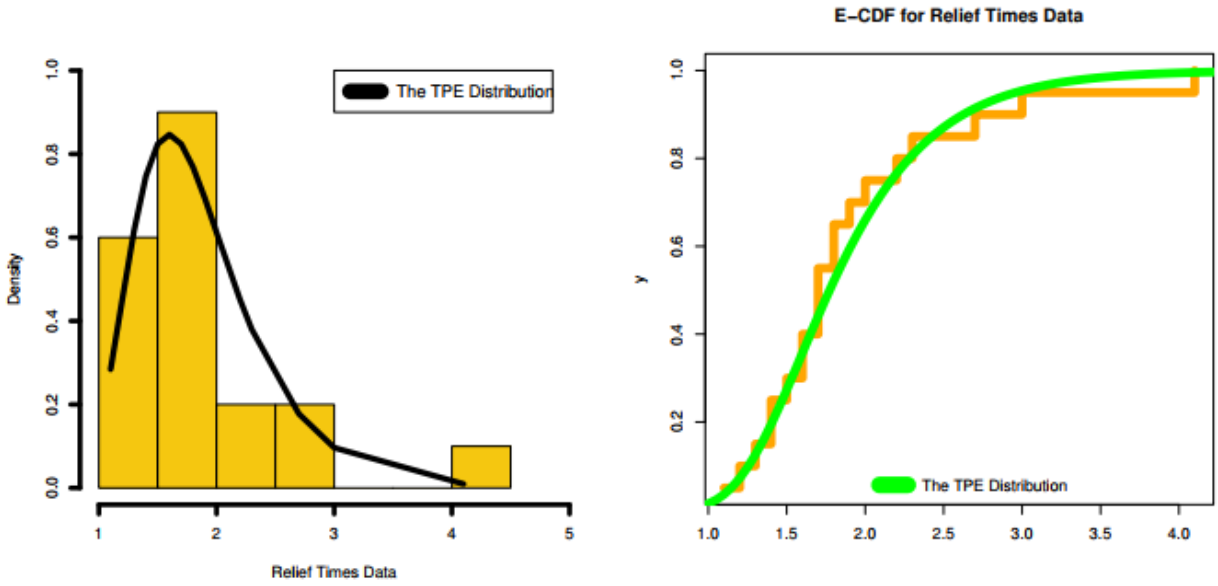


Figure 2: ES-PDF and ES-CDF for relief times data.

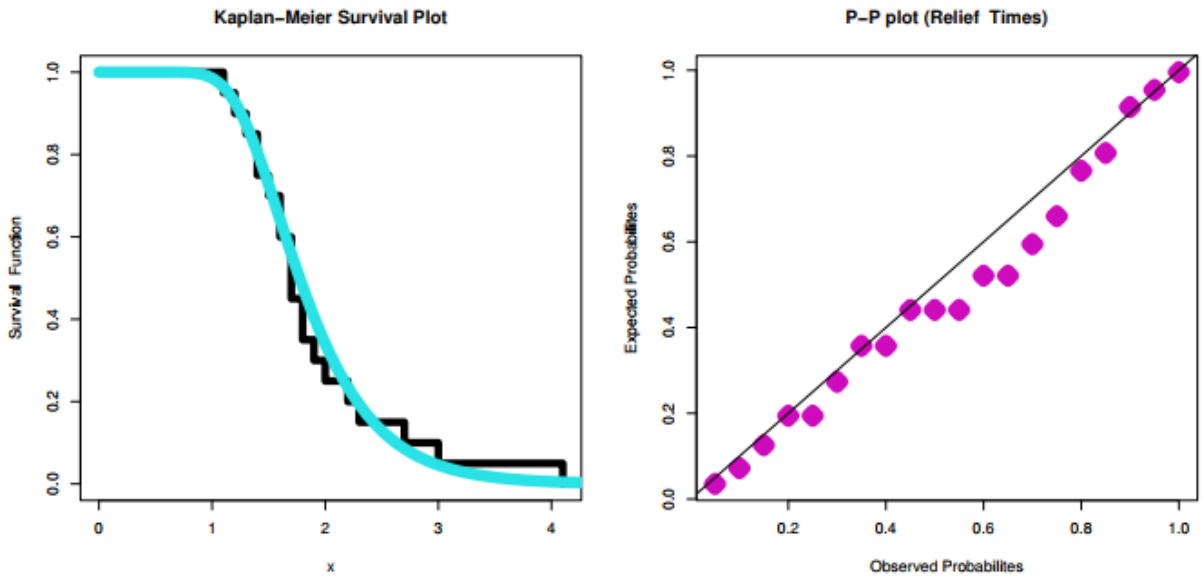
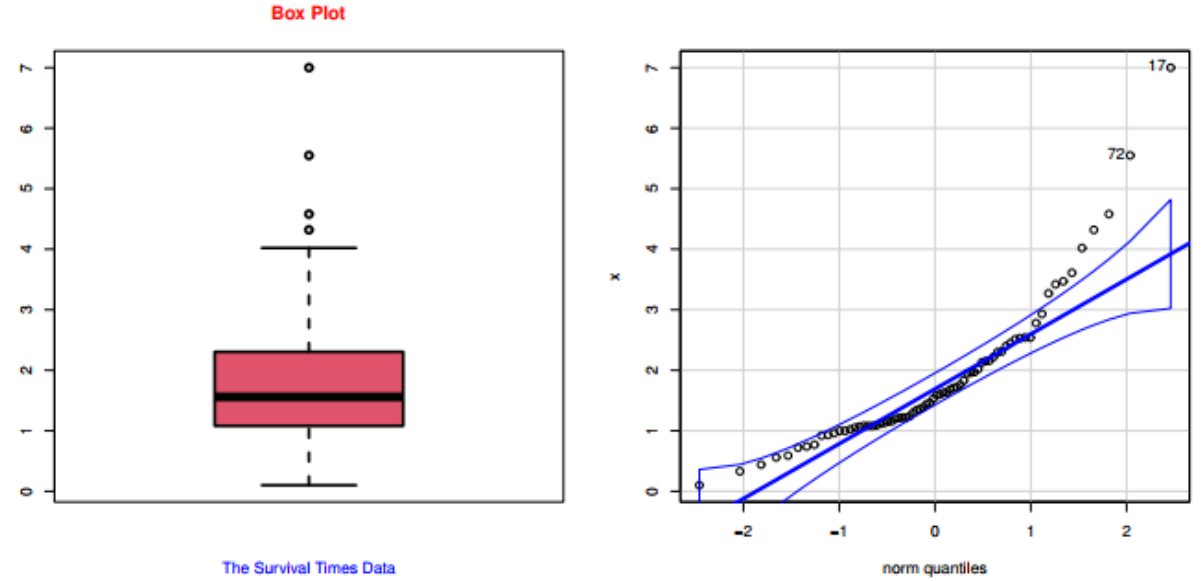


Figure 3: MM-S plot and P-P plot for relief times data.



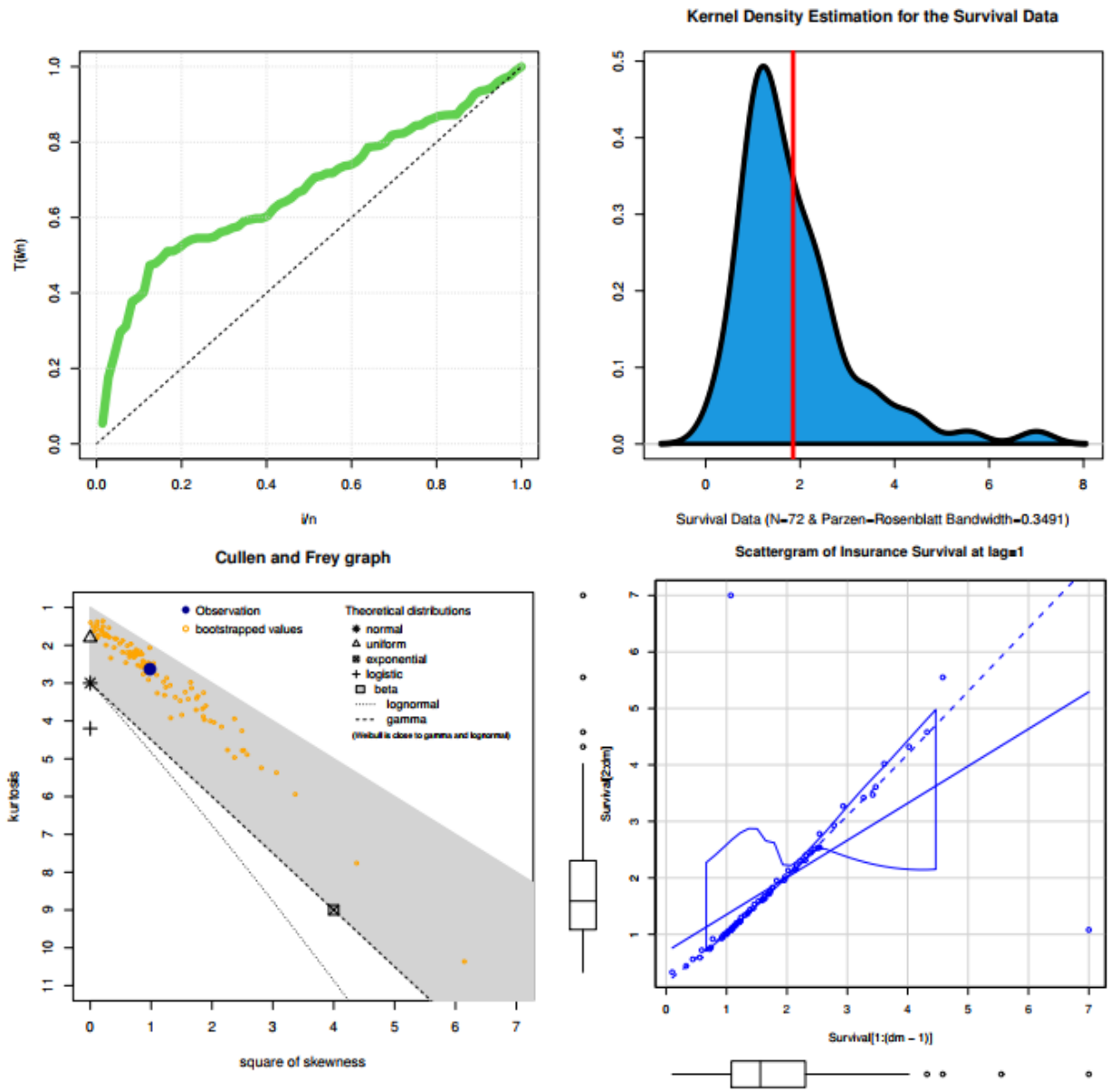


Figure 4: Describing the survival times data.

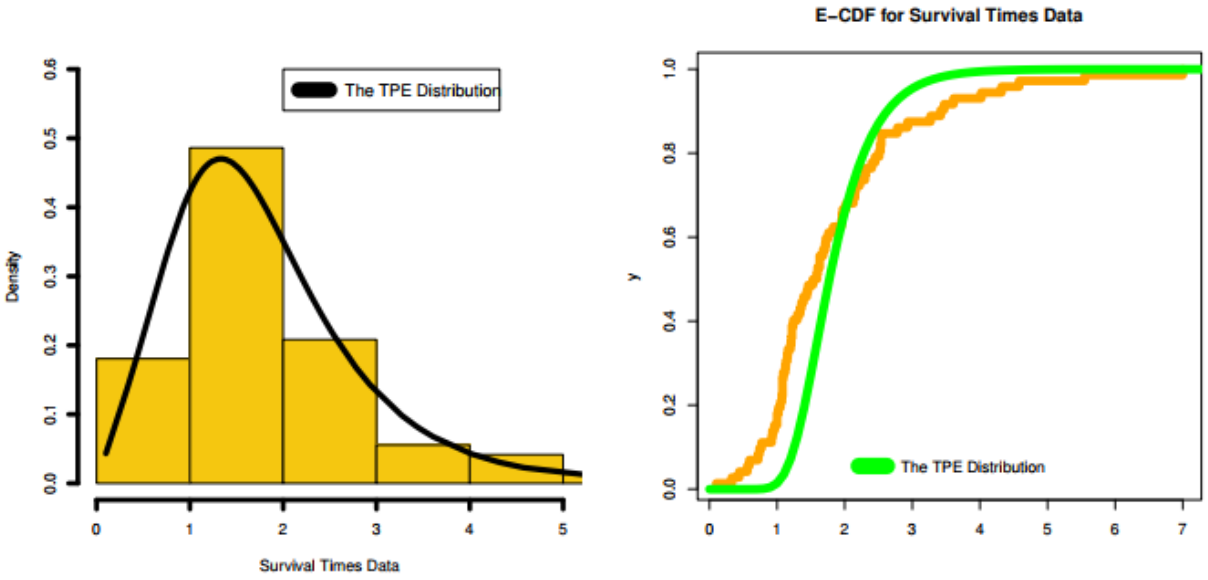


Figure 5: ES-PDF and ES-CDF for survival times data.

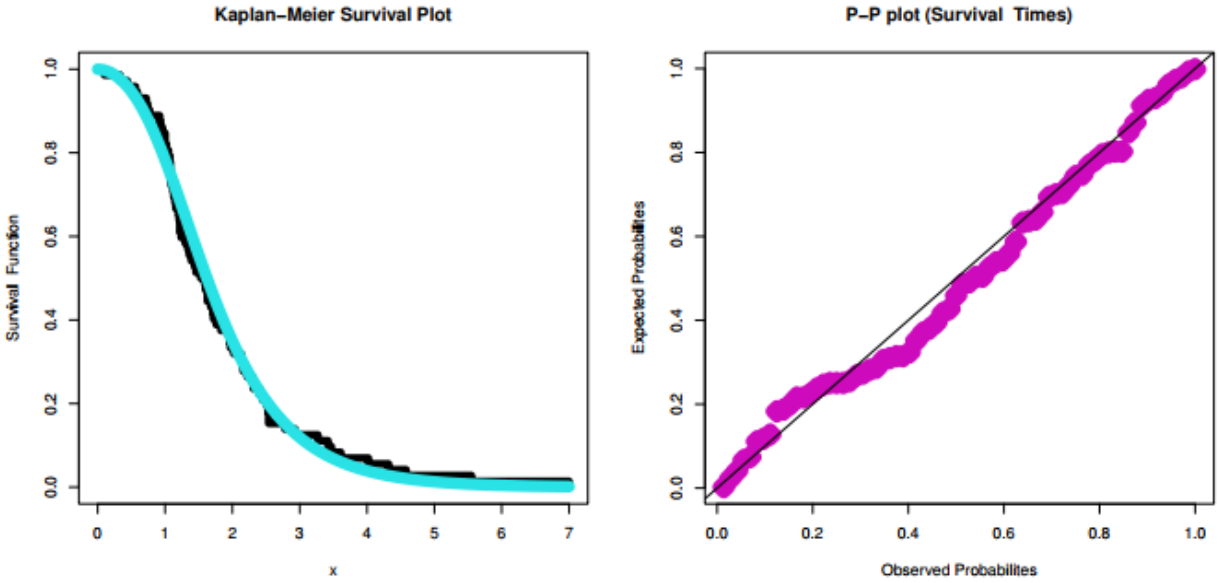


Figure 6: KM-S plot and P-P plot for survival times data.