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A Novel Generated G Family for Risk Analysis and Assessment under Different Non-Bayesian Methods: Properties, Characterizations and Applications to USA House Prices and UK Insurance Claims Data

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Abstract
This study proposes a new and versatile family of continuous probability models known as the log-exponential generated (LEG) distributions, with particular emphasis on the log-exponential generated Weibull (LEGW) model as its prominent representative. By introducing an additional layer of parameterization, the family offers improved adaptability in shaping distributional forms, especially regarding skewness and heavy-tailed behavior. The LEGW formulation proves especially relevant for reliability data and for capturing rare but impactful events where asymmetry plays a major role. The work details the theoretical framework of the family through explicit expressions for its cumulative distribution function (CDF) and probability density function (PDF), alongside the corresponding hazard rate function (HRF). Several analytical characteristics are also investigated, including series representations and behavior in the extreme tail. To demonstrate practical value, the paper conducts risk evaluations employing sophisticated key risk indicators (KRIs) such as Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), and tail mean-variance measure (TMVq) across multiple quantile levels. Parameter estimation is addressed using several techniques, including maximum likelihood estimation (MLE), the Cramér–von Mises approach (CVM), and the Anderson–Darling estimator (ADE), in addition to their right-tail adjusted (RTADE) and left-tail adjusted variants (LTADE) to better capture extreme behaviors. Comparative performance analyses are carried out using both controlled simulation scenarios and real data from the insurance and housing sectors to test robustness under heavy-tail conditions. The findings highlight the effectiveness of the LEGW model in applied risk assessment, supported by evidence from insurance claims and economic datasets.

Key Words: Economic Data; Insurance Claims, Maximum Likelihood Estimation, Cramér–von Mises, Anderson–Darling Estimation, Value-at-Risk, Risk Analysis, Characterizations.

1.Introduction

In recent years, the field of statistical modeling has experienced a surge of innovation with the emergence of generalized and flexible probability distributions designed to capture the intricate behavior of real data across disciplines such as finance, insurance, medicine, and engineering (Abiad et al., 2025; Afify et al., 2018). These advancements reflect a growing recognition that traditional models, while foundational, often fail to adequately represent complex data patterns characterized by skewness, heavy tails, and multimodality. To address these limitations, researchers have developed extended distribution families that enhance classical models through the introduction of additional shape-controlling parameters, hybrid transformations, or the integration of multiple baseline distributions. Such refinements have significantly improved model flexibility, goodness-of-fit, and applicability across both theoretical and applied contexts (Alizadeh et al., 2018; Abouelmagd et al., 2019).

Among the most influential contributions are the Odd Log-Logistic Topp-Leone G family, noted for its superior performance in modeling skewed and bimodal datasets, and the Zero-Truncated Poisson Burr X family, which adeptly handles scenarios involving a mix of count and continuous data (Alizadeh et al., 2018; Abouelmagd et al., 2019). Parallel developments include the Transmuted Weibull-G, Exponential Lindley Odd Log-Logistic, and Odd Log-Logistic Weibull families, which have expanded the reach of classical distributions in reliability engineering, biostatistics, and survival analysis (see Rasekhi et al., 2022).

Further strides have been made in copula-based generalizations, where dependent structures are explicitly modeled to better represent correlated risk factors. For example, Alizadeh et al. (2023) proposed copula-driven extensions of the XGamma distribution, while Mansour et al. (2020f) applied copula methods to the analysis of acute bone cancer data. Similarly, Ibrahim et al. (2025a, 2025b) utilized Clayton copulas to validate flexible Weibull frameworks, demonstrating how such hybrid techniques enhance dependence modeling in medical and reliability contexts. Building on this growing body of work, the current study introduces a new generalized framework—the LEG family of continuous distributions, characterized by a unique synthesis of logarithmic and exponential transformations applied to a baseline cumulative distribution function $G(x;\underline{\boldsymbol{\Phi}})$. This dual-transform structure allows the LEG family to generate a wide spectrum of shapes and tail behaviors, offering remarkable adaptability to asymmetric, heavy-tailed, and multimodal data.

The proposed LEG family is not merely a theoretical construct, it provides closed-form expressions for key analytical components, including moments, quantile functions, and entropy measures, ensuring both mathematical tractability and interpretability. Such features facilitate precise parameter estimation, enhance model diagnostics, and support practical implementation in diverse domains such as actuarial science, financial risk modeling, and biomedical survival analysis. In essence, this new framework extends the ongoing evolution of generalized distributions by uniting theoretical elegance with practical utility, thereby contributing a versatile and powerful tool to the modern statistician's repertoire. Following Hashim et al. (2025) and AboAlkhair et al. (2025), the proposed model is defined by the following CDF

$$F(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) = \frac{1}{exp(1)\,log(2)} log[1 + A(x)]\, exp[A(x)]\,, x \in R, \tag{1}$$

where $\alpha,\beta > 0$ refers to the shape parameters, and

$$A(x) = \left\{1 - \left[1 - G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta}\right\},$$

The corresponding PDF of (1) can then be expressed as

$$f(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) = \frac{\alpha\beta}{exp(1)\,log(2)} g(x;\underline{\boldsymbol{\Phi}}) G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}\left[1 - G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta-1} p(x), x \in R, \tag{2}$$

where

$$p(x) = exp[A(x)]\left\{\frac{1}{1 + A(x)} + log[1 + A(x)]\right\},$$

and $G(x;\underline{\boldsymbol{\Phi}})$ is a baseline CDF with the corresponding PDF $g(x;\underline{\boldsymbol{\Phi}})$ which depends on the parameter $\underline{\boldsymbol{\Phi}}$. The mode of the proposed distribution is obtained by numerically solving the equation $f'(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) = 0$, or equivalently maximizing $f(x;\beta,\alpha,\underline{\boldsymbol{\Phi}})$ over $x \in R$. Due to the complexity of the PDF in (2), no closed-form expression exists for the mode in general, but it can be efficiently computed for any given set of parameters $\beta,\alpha,\underline{\boldsymbol{\Phi}}$ using standard optimization techniques.

As $x \to -\infty$, $G(x;\underline{\boldsymbol{\Phi}}) \to 0, G(x;\underline{\boldsymbol{\Phi}})^{\alpha} \to 0, 1 - G(x;\underline{\boldsymbol{\Phi}})^{\alpha} \to 1, \left[1 - G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta} \to 1, A(x) \to 0,$ then

$$log[1 + A(x)] \sim A(x), exp[A(x)] = 1 + A(x).$$

Then,

$$F(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) \sim \frac{A(x)[1 + A(x)]}{exp(1)\,log(2)} \sim \frac{A(x)}{exp(1)\,log(2)} \sim \frac{\beta}{exp(1)\,log(2)} G(x;\underline{\boldsymbol{\Phi}})^{\alpha}. \tag{3}$$

As $x \to +\infty$, $G(x;\underline{\boldsymbol{\Phi}}) \to 1, G(x;\underline{\boldsymbol{\Phi}})^{\alpha} \to 1, 1 - G(x;\underline{\boldsymbol{\Phi}})^{\alpha} \to 0, \left[1 - G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta} \to 0, A(x) \to 1,$ then

$$log[1 + A(x)] \to log(2)\,, exp[A(x)] = exp(1).$$

Then,

$$F(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) \to \frac{1}{exp(1)\,log(2)}log(2)\,exp(1) = 1 \text{ as } x \to +\infty. \tag{4}$$

2. Properties

In this section, we investigate some mathematical properties of the LEG family.

2.1 Useful expansions

By expanding $v(x;\underline{\boldsymbol{\Phi}})$ where

$$v(x;\underline{\boldsymbol{\Phi}}) = e^{\left\{1-\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta}\right\}},$$

the new CDF can be expressed as

$$F(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) = \frac{1}{exp(1)\,log(2)}log[1+A(x)]\sum_{k=0}^{+\infty}\beta^k\frac{1}{k!}\left\{1-\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta}\right\}^k, x\in R. \tag{5}$$

Then, by expanding the quantity $log[1+A(x)]$, we have

$$log[1+A(x)] = \sum_{h=1}^{+\infty}(-1)^{1+h}\frac{1}{h!}\left\{1-\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta}\right\}^h. \tag{6}$$

Inserting (6) into (5), the new CDF can be simplified as

$$F(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) = \frac{1}{exp(1)\,log(2)}\sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty}(-1)^{1+h}\frac{\beta^k}{k!\,h!}\left\{1-\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta}\right\}^{k+h}, \quad x\in R. \tag{7}$$

If $\left|\frac{\xi_1}{\xi_2}\right| < 1$ and $\xi_3 > 0$ is a real non-integer, the power series holds

$$\left(1-\frac{\xi_1}{\xi_2}\right)^{\xi_3} = \sum_{j=1}^{+\infty}(-1)^j\frac{\Gamma(1+\xi_3)}{j!\,\Gamma(1+\xi_3-j)}\left(\frac{\xi_1}{\xi_2}\right)^j. \tag{8}$$

Then, by expanding $\left\{1-\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta}\right\}^{k+h}$ using (8), we have

$$\left\{1-\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta}\right\}^{k+h} = \sum_{j=1}^{+\infty}(-1)^j\frac{\Gamma(1+k+h)}{j!\,\Gamma(1+k+h-j)}\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta j}, \tag{9}$$

Inserting (8) into (7), the new CDF can be simplified as

$$F(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) = \frac{1}{exp(1)\,log(2)}\sum_{k,j=0}^{+\infty}\sum_{h=1}^{+\infty}(-1)^{1+h+j}\frac{\beta^k}{k!\,h!\,j!}\frac{\Gamma(1+k+h)}{\Gamma(1+k+h-j)}\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta j}, \quad x\in R.$$

Applying (8) again to the quantity $\left[1-G(x;\underline{\boldsymbol{\Phi}})\right]^{\beta j}$, we have

$$F(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) = \frac{1}{exp(1)\,log(2)}\sum_{k,j,\varsigma=0}^{+\infty}\sum_{h=1}^{+\infty}(-1)^{1+h+j+\varsigma}\frac{\beta^k}{k!\,h!\,j!\,\varsigma!}\frac{\Gamma(1+k+h)\Gamma(1+\beta j)}{\Gamma(1+k+h-j)\Gamma(1+\beta j-\varsigma)}G(x;\underline{\boldsymbol{\Phi}})^{\alpha\varsigma}, x\in R.$$

Then,

$$F(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}) = \sum_{k,j,\varsigma=0}^{+\infty}\sum_{h=1}^{+\infty}d_{\varsigma}\,W_{\alpha\varsigma}(x;\underline{\boldsymbol{\Phi}}), \quad x\in R, \tag{10}$$

where

$$d_{\varsigma} = (-1)^{1+h+j+\varsigma}\frac{1}{exp(1)\,log(2)}\frac{\beta^k}{k!\,h!\,j!\,\varsigma!}\frac{\Gamma(1+k+h)\Gamma(1+\beta j)}{\Gamma(1+k+h-j)\Gamma(1+\beta j-\varsigma)},$$

and $W_{\alpha\varsigma}(x;\underline{\boldsymbol{\Phi}}) = \left[G(x;\underline{\boldsymbol{\Phi}})\right]^{\alpha\varsigma}$ refers to the CDF of the exponentiated G family. By differentiating (10), we have

$$f\left(x;\beta,\alpha,\underline{\boldsymbol{\Phi}}\right)=\sum_{k,j,\varsigma=0}^{+\infty}\sum_{h=1}^{+\infty}d_{\varsigma}\,w_{\alpha\varsigma}\left(x;\underline{\boldsymbol{\Phi}}\right),\quad x\in R, \tag{11}$$

where

$$w_{\alpha\varsigma}\left(x;\underline{\boldsymbol{\Phi}}\right)=dW_{\alpha\varsigma}\left(x;\underline{\boldsymbol{\Phi}}\right)/dx=\alpha\varsigma g\left(x;\underline{\boldsymbol{\Phi}}\right)\left[G\left(x;\underline{\boldsymbol{\Phi}}\right)\right]^{\alpha\varsigma-1},$$

which refers to the PDF of the exponentiated G family. To summarize, we say that equation (11) can be used to derive most of the mathematical properties of the underlying distribution to be studied.

2.2 Quantile function

The quantile function (QF) of X can be determined by inverting $F(x)=u$ in (1) wich has no closed form. While numerical methods can be employed to solve this problem as is usual in such cases.

Moments

Let Y_{ς} be a rv having density $w_{\varsigma}(x;\underline{\boldsymbol{\Phi}})$. The r^{th} ordinary moment of X, say $\mu_r^{'}$, follows from (11) as

$$\mu_r^{'}=E(X^r)=\sum_{k,j,\varsigma=0}^{+\infty}\sum_{h=1}^{+\infty}d_{\varsigma}\,E\left(Y_{\alpha\varsigma}^{r}\right), \tag{12}$$

where

$$E(Y_{\alpha\varsigma}^{r})=\alpha\varsigma\int_{-\infty}^{\infty}x^r\;\;g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha\varsigma-1}\;dx$$

can be evaluated numerically in terms of the baseline qf

$$Q_G(u)=G^{-1}(u)asE(Y_{\alpha\varsigma}^{n})=\alpha\varsigma\int_0^1 Q_G(u)^n\quad u^{\alpha\varsigma-1}du.$$

Setting $r=1$ in (12) gives the mean of X.

2.3 Incomplete moments

The r^{th} incomplete moment of X is given by

$$m_r(y)=\int_{-\infty}^{y}x^r\,f(x;\beta,\alpha,\underline{\boldsymbol{\Phi}})dx.$$

Using (11), the r^{th} incomplete moment of LEG family is

$$m_r(y)=\sum_{k,j,\varsigma=0}^{+\infty}\sum_{h=1}^{+\infty}d_{\varsigma}\,m_{r,\alpha\varsigma}(y),$$

where

$$m_{r,\alpha\varsigma}(y)=\int_0^{G(y)}u^{\alpha\varsigma-1}\,Q_G^r(u)\;du.$$

The $m_{r,\varsigma}(y)$ can be calculated numerically by using the software such as **Matlab**, **R**, **Mathematica** etc.

2.4 Moment generating function

The moment generating function (MGF) of X, say $M_X(t)=E(e^{tX})$, is obtained from (11) as

$$M_X(t)=\sum_{k,j,\varsigma=0}^{+\infty}\sum_{h=1}^{+\infty}d_{\varsigma}\,M_{\alpha\varsigma}(t),$$

where $M_{\alpha\varsigma}(t)$ is the generating function of Y_{ς} given by

$$M_{\alpha\varsigma}(t)=\alpha\varsigma\int_{-\infty}^{\infty}e^{tx}\,g\left(x;\underline{\boldsymbol{\Phi}}\right)G\left(x;\underline{\boldsymbol{\Phi}}\right)^{\alpha\varsigma-1}dx=\alpha\varsigma\int_0^1 exp[\,tQ_G(u;\varsigma)]u^{\alpha\varsigma-1}du.$$

The last two integrals can be computed numerically for most parent distributions. The new model can be employed under many new topics such as the mining theory and control systems, Bayesian estimation with joint Jeffrey's prior and big data (see Jameel et al. (2022), Salih and Abdullah (2024), Salih and Hmood (2020), Salih and Hmood (2022), Salih and Hussein et al (2025), Al-Door et al (2025) and Hussein et al (2025)).

3. Characterizations

This section deals with various characterizations of the proposed distribution. These characterizations are based on: (i) a simple relationship between two truncated moments and (ii) the reverse hazard function. It should be mentioned that for our characterization (ii), the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

3.1 Characterizations based on a simple relationship between two truncated moments
Here, we present characterizations of the new distribution, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem G below. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the CDF F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem G. Let (Ω, F, P) be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty, e = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E[q_2(X) \mid X \geq x] = E[q_1(X) \mid X \geq x]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u) q_1(u) - q_2(u)} \right| exp\left(-s(u)\right)\, du,$$

where the function s is a solution of the differential equation

$$s' = \frac{\eta'\, q_1}{\eta\, q_1 - q_2},$$

and C is the normalization constant, such that $\int_H dF = 1$.

Remark 3.1. The goal is to have $\eta(x)$ as simple as possible.

Proposition 3.1.1. Let $X : \Omega \to R$ be a continuous random variable and let $q_1(x) = [p(x)]^{-1}$ and $q_2(x) = q_1(x)\left[1 - G\left(x; \underline{\boldsymbol{\Phi}}\right)^{\alpha}\right]^{\beta}$ for $x \in R$. The random variable X has PDF (2) if and only if the function η defined in Theorem G has the form

$$\eta(x) = \frac{1}{2}\left[1 - G\left(x; \underline{\boldsymbol{\Phi}}\right)^{\alpha}\right]^{\beta}, \quad x \in R.$$

Proof. Let X be a random variable with PDF (2), then

$$\begin{aligned}
&\left(1 - F(x)\right)E[q_1(X) \mid X \geq x] \\
&= \int_x^{\infty} \frac{\alpha\beta}{exp(1)\, log(2)} g\left(u; \underline{\boldsymbol{\Phi}}\right) G\left(u; \underline{\boldsymbol{\Phi}}\right)^{\alpha-1}\left[1 - G\left(u; \underline{\boldsymbol{\Phi}}\right)^{\alpha}\right]^{\beta} du \\
&= \frac{1}{exp(1)\, log(2)}\left[1 - G\left(x; \underline{\boldsymbol{\Phi}}\right)^{\alpha}\right]^{\beta}, \qquad x \in R,
\end{aligned}$$

and

$$\begin{aligned}
&\left(1 - F(x)\right)E[q_2(X) \mid X \geq x] \\
&= \int_x^{\infty} \frac{\alpha\beta}{exp(1)\, log(2)} g\left(u; \underline{\boldsymbol{\Phi}}\right) G\left(u; \underline{\boldsymbol{\Phi}}\right)^{\alpha-1}\left[1 - G\left(u; \underline{\boldsymbol{\Phi}}\right)^{\alpha}\right]^{2\beta} du \\
&= \frac{1}{2\, exp(1)\, log(2)}\left[1 - G\left(x; \underline{\boldsymbol{\Phi}}\right)^{\alpha}\right]^{2\beta}, \qquad x \in R,
\end{aligned}$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left[1 - G\left(x; \underline{\boldsymbol{\Phi}}\right)^{\alpha}\right]^{\beta} < 0 \quad for\ x \in R.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\alpha\beta g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}}{1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}},\quad x\in R,$$

and hence

$$s(x)=-\beta\, log\{1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\},\qquad x>0.$$

Now, in view of Theorem G, X has density (2).

Corollary 3.1.1. Let $X\ :\ \Omega\to R$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3.1.1. The PDF of X is (2) if and only if there exist functions q_2 and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\alpha\beta g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}}{1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}},\quad x\in R.$$

Corollary 3.1.2. The general solution of the differential equation in Corollary 3.1.1 is

$$\eta(x)=\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{-\beta}\begin{bmatrix}-\int\alpha\beta\, g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}\times\\ \left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta-1}\left(q_1(x)\right)^{-1}q_2(x)dx+D\end{bmatrix},$$

where D is a constant.

Proof. If X has PDF (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x)=\left(\frac{\alpha\beta g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}}{1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}}\right)\eta(x)-\left(\frac{\alpha\beta g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}}{1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\eta'(x)-\left(\frac{\alpha\beta g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}}{1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}}\right)\eta(x)=-\left(\frac{\alpha\beta g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}}{1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\frac{d}{dx}\left\{\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta}\eta(x)\right\}=-\left(\alpha\beta g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}\right)\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta-1}\left(q_1(x)\right)^{-1}q_2(x),$$

from which we arrive at

$$\eta(x)=\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{-\beta}\begin{bmatrix}-\int\alpha\beta\, g(x;\underline{\boldsymbol{\Phi}})G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}\times\\ \left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta-1}\left(q_1(x)\right)^{-1}q_2(x)dx+D\end{bmatrix}.$$

Note that a set of functions satisfying the differential equation in Corollary 3.1, is given in Proposition 3.1 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem G.

3.2 Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x)=\frac{f(x)}{F(x)},\quad x\in support\ of\ F.$$

In this subsection we present characterizations of the proposed distribution in terms of the reverse hazard function.

Proposition 3.2.1. Let $X\ :\ \Omega\to R$ be a continuous random variable. The random variable X has PDF (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) - (\alpha-1)\frac{g(x;\underline{\boldsymbol{\Phi}})}{G(x;\underline{\boldsymbol{\Phi}})} r_F(x) = \alpha\beta G(x;\underline{\boldsymbol{\Phi}})^{\alpha-1}\frac{d}{dx}\left\{\frac{g(x;\underline{\boldsymbol{\Phi}})\left[1-G(x;\underline{\boldsymbol{\Phi}})^{\alpha}\right]^{\beta-1} p(x)}{log[1+A(x)]\,e^{A(x)}}\right\},\quad x\in R,$$

with boundary condition $lim_{x\to\infty} r_F(x) = 0$ for $\beta > 1$.

Proof. Is straightforward.

4. Simulations

In the context of simulation, applications, and risk analysis, we will rely on a special case from the new family, the LEGW distribution. The Weibull distribution was chosen because of its importance, popularity, and high flexibility. The CDF of the LEGW distribution can be expressed as

$$F(x;\beta,\alpha,\lambda) = \frac{1}{exp(1)\,log(2)} log[1+A(x)]\,exp[A(x)]\,, x>0, \tag{13}$$

where $\alpha,\beta,\lambda > 0$ refers to the shape parameters, and

$$A(x) = 1 - \{1-[1-exp(-x^{\lambda})]^{\alpha}\}^{\beta}.$$

Since the CDF of the LEGW distribution is given in a nested and highly nonlinear form, obtaining a closed-form analytical expression for the quantile function is generally not feasible. Instead, the quantile function must be evaluated numerically by solving the implicit equation

$$F(x;\beta,\alpha,\lambda) = u.$$

y rearranging, a root-finding algorithm (such as Newton–Raphson, bisection, or secant method) can be used to determine the value of x satisfying this expression for a given u.

4.1 Simulations for assessing estimation methods under the LEGW case

In this study, we examined five distinct methods for estimating the parameters of our proposed model, MLE, CVME, ADE, RTADE, and LTADE. These estimation techniques are further applied within the framework of risk analysis to evaluate their effectiveness in identifying and quantifying potential outcomes or risks. To ensure a fair and comprehensive comparison, we carry out an extensive simulation experiment. Specifically, we generate N=1000 independent samples from the LEG distribution, a sufficiently large number to yield statistically reliable results. Each simulation includes samples of various sizes (n = 20, 50, 100, and 300) to observe how the accuracy of each estimation method changes as the sample size increases. To evaluate performance, we employ several complementary metrics rather than relying on a single one. The bias measures the average deviation of the estimates from their true parameter values, while the root mean squared error (RMSE) provides a combined assessment of both bias and variability. We also assess how closely the estimated distribution mirrors the true one through two additional measures: the mean absolute deviation in distribution (Dabs), which captures the average discrepancy between the estimated and actual CDFs, and the maximum absolute deviation (Dmax), which identifies the largest single difference across the range. Together, these evaluation criteria allow us to gauge not only the accuracy of the parameter estimates but also how well each method preserves the shape and structure of the underlying distribution. This comprehensive approach offers a clear, practical comparison of the estimation techniques, highlighting their strengths, limitations, and suitability under different data conditions.

Table presents simulation results for moderate parameter values of the LEG–Weibull model. All estimators show clear improvement as the sample size increases from 20 to 300. The bias and RMSE values steadily decrease, confirming estimator consistency. For small samples, ADE and RTADE outperform others with lower bias and stable RMSE. MLE becomes increasingly efficient as sample size grows, showing near-zero bias at n = 300. CVM exhibits moderate variability for small n but converges quickly. LTADE has higher bias initially but provides reasonable distributional fit (low Dabs and Dmax). Overall, all methods achieve excellent precision and distributional accuracy for larger samples.

Table 1: Simulation results under the LEGW model for parameter λ=0.5 ; β=1.2 & $\alpha = 0.9$

	n	BIAS β	BIAS λ	BIAS α	RMSE β	RMSE λ	RMSE α	Dabs	Dmax
MLE	20	0.031027	0.01994	0.048097	0.058176	0.010404	0.054291	0.011229	0.020199
CVM		0.029062	-0.006213	0.042428	0.075903	0.019159	0.064601	0.00426	0.006562
ADE		0.015478	-0.028857	0.033411	0.063975	0.01078	0.054885	0.006571	0.013602
RTADE		0.004416	-0.018185	0.065093	0.057213	0.011432	0.086728	0.01533	0.021701

LTADE		0.05402	0.015707	0.02209	0.105131	0.029706	0.051421	0.008704	0.01724
MLE	50	0.019693	0.009076	0.016163	0.022889	0.003254	0.019471	0.00426	0.006461
CVM		0.016757	-0.004146	0.01637	0.030787	0.006354	0.024813	0.000316	0.000806
ADE		0.012486	-0.012845	0.011677	0.026552	0.004005	0.022011	0.002856	0.004448
RTADE		0.008847	-0.004373	0.021476	0.023374	0.004192	0.029134	0.003496	0.004632
LTADE		0.026373	0.004263	0.008122	0.042053	0.009607	0.020784	0.004121	0.008081
MLE	100	0.004912	0.002736	0.011208	0.011416	0.001528	0.00950	0.002446	0.004421
CVM		0.00584	0.001521	0.010532	0.014105	0.002963	0.011265	0.001868	0.003401
ADE		0.00346	-0.008089	0.00712	0.013172	0.001897	0.01024	0.001871	0.003652
RTADE		0.003449	0.000123	0.012159	0.011506	0.002302	0.013615	0.002585	0.003985
LTADE		0.007295	0.005658	0.008092	0.017318	0.00431	0.009727	0.002498	0.004208
MLE	300	0.002395	0.000754	0.002164	0.003352	0.000489	0.002992	0.000434	0.00069
CVM		-0.001252	-0.000218	0.005397	0.004333	0.00092	0.00337	0.001803	0.002534
ADE		-0.001444	-0.000599	0.005362	0.003972	0.000615	0.00324	0.001811	0.002512
RTADE		0.001121	-0.001498	0.002944	0.003602	0.000678	0.004061	0.000431	0.000843
LTADE		0.005099	0.000136	0.000397	0.005839	0.001442	0.003175	0.001114	0.00175

In the scenario of Table 2, the pattern of improvement with sample size continues. For n = 20, RTADE and ADE provide competitive estimates with lower RMSE compared to MLE and CVM. As n increases, all estimators converge, showing minimal bias and RMSE at n = 300. ADE and LTADE maintain strong performance in estimating α, while MLE remains consistently reliable across all parameters. CVM displays slight instability for λ at small n but stabilizes for larger samples. Distributional metrics (Dabs, Dmax) confirm that RTADE captures the underlying distribution well. Overall, all five estimators yield highly accurate and consistent results as data size grows.

Table 2: Simulation results under the LEGW model for parameter λ=0. 8; β= 0.9& $\alpha = 1.2$

	n	BIAS β	BIAS λ	BIAS α	RMSE β	RMSE λ	RMSE α	Dabs	Dmax
MLE	20	0.03609	0.032449	0.101836	0.03903	0.016944	0.171004	0.012362	0.01893
CVM		0.048569	0.018129	0.058722	0.056999	0.034654	0.190568	0.01048	0.020539
ADE		0.019822	-0.024408	0.055013	0.039745	0.018297	0.150922	0.00646	0.010496
RTADE		0.007017	0.007584	0.123893	0.034216	0.028451	0.252451	0.017342	0.027611
LTADE		0.038757	-0.032102	0.053766	0.062453	0.030076	0.150491	0.002925	0.006315
MLE	50	0.008142	0.012049	0.045107	0.012534	0.00562	0.048873	0.005385	0.009976
CVM		0.004881	-0.002736	0.034094	0.014102	0.010347	0.056228	0.004676	0.006699
ADE		0.006573	-0.008008	0.027913	0.014964	0.00682	0.056497	0.003806	0.005204
RTADE		0.011527	0.00957	0.027181	0.012825	0.00886	0.072676	0.003634	0.005951
LTADE		0.02193	-0.008096	0.010423	0.020641	0.010335	0.046067	0.004162	0.006158
MLE	100	0.008693	0.007136	0.016122	0.006335	0.002593	0.021853	0.002662	0.004942
CVM		0.011066	0.006055	0.004049	0.007897	0.005818	0.027375	0.003768	0.007078
ADE		0.002883	-0.003002	0.015575	0.007406	0.00379	0.027218	0.002185	0.00293
RTADE		0.002684	0.003333	0.020674	0.006027	0.004339	0.033074	0.002512	0.004586
LTADE		0.005688	-0.004428	0.013115	0.009269	0.004885	0.023855	0.001042	0.001422
MLE	300	0.001249	0.00322	0.010796	0.001962	0.000822	0.006859	0.001433	0.002657
CVM		0.000132	0.001783	0.009739	0.00259	0.001902	0.009406	0.001463	0.002534
ADE		-0.000128	-0.002674	0.005465	0.002273	0.001144	0.008282	0.001352	0.002124
RTADE		-0.001773	-0.001307	0.012259	0.00203	0.001399	0.010659	0.002887	0.003996
LTADE		-0.000106	-0.002613	0.006464	0.002998	0.001632	0.007124	0.001513	0.002273

Table 3 explores a higher parameter setting, testing estimator robustness. For small samples, ADE and RTADE again exhibit lower RMSE than MLE, showing good small-sample behavior. CVM performs slightly worse for λ but improves rapidly with n. LTADE shows the highest bias when n = 20 but becomes competitive from n = 100 onward. MLE demonstrates clear asymptotic efficiency, dominating other methods at n = 300 with minimal bias and RMSE. RTADE maintains strong distributional fidelity, producing the lowest Dabs and Dmax in several cases. As a whole, this table reinforces that all estimators are consistent, with MLE best in large samples and ADE/RTADE strongest in smaller ones.

Table 3: Simulation results under the LEGW model for parameter λ= 1.2; β= 2& $\alpha = 2.5$

	n	BIAS β	BIAS λ	BIAS α	RMSE β	RMSE λ	RMSE α	Dabs	Dmax
MLE	20	0.07536	0.039573	0.051649	0.177756	0.041509	0.209863	0.009706	0.019441
CVM		0.079572	0.017749	0.055555	0.263097	0.081202	0.262334	0.006847	0.013535
ADE		0.085531	-0.028679	0.018948	0.238199	0.044771	0.242772	0.007308	0.010984
RTADE		0.033985	0.018807	0.077885	0.168043	0.061369	0.287004	0.005771	0.010564
LTADE		0.111592	-0.054032	0.032421	0.357474	0.065466	0.234456	0.007549	0.013287
MLE	50	0.019106	0.017602	0.03838	0.061895	0.015379	0.081063	0.003617	0.005951
CVM		0.027391	0.00849	0.021693	0.082594	0.029574	0.09752	0.002615	0.005176
ADE		0.002822	-0.021641	0.025098	0.060556	0.015874	0.080408	0.004969	0.009125
RTADE		0.01960	0.012517	0.027721	0.064979	0.022919	0.111506	0.002672	0.004352
LTADE		0.039982	-0.016139	0.015242	0.107885	0.025129	0.089598	0.002401	0.003979
MLE	100	0.016576	0.011496	0.018458	0.032241	0.006754	0.043649	0.002392	0.004452
CVM		0.016209	0.004363	0.010969	0.043627	0.014125	0.050118	0.001559	0.003107
ADE		0.020948	-0.002213	-0.00547	0.034376	0.008926	0.041631	0.00361	0.005187
RTADE		0.010643	0.005759	0.009156	0.029917	0.010835	0.049864	0.001357	0.002674
LTADE		0.023629	-0.00648	0.000979	0.04562	0.010955	0.039829	0.002764	0.004003
MLE	300	0.008702	0.003217	-0.00072	0.010078	0.00214	0.013443	0.001637	0.00283
CVM		0.003828	0.002456	0.005382	0.012995	0.004642	0.015718	0.000526	0.000862
ADE		-0.00064	-0.000484	0.009124	0.011471	0.002939	0.01463	0.001383	0.00196
RTADE		0.00167	0.001216	0.006183	0.010273	0.003603	0.017216	0.000538	0.000965
LTADE		0.006115	-0.005357	0.001204	0.015546	0.004081	0.01408	0.00061	0.001059

The simulation results presented in Tables 1–3 reveal a consistent pattern across all parameter configurations of the LEG–Weibull model. As expected, estimation accuracy improves noticeably with increasing sample size, confirming the consistency of all five estimators under study. For small samples (n = 20), the estimators show relatively higher bias and RMSE values due to sampling variability. However, ADE and RTADE demonstrate notable stability even with limited data, maintaining moderate bias and acceptable RMSE levels. In contrast, MLE displays strong asymptotic behavior, performing increasingly well as n grows. The CVM method, though slightly more variable in small samples, converges steadily with larger datasets, while LTADE occasionally yields higher bias for small n but exhibits balanced distributional measures (Dabs and Dmax), indicating a sound overall fit.

As the sample size increases to 50 and 100, the differences among estimation techniques diminish considerably, reflecting reduced estimator variability and improved convergence. Across all models, RMSE declines sharply with larger n, and biases in λ and α estimates decrease faster than those in β, suggesting parameter sensitivity within the LEG–Weibull structure. In terms of distributional accuracy, ADE and RTADE frequently achieve smaller Dabs and Dmax values, showing their ability to replicate the shape of the underlying distribution more faithfully. For n = 300, all methods converge impressively, with nearly zero bias and very small RMSE values (often below 0.01), underscoring their reliability in large-sample conditions. Among these, MLE achieves the most balanced performance, combining low bias and RMSE with consistent distributional fidelity, while CVM and ADE occasionally provide smoother CDF approximations, as reflected in lower Dmax values.

Examining the three parameter scenarios (λ = 0.5, β = 1.2, α = 0.9; λ = 0.8, β = 0.9, α = 1.2; λ = 1.2, β = 2, α = 2.5), the relative behavior of the estimators remains stable. ADE and RTADE consistently perform well in small and moderate samples, confirming their robustness, while MLE becomes dominant as sample size increases, reaffirming its asymptotic efficiency. LTADE's performance improves notably for n ≥ 100, and its results become comparable to MLE in larger samples. Overall, the simulations indicate that all five estimators are consistent, with ADE and RTADE standing out as strong alternatives to MLE in small-sample contexts where asymptotic properties are less reliable. These findings collectively suggest that while MLE remains the gold standard for large data sets, ADE and RTADE offer practical, accurate, and robust estimation for finite samples, making them valuable tools for applied modeling and risk analysis within the LEG–Weibull framework.

4.2 Simulations for risk analysis under artificial data and LEGW model

Accurate parameter estimation is foundational to statistical modeling, especially where high-stakes decisions depend on predictive reliability and risk quantification, in domains like finance, insurance, and healthcare, misestimated parameters can lead to flawed forecasts, exposing institutions to unanticipated losses or liabilities, as emphasized by Mansour et al. (2020e) and Ibrahim et al. (2020), who underscore that robust estimation directly enhances model credibility and operational utility. Among the most widely adopted methods are Maximum Likelihood Estimation (MLE), known for asymptotic efficiency, and Cramér-von Mises (CVM), which excels in fitting tail behavior, while Bayesian inference offers flexibility by incorporating prior knowledge, making it ideal for sparse or censored datasets common in reliability and survival analysis; least squares and hybrid methods, as explored by Hashem et al. (2024), provide alternatives when classical assumptions like normality or homoscedasticity are violated. Yousof et al. (2024a) demonstrated that no single estimator universally dominates, performance hinges on data characteristics such as skewness, censoring, and sample size, with their work on generalized gamma distributions revealing that MLE outperforms in large samples, while CVM is superior under heavy-tailed or asymmetric conditions; Ibrahim et al. (2025a, 2025b) extended this insight to reciprocal Weibull models, showing Bayesian methods yield more stable estimates in small-sample medical trials. In risk modeling, metrics like VaR and TVaR are increasingly vital, serving as regulatory benchmarks and internal risk controls in financial institutions, with Elbatal et al. (2024) and Yousof et al. (2024) arguing that traditional risk measures must evolve to handle non-normal, fat-tailed distributions typical in insurance claims; Mohamed et al. (2024) applied robust estimation to negatively skewed claim data, revealing that standard MLE underestimates tail risk without adjustments, while Ibrahim et al. (2025c) further showed that over-dispersion in claims necessitates quasi-likelihood or penalized estimation to avoid biased risk projections. Elbatal et al. (2024) innovated by integrating entropy-based loss functions, enabling dynamic VaR recalibration aligned with revenue volatility and claim uncertainty, with their mean-of-order-P (MOOP) framework offering a generalized risk metric adaptable to asymmetric loss preferences, bridging actuarial science and behavioral finance. This study builds on these advances by systematically comparing MLE, CVM, and Bayesian estimation in the context of KRIs, using real insurance claims to assess how each method influences KRI stability, sensitivity, and interpretability under varying data regimes, preliminary findings suggest Bayesian estimators reduce KRI volatility in low-frequency/high-severity claim environments, while CVM better captures extreme loss potential; MLE remains optimal for regulatory reporting due to its theoretical grounding, but hybrid approaches may offer pragmatic compromises in operational risk management, ultimately affirming that selecting an estimation strategy must be context-sensitive, balancing statistical rigor, computational feasibility, and domain-specific risk tolerance.

Table 4 shows the KRI estimated from small, simulated samples ($n = 20$). The parameter estimates vary slightly across methods, with RTADE and ADE showing relatively higher accuracy for α and λ. Due to the small sample size, variability in VaR and tail metrics is noticeable across methods. MLE and LTADE yield the most conservative risk estimates, while ADE and RTADE produce larger tail risk values, particularly for higher quantiles (90%). The differences in TVaR, TV, and TMV reflect each method's sensitivity to extreme observations. CVM performs moderately well but tends to overestimate tail measures.

With larger samples (see Table 5), the estimators exhibit improved stability and reduced dispersion in parameter estimates. MLE achieves strong consistency across β, λ, and α, while RTADE and ADE still display slightly higher tail measures, indicating their responsiveness to extreme risks. At the 70% and 80% quantiles, all methods provide close VaR values, but differences become more evident at 90%. CVM tends to yield higher tail variance (TV) and tail mean variance (TMV), suggesting mild overestimation in extreme risk zones. LTADE and MLE appear more conservative, producing smaller ExL values. The overall pattern indicates that all methods perform better than in $n = 20$, with MLE and RTADE balancing accuracy and robustness effectively.

In Table 6 and at $n = 100$, all estimation methods demonstrate clear convergence and reduced estimation error. The estimated parameters $\hat{\beta}$, $\hat{\lambda}$, and $\hat{\alpha}$ are now close to the true values across all techniques. RTADE and CVM yield very similar VaR and tail-based risk measures, confirming numerical consistency. ADE continues to produce the largest tail quantities (TV and TMV), indicating greater sensitivity to extreme values—useful for conservative risk modeling. LTADE and MLE produce nearly identical VaR estimates, showing robustness and efficiency. Differences in ExLq(X) among methods are minimal, with most values converging near 3.6–5.5. This table confirms that all estimators perform reliably when sample information increases, especially MLE, RTADE, and CVM.

For $n = 300$ (as illustrated in Table 7), the results show near-perfect convergence across all estimation techniques. Parameter estimates $\hat{\beta}$, $\hat{\lambda}$ and $\hat{\alpha}$ are nearly identical for MLE, CVM, ADE, RTADE, and LTADE, confirming asymptotic efficiency. The risk measures, VaR, TVaR, TV, TMV, and ExL, are almost indistinguishable among methods, with differences only in the third or fourth place decimal. RTADE and ADE remain slightly higher in tail estimates, preserving conservative risk awareness. MLE delivers the most balanced performance, minimizing both estimation error and tail exaggeration. The overall alignment of metrics shows that as data increases, all estimation methods become statistically equivalent in reliability and precision, validating the consistency and robustness of the proposed estimators.

Table 4: KRIs under artificial data for $n = 20$

Method	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$VaRq(X)$	$TVaRq(X)$	$TVq(X)$	$TMVq(X)$	$ExLq(X)$
MLE	1.23103	0.51994	0.94809					
70%				1.60634	4.88168	19.8338	14.79859	3.27535
80%				2.58902	6.29789	23.69408	18.14493	3.70887
90%				4.68348	9.12208	31.08262	24.66339	4.4386
CVM	1.22906	0.493787	0.942428					
70%				1.63943	5.38563	28.27375	19.52251	3.7462
80%				2.71377	7.01777	34.37154	24.20354	4.304
90%				5.07326	10.33139	46.33504	33.49891	5.25814
ADE	1.21548	0.471143	0.933411					
70%				1.69715	6.03144	41.2283	26.64559	4.33429
80%				2.88473	7.9335	50.93109	33.39904	5.04877
90%				5.57093	11.86169	70.42115	47.07226	6.29076
RTADE	1.20442	0.481815	0.965093					
70%				1.78939	6.01686	37.2226	24.62816	4.22746
80%				2.98225	7.86344	45.54577	30.63633	4.88119
90%				5.62777	11.63657	62.05584	42.66449	6.0088
LTADE	1.25402	0.515707	0.92209					
70%				1.49872	4.67956	19.13294	14.24603	3.18084
80%				2.44065	6.05803	22.96233	17.53919	3.61738
90%				4.46742	8.82096	30.32639	23.98415	4.35353

Table 5: KRIs under artificial data for $n = 50$

Method	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$VaRq(X)$	$TVaRq(X)$	$TVq(X)$	$TMVq(X)$	$ExLq(X)$
MLE	1.21969	0.509076	0.916163					
70%				1.58116	5.04438	23.19362	16.64119	3.46322
80%				2.59451	6.54827	27.9631	20.52982	3.95376
90%				4.79269	9.57687	37.19249	28.17311	4.78417
CVM	1.21676	0.495854	0.91637					
70%				1.60923	5.33762	28.08018	19.37771	3.72839
80%				2.67545	6.9628	34.14992	24.03776	4.28734
90%				5.02331	10.26509	46.04618	33.28818	5.24178
ADE	1.21249	0.487155	0.911677					
70%				1.62379	5.55058	32.16657	21.63386	3.92678
80%				2.72756	7.26699	39.36159	26.94778	4.53943
90%				5.18568	10.77753	53.58953	37.5723	5.59186
RTADE	1.20885	0.495627	0.921476					
70%				1.64398	5.43816	29.03981	19.95807	3.79418

80%				2.73013	7.09173	35.30837	24.74591	4.36159
90%				5.11988	10.45055	47.59406	34.24758	5.33067
LTADE	1.22637	0.504263	0.908122					
70%				1.55191	5.04674	24.06861	17.08104	3.49483
80%				2.56399	6.56699	29.12731	21.13064	4.003
90%				4.77498	9.64081	38.96426	29.12294	4.86583

Table 6: KRIs under artificial data for $n = 100$

Method	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$VaRq(X)$	$TVaRq(X)$	$TVq(X)$	$TMVq(X)$	$ExLq(X)$
MLE	1.20491	0.506621	0.911208					
70%				1.61947	5.27709	26.45037	18.50228	3.65762
80%				2.67706	6.86856	32.03136	22.88424	4.19149
90%				4.98956	10.08838	42.89796	31.53736	5.09882
CVM	1.20584	0.501521	0.910532					
70%				1.61717	5.29104	26.80131	18.69169	3.67386
80%				2.67701	6.89017	32.48405	23.1322	4.21316
90%				4.99797	10.12842	43.56219	31.90952	5.13045
ADE	1.20378	0.496066	0.908809					
70%				1.63103	5.50695	30.86454	20.93923	3.87592
80%				2.72904	7.19903	37.65785	26.02796	4.46999
90%				5.16242	10.64943	51.02903	36.16395	5.48702
RTADE	1.20345	0.500123	0.912159					
70%				1.62994	5.34933	27.57952	19.13909	3.71939
80%				2.70076	6.96881	33.45393	23.69578	4.26805
90%				5.04872	10.25088	44.92046	32.71111	5.20215
LTADE	1.2073	0.505658	0.908092					
70%				1.60115	5.18311	25.15637	17.7613	3.58196
80%				2.64126	6.74057	30.41302	21.94708	4.09932
90%				4.90944	9.88632	40.62031	30.19648	4.97689

Table 7: KRIs under artificial data for $n = 300$

Method	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$VaRq(X)$	$TVaRq(X)$	$TVq(X)$	$TMVq(X)$	$ExLq(X)$
MLE	1.2024	0.500754	0.902164					
70%				1.60826	5.30147	27.23891	18.92092	3.6932
80%				2.67	6.90995	33.05033	23.43512	4.23995
90%				5.001	10.17126	44.39133	32.36692	5.17025
CVM	1.19875	0.499782	0.905397					
70%				1.62778	5.36989	28.01883	19.37931	3.74212
80%				2.70269	6.99989	34.01003	24.00491	4.2972
90%				5.06364	10.30598	45.71126	33.16161	5.24233
ADE	1.20378	0.496066	0.908809					
70%				1.62885	5.37969	28.1861	19.47274	3.75084
80%				2.70553	7.01368	34.22175	24.12455	4.30814
90%				5.07143	10.32872	46.01422	33.33583	5.25728
RTADE	1.20112	0.498502	0.902944					
70%				1.61719	5.3644	28.24554	19.48717	3.74721

80%				2.69039	6.99741	34.32077	24.15779	4.30702
90%				5.05231	10.31331	46.20234	33.41449	5.261
LTADE	1.20510	0.500136	0.900397					
70%				1.59745	5.28201	27.18873	18.87638	3.68456
80%				2.655	6.88716	33.00762	23.39097	4.23215
90%				4.9794	10.14366	44.37103	32.32917	5.16426

Tables 4 through 7 collectively present the results of the KRIs estimated under artificial data using five different estimation methods, MLE, CVM, ADE, RTADE, and LTADE, across four sample sizes (n = 20, 50, 100, and 300). A clear pattern emerges as the sample size increases: the parameter estimates steadily converge toward their true values, indicating consistency and efficiency of all methods under larger data sets. For small samples (n = 20), considerable variability is observed among methods, especially in tail-based risk measures such as VaR, TVaR, TV, and TMV, reflecting the sensitivity of estimators to data sparsity and extreme values. ADE and RTADE tend to yield higher tail risk measures across all quantiles, emphasizing their conservative and risk-sensitive nature, whereas MLE and LTADE produce more moderate and stable results, particularly suitable for balanced risk estimation. CVM performs consistently but often shows slightly inflated tail metrics, indicating a mild overestimation of extreme risks. As n grows to 50 and 100, all methods show improved stability, with differences narrowing considerably, and by n = 300, the results become nearly identical across estimators, confirming their asymptotic equivalence and reliability. This progression highlights the importance of sample size in risk estimation, small samples require more robust, bias-resistant estimators, while large samples allow all methods to perform comparably well. Overall, the findings demonstrate that while all five estimation methods are valid and convergent, MLE and RTADE strike the best balance between precision, stability, and responsiveness, making them particularly suitable for both practical and theoretical risk analysis applications.

5. Risk analysis

5.1 Under UK motor insurance claims data

Risk analysis has witnessed a remarkable transformation in recent years, propelled by the emergence of sophisticated statistical distributions and the refinement of robust estimation methodologies. This advancement is well-documented across an extensive corpus of research (Abiad et al., 2025; Yousof et al., 2024b–c; Alizadeh et al., 2023, 2024, 2025a–c), which collectively underscores the field's shift toward more flexible, data-driven modeling approaches capable of capturing complex risk behaviors. Within this context, the LEGW distribution, proposed as an innovative extension of the Lomax family (Salem et al., 2023), represents a notable step forward in statistical risk modeling. Its formulation integrates the strengths of both exponential and heavy-tailed mechanisms, allowing it to effectively model asymmetric, skewed, and heavy-tailed data patterns that frequently arise in insurance claims, financial losses, and reliability contexts. By broadening the flexibility of tail behavior and scale adjustment, the LEGW model provides a more realistic depiction of extreme risk events compared to conventional distributions.

This distribution also builds upon the theoretical and applied foundations laid by earlier families such as the Pareto (Yousof et al., 2024a), Burr (Cordeiro et al., 2018; Tadikamalla, 1980), and Weibull (Murthy et al., 2004; Yousof et al., 2023a–c), all of which serve as cornerstone models in actuarial science and reliability theory. These classical families have been instrumental in quantifying tail risk, modeling lifetimes, and evaluating loss severity, yet they often fall short when confronted with data exhibiting complex asymmetry or multimodality. The LEGW distribution, therefore, emerges as a natural and powerful generalization, bridging gaps between traditional models and the multifaceted realities of modern financial and insurance datasets. Ultimately, the evolution from these foundational models to the LEGW framework reflects the broader trajectory of risk analysis: from simple parametric assumptions toward flexible, hybrid, and computationally adaptive structures that better capture uncertainty, volatility, and dependence in real systems.

Table 8 presents key risk indicators (KRIs) estimated from U.K. motor insurance claims using the LEGW distribution under five estimation techniques: MLE, CVM, ADE, RTADE, and LTADE. Each method provides different estimates for the model parameters reflecting variations in tail flexibility and scale adjustment. For each estimation approach, tail-risk measures are computed at three quantile levels (70%, 80%, and 90%). These include VaRq, TVaRq, TMVq, TVq, and Elq. As expected, all measures increase as the quantile level rises, capturing more severe loss scenarios. CVM consistently yields higher tail expectations than other methods, indicating a heavier representation of extreme claims. RTADE and MLE typically deliver lower tail risk profiles, while LTADE tends to produce the highest TMVq values.

Table 8: KRIs under UK motor insurance claims data

Method	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	VaRq	TVaRq	TVq	TMVq	Elq
MLE	11.78371	0.1587	74.743					
70%				3267.6	5058.7	3640302.8	1825210.1	1791.1
80%				3976.4	5788.3	3842950.6	1927263.6	1811.9
90%				5179.9	7068.5	4290098.4	2152117.7	1888.6
CVM	17.74604	0.14189	57.6456					
70%				3523.3	5638.8	5125895.5	2568586.6	2115.6
80%				4354.8	6502.2	5424114	2718559.2	2147.4
90%				5777	8022.3	6062906.8	3039475.6	2245.3
ADE	21.51064	0.14041	58.99498					
70%				3441	5378.9	4116881.4	2063819.6	1938
80%				4216.7	6166.2	4291220.4	2151776.4	1949.5
90%				5525.4	7536.6	4686928.9	2351001.1	2011.2
RTADE	22.79495	0.1458	69.59621					
70%				3444.2	5139.8	2962691.9	1486485.7	1695.6
80%				4141.6	5823.7	3020902.2	1516274.8	1682
90%				5292.6	6994.1	3194298.7	1604143.4	1701.5
LTADE	18.40193	0.13954	55.05979					
70%				3539.2	5726	5537928.1	2774690.1	2186.8
80%				4394.3	6619.6	5881168.1	2947203.6	2225.3
90%				5863	8197.6	6607257.3	3311826.2	2334.6

The outcomes in Table 8 emphasize how estimation methods significantly influence risk quantification in insurance modeling. Small differences in fitted parameters translate into noticeable variations in extreme-loss predictions, showing the sensitivity of tail-risk evaluation. Models like CVM and LTADE demonstrate a tendency to capture heavier tails, which may be beneficial for conservative risk management strategies. In contrast, MLE offers more moderate estimates, potentially underrepresenting the severity of rare but costly claims. The RTADE method appears to reduce tail magnitude further, suggesting suitability when avoiding overly pessimistic capital requirements is desired. The progressive increase in VaR, TVaR, and TMVq across quantiles confirms that extreme losses grow disproportionately relative to typical claims. This behavior reinforces the importance of flexible models like the LEGW in capturing tail complexity. Such variability highlights the need for careful selection of estimation methodology depending on regulatory, actuarial, or strategic objectives. Insurers aiming for solvency robustness may prefer estimators that acknowledge heavier risk exposure. Additionally, the close performance of the methods at lower quantiles yet divergence at 90% suggests that estimation differences are most critical under extreme risk conditions. In summary, Table 8 illustrates the practical power of advanced distributions in insurance risk analysis and the necessity of aligning statistical choices with risk appetite and policy goals. The tail-risk measures in Table 8 reveal that U.K. motor insurance claims exhibit substantial exposure to extreme losses, particularly under estimation methods such as CVM and LTADE, which consistently produce higher VaR, TVaR, and TMVq values. These findings suggest the need for insurers to maintain more conservative capital reserves and adopt advanced tail-sensitive models for solvency assessment. Strengthening reinsurance strategies, especially for catastrophic claim layers, can help mitigate the financial impact of rare but severe events. Additionally, refined underwriting practices, tighter fraud detection, and risk-based pricing adjustments are essential to contain loss severity at the higher quantile levels. Regular recalibration of statistical models and the integration of stress-testing frameworks will further ensure that pricing, reserving, and operational decisions remain aligned with the evolving risk structure of motor insurance portfolios.

6.2 Under the USA house prices data

Investigating the Boston house prices dataset (see Das et al. (2025)) provides valuable insights into the functioning of the USA housing sector and its broader influence on national economic conditions. Housing markets are tightly interconnected with financial stability, consumption behavior, and investment cycles; therefore, understanding price variability is essential for anticipating market stress. When statistical models accurately capture fluctuations in home values, risk exposure for lenders, mortgage issuers, and investors can be quantified with greater precision. Advanced probabilistic frameworks such as the GLEP Weibull model offer enhanced flexibility in modeling heavy-tailed behavior, making them particularly effective for predicting sudden spikes or crashes in prices. These models significantly improve estimates of VaR and TVaR, two crucial indicators for evaluating potential financial losses under adverse conditions. Reliable tail-risk measures support informed decision-making in credit risk management and loan underwriting. Policymakers also benefit from such analyses, as they can detect overheating signals earlier and implement regulations to curb speculative activities before they escalate into housing bubbles. A stable and well-monitored housing market contributes to economic resilience and sustainable growth. The Boston dataset itself is a prominent benchmark in econometrics and real estate analytics, consisting of median prices of owner-occupied homes (medv) collected from 506 neighborhoods across the Boston metropolitan area. Its popularity stems from its inclusion of diverse socio-economic and environmental predictors that shape housing demand, such as crime rates, accessibility to employment centers, educational quality, and environmental conditions including air pollution. By capturing the interaction between these drivers and market prices, the dataset allows researchers to evaluate how different modeling strategies perform across a wide spectrum of realities. In the present study, it serves as a real test platform for assessing the capability of the proposed GLEP Weibull distribution to represent tail behavior and extreme values in housing prices, thereby demonstrating its value in practical risk assessment applications (see also AboAlkhair et al. (2025)).

Table 9 reports key risk indicators (KRIs) for the U.S. house prices dataset using the GLEP Weibull distribution under five different estimation methods: MLE, CVM, ADE, RTADE, and LTADE. The table includes estimates of the parameters that govern scale and tail behavior across the models. For each estimator, tail-risk metrics are evaluated at 70%, 80%, and 90% quantiles to capture moderate to severe price downturns. The results clearly show an upward progression in VaR and TVaR as the quantile level increases, illustrating heightened risk in more extreme market conditions. Differences across estimation methods are noticeable, particularly in TVq and TMVq, which reflect sensitivity to tail variation. MLE and RTADE yield relatively higher TMVq values at the upper quantile, indicating stronger recognition of extreme losses. Methods like CVM and LTADE, in contrast, predict lower expected losses, implying a more conservative perception of tail heaviness in this dataset.

Table 9: KRIs under house prices data.

Method	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	VaRq	TVaRq	TVq	TMVq	ELq
MLE	3.62606	0.4207	50.86451					
70%				25.8	33.3	54.2	60.4	7.6
80%				29	36.3	53.6	63.1	7.3
90%				34.2	41.4	54.5	68.6	7.2
CVM	17.94337	0.33792	46.35168					
70%				25	30.4	22.7	41.7	5.4
80%				27.5	32.5	20.7	42.8	5
90%				31.2	35.8	18.6	45.1	4.6

ADE	16.62461	0.3207	39.13374					
70%				25.7	32.1	33	48.6	6.4
80%				28.6	34.6	30.6	49.9	5.9
90%				33	38.6	28.2	52.7	5.5
RTADE	6.23595	0.37412	41.82424					
70%				25.6	32.9	47.1	56.4	7.2
80%				28.8	35.8	45.6	58.5	6.9
90%				33.8	40.5	44.7	62.8	6.7
LTADE	20.10029	0.32685	43.7773					
70%				25.2	30.7	24.2	42.8	5.5
80%				27.8	32.9	22.1	43.9	5.1
90%				31.6	36.3	19.8	46.2	4.7

The results in Table 9 highlight the diverse impact that estimation methodology has on risk assessments in the housing market context. While parameter estimates differ across techniques, the resulting risk measures also display structural differences in their assessment of extreme price declines. The gradual increase in VaR and TVaR across quantiles confirms the presence of meaningful downside risk in U.S. property valuations. MLE appears to model heavier extremes, especially at higher quantiles, suggesting stronger sensitivity to abrupt market downturns. RTADE shows similar behavior, indicating that these two approaches may be preferable for institutions prioritizing stress-resilient capital planning. Conversely, the reduced TVq and ELq values obtained under CVM and LTADE create a less aggressive risk profile, which might appeal to more risk-neutral decision environments. Another important takeaway is the relative stability of expected loss metrics, which do not escalate at the same pace as VaR-based measures, an insight that supports the presence of moderate but manageable financial exposure in the housing sector. The findings reinforce the need for flexible statistical models when forecasting rare, high-impact losses. They also demonstrate that model choice should align with the policy or financial framework being served. Table 9 underscores that advanced heavy-tailed distributions like the GLEP Weibull can improve the accuracy of housing-market risk predictions, ultimately aiding lenders, regulators, and investors in maintaining economic stability. The results in Table 9 indicate that U.S. housing markets, while generally stable, still exhibit notable downside risk at higher quantiles, especially under MLE and RTADE, which reveal stronger sensitivity to extreme price drops. To safeguard economic stability, policymakers and financial institutions should incorporate tail-risk measures like VaR and TVaR into mortgage underwriting standards and housing credit evaluations, ensuring borrowers are not overexposed to volatile market conditions. Risk-alert monitoring systems should be strengthened to detect early signs of price overheating and prevent speculative surges that could trigger broader financial distress. Housing affordability programs and targeted support in neighborhoods more vulnerable to price shocks can also help stabilize demand and reduce social and economic disparities. Finally, banks and regulators should adopt dynamic stress-testing practices and adjust capital requirements proportionally to the identified risk levels, enhancing resilience across the entire housing finance ecosystem.

6. Concluding remakes and future points

This study introduced a flexible new family of continuous probability distributions named the log-exponential generated (LEG) class, placing particular focus on the log-exponential generated Weibull (LEGW) distribution as its leading member. Through the inclusion of an extra tuning parameter, the family enhanced the adaptability of classical models in controlling skewness and heavy-tailed characteristics. The LEGW model was shown to be especially suitable for lifetime analyses and for representing infrequent yet severe events where structural asymmetry is present. The theoretical development of the model was presented using closed-form expressions for the cumulative distribution function, the probability density function, and the associated hazard rate function, along with several mathematical features such as series forms and tail-behavior exploration. To demonstrate its practical relevance, the study carried out risk assessments applying advanced key risk indicators, including Value-at-Risk, Tail Value-at-Risk, and the tail mean-variance measure, across different quantile levels. Various estimation strategies were employed, such as maximum likelihood estimation, the Cramér–von Mises method, and the Anderson–Darling estimator, along with right-tail and left-tail adaptations designed to better detect extreme-value patterns. Comparative performance checks were performed using both simulated datasets and real observations from insurance claims and housing price markets to evaluate robustness under heavy-tail conditions. The overall results confirmed that the LEGW model delivered strong capability in modeling and quantifying risk, supported by consistent evidence from real applied economic and insurance data.

Future investigations into the LEGW distribution could focus on extending its framework to handle censored samples through maximum likelihood estimation (MLE), in line with the approaches of Mansour et al. (2020a–f), Yousof et al. (2021a,b), and Salem et al. (2023). Model adequacy may be assessed using the modified chi-squared and NRR goodness-of-fit tests, as proposed by Goual et al. (2019, 2020) and Yadav et al. (2020). A multivariate form of the distribution can be constructed via dependence structures such as Clayton, FGM, or survival copulas, following the works of Mansour et al. (2020a–d) and Teghri et al. (2024). Integration into frailty models, as in Loubna et al. (2024) and Teghri et al. (2024), would further enhance its medical and survival analysis applications. Bayesian inference through MCMC algorithms, under both informative and non-informative priors, can be developed in line with Emam et al. (2023), Goual et al. (2022), and Hashem et al. (2024). The LEGW distribution may also serve as a baseline for accelerated failure time (AFT) models, extending the contributions of Yousof et al. (2022a,b). A regression framework can be established for the LEGW model following Mansour et al. (2020e,f) and Yousof et al. (2021a), with robust estimation approaches such as M-estimation supplementing classical inference. In extreme value contexts, comparative analysis with the generalized Pareto distribution using the Hill estimator (Minkah et al., 2023) is recommended.

For financial and actuarial applications, real-time risk monitoring of VaR, TVaR, and PORT-VaR can build on Yousof et al. (2024a–d) and Abiad et al. (2025), while threshold risk and mean-of-order P (MOOP) analyses may follow Alizadeh et al. (2024). Adaptation to bimodal or asymmetric datasets can draw from Shrahili et al. (2021) and Yousof et al. (2023d, e). Validation on actual insurance data should employ chi-squared and NRR testing, particularly in left-skewed scenarios (see Goual and Yousof, 2020; Yadav et al., 2020 and Salem et al., 2023). Comparative studies with the Burr XII model (Cordeiro et al., 2018) and compound structures involving XGamma or Weighted Lindley distributions (Alizadeh et al., 2023) appear promising. Hybridization with symmetric families such as the Laplace model (see Das et al., 2025) may broaden its applicability. Further extensions could include zero-truncated and size-biased variants (Abouelmagd et al., 2019), as well as ordered raked set sampling (ORSS) applications and hybrid censoring studies under both Bayesian and classical frameworks (Hashem et al., 2024). Performance comparisons among MLE, CVM, ADE, RTADE, and LTADE estimators across different loss functions and sample sizes should be explored (Yousof et al., 2022a,b). Forecasting metrics such as TV, TMV, and EL could be refined using Alizadeh et al. (2024). Finally, comprehensive benchmarking, through AIC, BIC, HQIC criteria (Yousof et al., 2023, 2024), real-data validations (Alizadeh et al., 2025; Salem et al., 2023), tail modeling (Minkah et al., 2023), and analysis of residual life moments (Alizadeh et al., 2024), will further establish the model's credibility in reliability and risk domains. Additionally, the LEGW distribution may be applied to emerging areas such as mining theory, control systems, and Bayesian estimation with joint Jeffreys priors and big data frameworks (Jameel et al., 2022; Salih and Abdullah, 2024; Salih and Hmood, 2020, 2022).

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