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A Bivariate Cubic Transmuted Weibull Distribution Properties and Application

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Abstract. Lifetime distributions play a key role in statistical modeling, with extensive applications across biostatistics, reliability engineering, and survival analysis. This paper introduces a novel and flexible bivariate lifetime model, termed the *Bivariate Cubic Transmuted Weibull Distribution (BCTWD)*, which extends the transmuted Weibull framework proposed by Alsalafi et al. (2025) by incorporating a cubic transmutation mechanism to enhance modeling flexibility and capture complex dependence structures. Existing bivariate Weibull models cannot simultaneously accommodate flexible marginal tail behavior and complex dependence structures, limiting their applicability in scenarios with heterogeneous failure patterns. The theoretical foundations of the proposed BCTWD are rigorously developed, including its joint and marginal probability density and cumulative distribution functions, along with essential statistical and reliability properties. Parameter estimation is performed using both the Maximum Likelihood (ML) and Inference Functions for Margins (IFM) methods, whose performances are systematically evaluated through simulation experiments. The simulation outcomes indicate that the estimators are, For n=200, the maximum absolute bias for shape parameters are 0.048, and the maximum MSE is 0.29, indicating satisfactory finite-sample performance, particularly for the shape parameters under heavy-tailed scenarios.
An empirical application to bilateral eye failure time data from a diabetic retinopathy study demonstrates the practical utility of the proposed model. Based on the maximum likelihood estimates and model selection criteria, including AIC, AICc, and BIC, the BCTWD achieves superior goodness-of-fit compared with the Bivariate Transmuted Weibull (BTW) and Bivariate Weibull (BW) distributions. While the BCTWD exhibits slightly greater parameter variability due to its added flexibility, it provides the most accurate representation of the data, confirming its effectiveness in modeling dependent lifetimes.
Overall, the BCTWD enriches the family of multivariate lifetime distributions by offering enhanced adaptability and interpretability, making it a valuable tool for applications in reliability analysis, biostatistics, and survival modeling.
Keywords: The bivariate cubic transmuted Weibull, Weibull distribution, Maximum likelihood estimation, Inference functions for matrices.

1 Introduction

The Weibull distribution, introduced by Weibull (1951), is one of the most widely used lifetime distributions in reliability engineering and survival analysis due to its flexibility in modeling various types of hazard rate functions. The cumulative distribution function (CDF) of the classical Weibull distribution is given by:

$$F(x;\lambda,k)=\begin{cases}1-\exp\left[-\left(\frac{x}{\lambda}\right)^k\right], & x\geq 0,\\ 0, & x<0\end{cases} \tag{1}$$

where $\lambda > 0$ is the scale parameter and $k > 0$ is the shape parameter. Bivariate lifetime data arise naturally in numerous practical applications. In medical research, paired organs, such as the eyes, kidneys, and lungs, exhibit correlated failure times. In reliability engineering, twin components operating under similar stress conditions demonstrate dependent degradation patterns. Traditional independent lifetime models fail to capture such dependencies, potentially leading to misleading inferences. Although univariate Weibull distributions have been extensively studied, modeling bivariate lifetime data requires careful consideration of both marginal behavior and dependence structure.

Several extensions of the Weibull distribution have been introduced to accommodate more complex data patterns. Among these, the transmuted Weibull (TW) distribution, introduced by Aryal and Tsokos (2011), employs a quadratic rank transmutation map and is defined by CDF:

$$F(x)=(1+\lambda)G(x)-\lambda G^2(x), \tag{2}$$

where $G(x)$ is the Weibull CDF baseline and $\lambda\in[-1,1]$ is the transmutation parameter. This transformation improves the modeling of skewness and tail behavior. More recently, Khan et al. (2017) introduced the transmuted Weibull distribution and studied various structural properties with an application to two real datasets, Pobočíková et al. (2018) used the quadratic rank transmutation map to develop the TW distribution.

To further enhance flexibility, Rahman et al. (2019) proposed the Cubic Transmuted Weibull (CTW) distribution with the CDF:

$$F(x)=(1+\lambda_1)G(x)+(\lambda_2-\lambda_1)G^2(x)-\lambda_2G^3(x), \tag{3}$$

where $\lambda_1,\lambda_2\in[-1,1]$ and $-2\leq\lambda_1+\lambda_2\leq 1$. This model allows for modeling complex bimodal and hazard rate data more effectively. AL-Kadim and Mohammed (2017) studied a cubic transmuted Weibull distribution CTWD, and discussed the order statistic and some of the statistical properties. Tushar et al. (2024) proposed a new cubic transmuted inverse Weibull (CTIW) distribution by adding an extra parameter to the well-known inverse Weibull (IW) distribution, also discussing explicit expansions of moments, quantile, and the generating functions, reliability analysis, etc. The distribution of order statistics for the CTIW distribution is also obtained, along with a numerical computation of the mean and variance of different order statistics for different sample sizes and for different values of the parameters. The model parameters were estimated using the maximum likelihood estimation method. A simulation study has been conducted to see the consistency of the estimation process. Aslam et al. (2018) introduced a new class of probability distributions, namely the *cubic transmuted G family*, which extends the flexibility of the traditional transmuted G framework. Within this context, they further examined specific subclasses of the proposed family, referred to as the *bivariate cubic transformed (BCT) distribution*. The joint CDF of the BCT distribution is defined as

$$F_{X,Y}(x,y)=\lambda_1G(x,y)^{\alpha}\ +\ (\lambda_2-\lambda_1)G(x,y)^{2\alpha}\ +\ (1-\lambda_2)G(x,y)^{3\alpha}, \tag{4}$$

where $G(x,y)$ denotes the joint distribution function of the model.

Extending these ideas into the multivariate domain, Alsalafi et al. (2025) introduced the bivariate cubic transmuted (BCT) family of distributions. The BCT distribution allows flexible modeling of the joint behavior of two dependent random variables. The joint CDF is given by:

$$\begin{aligned}F_{X,Y}(x,y)=G_1(x)G_2(y)[&1+\lambda_1+\lambda_3+\lambda_5+(\lambda_2-\lambda_1)G_1(x)-(\lambda_2+\lambda_5)G_1^2(x)\\ &+(\lambda_4-\lambda_3)G_2(y)-(\lambda_4+\lambda_5)G_2^2(y)+\lambda_5G_1^2(x)G_2^2(y)],\end{aligned} \tag{5}$$

where $G_1(x)$ and $G_2(y)$ are the marginal baseline CDFs of variables X and Y, respectively,
$(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)\in[-1,1]$ under these conditions:$-2\leq\lambda_1+\lambda_3+\lambda_5\leq 0$,$-1\leq\lambda_2-\lambda_1\leq 1$,$-1\leq\lambda_2+\lambda_5\leq 1$, $-1\leq\lambda_3-\lambda_4\leq 1$ and $-1\leq\lambda_4+\lambda_5\leq 1$.

This bivariate extension captures a wide range of dependency structures and is particularly suitable for applications in reliability and survival analysis, where modeling joint lifetimes is crucial.

The rest of the article is described as follows: Section 1: The remainder of this paper is organized as follows. Section 2 introduces the BCTWD and derives its distributional properties. Section 3 discusses statistical properties, including marginal and conditional distributions. Section 4 examines the reliability and hazard rate functions. Section 5 presents parameter estimation via maximum likelihood and IFM methods. Section 6 evaluates the performance of the estimator through simulation studies. Section 7 applies the model to the optical failure time data. Section 8 concludes with a discussion of future research directions.

2 The Bivariate cubic Transmuted Weibull Distribution

Despite these developments, existing Bivariate Weibull models face limitations: (i) The bivariate Weibull (BW) distribution assumes independence in marginals, failing to model complex dependencies. (ii) The bivariate transmuted Weibull (BTW) employs only quadratic transmutation, limiting its ability to capture certain hazard rate patterns. (iii) Most existing models cannot simultaneously accommodate bimodal marginal distributions and flexible tail dependencies.

This paper addresses these gaps by proposing the BCTWD with five dependencies parameters. The BCTWD is proposed using the univariate Weibull distribution that has the following CDFs:

$$G_1(x) = 1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}}; x, \beta_1, \alpha_1 \geqslant 0 \tag{6}$$

and

$$G_2(y) = 1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}}; x, \beta_2, \alpha_2 \geqslant 0 \tag{7}$$

Using the CDFs by (5), the distribution function of the proposed BCTWD is

$$\begin{aligned} F_{BCTW}(x, y) = {} & (1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})\big[1 + \lambda_1 + \lambda_3 + \lambda_5 \\ & + (\lambda_2 - \lambda_1)(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}}) - (\lambda_2 + \lambda_5)(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})^2 \\ & - (\lambda_3 - \lambda_4)(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}}) - (\lambda_4 + \lambda_5)(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})^2 \\ & + \lambda_5(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})^2(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})^2\big], \end{aligned} \tag{8}$$

For $x, y, \alpha_1, \alpha_2, \beta_1 > 0$ and $\beta_2 > 0$

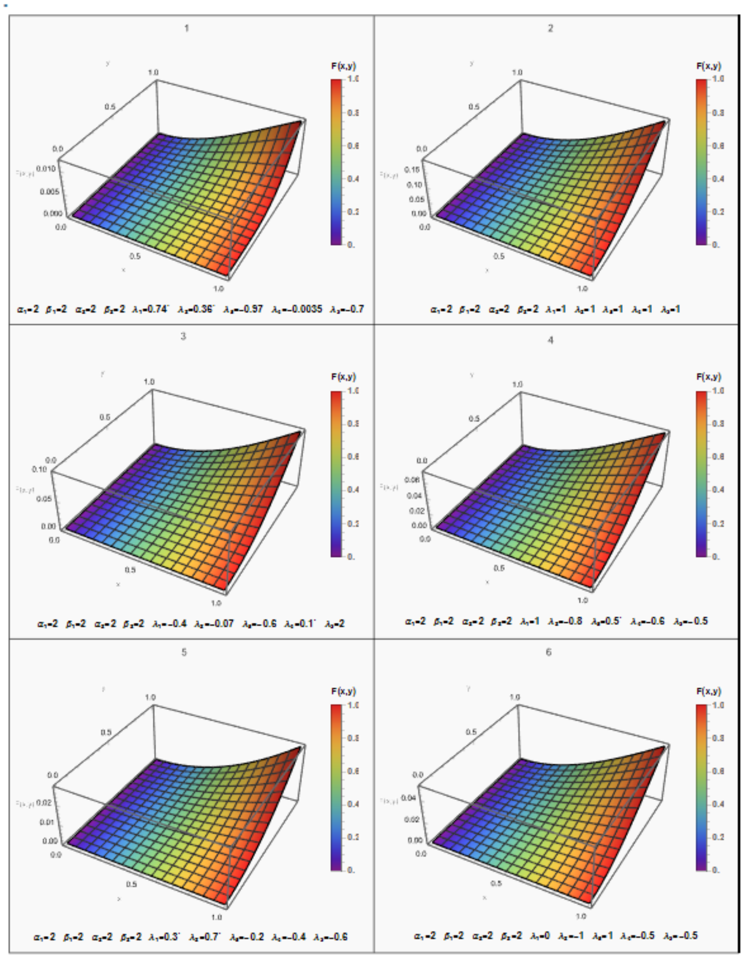

Figure 1: CDF of the BCTWD

Figure 1: The CDF surfaces of the proposed BCTWD are displayed for six different parameter configurations. In all cases, the CDF increases monotonically with respect to both variables, reaching its highest values at the upper-right corners of each panel, which is consistent with the theoretical properties of cumulative functions. The variations in surface curvature across the six plots highlight the influence of the shape and transmutation parameters on the dependence structure and overall geometry of the distribution. These visual patterns demonstrate the flexibility and adaptability of the BCTWD model to capture a wide range of dependence behaviors and joint lifetime relationships.

The joint probability density function (PDF) of the BCTWD is obtained by differentiating its cumulative distribution function CDF in (8) with respect to both variables as:

$$\begin{aligned} f_{BCTW}(x,y) = \frac{\alpha_2\alpha_1}{\beta_1^{\alpha_1}\beta_2^{\alpha_2}} x^{\alpha_1-1} y^{\alpha_2-1} e^{-(\frac{x}{\beta_1})^{\alpha_1}} e^{-(\frac{y}{\beta_2})^{\alpha_2}} \Big[& 1+\lambda_1+\lambda_3+\lambda_5 \\ & + 2(\lambda_2-\lambda_1)(1-e^{-(\frac{x}{\beta_1})^{\alpha_1}}) - 3(\lambda_2+\lambda_5)(1-e^{-(\frac{x}{\beta_1})^{\alpha_1}})^2 \\ & - 2(\lambda_3-\lambda_4)(1-e^{-(\frac{y}{\beta_2})^{\alpha_2}}) - 3(\lambda_4+\lambda_5)(1-e^{-(\frac{y}{\beta_2})^{\alpha_2}})^2 \\ & + 9\lambda_5(1-e^{-(\frac{x}{\beta_1})^{\alpha_1}})^2(1-e^{-(\frac{y}{\beta_2})^{\alpha_2}})^2 \Big]. \end{aligned} \tag{9}$$

For $(x,y)\epsilon(0,\infty)^2$, shape parameters $(\alpha_1,\alpha_2)>0$ and scale parameters $(\beta_1,\beta_2)>0$, $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$ are the transmutation parameters such that $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)\in[-1,1]$ under these conditions:$-2\le\lambda_1+\lambda_3+\lambda_5\le 0$,$-1\le\lambda_2-\lambda_1\le 1$,$-1\le\lambda_2+\lambda_5\le 1$, $-1\le\lambda_3-\lambda_4\le 1$ and $-1\le\lambda_4+\lambda_5\le 1$.

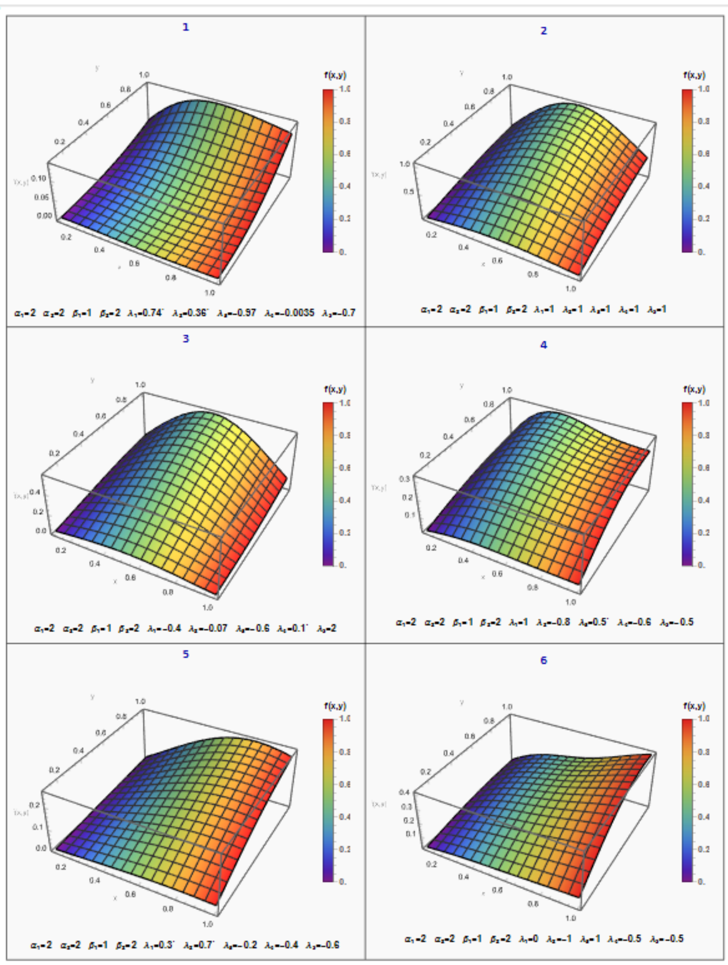

Figure 2: PDF of the BCTWD

Figure 2: PDF surfaces of the proposed distributions are displayed for six distinct parameter configurations. In all cases, the density values are concentrated near the upper-right regions of the surfaces, reflecting higher joint probabilities for larger values of the random variables. The curvature and slope variations across the plots illustrate the sensitivity of the BCTWD to changes in the shape and transmutation parameters, confirming its flexibility in capturing diverse dependence structures. Overall, the plots demonstrate that the BCTWD model provides a versatile framework for modeling bivariate lifetime data with varying degrees of association and tail behavior.

The distribution and density functions of the proposed BCTWD, given in (8) and (9), immediately provide the following special cases:

1. The CDFS and PDFS of the bivariate cubic transmuted exponential (BCTE for short) distribution are obtained by using $\alpha_1 = \alpha_2 = 1$ and are, respectively

$$
\begin{aligned}
F_{BCTE}(x,y) = (1-e^{-(\frac{x}{\beta_1})})(1-e^{-(\frac{y}{\beta_2})})\Big[&1+\lambda_1+\lambda_3+\lambda_5 \\
&+(\lambda_2-\lambda_1)(1-e^{-(\frac{x}{\beta_1})})-(\lambda_2+\lambda_5)(1-e^{-(\frac{x}{\beta_1})})^2 \\
&-(\lambda_3-\lambda_4)(1-e^{-(\frac{y}{\beta_2})})-(\lambda_4+\lambda_5)(1-e^{-(\frac{y}{\beta_2})})^2 \\
&+\lambda_5(1-e^{-(\frac{x}{\beta_1})})^2(1-e^{-(\frac{y}{\beta_2})})^2\Big],
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
f_{BCTE}(x,y) = \frac{1}{\beta_1\beta_2} e^{-(\frac{x}{\beta_1})} e^{-(\frac{y}{\beta_2})} \Big[& 1 + \lambda_1 + \lambda_3 + \lambda_5 \\
& + 2(\lambda_2 - \lambda_1)(1 - e^{-(\frac{x}{\beta_1})}) - 3(\lambda_2 + \lambda_5)(1 - e^{-(\frac{x}{\beta_1})})^2 \\
& - 2(\lambda_3 - \lambda_4)(1 - e^{-(\frac{y}{\beta_2})}) - 3(\lambda_4 + \lambda_5)(1 - e^{-(\frac{y}{\beta_2})})^2 \\
& + 9\lambda_5(1 - e^{-(\frac{x}{\beta_1})})^2(1 - e^{-(\frac{y}{\beta_2})})^2 \Big].
\end{aligned} \tag{11}
$$

2. The CDFs and PDFs of the bivariate cubic transmuted Rayleigh distribution (BCTR for short) are obtained using $\alpha_1 = \alpha_2 = 2$ and are, respectively

$$
\begin{aligned}
F_{BCTR}(x,y) = (1 - e^{-(\frac{x}{\beta_1})^2})(1 - e^{-(\frac{y}{\beta_2})^2}) \big[& 1 + \lambda_1 + \lambda_3 + \lambda_5 \\
& + (\lambda_2 - \lambda_1)(1 - e^{-(\frac{x}{\beta_1})^2}) - (\lambda_2 + \lambda_5)(1 - e^{-(\frac{x}{\beta_1})^2})^2 \\
& - (\lambda_3 - \lambda_4)(1 - e^{-(\frac{y}{\beta_2})^2}) - (\lambda_4 + \lambda_5)(1 - e^{-(\frac{y}{\beta_2})^2})^2 \\
& + \lambda_5(1 - e^{-(\frac{x}{\beta_1})^2})^2(1 - e^{-(\frac{y}{\beta_2})^2})^2 \big],
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
f_{BCTR}(x,y) = \frac{4}{\beta_1^2\beta_2^2} xy e^{-(\frac{x}{\beta_1})^2} e^{-(\frac{y}{\beta_2})^2} \big[& 1 + \lambda_1 + \lambda_3 + \lambda_5 \\
& + 2(\lambda_2 - \lambda_1)(1 - e^{-(\frac{x}{\beta_1})^2}) - 3(\lambda_2 + \lambda_5)(1 - e^{-(\frac{x}{\beta_1})^2})^2 \\
& - 2(\lambda_3 - \lambda_4)(1 - e^{-(\frac{y}{\beta_2})^2}) - 3(\lambda_4 + \lambda_5)(1 - e^{-(\frac{y}{\beta_2})^2})^2 \\
& + 9\lambda_5(1 - e^{-(\frac{x}{\beta_1})^2})^2(1 - e^{-(\frac{y}{\beta_2})^2})^2 \big].
\end{aligned} \tag{13}
$$

***Special cases**
By setting specific values for these parameters, several well-known and extended distributions can be obtained, as shown in the following tables.

Table 1: Distributions Obtained by Parameter Changes (Part 1)

Distribution	Parameters	Description
Bivariate Exponential	$\alpha_1 = \alpha_2 = 1,\ \lambda_i = 0$	Classical bivariate exponential
Bivariate Rayleigh	$\alpha_1 = \alpha_2 = 2,\ \lambda_i = 0$	Classical bivariate Rayleigh
Bivariate Weibull	General $\alpha_1, \alpha_2 > 0,\ \lambda_i = 0$	Standard independent Weibull
BCTW	General $\alpha_1, \alpha_2,\ \lambda_i \neq 0$	Bivariate Cubic Transmuted Weibull
BCTR	$\alpha_1 = \alpha_2 = 2,\ \lambda_i \neq 0$	Cubic Transmuted Rayleigh
BCTE	$\alpha_1 = \alpha_2 = 1,\ \lambda_i \neq 0$	Cubic Transmuted Exponential

Table 2: Additional Derived Distributions (Part 2)

Distribution	Parameters	Athor(s)(Year)
Generalized Exponential	$\alpha_1 = \alpha_2 = 1,$ some $\lambda_i \neq 0$	Gupta and Kundu (1999)
Transmuted Rayleigh	$\alpha_1 = \alpha_2 = 2,$ some $\lambda_i \neq 0$	Merovci (2013)
Independent Weibull	$\lambda_i = 0, f(x,y) = f(x)f(y)$	Darwish et al. (2021)
Independent Exponential	$\alpha_1 = \alpha_2 = 1,\ \lambda_i = 0$	Kotz and Singpurwalla (1999)
Independent Rayleigh	$\alpha_1 = \alpha_2 = 2,\ \lambda_i = 0$	Salo et al. (2006)

Table 3: CDFs of Derived Models

Distribution	Parameters	CDF Expression
Bivariate Exponential	$\alpha_1 = \alpha_2 = 1,\ \lambda_i = 0$	$(1 - e^{-x/\beta_1})(1 - e^{-y/\beta_2})$
Bivariate Rayleigh	$\alpha_1 = \alpha_2 = 2,\ \lambda_i = 0$	$(1 - e^{-(x/\beta_1)^2})(1 - e^{-(y/\beta_2)^2})$
Weibull	General $\alpha_1, \alpha_2,\ \lambda_i = 0$	$(1 - e^{-(x/\beta_1)^{\alpha_1}})(1 - e^{-(y/\beta_2)^{\alpha_2}})$
BCTW	General $\alpha_1, \alpha_2,\ \lambda_i \neq 0$	$F_{\mathrm{BCTW}}(x, y)$
BCTR	$\alpha_1 = \alpha_2 = 2,\ \lambda_i \neq 0$	$F_{\mathrm{BCTW}}(x, y)$
BCTE	$\alpha_1 = \alpha_2 = 1,\ \lambda_i \neq 0$	$F_{\mathrm{BCTW}}(x, y)$
Generalized Exponential	$\alpha_1 = \alpha_2 = 1,\ \lambda_i \neq 0$	Derived from $F_{\mathrm{BCTW}}(x, y)$
Independent Weibull	$\lambda_i = 0$	$F(x, y) = F(x)F(y)$
Independent Exponential	$\alpha_1 = \alpha_2 = 1,\ \lambda_i = 0$	Same as bivariate exponential
Independent Rayleigh	$\alpha_1 = \alpha_2 = 2,\ \lambda_i = 0$	Same as bivariate Rayleigh

3 Statistical Properties

This section contains some useful statistical properties of the BCTWD outside of the special cases. These properties include marginal distributions, conditional distributions, conditional moments, product moments, ratio moments, reliability function, hazard rate function, and random number generation.

3.1 The Marginal Distributions

The marginal CDFs of X and Y for BCTWD are derived using the marginal CDFs of X and Y from the BCT family of distributions proposed by Alsalafi et al. (2025). The marginal CDFs of X and Y are, respectively, given as follows:

$$F_{CTW}(x) = (1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})[1 + \lambda_1 e^{-(\frac{x}{\beta_1})^{\alpha_1}} + \lambda_2 e^{-(\frac{x}{\beta_1})^{\alpha_1}}(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})], \tag{14}$$

where $(\lambda_1, \lambda_2) \in [-1, 1]$ and $-1 \leqslant \lambda_1 + \lambda_2 \leqslant 1$, and

$$F_{CTW}(y) = (1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})[1 + \lambda_3 e^{-(\frac{y}{\beta_2})^{\alpha_2}} + \lambda_4 e^{-(\frac{y}{\beta_2})^{\alpha_2}}(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})], \tag{15}$$

where $(\lambda_3, \lambda_4) \in [-1, 1]$ and $-1 \leqslant \lambda_3 + \lambda_4 \leqslant 1$.
It can be easily seen that both marginal distribution functions are cubic transmuted Weibull distributions (CTWs), proposed by Rahman et al. (2019) where β_1, α_1 correspond to λ, k in Rahman et al. (2019).
The marginal distribution functions for the special distributions provided by the BCTWD is given below.

1. The marginal CDFs of X and Y for the BCTE distribution are obtained by using $\alpha_1 = \alpha_2 = 1$ and are, respectively in (14) and (15), given as

$$F_{CTE}(x) = (1 - e^{-(\frac{x}{\beta_1})})[1 + \lambda_1 e^{-(\frac{x}{\beta_1})} + \lambda_2 e^{-(\frac{x}{\beta_1})}(1 - e^{-(\frac{x}{\beta_1})})], \tag{16}$$

   and

$$F_{CTE}(y) = (1 - e^{-(\frac{y}{\beta_2})})[1 + \lambda_3 e^{-(\frac{y}{\beta_2})} + \lambda_4 e^{-(\frac{y}{\beta_2})}(1 - e^{-(\frac{y}{\beta_2})})]. \tag{17}$$

   It can be easily seen that both marginal distribution functions are cubic transmuted exponential (CTE for short) distributions, proposed by Rahman et al. (2018).

2. The marginal CDFs of X and Y for the BCTR distribution are obtained using $\alpha_1 = \alpha_2 = 2$ and are, respectively, in (14) and (15), given as

$$F_{CTR}(x) = (1 - e^{-(\frac{x}{\beta_1})^2})[1 + \lambda_1 e^{-(\frac{x}{\beta_1})^2} + \lambda_2 e^{-(\frac{x}{\beta_1})^2}(1 - e^{-(\frac{x}{\beta_1})^2})], \tag{18}$$

and

$$F_{CTR}(y) = (1 - e^{-(\frac{y}{\beta_2})^2})[1 + \lambda_3 e^{-(\frac{y}{\beta_2})^2} + \lambda_4 e^{-(\frac{y}{\beta_2})^2}(1 - e^{-(\frac{y}{\beta_2})^2})]. \tag{19}$$

It is easily seen that the marginal distribution functions are cubic transmuted Rayleigh distributions (CTRs), discussed by Rahman (2022)

The marginal density functions of X and Y for the BCTWD are, respectively.

$$f_{CTW}(x) = \frac{\alpha_1}{\beta_1^{\alpha_1}} x^{\alpha_1-1} e^{-(\frac{x}{\beta_1})^{\alpha_1}} [1 - \lambda_1(1 - 2e^{-(\frac{x}{\beta_1})^{\alpha_1}}) - \lambda_2(1 - 4e^{-(\frac{x}{\beta_1})^{\alpha_1}} + 3e^{-2(\frac{x}{\beta_1})^{\alpha_1}})] \tag{20}$$

and

$$f_{CTW}(y) = \frac{\alpha_2}{\beta_2^{\alpha_2}} y^{\alpha_2-1} e^{-(\frac{y}{\beta_2})^{\alpha_2}} [1 - \lambda_3(1 - 2e^{-(\frac{y}{\beta_2})^{\alpha_2}}) - \lambda_4(1 - 4e^{-(\frac{y}{\beta_2})^{\alpha_2}} + 3e^{-2(\frac{y}{\beta_2})^{\alpha_2}})]. \tag{21}$$

The marginal density functions of special cases are easily written from above and are given as:

1. The marginal PDFs of X and Y for the BCTE distribution are obtained using $\alpha_1 = \alpha_2 = 1$ and are, respectively, in (20) and (21), given as:

$$f_{CTE}(x) = \frac{1}{\beta_1} e^{-(\frac{x}{\beta_1})} \left[1 - \lambda_1(1 - 2e^{-(\frac{x}{\beta_1})}) - \lambda_2(1 - 4e^{-(\frac{x}{\beta_1})} + 3e^{-2(\frac{x}{\beta_1})})\right] \tag{22}$$

and

$$f_{CTE}(y) = \frac{1}{\beta_2} e^{-(\frac{y}{\beta_2})} \left[1 - \lambda_3(1 - 2e^{-(\frac{y}{\beta_2})}) - \lambda_4(1 - 4e^{-(\frac{y}{\beta_2})} + 3e^{-2(\frac{y}{\beta_2})})\right]. \tag{23}$$

2. The marginal PDFs of X and Y for the BCTR distribution are obtained using $\alpha_1 = \alpha_2 = 2$ and are, respectively, in (20) and (21), given as:

$$f_{CTR}(x) = \frac{2}{\beta_1^2} x e^{-(\frac{x}{\beta_1})^2} [1 - \lambda_1(1 - 2e^{-(\frac{x}{\beta_1})^2}) - \lambda_2(1 - 4e^{-(\frac{x}{\beta_1})^2} + 3e^{-2(\frac{x}{\beta_1})^2})] \tag{24}$$

and

$$f_{CTR}(y) = \frac{2}{\beta_2^2} y e^{-(\frac{y}{\beta_2})^2} [1 - \lambda_3(1 - 2e^{-(\frac{y}{\beta_2})^2}) - \lambda_4(1 - 4e^{-(\frac{y}{\beta_2})^2} + 3e^{-2(\frac{y}{\beta_2})^2})]. \tag{25}$$

3.2 The Conditional Distributions

Alsalafi et al. (2025) Introduced the conditional distribution of X given Y = y and Y given X=x where $g_1(x)$ and $g_2(y)$ denote the marginal PDFs corresponding to $G_1(x)$ and $G_2(y)$, respectively. For the BCTWD, the CTW densities are given in equations (20) and (21). The bivariate cubic transmuted family of distributions is :

$$f_{X|y}(x, y) = \frac{g_1(x)\delta(\phi)}{[1 + \lambda_3 - 2\lambda_3 G_2(y) + 2\lambda_4 G_2(y) - 3\lambda_4 G_2^2(y)]} \tag{26}$$

Where

$$\begin{aligned}\delta(\phi) = [&1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) + 2(\lambda_4 - \lambda_3)G_2(y) \\ &- 3(\lambda_4 + \lambda_5)G_2^2(y) + 9\lambda_5 G_1^2(x)G_2^2(y)]\end{aligned}$$

Again, the conditional PDF of Y given X = x is obtained by using

$$f_{Y|x}(y, x) = \frac{g_2(y)\delta(\phi)}{[1 + \lambda_1 - 2\lambda_1 G_1(x) + 2\lambda_2 G_1(x) - 3\lambda_2 G_1^2(x)]} \tag{27}$$

Where

$$\delta(\phi) = [1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)G_1(x) - 3(\lambda_2 + \lambda_5)G_1^2(x) + 2(\lambda_4 - \lambda_3)G_2(y) - 3(\lambda_4 + \lambda_5)G_2^2(y) + 9\lambda_5 G_1^2(x)G_2^2(y)]$$

the conditional PDFs of X given $Y = y$ for the BCTWD are obtained by using (9) and (21) in (26). The conditional distribution of X given $Y = y$ is

$$\begin{aligned} f_{BCTW}(x|y) = \frac{\alpha_1}{\Delta_{BCTW}(y)\beta_1^{\alpha_1}} x^{\alpha_1-1} e^{-(\frac{x}{\beta_1})^{\alpha_1}} \big[\eta + 2\delta_2(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}}) \\ - 3\delta_6(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})^2 - 2\delta_4(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}}) \\ - 3\delta_7(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})^2 + 9\lambda_5(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})^2(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})^2\big], \end{aligned} \tag{28}$$

Where $\eta = 1 + \lambda_1 + \lambda_3 + \lambda_5$, $\delta_2 = \lambda_2 - \lambda_1$,$\delta_4 = \lambda_3 - \lambda_4$, $\delta_6 = \lambda_2 + \lambda_5$, $\delta_7 = \lambda_4 + \lambda_5$ and

$$\Delta_{BCTW}(y) = [1 - \lambda_3(1 - 2e^{-(\frac{y}{\beta_2})^{\alpha_2}}) - \lambda_4(1 - 4e^{-(\frac{y}{\beta_2})^{\alpha_2}} + 3e^{-2(\frac{y}{\beta_2})^{\alpha_2}})].$$

The conditional distribution of X given $Y = y$ for the special distributions obtained from the BCTWD is:

1. The conditional distribution of X given $Y = y$ for the BCTE distribution is obtained by using $\alpha_1 = \alpha_2 = 1$ in (28) as:

$$\begin{aligned} f_{BCTE}(x|y) = \frac{1}{\Delta_{BCTE}(y)\beta_1} e^{-(\frac{x}{\beta_1})} \big[\eta + 2\delta_2(1 - e^{-(\frac{x}{\beta_1})}) \\ - 3\delta_6(1 - e^{-(\frac{x}{\beta_1})})^2 - 2\delta_4(1 - e^{-(\frac{y}{\beta_2})}) \\ - 3\delta_7(1 - e^{-(\frac{y}{\beta_2})})^2 + 9\lambda_5(1 - e^{-(\frac{x}{\beta_1})})^2(1 - e^{-(\frac{y}{\beta_2})})^2\big], \end{aligned} \tag{29}$$

Where

$$\Delta_{BCTE}(y) = [1 - \lambda_3(1 - 2e^{-(\frac{y}{\beta_2})}) - \lambda_4(1 - 4e^{-(\frac{y}{\beta_2})} + 3e^{-2(\frac{y}{\beta_2})})].$$

2. The conditional distribution of X given $Y = y$ for the BCTR distribution is obtained by using $\alpha_1 = \alpha_2 = 2$ in (28) and is given as:

$$\begin{aligned} f_{BCTR}(x|y) = \frac{2}{\Delta_{BCTR}(y)\beta_1^2} x e^{-(\frac{x}{\beta_1})^2} \big[\eta + 2\delta_2(1 - e^{-(\frac{x}{\beta_1})^2}) \\ - 3\delta_6(1 - e^{-(\frac{x}{\beta_1})^2})^2 - 2\delta_4(1 - e^{-(\frac{y}{\beta_2})^2}) \\ - 3\delta_7(1 - e^{-(\frac{y}{\beta_2})^2})^2 + 9\lambda_5(1 - e^{-(\frac{x}{\beta_1})^2})^2(1 - e^{-(\frac{y}{\beta_2})^2})^2\big], \end{aligned} \tag{30}$$

Where

$$\Delta_{BCTR}(y) = [1 - \lambda_3(1 - 2e^{-(\frac{y}{\beta_2})^2}) - \lambda_4(1 - 4e^{-(\frac{y}{\beta_2})^2} + 3e^{-2(\frac{y}{\beta_2})^2})].$$

The conditional density function of Y given $X = x$ is easily obtained by using (9) and (20) in (27) and is

$$\begin{aligned} f_{BCTW}(y|x) = \frac{\alpha_2}{\Delta_{BCTW}(x)\beta_2^{\alpha_2}} y^{\alpha_2-1} e^{-(\frac{y}{\beta_2})^{\alpha_2}} \big[\eta + 2\delta_2(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}}) \\ - 3\delta_6(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})^2 - 2\delta_4(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}}) \\ - 3\delta_7(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})^2 + 9\lambda_5(1 - e^{-(\frac{x}{\beta_1})^{\alpha_1}})^2(1 - e^{-(\frac{y}{\beta_2})^{\alpha_2}})^2\big], \end{aligned} \tag{31}$$

Where $\eta = 1 + \lambda_1 + \lambda_3 + \lambda_5$, $\delta_2 = \lambda_2 - \lambda_1$,$\delta_4 = \lambda_3 - \lambda_4$, $\delta_6 = \lambda_2 + \lambda_5$, $\delta_7 = \lambda_4 + \lambda_5$ and

$$\Delta_{BCTW}(x) = [1 - \lambda_1(1 - 2e^{-(\frac{x}{\beta_1})^{\alpha_1}}) - \lambda_2(1 - 4e^{-(\frac{x}{\beta_1})^{\alpha_1}} + 3e^{-2(\frac{x}{\beta_1})^{\alpha_1}})]$$

The conditional distribution of Y given $X = x$ for the special cases arising from the BCTW distribution are:

1. The conditional distribution of Y given $X = x$ for the BCTE distribution is obtained by using $\alpha_1 = \alpha_2 = 1$ in (31) as:

$$\begin{aligned} f_{BCTE}(y|x) = & \frac{1}{\Delta_{BCTE}(x)\beta_2} e^{-(\frac{y}{\beta_2})} \big[\eta + 2\delta_2(1 - e^{-(\frac{x}{\beta_1})}) \\ & - 3\delta_6(1 - e^{-(\frac{x}{\beta_1})})^2 - 2\delta_4(1 - e^{-(\frac{y}{\beta_2})}) \\ & - 3\delta_7(1 - e^{-(\frac{y}{\beta_2})})^2 + 9\lambda_5(1 - e^{-(\frac{x}{\beta_1})})^2(1 - e^{-(\frac{y}{\beta_2})})^2\big], \end{aligned} \tag{32}$$

Where

$$\Delta_{BCTE}(x) = [1 - \lambda_1(1 - 2e^{-(\frac{x}{\beta_1})}) - \lambda_2(1 - 4e^{-(\frac{x}{\beta_1})} + 3e^{-2(\frac{x}{\beta_1})})]$$

2. The conditional distribution of Y given $X = x$ for the BCTR distribution is obtained by using $\alpha_1 = \alpha_2 = 2$ in (31) as:

$$\begin{aligned} f_{BCTR}(y|x) = & \frac{2}{\Delta_{BCTR}(x)\beta_2^2} y e^{-(\frac{y}{\beta_2})^2} \big[\eta + 2\delta_2(1 - e^{-(\frac{x}{\beta_1})^2}) \\ & - 3\delta_6(1 - e^{-(\frac{x}{\beta_1})^2})^2 - 2\delta_4(1 - e^{-(\frac{y}{\beta_2})^2}) \\ & - 3\delta_7(1 - e^{-(\frac{y}{\beta_2})^2})^2 + 9\lambda_5(1 - e^{-(\frac{x}{\beta_1})^2})^2(1 - e^{-(\frac{y}{\beta_2})^2})^2\big], \end{aligned} \tag{33}$$

Where

$$\Delta_{BCTR}(x) = [1 - \lambda_1(1 - 2e^{-(\frac{x}{\beta_1})^2}) - \lambda_2(1 - 4e^{-(\frac{x}{\beta_1})^2} + 3e^{-2(\frac{x}{\beta_1})^2})]$$

The conditional distributions are useful in obtaining the conditional moments of the distribution, which will be obtained in the following.

3.3 Conditional Moments

The moments provide useful information about the location, scale, and shape of a distribution. The conditional moments are useful in studying the behavior of one random variable under certain conditions on the other variable. The conditional moments for the bivariate cubic transmuted family distributions contain moments of the baseline distribution and moments of order statistics for the baseline distribution. The raw moments and moments of order statistics from the Weibull distribution are given in the following result.

Result: The r^{th} raw moment of a Weibull random variable, Z, with shape parameter α and scale parameter β, and its second order statistics in a sample of size 2 are, respectively, given as:

$$\mu_Z^r = E(Z^r) = \beta^r \Gamma(\frac{r}{\alpha} + 1) \tag{34}$$

and

$$\mu_{Z(2:2)}^r = E(Z_{(2:2)}^r) = 2\beta^r \Gamma(\frac{r}{\alpha} + 1)[1 - 2^{-(\frac{r}{\alpha}+1)}]. \tag{35}$$

$$\mu_{Z(3:3)}^r = E(Z_{(3:3)}^r) = 6\beta^r \Gamma(\frac{r}{\alpha} + 1)[1 - 2^{-(\frac{r}{\alpha})} + 3^{-(\frac{r}{\alpha}+1)}]. \tag{36}$$

The conditional moment of X given $Y = y$ is shown in the following theorem.

Theorem 1. *If random variables X and Y have a joint BCTWD, then the r^{th} conditional moment of X given $Y = y$ is*

$$\begin{aligned} \mu_{X|y}^r = E(X^r \mid y) = & \frac{1}{\Psi(y)} \beta_1^r \Gamma\left(\frac{r}{\alpha_1} + 1\right) \Big\{A_0 + A_1 t + A_2 t^2 \\ & + 2(\lambda_2 - \lambda_1)\left(1 - 2^{-\left(\frac{r}{\alpha_1}+1\right)}\right) - 6\left(\lambda_2 - 2\lambda_5 + 6\lambda_5 t - 3\lambda_5 t^2\right)\left(1 - 2^{-\frac{r}{\alpha_1}} + 3^{-\left(\frac{r}{\alpha_1}+1\right)}\right)\Big\}, \end{aligned} \tag{37}$$

where

$$t = e^{-(y/\beta_2)^{\alpha_2}},$$

$$A_0 = 1 + \lambda_1 - \lambda_4 - \lambda_3 - 2\lambda_5,$$

$$A_1 = 4\lambda_4 + 2\lambda_3 + 6\lambda_5,$$

$$A_2 = -3(\lambda_4 + \lambda_5),$$

and

$$\Psi(y) = (1 + \lambda_3) + 2(\lambda_4 - \lambda_3)\big(1 - t\big) - 3\lambda_4\big(1 - 2t + t^2\big).$$

Proof. *The conditional moment r^{th} of X given $Y = y$ when X and Y have the BCT family of distributions introduced by Alsalafi et al. (2025) as:*

$$E(X^r/y) = \frac{1}{\Phi(y)}\Big[\big(\eta + 2(\lambda_4 - \lambda_3)G_2(y) - 3(\lambda_4 + \lambda_5)G_2^2(y)\big)\mu_x^r + (\lambda_2 - \lambda_1)\mu_{x(2:2)}^r - \big(\lambda_2 + \lambda_5 - 3\lambda_5 G_2^2(y)\big)\mu_{x(3:3)}^r\Big],$$

where $\Phi(y) = (1 + \lambda_3) + 2(\lambda_4 - \lambda_3)G_2(y) - 3\lambda_4 G_2^2(y)$, $\eta = 1 + \lambda_1 + \lambda_3 + \lambda_5$, Now, for the BCTWD, the random variable X is $W(\alpha_1, \beta_1)$ and the random variable Y is $W(\alpha_2, \beta_2)$ hence, using (34),(35) and (36)the r^{th} raw moment of X and $r^t h$ raw moment of larger observation in a sample of size 2 and 3 from X are, respectively

$$\mu_X^r = E(X^r) = \beta_1^r \Gamma(\frac{r}{\alpha_1} + 1) \tag{38}$$

and

$$\mu_{X(2:2)}^r = E(X_{(2:2)}^r) = 2\beta_1^r \Gamma(\frac{r}{\alpha_1} + 1)[1 - 2^{-(\frac{r}{\alpha_1}+1)}]. \tag{39}$$

$$\mu_{X(3:3)}^r = E(X_{(3:3)}^r) = 6\beta_1^r \Gamma(\frac{r}{\alpha_1} + 1)[1 - 2^{-(\frac{r}{\alpha_1})} + 3^{-(\frac{r}{\alpha_1}+1)}]. \tag{40}$$

In addition, $G_2(y)$ is given in (7) and hence

4 The Bivariate Reliability and Hazard Rate Functions

The bivariate reliability function of X and Y is obtained using (6) and (7). Following Alsalafi et al. (2025), the reliability function of the BCT family is given by

$$\begin{aligned} R(x, y) = {} & 1 + G_1(x)\big(\eta G_2(y) - \delta_1\big) + \delta_2 G_1^2(x)\big(1 + G_2(y)\big) + G_1^3(x)\big(\lambda_2 - \delta_6 G_2(y)\big) \\ & - \delta_3 G_2(y) + \delta_4 G_2^2(y)\big(1 - G_1(x)\big) + G_2^3(y)\Big(\lambda_4 - \delta_7 G_1(x) + \lambda_5 G_1^3(x)\Big), \end{aligned} \tag{41}$$

In (41), the expression for the bivariate reliability function of BCTWD is given by

$$\begin{aligned} R(x_i, y_i) = {} & 1 + \big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big)\Big(\eta\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big) - \delta_1\Big) \\ & + \delta_2\big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big)^2\big(2 - e^{-(y_i/\beta_2)^{\alpha_2}}\big) \\ & + \big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big)^3\Big(\lambda_2 - \delta_6\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big)\Big) \\ & - \delta_3\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big) + \delta_4\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big)^2 e^{-(x_i/\beta_1)^{\alpha_1}} \\ & + \big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big)^3\Big(\lambda_4 - \delta_7\big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big) + \lambda_5\big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big)^3\Big), \end{aligned} \tag{42}$$

where $\delta_1 = (1+\lambda_1)$,$\delta_2 = (\lambda_2-\lambda_1)$, $\delta_3 = (1+\lambda_3)$, $\delta_4 = (\lambda_3-\lambda_4)$, $\eta = (1+\lambda_1+\lambda_3+\lambda_5)$, $\delta_6 = (\lambda_2+\lambda_5)$, $\delta_7 = (\lambda_4+\lambda_5)$.

The bivariate hazard rate function of X and Y is obtained using the bivariate density function (11) and the bivariate reliability function (42), which for the bivariate hazard rate function of the BCTWD is

$$h(x_i, y_i) = \frac{\alpha_1\alpha_2}{\beta_1^{\alpha_1}\beta_2^{\alpha_2}} x_i^{\alpha_1-1} y_i^{\alpha_2-1} e^{-(x_i/\beta_1)^{\alpha_1}-(y_i/\beta_2)^{\alpha_2}} \frac{B_i}{R(x_i, y_i)}, \tag{43}$$

where

$$\begin{aligned} B_i =\ & 1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)\left(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\right) - 3(\lambda_2 + \lambda_5)\left(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\right)^2 \\ & - 2(\lambda_3 - \lambda_4)\left(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\right) - 3(\lambda_4 + \lambda_5)\left(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\right)^2 \\ & + 9\lambda_5\left(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\right)^2\left(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\right)^2. \end{aligned}$$

The hazard rate function characterizes the instantaneous failure rate in time (x, y) given survival up to that point. In our optical data application, an increasing bivariate hazard rate would indicate that as both eyes age, the risk of simultaneous failure accelerates, consistent with progressive diseases such as diabetic retinopathy. Conversely, a decreasing hazard might suggest early-life defects that stabilize over time.

The plot of the hazard rate function BCTWD for $\beta_1 = 1; \beta_2 = 2; \lambda_1 = 0.74; \lambda_2 = 0.36; \lambda_3 = -0.97; \lambda_4 = -0.0035; \lambda_5 = -0.7$ for various values of α_1 and α_2 are given in Figure 3 below.

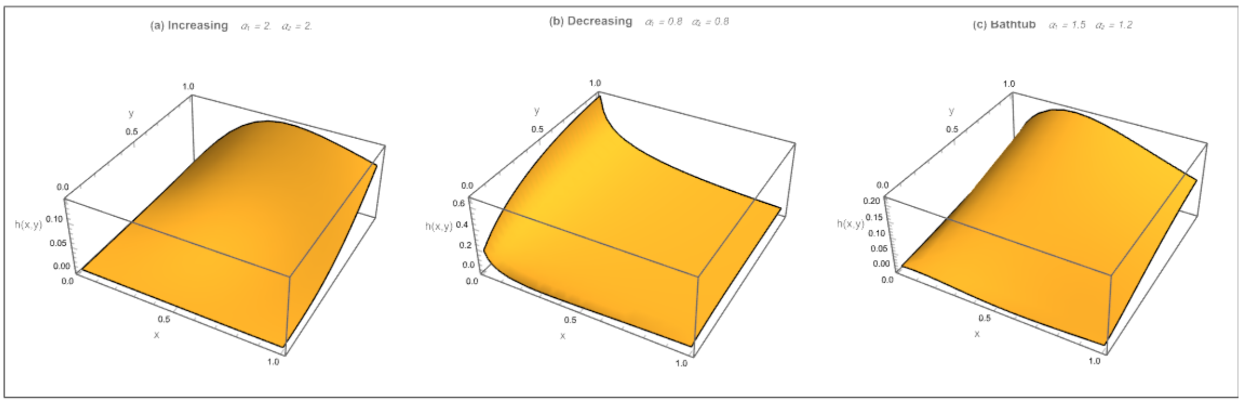

Figure 3: Joint hazard rate function of the BCTWD

The plots of the hazard rate function indicate that BCTWD has both increasing, decreasing, and bathtub-shaped trends.

4.1 The Mean Residual Life (MRL) Function

The mean residual life (MRL) function provides an important reliability measure that quantifies the expected additional lifetime of a system or component, given that it has survived up to a certain point. For the bivariate case, the MRL of X given that both $X > x$ and $Y > y$ is defined as:

$$MRL(x, y) = E[X - x \mid X > x, Y > y] = \frac{\int_x^\infty R(t, y)\, dt}{R(x, y)}, \tag{44}$$

where $R(x, y)$ denotes the joint survival function of the random variables X and Y. The numerator $\int_x^\infty R(t, y)\, dt$ represents the area under the survival curve beyond the point x, indicating the cumulative probability of survival beyond x, while the denominator $R(x, y)$ corresponds to the conditional survival probability at the point (x, y).
For the BCTWD, this requires numerical integration, given the complexity of R(x,y) in equation (42).

5 Estimation of Parameters

In this section, we introduce different estimation methods used to estimate the parameters of the BCTWD, such as the maximum likelihood estimation (MLE) and inference functions for margins (IFM).

5.1 Maximum Likelihood Estimation (MLE)

In this section, the parameters of the BCTWD are estimated using the Maximum Likelihood Estimation (MLE) method. Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be a random sample of size n drawn from the BCTWD. The likelihood function is given by:

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f_{\text{BCTW}}(x_i, y_i), \tag{45}$$

where $\boldsymbol{\theta} = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ denotes the vector of unknown parameters and $f_{\text{BCTW}}(x, y)$ is the joint probability density function of the BCTWD, given in (9). The likelihood function is, therefore,

$$\begin{aligned} L = {} & \frac{\alpha_2\alpha_1}{\beta_1^{\alpha_1}\beta_2^{\alpha_2}}\Big(\prod_{i=1}^{n} x_i^{\alpha_1-1} y_i^{\alpha_2-1}\Big) e^{-\sum_{i=1}^{n}(\frac{x_i}{\beta_1})^{\alpha_1}} e^{-\sum_{i=1}^{n}(\frac{y_i}{\beta_2})^{\alpha_2}} \times \prod_{i=1}^{n}\Bigg[1 + \lambda_1 + \lambda_3 + \lambda_5 \\ & + 2(\lambda_2 - \lambda_1)(1 - e^{-(\frac{x_i}{\beta_1})^{\alpha_1}}) - 3(\lambda_2 + \lambda_5)(1 - e^{-(\frac{x_i}{\beta_1})^{\alpha_1}})^2 \\ & - 2(\lambda_3 - \lambda_4)(1 - e^{-(\frac{y_i}{\beta_2})^{\alpha_2}}) - 3(\lambda_4 + \lambda_5)(1 - e^{-(\frac{y_i}{\beta_2})^{\alpha_2}})^2 \\ & + 9\lambda_5(1 - e^{-(\frac{x_i}{\beta_1})^{\alpha_1}})^2(1 - e^{-(\frac{y_i}{\beta_2})^{\alpha_2}})^2\Bigg], \end{aligned} \tag{46}$$

The log-likelihood function

$$\begin{aligned} \ell = {} & n(\ln\alpha_1 + \ln\alpha_2 - \alpha_1\ln\beta_1 - \alpha_2\ln\beta_2) + (\alpha_1 - 1)\sum_{i=1}^{n}\ln x_i + (\alpha_2 - 1)\sum_{i=1}^{n}\ln y_i \\ & - \sum_{i=1}^{n}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1} - \sum_{i=1}^{n}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2} + \sum_{i=1}^{n}\ln\Bigg[1 + \lambda_1 + \lambda_3 + \lambda_5 \\ & \quad + 2(\lambda_2 - \lambda_1)\Big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\Big) - 3(\lambda_2 + \lambda_5)\Big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\Big)^2 \\ & \quad - 2(\lambda_3 - \lambda_4)\Big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\Big) - 3(\lambda_4 + \lambda_5)\Big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\Big)^2 \\ & \quad + 9\lambda_5\Big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\Big)^2\Big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\Big)^2\Bigg]. \end{aligned} \tag{47}$$

The parameters of the proposed BCTWD are estimated using MLE. The log-likelihood function given in (47) is maximized numerically using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) quasi-Newton optimization algorithm implemented in the maxLik and bbmle packages in R Henningsen and Toomet (2011). Convergence is declared when the relative change in log-likelihood is less than 10^{-8} and the gradient norm is below 10^{-6}, or when the maximum number of iterations (1000) is reached.
Initial values are obtained as follows:
(i) $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are set to marginal MLE estimates from fitting the univariate Weibull distributions to X and Y separately;
(ii) are similarly obtained from marginal fits;
(iii) $\hat{\beta}_1$ and $\hat{\beta}_2$ and all λ parameters are initialized at zero, representing independence. Sensitivity analysis confirms that the estimates are robust to alternative starting values.

The MLEs of $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are obtained by maximizing the log-likelihood function (47). The derivatives with respect to unknown parameters are given by

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha_1} &= \frac{n}{\alpha_1} - n\ln\beta_1 + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\ln\left(\frac{x_i}{\beta_1}\right)\\ &+ \sum_{i=1}^{n}\frac{1}{B_i}\Bigg\{2(\lambda_2-\lambda_1)e^{-(x_i/\beta_1)^{\alpha_1}}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\ln\left(\frac{x_i}{\beta_1}\right)\\ &- 6(\lambda_2+\lambda_5)(1-e^{-(x_i/\beta_1)^{\alpha_1}})e^{-(x_i/\beta_1)^{\alpha_1}}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\ln\left(\frac{x_i}{\beta_1}\right)\\ &+ 18\lambda_5(1-e^{-(x_i/\beta_1)^{\alpha_1}})^2(1-e^{-(y_i/\beta_2)^{\alpha_2}})^2e^{-(x_i/\beta_1)^{\alpha_1}}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\ln\left(\frac{x_i}{\beta_1}\right)\Bigg\},\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha_2} &= \frac{n}{\alpha_2} - n\ln\beta_2 + \sum_{i=1}^{n}\ln y_i - \sum_{i=1}^{n}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\ln\left(\frac{y_i}{\beta_2}\right)\\ &+ \sum_{i=1}^{n}\frac{1}{B_i}\Bigg\{-2(\lambda_3-\lambda_4)e^{-(y_i/\beta_2)^{\alpha_2}}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\ln\left(\frac{y_i}{\beta_2}\right)\\ &- 6(\lambda_4+\lambda_5)(1-e^{-(y_i/\beta_2)^{\alpha_2}})e^{-(y_i/\beta_2)^{\alpha_2}}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\ln\left(\frac{y_i}{\beta_2}\right)\\ &+ 18\lambda_5(1-e^{-(x_i/\beta_1)^{\alpha_1}})^2(1-e^{-(y_i/\beta_2)^{\alpha_2}})^2\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\ln\left(\frac{y_i}{\beta_2}\right)\Bigg\},\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta_1} &= -\frac{n\alpha_1}{\beta_1} - \sum_{i=1}^{n}\frac{\alpha_1}{\beta_1}x_i\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\\ &+ \sum_{i=1}^{n}\frac{1}{B_i}\Bigg[-2(\lambda_2-\lambda_1)e^{-(x_i/\beta_1)^{\alpha_1}}\frac{\alpha_1}{\beta_1}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\\ &+ 6(\lambda_2+\lambda_5)(1-e^{-(x_i/\beta_1)^{\alpha_1}})e^{-(x_i/\beta_1)^{\alpha_1}}\frac{\alpha_1}{\beta_1}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\\ &- 18\lambda_5(1-e^{-(x_i/\beta_1)^{\alpha_1}})^2(1-e^{-(y_i/\beta_2)^{\alpha_2}})^2e^{-(x_i/\beta_1)^{\alpha_1}}\frac{\alpha_1}{\beta_1}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\Bigg],\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta_2} &= -\frac{n\alpha_2}{\beta_2} - \sum_{i=1}^{n}\frac{\alpha_2}{\beta_2}y_i\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\\ &+ \sum_{i=1}^{n}\frac{1}{B_i}\Bigg[-2(\lambda_3-\lambda_4)e^{-(y_i/\beta_2)^{\alpha_2}}\frac{\alpha_2}{\beta_2}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\\ &+ 6(\lambda_4+\lambda_5)(1-e^{-(y_i/\beta_2)^{\alpha_2}})e^{-(y_i/\beta_2)^{\alpha_2}}\frac{\alpha_2}{\beta_2}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\\ &- 18\lambda_5(1-e^{-(x_i/\beta_1)^{\alpha_1}})^2(1-e^{-(y_i/\beta_2)^{\alpha_2}})^2e^{-(y_i/\beta_2)^{\alpha_2}}\frac{\alpha_2}{\beta_2}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\Bigg],\end{aligned} \tag{51}$$

$$\frac{\partial \ell}{\partial \lambda_1} = \sum_{i=1}^{n}\frac{1-2(1-e^{-(x_i/\beta_1)^{\alpha_1}})}{B_i}, \tag{52}$$

$$\frac{\partial \ell}{\partial \lambda_2} = \sum_{i=1}^{n}\frac{2(1-e^{-(x_i/\beta_1)^{\alpha_1}})-3(1-e^{-(x_i/\beta_1)^{\alpha_1}})^2}{B_i}, \tag{53}$$

$$\frac{\partial \ell}{\partial \lambda_3} = \sum_{i=1}^{n} \frac{1 - 2(1 - e^{-(y_i/\beta_2)^{\alpha_2}})}{B_i}, \tag{54}$$

$$\frac{\partial \ell}{\partial \lambda_4} = \sum_{i=1}^{n} \frac{-2(1 - e^{-(y_i/\beta_2)^{\alpha_2}}) - 3(1 - e^{-(y_i/\beta_2)^{\alpha_2}})^2}{B_i}, \tag{55}$$

$$\frac{\partial \ell}{\partial \lambda_5} = \sum_{i=1}^{n} \frac{1 - 3(1 - e^{-(x_i/\beta_1)^{\alpha_1}})^2 - 3(1 - e^{-(y_i/\beta_2)^{\alpha_2}})^2 + 9(1 - e^{-(x_i/\beta_1)^{\alpha_1}})^2(1 - e^{-(y_i/\beta_2)^{\alpha_2}})^2}{B_i}, \tag{56}$$

where

$$\begin{aligned} B_i = {} & 1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)(1 - e^{-(x_i/\beta_1)^{\alpha_1}}) - 3(\lambda_2 + \lambda_5)(1 - e^{-(x_i/\beta_1)^{\alpha_1}})^2 \\ & - 2(\lambda_3 - \lambda_4)(1 - e^{-(y_i/\beta_2)^{\alpha_2}}) - 3(\lambda_4 + \lambda_5)(1 - e^{-(y_i/\beta_2)^{\alpha_2}})^2 \\ & + 9\lambda_5(1 - e^{-(x_i/\beta_1)^{\alpha_1}})^2(1 - e^{-(y_i/\beta_2)^{\alpha_2}})^2. \end{aligned}$$

The maximum likelihood estimator of the parameters $\boldsymbol{\theta} = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is obtained by equating the above derivatives to zero and numerically solving the resulting equations. It is a well-known fact that $n \to \infty$, the asymptotic distribution of the MLE $(\hat{\alpha_1}, \hat{\alpha_2}, \hat{\beta_1}, \hat{\beta_2}, \hat{\lambda_1}, \hat{\lambda_2}, \hat{\lambda_3}, \hat{\lambda_4}, \hat{\lambda_5})$ is given by

$$\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \\ \hat{\lambda}_4 \\ \hat{\lambda}_5 \end{pmatrix} \sim N\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & \dots & V_{19} \\ V_{21} & V_{22} & \dots & V_{29} \\ \vdots & \vdots & \ddots & \vdots \\ V_{91} & V_{92} & \dots & V_{99} \end{pmatrix} \right)$$

$$V^{-1} = -E \begin{bmatrix}
\frac{\partial^2 \ell}{\partial \alpha_1^2} & \frac{\partial^2 \ell}{\partial \alpha_1 \partial \alpha_2} & \frac{\partial^2 \ell}{\partial \alpha_1 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \alpha_1 \partial \beta_2} & \frac{\partial^2 \ell}{\partial \alpha_1 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \alpha_1 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \alpha_1 \partial \lambda_3} & \frac{\partial^2 \ell}{\partial \alpha_1 \partial \lambda_4} & \frac{\partial^2 \ell}{\partial \alpha_1 \partial \lambda_5} \\
\frac{\partial^2 \ell}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 \ell}{\partial \alpha_2^2} & \frac{\partial^2 \ell}{\partial \alpha_2 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \alpha_2 \partial \beta_2} & \frac{\partial^2 \ell}{\partial \alpha_2 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \alpha_2 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \alpha_2 \partial \lambda_3} & \frac{\partial^2 \ell}{\partial \alpha_2 \partial \lambda_4} & \frac{\partial^2 \ell}{\partial \alpha_2 \partial \lambda_5} \\
\frac{\partial^2 \ell}{\partial \beta_1 \partial \alpha_1} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \alpha_2} & \frac{\partial^2 \ell}{\partial \beta_1^2} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_2} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \lambda_3} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \lambda_4} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \lambda_5} \\
\frac{\partial^2 \ell}{\partial \beta_2 \partial \alpha_1} & \frac{\partial^2 \ell}{\partial \beta_2 \partial \alpha_2} & \frac{\partial^2 \ell}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \beta_2^2} & \frac{\partial^2 \ell}{\partial \beta_2 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \beta_2 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \beta_2 \partial \lambda_3} & \frac{\partial^2 \ell}{\partial \beta_2 \partial \lambda_4} & \frac{\partial^2 \ell}{\partial \beta_2 \partial \lambda_5} \\
\frac{\partial^2 \ell}{\partial \lambda_1 \partial \alpha_1} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \alpha_2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \beta_2} & \frac{\partial^2 \ell}{\partial \lambda_1^2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_3} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_4} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_5} \\
\frac{\partial^2 \ell}{\partial \lambda_2 \partial \alpha_1} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \alpha_2} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \beta_2} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_2^2} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_3} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_4} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_5} \\
\frac{\partial^2 \ell}{\partial \lambda_3 \partial \alpha_1} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \alpha_2} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \beta_2} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_3^2} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_4} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_5} \\
\frac{\partial^2 \ell}{\partial \lambda_4 \partial \alpha_1} & \frac{\partial^2 \ell}{\partial \lambda_4 \partial \alpha_2} & \frac{\partial^2 \ell}{\partial \lambda_4 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \lambda_4 \partial \beta_2} & \frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_3} & \frac{\partial^2 \ell}{\partial \lambda_4^2} & \frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_5} \\
\frac{\partial^2 \ell}{\partial \lambda_5 \partial \alpha_1} & \frac{\partial^2 \ell}{\partial \lambda_5 \partial \alpha_2} & \frac{\partial^2 \ell}{\partial \lambda_5 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \lambda_5 \partial \beta_2} & \frac{\partial^2 \ell}{\partial \lambda_5 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_5 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_5 \partial \lambda_3} & \frac{\partial^2 \ell}{\partial \lambda_5 \partial \lambda_4} & \frac{\partial^2 \ell}{\partial \lambda_5^2}
\end{bmatrix},$$

which is a Fisher information matrix. The entries V^{-1} are given in the Appendix.
An approximate 95% two-sided confidence intervals for $(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ are respectively given by

$$\hat{\theta}_j \pm Z_{0.025}\sqrt{V_{jj}}, \quad j = 1, \dots, 9,$$

Here, V is the estimated variance-covariance matrix of $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4, \hat{\lambda}_5)$.

5.2 Inference Functions for Marginals (IFM)

The IFM method offers computational advantages over full MLE. By estimating marginal parameters separately in the first stage, IFM reduces the dimensionality of the optimization problem from 9 parameters to 4 marginal parameters

plus 5 dependence parameters in two stages. This two-stage approach typically converges more reliably, particularly when sample sizes are small or dependence is weak. Additionally, IFM permits use of any consistent estimators for marginals. Joe and Xu (1996) examined a computationally intensive approach, highlighting that the choice of marginals can affect the estimation of copula parameters. To address this, Joe (2005) proposed a parametric two-step estimation method. In the first step, each marginal distribution is estimated separately.
The marginal density functions of X and Y for BCTWD are in (20) and (21), respectively. The log-likelihood for a sample $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are:

$$\begin{aligned}\ell_X(\alpha_1,\beta_1) &= \sum_{i=1}^n \ln f_{CTW}(x_i;\alpha_1,\beta_1,\lambda_1,\lambda_2)\\ &= n\ln(\alpha_1) - n\alpha_1\ln(\beta_1) + (\alpha_1-1)\sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\\ &+ \sum_{i=1}^n \ln\left[1-\lambda_1\left(1-2e^{-(\frac{x_i}{\beta_1})^{\alpha_1}}\right)-\lambda_2\left(1-4e^{-(\frac{x_i}{\beta_1})^{\alpha_1}}+3e^{-2(\frac{x_i}{\beta_1})^{\alpha_1}}\right)\right].\end{aligned}$$

and

$$\begin{aligned}\ell_Y(\alpha_2,\beta_2) &= \sum_{i=1}^n \ln f_{CTW}(y_i;\alpha_2,\beta_2,\lambda_2,\lambda_3)\\ &= n\ln(\alpha_2) - n\alpha_2\ln(\beta_2) + (\alpha_2-1)\sum_{i=1}^n \ln(y_i) - \sum_{i=1}^n\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\\ &+ \sum_{i=1}^n \ln\left[1-\lambda_3\left(1-2e^{-\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}}\right)-\lambda_4\left(1-4e^{-\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}}+3e^{-2\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}}\right)\right].\end{aligned}$$

We now present the partial derivatives of the log-density with respect to parameters:

$$\begin{aligned}\frac{\partial\ell_X}{\partial\alpha_1} &= \frac{n}{\alpha_1} - n\ln\beta_1 + \sum_{i=1}^n \ln x_i - \sum_{i=1}^n t_i\, r_i\\ &+ \sum_{i=1}^n \frac{1}{A_i}\left[-\lambda_1\Big(-2\cdot\big(-e^{-t_i}\, t_i r_i\big)\Big) - \lambda_2\Big(-4\cdot\big(-e^{-t_i}\, t_i r_i\big) + 3\cdot\big(-2e^{-2t_i}\, t_i r_i\big)\Big)\right],\end{aligned}$$

$$\frac{\partial\ell_X}{\partial\beta_1} = -\frac{n\alpha_1}{\beta_1} + \sum_{i=1}^n \frac{\alpha_1}{\beta_1}t_i + \sum_{i=1}^n \frac{1}{A_i}\left[-\lambda_1\Big(-2\,e^{-t_i}\,\frac{\alpha_1}{\beta_1}t_i\Big) - \lambda_2\Big(-4\,e^{-t_i}\,\frac{\alpha_1}{\beta_1}t_i + 6\,e^{-2t_i}\,\frac{\alpha_1}{\beta_1}t_i\Big)\right],$$

$$\frac{\partial\ell_X}{\partial\lambda_1} = -\sum_{i=1}^n \frac{(1-2e^{-t_i})}{A_i},$$

$$\frac{\partial\ell_X}{\partial\lambda_2} = -\sum_{i=1}^n \frac{(1-4e^{-t_i}+3e^{-2t_i})}{A_i}.$$

Where

$$t_i = \left(\frac{x_i}{\beta_1}\right)^{\alpha_1}, \qquad r_i = \ln\left(\frac{x_i}{\beta_1}\right),$$

$$A_i = 1-\lambda_1\big(1-2e^{-t_i}\big) - \lambda_2\big(1-4e^{-t_i}+3e^{-2t_i}\big).$$

And

$$\frac{\partial\ell_Y}{\partial\alpha_2} = \frac{n}{\alpha_2} - n\ln(\beta_2) + \sum_{i=1}^n \ln(y_i) - \sum_{i=1}^n u_i s_i - \sum_{i=1}^n \frac{1}{W_i}\Big[\lambda_3\Big(-2e^{-u_i}\,u_i s_i\Big) + \lambda_4\Big(-4e^{-u_i}\,u_i s_i + 6e^{-2u_i}\,u_i s_i\Big)\Big],$$

$$\frac{\partial \ell_Y}{\partial \beta_2} = -\frac{n\alpha_2}{\beta_2} + \sum_{i=1}^{n} \frac{\alpha_2}{\beta_2} u_i - \sum_{i=1}^{n} \frac{1}{W_i}\Big[\lambda_3\Big(-2e^{-u_i}\,\frac{\alpha_2}{\beta_2}u_i\Big) + \lambda_4\Big(-4e^{-u_i}\,\frac{\alpha_2}{\beta_2}u_i + 6e^{-2u_i}\,\frac{\alpha_2}{\beta_2}u_i\Big)\Big],$$

$$\frac{\partial \ell_Y}{\partial \lambda_3} = -\sum_{i=1}^{n} \frac{1-2e^{-u_i}}{W_i},$$

$$\frac{\partial \ell_Y}{\partial \lambda_4} = -\sum_{i=1}^{n} \frac{1-4e^{-u_i}+3e^{-2u_i}}{W_i}.$$

Where

$$u_i = \left(\frac{y_i}{\beta_2}\right)^{\alpha_2}, \qquad s_i = \ln\left(\frac{y_i}{\beta_2}\right),$$

$$W_i = 1 - \lambda_3(1-2e^{-u_i}) - \lambda_4(1-4e^{-u_i}+3e^{-2u_i}).$$

Then, in the second step the dependence parameter is estimated by maximizing the log-likelihood function of the copula density using the ML estimates of the marginal $\hat{F}_{CTW}(x_i;\alpha_1,\beta_1,\lambda_1,\lambda_2)$, $\hat{F}_{CTW}(y_i;\alpha_2,\beta_2,\lambda_2,\lambda_3)$
. Considering the (9), the log-likelihood function of a Weibull distribution is defined as:

$$\begin{aligned}\ell(\theta) &= n(\ln\alpha_1 + \ln\alpha_2) - n(\alpha_1\ln\beta_1 + \alpha_2\ln\beta_2)\\ &\quad + (\alpha_1-1)\sum_{i=1}^{n}\ln x_i + (\alpha_2-1)\sum_{i=1}^{n}\ln y_i\\ &\quad - \sum_{i=1}^{n}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1} - \sum_{i=1}^{n}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}.\end{aligned}$$

The MLEs $(\alpha_1,\alpha_2,\beta_1,\beta_2)$can be obtained by solving simultaneously the likelihood equations. Considering the previous step, the IFM estimate of a BCTWD is defined as:

$$\begin{aligned}\ell_{\text{dep}}(\lambda) = \sum_{i=1}^{n}\ln\Bigg\{&1+\lambda_1+\lambda_3+\lambda_5\\ &+2(\lambda_2-\lambda_1)\Big(1-e^{-(x_i/\beta_1)^{\alpha_1}}\Big)\\ &-3(\lambda_2+\lambda_5)\Big(1-e^{-(x_i/\beta_1)^{\alpha_1}}\Big)^2\\ &-2(\lambda_3-\lambda_4)\Big(1-e^{-(y_i/\beta_2)^{\alpha_2}}\Big)\\ &-3(\lambda_4+\lambda_5)\Big(1-e^{-(y_i/\beta_2)^{\alpha_2}}\Big)^2\\ &+9\lambda_5\Big(1-e^{-(x_i/\beta_1)^{\alpha_1}}\Big)^2\Big(1-e^{-(y_i/\beta_2)^{\alpha_2}}\Big)^2\Bigg\}.\end{aligned}$$

The derivatives with respect to unknown parameters are given by

$$\frac{\partial \ell_{\text{dep}}}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{1 - 2\big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big)}{B_i},$$

$$\frac{\partial \ell_{\text{dep}}}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{2\big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big) - 3\big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big)^2}{B_i},$$

$$\frac{\partial \ell_{\text{dep}}}{\partial \lambda_3} = \sum_{i=1}^{n} \frac{1 - 2\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big)}{B_i},$$

$$\frac{\partial \ell_{\text{dep}}}{\partial \lambda_4} = \sum_{i=1}^{n} \frac{2\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big) - 3\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big)^2}{B_i},$$

$$\frac{\partial \ell_{\text{dep}}}{\partial \lambda_5} = \sum_{i=1}^{n} \frac{1 - 3\big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big)^2 - 3\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big)^2 + 9\big(1 - e^{-(x_i/\beta_1)^{\alpha_1}}\big)^2\big(1 - e^{-(y_i/\beta_2)^{\alpha_2}}\big)^2}{B_i}.$$

where

$$\begin{aligned} B_i = {} & 1 + \lambda_1 + \lambda_3 + \lambda_5 + 2(\lambda_2 - \lambda_1)(1 - e^{-(x_i/\beta_1)^{\alpha_1}}) - 3(\lambda_2 + \lambda_5)(1 - e^{-(x_i/\beta_1)^{\alpha_1}})^2 \\ & - 2(\lambda_3 - \lambda_4)(1 - e^{-(y_i/\beta_2)^{\alpha_2}}) - 3(\lambda_4 + \lambda_5)(1 - e^{-(y_i/\beta_2)^{\alpha_2}})^2 \\ & + 9\lambda_5(1 - e^{-(x_i/\beta_1)^{\alpha_1}})^2(1 - e^{-(y_i/\beta_2)^{\alpha_2}})^2. \end{aligned}$$

There is no closed-form expression for the MLE, and its computation has to be performed numerically using a nonlinear optimization algorithm.

6 Simulation Study

In this section, a simulation study is conducted to evaluate the performance of the MLE procedure. Furthermore, BCTWD has been applied to real-life data sets to investigate its applicability.

6.1 Simulation Scenarios

To assess the performance of the parameter estimates, we consider five simulation scenarios. Each scenario is defined by a distinct set of true parameter values $(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$. The corresponding values are presented in Table 4.

Table 4: True parameter values under each simulation scenario

Scenario	α_1	α_2	β_1	β_2	λ_1	λ_2	λ_3	λ_4	λ_5
Baseline	2.0	3.0	1.0	1.5	0.05	0.03	0.02	-0.04	-0.08
Large Variation	1.2	4.5	0.8	2.5	0.01	0.02	-0.03	0.01	0.005
Heavy Tail	0.8	1.0	1.0	1.0	0.10	0.05	0.10	-0.10	-0.25
Near Independence	2.5	2.5	1.2	1.2	0.00	0.00	0.00	0.00	0.00
Strong Dependence	3.0	2.0	1.5	2.0	0.20	0.15	-0.10	-0.15	-0.10

Note: Parameter values are chosen to represent realistic scenarios: Baseline reflects moderate hazard rates and weak dependence typical in biomedical data; Large Variation captures heterogeneous populations with differing frailties; Heavy Tail represents distributions with substantial early failures ($\alpha_i \leqslant 1$); Near Independence tests estimator performance when ($\lambda_i \approx 0$); Strong Dependence examines highly correlated failure times.

6.2 Performance Evaluation Metrices

To evaluate estimator performance, we compute bias and mean squared error (MSE) as follows:

$$Bias(\hat{\Theta}) = \bar{\hat{\Theta}} - \Theta,$$

$$MSE(\hat{\Theta}) = Var(\hat{\Theta}) + (Bias(\hat{\Theta}))^2,$$

where $\Theta = (a_1, a_2, b_1, b_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, $\hat{\Theta} = (\hat{a_1}, \hat{a_2}, \hat{b_1}, \hat{b_2}, \hat{\lambda_1}, \hat{\lambda_2}, \hat{\lambda_3}, \hat{\lambda_4}, \hat{\lambda_5})$ and $\bar{\hat{\Theta}}$ is the mean of Θ. We consider different sample sizes: $n = 20, 50, 100$ and 200 set the number of Monte Carlo replications at $m = 1000$.

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Table 5: Simulation Results for Baseline Parameters for BCTWD using (MLE) method

	n=20			n=50			n=100			n=200		
Parameters	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE
a1	2.1306	0.1306	0.5011	2.0263	0.0263	0.3821	1.9678	-0.0322	0.3357	1.9726	-0.0274	0.2922
a2	3.1752	0.1752	0.7270	3.0289	0.0289	0.5549	2.9548	-0.0452	0.5133	2.9472	-0.0528	0.4589
b1	0.9979	-0.0021	0.2237	1.0303	0.0303	0.2325	0.9979	-0.0021	0.2095	1.0021	0.0021	0.1809
b2	1.4790	-0.0210	0.2205	1.5039	0.0039	0.2231	1.4846	-0.0154	0.2152	1.4916	-0.0084	0.1975
λ_1	0.0978	0.0478	0.7028	0.1614	0.1114	0.6706	0.0793	0.0293	0.6057	0.0861	0.0361	0.5623
λ_2	-0.1395	-0.1695	0.7089	-0.2123	-0.2423	0.6560	-0.2509	-0.2809	0.6084	-0.2176	-0.2476	0.5561
λ_3	0.0743	0.0543	0.7087	0.1371	0.1171	0.6524	0.0805	0.0605	0.6378	0.0941	0.0741	0.5831
λ_4	-0.1927	-0.1527	0.7267	-0.2765	-0.2365	0.6625	-0.2921	-0.2521	0.6144	-0.2667	-0.2267	0.5385
λ_5	-0.0314	0.0486	0.6045	-0.0097	0.0703	0.3880	0.0086	0.0886	0.2346	0.0038	0.0838	0.1604

Table 6: Simulation Results for Large Variation Parameters for BCTWD using (MLE) method

	n=20			n=50			n=100			n=200		
Parameters	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE
a1	1.2911	0.0911	0.3038	1.2099	0.0099	0.2415	1.1736	-0.0264	0.2082	1.1762	-0.0238	0.1796
a2	4.6672	0.1672	1.0216	4.4344	-0.0656	0.8224	4.4234	-0.0766	0.7504	4.4008	-0.0992	0.6937
b1	0.8245	0.0245	0.2948	0.8335	0.0335	0.3087	0.8097	0.0097	0.2842	0.8130	0.0130	0.2499
b2	2.4600	-0.0400	0.2634	2.4707	-0.0293	0.2588	2.4732	-0.0268	0.2393	2.4858	-0.0142	0.2241
λ_1	0.1478	0.1378	0.6818	0.1323	0.1223	0.6707	0.0515	0.0415	0.6436	0.0705	0.0605	0.5702
λ_2	-0.1885	-0.2085	0.7205	-0.2960	-0.3160	0.6601	-0.2402	-0.2602	0.6335	-0.2212	-0.2412	0.5435
λ_3	0.0024	0.0324	0.7273	0.0452	0.0752	0.6839	0.0500	0.0800	0.6401	0.0719	0.1019	0.5923
λ_4	-0.1862	-0.1962	0.7366	-0.2456	-0.2556	0.6599	-0.2673	-0.2773	0.6184	-0.2353	-0.2453	0.5218
λ_5	0.0104	0.0054	0.5727	0.0214	0.0164	0.3779	0.0113	0.0063	0.2375	0.0033	-0.0017	0.1611

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Table 7: Simulation Results for Heavy Tail Parameters for BCTWD using (MLE) method

	n=20			n=50			n=100			n=200		
Parameters	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE
a1	0.8643	0.0643	0.2133	0.8159	0.0159	0.1547	0.7839	-0.0161	0.1405	0.7740	-0.0260	0.1232
a2	1.0684	0.0684	0.2577	1.0170	0.0170	0.1910	0.9769	-0.0231	0.1757	0.9798	-0.0202	0.1503
b1	1.1199	0.1199	0.5654	1.1201	0.1201	0.5505	1.0540	0.0540	0.5362	1.0362	0.0362	0.4831
b2	1.0424	0.0424	0.4374	1.0623	0.0623	0.4419	1.0351	0.0351	0.4426	1.0139	0.0139	0.3810
λ_1	0.1852	0.0852	0.6783	0.1581	0.0581	0.6522	0.0668	-0.0332	0.6344	0.0254	-0.0746	0.5813
λ_2	-0.1289	-0.1789	0.7243	-0.2247	-0.2747	0.6792	-0.2714	-0.3214	0.6262	-0.1993	-0.2493	0.5784
λ_3	0.1285	0.0285	0.6875	0.1306	0.0306	0.6561	0.0737	-0.0263	0.6400	0.0676	-0.0324	0.5740
λ_4	-0.1802	-0.0802	0.7136	-0.2394	-0.1394	0.6745	-0.2409	-0.1409	0.6263	-0.2577	-0.1577	0.5350
λ_5	-0.0461	0.2039	0.5881	-0.0299	0.2201	0.3930	0.0017	0.2517	0.2535	0.0080	0.2580	0.1740

Table 8: Simulation Results for Near Independence Parameters for BCTWD using (MLE) method

	n=20			n=50			n=100			n=200		
Parameters	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE
a1	2.1592	0.1592	0.2868	2.0143	0.0143	0.1791	1.9862	-0.0138	0.1333	1.9611	-0.0389	0.1141
a2	3.2346	0.2346	0.6003	3.0045	0.0045	0.3687	2.9629	-0.0371	0.2910	2.9413	-0.0587	0.2399
b1	1.0747	0.0747	0.0822	1.0732	0.0732	0.0968	1.0733	0.0733	0.0906	1.0594	0.0594	0.0829
b2	1.5459	0.0459	0.0692	1.5621	0.0621	0.0953	1.5718	0.0718	0.0907	1.5448	0.0448	0.0732
λ_1	0.4070	0.3570	0.7775	0.3591	0.3091	0.6777	0.3177	0.2677	0.5847	0.2293	0.1793	0.4783
λ_2	-0.0225	-0.0525	0.9448	-0.2239	-0.2539	0.8891	-0.2478	-0.2778	0.7447	-0.2202	-0.2502	0.5456
λ_3	0.3506	0.3306	0.7746	0.3468	0.3268	0.7149	0.3216	0.3016	0.6021	0.2270	0.2070	0.4538
λ_4	-0.1184	-0.0784	0.9191	-0.1899	-0.1499	0.8501	-0.2301	-0.1901	0.7265	-0.2374	-0.1974	0.5242
λ_5	-0.0340	0.0460	0.6633	-0.0416	0.0384	0.3149	-0.0386	0.0414	0.1288	-0.0299	0.0501	0.0560

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Table 9: Simulation Results for Strong Dependence Parameters for BCTWD using (MLE) method

	n=20			n=50			n=100			n=200		
Parameters	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE	Estimates	Bias	MSE
a1	3.2296	0.2296	0.5609	3.0408	0.0408	0.3527	2.9961	-0.0039	0.2854	2.9518	-0.0482	0.2356
a2	2.1332	0.1332	0.2511	1.9954	-0.0046	0.1653	1.9489	-0.0511	0.1378	1.9345	-0.0655	0.1097
b1	1.6174	0.1174	0.0871	1.6140	0.1140	0.0990	1.5996	0.0996	0.0931	1.5913	0.0913	0.0854
b2	2.0927	0.0927	0.2792	2.0831	0.0831	0.3406	2.0921	0.0921	0.3585	2.0713	0.0713	0.2966
λ_1	0.5622	0.3622	0.6718	0.4811	0.2811	0.5878	0.4008	0.2008	0.4925	0.3229	0.1229	0.4195
λ_2	0.1858	0.0358	0.9108	-0.0733	-0.2233	0.9317	-0.1818	-0.3318	0.7968	-0.1322	-0.2822	0.5949
λ_3	0.3211	0.4211	0.8906	0.2718	0.3718	0.7473	0.2377	0.3377	0.6487	0.1584	0.2584	0.5028
λ_4	-0.1282	0.0218	0.8978	-0.2892	-0.1392	0.8019	-0.2926	-0.1426	0.6832	-0.2226	-0.0726	0.4810
λ_5	-0.0952	0.0048	0.6745	-0.0409	0.0591	0.3483	-0.0339	0.0661	0.1362	-0.0129	0.0871	0.0539

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The simulation results indicate that the estimators of the BCTWD model are approximately unbiased for marginal parameters (a_1, a_2, b_1, b_2), with absolute bias ¡ 0.17 across all scenarios and sample sizes. However, dependence parameters $(\lambda_1 - \lambda_5)$ exhibit substantial bias for small samples ($n \leq 50$), particularly in heavy-tailed scenarios. For n=20 under Heavy Tail, λ_5 shows relative bias exceeding 80%. Bias decreases markedly as sample size increases, with n=200 yielding satisfactory performance.

6.3 IFM Performance

The inference functions for the margins (IFM) method are employed to estimate the parameters of the BCTWD through a Monte Carlo simulation study. The estimation procedure is implemented in two stages using the R packages `maxLik` and `stats4`. In the first stage, the marginal Weibull parameters are estimated separately by maximum likelihood. In the second stage, the dependence parameters are estimated by maximizing the corresponding log-likelihood function conditional on the estimated marginal parameters. The sample size is $n = 200$ with the number of Monte Carlo replications set at $m = 1000$.

Table 10: Simulation Results for IFM Estimates, Bias, and MSE for BCTWD model across scenarios

Scenario	Parameter	Estimate	Bias	MSE
	α_1	2.05	0.05	0.02
	β_1	3.08	0.08	0.01
	α_2	1.01	0.01	0.00
	β_2	1.51	0.01	0.01
Baseline	λ_1	-0.17	-0.22	0.06
	λ_2	0.06	0.03	0.02
	λ_3	-0.18	-0.20	0.04
	λ_4	0.06	0.10	0.03
	λ_5	0.01	0.00	0.01
	α_1	1.21	0.01	0.00
	β_1	4.53	0.03	0.08
	α_2	0.80	0.00	0.00
	β_2	2.47	-0.03	0.05
Large Variation	λ_1	-0.17	-0.18	0.04
	λ_2	0.04	0.02	0.02
	λ_3	-0.16	-0.13	0.02
	λ_4	0.03	0.02	0.03
	λ_5	0.01	0.00	0.01
	α_1	0.81	0.01	0.00
	β_1	0.98	-0.02	0.01
	α_2	1.00	0.00	0.00
	β_2	1.00	0.00	0.01
Heavy Tail	λ_1	0.03	-0.07	0.01
	λ_2	-0.05	-0.10	0.04
	λ_3	0.02	-0.08	0.02
	λ_4	-0.03	0.07	0.03
	λ_5	0.01	-0.04	0.01
	α_1	2.54	0.04	0.02
	β_1	2.51	0.01	0.00
	α_2	1.21	0.01	0.01
	β_2	1.21	0.01	0.00
Near Independence	λ_1	0.01	0.01	0.01
	λ_2	-0.01	-0.01	0.03
	λ_3	0.01	0.01	0.01
	λ_4	-0.01	-0.01	0.03
	λ_5	0.01	0.01	0.01
	α_1	3.02	0.02	0.03
	β_1	2.00	-0.00	0.00
	α_2	1.51	0.01	0.01
	β_2	2.00	0.00	0.01
Strong Dependence	λ_1	0.02	-0.18	0.04
	λ_2	-0.03	-0.18	0.06
	λ_3	0.03	0.13	0.03
	λ_4	-0.04	0.11	0.05
	λ_5	-0.00	0.10	0.02

In table (10), the IFM results show that the marginal parameters are estimated with high precision and minimal bias, confirming the stability of the model in all scenarios. However, dependence parameters exhibit larger bias and MSE, particularly under strong dependence, indicating increased estimation variability. Overall, the IFM method provides reliable marginal estimates; however, the accuracy improves with larger samples or weaker correlations.

7 Real Data Applications

In this section, real-data applications, such as BCTWD, are presented.

7.1 Data Description

To verify applicability, real-life applications have been conducted for the proposed BCTWD. Travis-Lumer et al. (2025): The dataset consists of n = 154 patients with paired measurements from left and right eyes. Each observation represents time (in months) from baseline examination until detection of diabetic retinopathy. Pseudo-observations for bivariate survival data. In the data set, X represents the time to the event for the left eye, and Y represents the time to the event for the right eye. The event of interest is the onset of moderate non-proliferative diabetic retinopathy (NPDR), as diagnosed by trained ophthalmologists using standard retinal photography. The times are measured from study enrollment to first detection during follow-up examinations.
Table (11) gives some descriptive statistics of the data.

7.2 Model Fitting

We have modeled data sets using the BCTWD alongside bivariate distributions. Maximum likelihood estimates of the model parameters are obtained using the maxLik R package. To evaluate the performance of BCTWD with competing distributions, the Akaike information criterion (AIC), Corrected Akaike Information Criterion (AICc), and the Bayesian information criterion (BIC) are calculated.

7.3 Model Comparison

For comparison, we have used the bivariate Weibull distribution Shahbaz et al. (2012), which involves four parameters, and the bivariate transformed Weibull distribution Darwish et al. (2021). The distributions are fitted by obtaining the maximum likelihood estimates of the parameters.

7.4 Diagnostic Checks

Table 11: Summary Statistics for Optical Data

	Min.	Q_1	Median	Mean	Q_3	Max.
X	0.30	18.73	40.17	36.89	55.13	74.93
Y	0.60	13.80	38.30	34.26	52.77	74.97

Table 12: MLEs and SEs for selected Distribution

Distribution	Parameter	Estimate	SE
BCTWD	α_1	1.5036	0.0999
	α_2	1.4641	0.1176
	β_1	31.7059	1.4675
	β_2	31.2191	1.3409
	λ_1	- 0.2115	0.2086
	λ_2	-0.9999	0.4784
	λ_3	0.0381	0.2951
	λ_4	-0.9999	0.5301
	λ_5	0.5867	0.0945
BTW	θ_1	43.4569	1.2310
	θ_2	32.7911	2.8286
	α_1	1.6511	0.1014
	α_2	1.3480	0.1009
	λ_1	0.0019	1.1883
	λ_2	0.0013	1.0559
	λ_3	0.0000	1.2518
BW	α_1	1.5812	0.0930
	α_2	0.7940	0.0434
	β_1	40.4103	0.1023
	β_2	1150.6867	0.9311

For BTW distribution: θ_1 and θ_2 are scale parameters corresponding to marginals X and Y respectively; α_1 , α_2 are shape parameters; λ_1, λ_2 and λ_3 are dependence parameters under the quadratic transmutation scheme Darwish et al. (2021)

Table 13: Selection Criteria for Selected Distributions

Distribution	LogLik	AIC	AICc	BIC
BCTWD	-1717.804	3453.607	3454.57	3483.156
BTW	-1767.482	3548.965	3549.557	3571.947
BW	-1866.861	3741.722	3741.93	3754.854

The descriptive statistics for the optical data (Table 11) show that both variables X and Y exhibit substantial variability. For the X variable, the values range from 0.30 to 74.93 with a median of 40.17 and a mean of 36.89, indicating a moderately right-skewed distribution with a few large observations. Similarly, the Y variable ranges from 0.60 to 74.97, with a median of 38.30 and a mean of 34.26, also suggesting moderate skewness and a wide spread. Overall, both variables exhibit considerable dispersion, reflecting the heterogeneity inherent in the optical dataset.

The results of MLE (Table 12). Based on likelihood-based criteria (AIC, BIC), the BCTWD provides an improved fit compared to BTW and BW models. However, the presence of boundary parameter estimates and large standard errors suggests that a reduced model with fewer dependence parameters may achieve a similar fit with greater parsimony. Future applications should consider model selection strategies such as stepwise elimination of nonsignificant dependence parameters.

Model selection criteria (Table 13) further support these findings. Although the BCTWD achieves the highest log-likelihood value (-1717.804), its AIC (3453.607), AICc (3454.57), and BIC (3483.156) are all lower than those of the BTW (AIC = 3548.965) and BW (AIC = 3741.722) models. This indicates that the BCTWD provides the best balance between model fit and complexity, as lower AIC, AICc, and BIC values denote superior model performance. Therefore, while the BCTWD model exhibits higher estimation variability, it nonetheless outperforms the BTW and

BW models in terms of likelihood-based criteria.

Overall, the analysis demonstrates that the BCTWD offers the most appropriate and flexible framework for modeling the optical data, effectively capturing both the marginal and joint lifetime dependencies. The results confirm that, despite its complex structure, the BCTWD achieves the best fit according to information criteria (AIC, AICc, and BIC), outperforming both the BTW and BW models.

8 Conclusions

In this study, a new and flexible family of lifetime distributions, termed the *Bivariate Cubic Transmuted Weibull Distribution (BCTWD)*, has been proposed and thoroughly investigated. The explicit forms of its joint and marginal probability density functions, as well as its cumulative distribution function, were derived and examined in detail. Key structural properties, including dependence behavior, reliability measures, and hazard rate characteristics, were also explored. Parameter estimation was carried out using the Maximum Likelihood (ML) method and the Inference Functions for Margins (IFM) technique, and their finite-sample performances were evaluated through extensive simulation experiments. The simulation results indicate that the estimators of the BCTWD model are approximately unbiased for the marginal parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, with absolute bias ¡ 0.17 for all scenarios and sample sizes. However, dependence parameters $(\lambda_1 - \lambda_5)$ exhibit substantial bias for small samples $(n \leq 50)$, particularly in heavy-tailed scenarios. For n=20 under Heavy Tail, λ_5 shows a relative bias exceeding 80%. Bias decreases markedly as sample size increases, with n=200 yielding satisfactory performance. An empirical application to optical data demonstrated the practical relevance of the proposed model. Comparison with the *Bivariate Transmuted Weibull (BTW)* and *Bivariate Weibull (BW)* models showed that the BCTWD achieves the best goodness-of-fit according to the likelihood-based criteria (AIC, AICc, and BIC), providing lower values than both alternative models. These findings confirm that the BCTWD model not only enhances flexibility in capturing complex dependence structures but also achieves superior model performance in real-data applications.

Overall, the proposed BCTWD enriches the toolbox of multivariate lifetime models, offering a powerful and adaptable framework for modeling dependent lifetimes in diverse areas such as biostatistics, reliability engineering, and survival analysis. Future research may extend this work by developing regression-based and Bayesian formulations of the BCTWD, as well as exploring its potential in high-dimensional dependence modeling.

References

1. AL-Kadim, K. A. and Mohammed, M. H. (2017). The cubic transmuted weibull distribution. *Journal of University of Babylon*, 3:862–876.
2. Alsalafi, A., Shahbaz, S., and Al-Turk, L. (2025). A new bivariate transmuted family of distributions: Properties and application. *European Journal of Pure and Applied Mathematics*, 18(2):5819–5819.
3. Aryal, G. R. and Tsokos, C. P. (2011). Transmuted weibull distribution: A generalization of the weibull probability distribution. *European Journal of pure and applied mathematics*, 4(2):89–102.
4. Aslam, M., Hussain, Z., and Asghar, Z. (2018). Cubic transmuted-g family of distributions and its properties. *Stochastics and Quality Control*, 33(2):103–112.
5. Darwish, J. A., AL-Turk, L. I., and Shahbaz, M. Q. (2021). The bivariate transmuted family of distributions: Theory and applications. *Computer Systems Science & Engineering*, 36(1).
6. Gupta, R. D. and Kundu, D. (1999). Theory & methods: Generalized exponential distributions. *Australian & New Zealand Journal of Statistics*, 41(2):173–188.
7. Henningsen, A. and Toomet, O. (2011). maxlik: A package for maximum likelihood estimation in r. *Computational Statistics*, 26(3):443–458.
8. Joe, H. (2005). Asymptotic efficiency of the two-stage estimation method for copula-based models. *Journal of Multivariate Analysis*, 94(2):401–419.
9. Joe, H. and Xu, J. J. (1996). The estimation method of inference functions for margins for multivariate models. Technical Report 166, Department of Statistics, University of British Columbia, Vancouver.
10. Khan, M. S., King, R., and Hudson, I. L. (2017). Transmuted weibull distribution: Properties and estimation. *Communications in Statistics-Theory and Methods*, 46(11):5394–5418.
11. Kotz, S. and Singpurwalla, N. D. (1999). On a bivariate distribution with exponential marginals. *Scandinavian Journal of Statistics*, 26(3):451–464.
12. Merovci, F. (2013). Transmuted rayleigh distribution. *Austrian Journal of statistics*, 42(1):21–31.
13. Pobočíková, I., Sedliačková, Z., and Michalková, M. (2018). Transmuted weibull distribution and its applications. In *MATEC Web of Conferences*, volume 157, page 08007. EDP Sciences.
14. Rahman, M. M. (2022). Cubic transmuted rayleigh distribution: Theory and application. *Austrian Journal of Statistics*, 51(3):164–177.
15. Rahman, M. M., Al-Zahrani, B., and Shahbaz, M. Q. (2018). A general transmuted family of distributions. *Pakistan Journal of Statistics and Operation Research*, 14(2):451–469.
16. Rahman, M. M., Al-Zahrani, B., and Shahbaz, M. Q. (2019). Cubic transmuted weibull distribution: properties and applications. *Annals of Data Science*, 6(1):83–102.
17. Salo, J., El-Sallabi, H. M., and Vainikainen, P. (2006). The distribution of the product of independent rayleigh random variables. *IEEE transactions on Antennas and Propagation*, 54(2):639–643.
18. Shahbaz, S., Shahbaz, M. Q., Rafiq, A., and Maria, A. (2012). On trivariate pseudo weibull distribution. *Acta Universitatis Apulensis*, 31:241–247.
19. Travis-Lumer, Y., Mandel, M., and Betensky, R. A. (2025). Pseudo-observations for bivariate survival data. *Biometrics*, 81(1):ujaf006.
20. Tushar, M. T., Shahbaz, S. H., Rahman, M. M., and Shahbaz, M. Q. (2024). A new cubic transmuted inverse weibull distribution: Theory and applications. *Pakistan Journal of Statistics and Operation Research*, pages 85–98.
21. Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18(3):293–297.