Pakistan Journal of Statistics and Operation Research

Point and Interval Estimation Techniques for the 2S-Lindley Distribution Under Type-II Censoring

Joul Kanjo[1], Akbar Asgharzadeh[1,*],
Mohammad Z. Raqab[2]

*Corresponding author: a.asgharzadeh@umz.ac.ir

1. Department of Statistics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar 47416-1407, Iran, joul.kanjo@gmail.com & a.asgharzadeh@umz.ac.ir
2. Department of Statistics & Operations Research, Kuwait University, Al-Shadadiyya, Kuwait
& Department of Mathematics, The University of Jordan, Amman 11942, Jordan, mraqab@ju.edu.jo

Abstract

Recently, Chesneau et al.(2020) introduced a new distribution called the 2S-Lindley distribution, which is based on the sum of two independent Lindley random variables with the same parameter. In this paper, we employ different methods to estimate the unknown parameter of the 2S-Lindley distribution using Type-II censored samples. These methods include the moment-based method, maximum likelihood estimation, the bootstrap method, and Bayesian inference. We provide both point and interval estimates for the parameter using each method. We also consider a real dataset that follows the 2S-Lindley distribution, for which various estimates are calculated and analyzed. Finally, we conduct a simulation study to illustrate and compare the effectiveness of these methods.

Key Words: Bayesian Estimation; Bootstrap Methods; Confidence Interval; Lindley Distribution; Point Estimation; 2S-Lindley Distribution.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

One of the statistical models that has attracted the attention of many statisticians in recent years is the Lindley distribution. This distribution is a competitor to the exponential distribution, but in terms of many mathematical and statistical properties, it is more flexible than the exponential distribution. In recent years, various generalizations of this distribution have appeared in the theory of distributions, which have been used in various fields. See, for example, Bakouch et al.(2012), Ghitany et al.(2013), Asgharzadeh et al.(2017a), Abdi et al.(2019) and Chesneau et al.(2023). Inferential methods for the Lindley distribution have been discussed by many authors, mostly including estimation and prediction with different types of data, e.g., Al-Mutairi et al.(2013), Gupta and Singh(2013), Valiollahi et al.(2017), Asgharzadeh et al.(2018, 2017b), and Goel and Krishna(2022).

Chesneau et al.(2020) introduced the 2S-Lindley distribution, which represents the sum of two independent and identically distributed random variables following the common Lindley distribution. The 2S-Lindley distribution with the parameter $\gamma > 0$, denoted by 2S-L(γ), has the following probability density function (pdf)

$$f_\gamma(x) = \frac{\gamma^4}{(1+\gamma)^2}\left(\frac{x^3}{6} + x^2 + x\right)e^{-\gamma x}, \ \ x > 0, \ \gamma > 0.$$

This pdf can be written as a combination of three gamma distribution pdfs as follows:

$$f_\gamma(x) = \frac{1}{(1+\gamma)^2}\Big[f_{GAM}(x,4,\gamma) + 2\gamma f_{GAM}(x,3,\gamma) + \gamma^2 f_{GAM}(x,2,\gamma)\Big],$$

where $f_{GAM}(x,k,\gamma)$ is the pdf of a gamma distribution with parameters k and γ, denoted as GAM(k,γ), given by

$$f_{GAM}(x,k,\gamma) = \frac{\gamma^k}{\Gamma(k)}x^{k-1}e^{-\gamma x}, \quad x > 0, \ k > 0, \ \gamma > 0,$$

where $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt$ denotes a gamma function.
The cumulative distribution function (cdf), for $x > 0$ and $\gamma > 0$, can be obtained as

$$F_\gamma(x) = 1 - \frac{1}{6(1+\gamma)^2}\Big[\gamma^3x^3 + (6\gamma^3+3\gamma^2)x^2 + (6\gamma^3+12\gamma^2+6\gamma)x + (6\gamma^2+12\gamma+6)\Big]e^{-\gamma x}.$$

Note that $f_\gamma(x)$ and $F_\gamma(x)$ can rewritten as

$$f_\gamma(x) = \frac{\gamma^4}{6(1+\gamma)^2}a(x)\, e^{-\gamma x}, \ \ x > 0, \ \gamma > 0. \tag{1}$$

and

$$F_\gamma(x) = 1 - \frac{1}{6(1+\gamma)^2}\Big[\gamma^3 a(x) + \gamma^2 b(x) + \gamma c(x) + 6\Big]e^{-\gamma x}, \tag{2}$$

where $a(x) = x^3 + 6x^2 + 6x, \ b(x) = 3x^2 + 12x + 6$ and $c(x) = 6x + 12$. Here, $b(x)$ and $c(x)$ are the first and second derivatives of $a(x)$, respectively.
The hazard rate function (hrf) is an important function in reliability, defined as the ratio of the pdf to the complementary cdf. For the 2S-L(γ) distribution, the hrf is given by (for $x > 0$ and $\gamma > 0$)

$$h(x) = \frac{f_\gamma(x)}{\bar{F}_\gamma(x)} = \frac{f_\gamma(x)}{1 - F_\gamma(x)} = \frac{\gamma^4 a(x)}{\gamma^3 a(x) + \gamma^2 b(x) + \gamma c(x) + 6}.$$

It can be shown that the hrf of this distribution increases at first and then remains almost constant. In the real world, there are some situations in reliability and survival analysis, where the failure rate behaves like this. For example, many electronic and mechanical devices may have a higher failure rate initially. After fixing the initial failures and the warranty period, these devices enter a stage where their failure rate is fixed for a long time until they fail. This stage is called the useful lifetime period. In some chronic diseases (such as some cancers), as the disease progresses in the early and middle stages, the probability of death increases initially. However, with proper management and early treatments, the disease may be controlled and the risk stabilized. Also, during the recovery times after heavy surgeries, the mortality rate due to infection and surgical complications is high at the beginning. But after a while, the risk stabilizes. Therefore, the 2S-Lindley distribution is suitable for modeling these types of lifetimes and recovery times.
Chesneau et al.(2020) studied the basic statistical and mathematical features of this distribution including moments, characteristic function, stochastic ordering, order statistics, and by analyzing four reliability and survival data sets, they showed the power of this distribution compared to Lindley and exponential distributions. Hamedani and Najaf(2021) showed that the assumption of "independence" of 2 random variables in the 2S-Lindley distribution can be replaced by the much weaker assumption of "sub-independence" of 2 random variables. They then gave some characteristics of the 2S-Lindley distribution. Recently, Chesneau et al.(2023) proposed the Poisson 2S-Lindley distribution by compounding the 2S-Lindley and Poisson distributions.

Let $X_1, X_2, ..., X_n$ be an independent and identically distributed (iid) sample from the 2S-L(γ) distribution with the pdf and cdf given in Eqs. 1 and 2, respectively. Suppose we are only able to observe the first m order statistics $X_{1:n} < X_{2:n} < \ldots < X_{m:n}$, commonly referred to Type-II censored sample. For the sake of simplicity in notation, we will denote the random variables $X_{1:n} < X_{2:n} < \ldots < X_{m:n}$ by $X_1 < X_2 < \ldots < X_m$. Our goal in this paper is to explore the inference of model parameters for the 2S-L(γ) distribution under Type-II censoring, covering both point and interval estimation. We investigate various techniques for estimating γ, including the moment-based method, the maximum likelihood method, bootstrap and Bayesian methods. Additionally, we evaluate the so obtained estimates and assess their effectiveness in practice. To our knowledge, no previous work has analyzed censored data based on the 2S-Lindley distribution.

The organization of the paper is as follows. Section 2 introduces the maximum likelihood method to derive the maximum likelihood estimate MLE of γ and provides approximate confidence intervals (CIs) based on the asymptotic normality of this estimator. Section 3 considers the moment-based method, obtaining a point estimate and an exact CI for γ. In Section 4, we examine the bootstrap method. Section 5 presents Bayes estimates and Bayesian credible intervals using Lindley's approximation and importance sampling techniques. Section 6 applies these techniques to practical data to show their application. A Monte Carlo simulation study is performed to assess the performance of the proposed techniques in Section 7. Finally, the findings and future insights are described in Section 8.

2. Maximum likelihood method

In this section, we derive the maximum likelihood estimator (MLE) of γ using a Type-II censored sample from the 2S-Lindley distribution. We then discuss the asymptotic normality of this MLE and proceed to obtain the corresponding CIs for the model parameter. For the MLE of γ for the case of a complete sample, see Chesneau et al.(2020).

2.1. Maximum likelihood estimation

The likelihood function of γ given the observed Type-II censored sample $x_1 < x_2 < \cdots < x_m$ from the 2S-L(γ) distribution is

$$L(\gamma, \mathbf{x}) = \frac{n!}{(n-m)!} \left\{ \prod_{i=1}^{m} f_\gamma(x_i) \right\} \left[1 - F_\gamma(x_m)\right]^{n-m}.$$

By using Eqs. (1) and (2), we have

$$L(\gamma, \mathbf{x}) \propto \frac{\gamma^{4m}}{(1+\gamma)^{2n}} \prod_{i=1}^{m} a(x_i)\, e^{-\gamma \sum_{i=1}^{m} x_i - (n-m)\gamma x_m} \Big[\gamma^3 a(x_m) + \gamma^2 b(x_m) + \gamma c(x_m) + 6\Big]^{n-m}. \tag{3}$$

The log-likelihood function is

$$\begin{aligned} l(\gamma, \mathbf{x}) = \ln L(\gamma, \mathbf{x}) =& Constant + 4m \ln\gamma - 2n\ln(1+\gamma) - \gamma \left[\sum_{i=1}^{m} x_i + (n-m)x_m\right] \\ &+ (n-m)\ln\Big[\gamma^3 a(x_m) + \gamma^2 b(x_m) + \gamma c(x_m) + 6\Big]. \end{aligned} \tag{4}$$

Now, the MLE of $\gamma, \widehat{\gamma}$, is computed by solving the likelihood equation:

$$\frac{d\, l(\gamma, \mathbf{x})}{d\gamma} = \frac{4m}{\gamma} - \frac{2n}{1+\gamma} - \sum_{i=1}^{m} x_i - (n-m)x_m + (n-m)\left(\frac{3\gamma^2 a(x_m) + 2\gamma b(x_m) + c(x_m)}{\gamma^3 a(x_m) + \gamma^2 b(x_m) + \gamma c(x_m) + 6}\right) = 0, \tag{5}$$

with respect to γ. Numerical methods can be used to solve the above likelihood equation. This equation can also be solved using some functions in mathematical or statistical software.

To verify that the MLE exists and is unique, let us define $\phi(\gamma) = \frac{d\, l(\gamma;\mathbf{x})}{d\gamma}$. The limits of $\phi(\gamma)$, when γ approaches 0 and ∞ respectively, are:

$$\lim_{\gamma\to 0} \phi(\gamma) = +\infty, \quad \lim_{\gamma\to\infty} \phi(\gamma) = -\sum_{i=1}^{m} x_i - (n-m)x_m < 0.$$

Also, after some algebra, it can be shown that

$$\phi'(\gamma) = \frac{d^2 l(\gamma, \mathbf{x})}{d\gamma^2} = \frac{(2n-4m)\gamma^2 - 8m\gamma - 4m}{\gamma^2(1+\gamma)^2} - (n-m) \times \left[\frac{3a_m^2\gamma^4 + 4a_m b_m \gamma^3 + 2b_m^2\gamma^2 + 2(b_m c_m - 18a_m)\gamma + (c_m^2 - 12b_m)}{(a_m\gamma^3 + b_m\gamma^2 + c_m\gamma + 6)^2} \right],$$

where $a_m = a(x_m),\ b_m = b(x_m)$ and $c_m = c(x_m)$ for the simplicity in the notations. Now, since $b_m c_m - 18a_m = 72(x_m + 1) > 0$ and $c_m^2 - 12b_m = 72 > 0$, we can conclude that $\phi'(\gamma) < 0$ when $2n - 4m < 0$ (or $m > \frac{n}{2}$). Therefore, when $m > \frac{n}{2}$, the function $\phi(\gamma)$ is a continuously decreasing function on the interval $(0, \infty)$, which decreases monotonically from $+\infty$ to a negative value. Hence, the MLE of γ, which is the root of the equation $\phi(\gamma) = 0$, exists and is unique when $m > \frac{n}{2}$. In other words, the existence and uniqueness of the MLE is guaranteed when the proportion of censored data is less than 50% of the sample size, which is usually the case in practice.

2.2. Approximate CI

The approximate CIs (ACIs) are obtained based on the asymptotic normality of the MLEs. According to Bartlett(1953), these CIs are known for providing the shortest intervals on average in large samples. When sample sizes are large, the MLE $\widehat{\gamma}$ approximately follows a normal distribution with mean γ and variance σ_γ^2. Here $\sigma_\gamma^2 = \frac{1}{I(\gamma)}$ and $I(\gamma) = -E\left(\frac{d^2 l(\gamma, \mathbf{X})}{d\gamma^2}\right)$, represents the expected Fisher information of the sample. Therefore, the approximate distribution of the pivotal quantity

$$\frac{\widehat{\gamma} - \gamma}{\sqrt{\frac{1}{I(\gamma)}}},$$

is a standard normal. This give a $100(1-\alpha)\%$ approximate CI for γ as

$$\left(\widehat{\gamma}_l^{ACI}, \widehat{\gamma}_u^{ACI}\right) = \widehat{\gamma} \pm z_{1-\frac{\alpha}{2}} \frac{1}{\sqrt{I(\gamma)}},$$

where $z_{1-\frac{\alpha}{2}}$ is the $\left(1 - \frac{\alpha}{2}\right) th$ percentile of the standard normal distribution.

We can approximate the expected Fisher information $I(\gamma)$ by the observed Fisher information $\hat{I}(\hat{\gamma}) = -\frac{d^2 l(\gamma; \mathbf{X})}{d\gamma^2} \mid_{\gamma=\hat{\gamma}}$, and construct the $100(1-\alpha)\%$ approximate CI as

$$\left(\widehat{\gamma}_l^{ACI}, \widehat{\gamma}_u^{ACI}\right) = \widehat{\gamma} \pm z_{1-\frac{\alpha}{2}} \frac{1}{\sqrt{\hat{I}(\hat{\gamma})}}, \tag{6}$$

see e.g., Casella and Berger(2024). The use of the observed Fisher information is superior to the expected Fisher information (see Efron and Hinkley(1978)). Note that from the results given in the previous section, the observed Fisher information is given by

$$\hat{I}(\hat{\gamma}) = -\frac{(2n-4m)\gamma^2 - 8m\gamma - 4m}{\gamma^2(1+\gamma)^2} \mid_{\gamma=\hat{\gamma}} + (n-m) \times \left[\frac{3a_m^2\gamma^4 + 4a_m b_m \gamma^3 + 2b_m^2\gamma^2 + 2(b_m c_m - 18a_m)\gamma + (c_m^2 - 12b_m)}{(a_m\gamma^3 + b_m\gamma^2 + c_m\gamma + 6)^2} \right]_{\gamma=\hat{\gamma}}.$$

It should be noted here that the lower bound of the approximate CI in (6) can be a negative value. Since $\gamma > 0$, we can propose a modified approximate CI as follows:

$$\left(\widehat{\gamma}_{l+}^{ACI}, \widehat{\gamma}_u^{ACI}\right) = \left(\left(\widehat{\gamma} - z_{1-\frac{\alpha}{2}} \frac{1}{\sqrt{\hat{I}(\hat{\gamma})}}\right)_+, \widehat{\gamma} + z_{1-\frac{\alpha}{2}} \frac{1}{\sqrt{\hat{I}(\hat{\gamma})}} \right),$$

where $k_+ = \max\{0, k\}$.

In an alternative approach to ensure that the lower bound of the approximate CI is not become a negative value, some logarithmic transformations can be used on the parameter. Here, we use the logarithmic transformation $\ln\gamma$. The MLE of $\ln\gamma$ is $\ln\hat{\gamma}$. Using the Delta Method, the mean and variance of $\ln\hat{\gamma}$ can be approximated by

$$E(\ln\hat{\gamma}) \approx \ln\gamma, \quad \text{and} \quad Var(\ln\hat{\gamma}) \approx \frac{1}{\gamma^2 I(\gamma)},$$

see e.g., Casella and Berger(2024). Now, since the approximate distribution of

$$\frac{\ln\widehat{\gamma} - \ln\gamma}{\sqrt{\frac{1}{\hat{\gamma}^2\hat{I}(\hat{\gamma})}}},$$

is a standard normal, we have

$$P\left(-z_{1-\frac{\alpha}{2}} < \frac{\ln\widehat{\gamma} - \ln\gamma}{\sqrt{\frac{1}{\hat{\gamma}^2\hat{I}(\hat{\gamma})}}} < z_{1-\frac{\alpha}{2}}\right) = 1-\alpha,$$

from which we get the $100(1-\alpha)\%$ approximate CI for γ using log transformed MLE as

$$\left(\widehat{\gamma}_l^{ACI^*}, \widehat{\gamma}_u^{ACI^*}\right) = \left(\widehat{\gamma}\ \exp\left(-z_{1-\frac{\alpha}{2}}\frac{1}{\hat{\gamma}\sqrt{\hat{I}(\hat{\gamma})}}\right), \widehat{\gamma}\ \exp\left(z_{1-\frac{\alpha}{2}}\frac{1}{\hat{\gamma}\sqrt{\hat{I}(\hat{\gamma})}}\right)\right). \tag{7}$$

3. Moment-based method

Here, we consider the moment-based method (or pivotal-based method) for estimating the unknown parameter γ from the 2S-L(γ) distribution under Type-II censored data.

3.1. Moment-based estimation

Moment-based estimator (MBE) is based on the use of the probability integral transformation (PTT) theorem and a well-known result for spacings of order statistics due to Sukhatme (1937). Suppose $X_1 < X_2 < \cdots < X_m$ is a Type-II censored sample from the 2S-L(γ) distribution with cdf $F_\gamma(x)$ in Eq. (2). Using the PTT theorem, the random variables $Y_1 < Y_2 < \cdots < Y_m$ with

$$\begin{aligned} Y_i &= -\ln\Big(1 - F(X_i,\gamma)\Big) \\ &= -\ln\left\{\frac{1}{6(1+\gamma)^2}\Big[\gamma^3 a(X_i) + \gamma^2 b(X_i) + \gamma c(X_i) + 6\Big]e^{-\gamma X_i}\right\} \\ &= \ln 6 + 2\ln(1+\gamma) + \gamma X_i - \ln\Big[\gamma^3 a(X_i) + \gamma^2 b(X_i) + \gamma c(X_i) + 6\Big], \quad i = 1, 2, ..., m, \end{aligned}$$

are a Type-II censored sample from a standard exponential distribution, Exp(1). In addition, the spacings $T_i = (n-i+1)(Y_i - Y_{i-1}),\ i = 1, ..., m$, with $Y_0 = 0$, are themselves iid random variables from the Exp(1) distribution.

To obtain the MBE of γ, denoted as $\widehat{\gamma}^{MBE}$, let us define $W(\gamma) \equiv W(\mathbf{X};\gamma) = 2\sum_{i=1}^m T_i$. It can be shown that

$$W(\gamma) = 2\sum_{i=1}^m T_i = 2\sum_{i=1}^m s_i\left(\ln 6 + 2\ln(1+\gamma) + \gamma x_i - \ln\Big[\gamma^3 a(X_i) + \gamma^2 b(X_i) + \gamma c(X_i) + 6\Big]\right), \tag{8}$$

where $s_1 = ... = s_{m-1} = 1$ and $s_m = n - m + 1$. It is also clear that the distribution of $W(\gamma)$ is a chi-square with $2m$ degrees of freedom, i.e., $W(\gamma) \sim \chi^2_{2m}$. Moreover, $W(\gamma)/2m$ converges in probability to 1. Then, we can obtain the MBE estimate of γ as the unique solution of the equation $W(\gamma)=2m$, or the equation

$$\sum_{i=1}^{m} s_i\left(\ln 6+2\ln(1+\gamma)+\gamma x_i-\ln\left[\gamma^3 a(x_i)+\gamma^2 b(x_i)+\gamma c(x_i)+6\right]\right)=m. \tag{9}$$

Let us now investigate that the Eq. (9) has exactly one solution. By differentiating and some calculations, it is easy to check that

$$\begin{aligned}
&\frac{d}{d\gamma}\left(\ln 6+2\ln(1+\gamma)+\gamma x_i-\ln\left[\gamma^3 a(x_i)+\gamma^2 b(x_i)+\gamma c(x_i)+6\right]\right)\\
&=\frac{2}{1+\gamma}+x_i-\frac{3\gamma^2 a(x_i)+2\gamma b(x_i)+c(x_i)}{\gamma^3 a(x_i)+\gamma^2 b(x_i)+\gamma c(x_i)+6}\\
&=\frac{(x_i^4+6x_i^3+6x_i^2)\gamma^4+(x_i^4+8x_i^3+12x_i^2)\gamma^3}{(1+\gamma)\left[(x_i^3+6x_i^2+6x_i)\gamma^3+3(x_i^2+4x_i+2)\gamma^2+6(x_i+2)\gamma+6\right]}\\
&\geq 0.
\end{aligned}$$

This shows that $\frac{dW(\gamma)}{d\gamma}\geq 0$, and therefore $W(\gamma)$ is monotonic increasing in $\gamma\in(0,+\infty)$. On the other hand, since $W(\gamma)$ is a continuous function with $\lim_{\gamma\to 0}W(\gamma)=0$ and $\lim_{\gamma\to+\infty}W(\gamma)=+\infty$, the equation $W(\gamma)=2m$ has a unique solution in $(0,+\infty)$. As a result, we can guarantee that MBE exists and is unique.

3.2. Exact confidence interval

Here, an exact CI for the parameter γ of the 2S-L(γ) distribution is investigated. To obtain an exact CI (ECI) for γ, consider the random variable

$$W(\gamma)=2\sum_{i=1}^{m} s_i\left(\ln 6+2\ln(1+\gamma)+\gamma X_i-\ln\left[\gamma^3 a(X_i)+\gamma^2 b(X_i)+\gamma c(X_i)+6\right]\right),$$

given in Eq. (8). This random variable is a pivotal quantity (or pivot) since it is a function of **X** and γ whose distribution does not depend on the parameter γ . We can use this pivot to construct an ECI for γ. Since $W(\gamma)\sim\chi^2_{2m}$, we have

$$P\left(\chi^2_{\alpha/2,2m}<W(\gamma)<\chi^2_{1-\alpha/2,2m}\right)=1-\alpha,$$

where $\chi^2_{\alpha/2,2m}$ and $\chi^2_{1-\alpha/2,2m}$ represent the lower and upper $\alpha/2$ percentage points of χ^2_{2m}, respectively. Since $W(\gamma)$ is a monotonic increasing function in γ, we can invert the above interval to obtain the $100(1-\alpha)\%$ ECI

$$\left(\widehat{\gamma}_l^{ECI},\widehat{\gamma}_u^{ECI}\right)=\left(W^{-1}\left(\chi^2_{\alpha/2,2m}\right),W^{-1}\left(\chi^2_{1-\alpha/2,2m}\right)\right), \tag{10}$$

where $W^{-1}(t)$ is the solution of γ for the equation $W(\gamma)=t$.

4. Bootstrap method

Bootstrap is a straightforward approach to approximate the sampling distribution of estimators, especially to investigate their biases and standard errors. This approach was first introduced by Efron and Hinkley(1978) and has received much attention in recent decades. For more details and developments, see Efron and Tibshirani(1993). The bootstrap method is based on resampling. In this method, first a random sample is taken from the target population and then resampling is done based on the original sample. Then, in each iteration, the desired estimator value is computed. The average of these values is used to approximate the bias of the estimator and the variance of these values is used to approximate the standard error of the estimator. Usually, the bootstrap method is used when the distribution of the estimators is not known and the sample size is small, and as a result large sample methods cannot be used.

Here we consider the percentile bootstrap method, which is a common and popular method. Then, we discuss the estimation of parameter γ using parametric bootstrap (PB) method. The BP method involves resampling from a known distribution, where the parameter is estimated from generated samples. The following algorithm is employed to obtain the BP estimates of γ of the 2S-L(γ) distribution:

1. Using the original sample $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, compute the MLE of γ , $\widehat{\gamma}$.
2. Based on the MLE $\widehat{\gamma}$ obtained in Step 1, generate a random sample of size n from the 2S-L($\widehat{\gamma}$) distribution and derive the first m order statistic(s), $\mathbf{x}^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)})$.
3. Calculate the MLE of γ based on $\mathbf{x}^{(1)}$, say $\widehat{\gamma}^{(1)}$.
4. Repeat the first two steps B times and obtain $\widehat{\gamma}^{(1)}, \widehat{\gamma}^{(2)}, \ldots, \widehat{\gamma}^{(B)}$.
5. Sort $\widehat{\gamma}^{(1)}, \widehat{\gamma}^{(2)}, ..., \widehat{\gamma}^{(B)}$ calculated in the previous step as $\widehat{\gamma}^{[1]}, \widehat{\gamma}^{[2]}, \ldots, \widehat{\gamma}^{[B]}$.

After that, a two-sided $100(1-\alpha)\%$ BP CI of γ is given by the form

$$\left(\widehat{\gamma}_l^{PBCI}, \widehat{\gamma}_u^{PBCI}\right) = \left(\widehat{\gamma}^{([B\alpha/2])}, \widehat{\gamma}^{([B(1-\alpha/2)])}\right). \tag{11}$$

Also, using the bootstrap estimates $\widehat{\gamma}^{(1)}, \widehat{\gamma}^{(2)}, ..., \widehat{\gamma}^{(B)}$, the biase and standard error of the MLE $\widehat{\gamma}$ can be estimated as

$$\widehat{Bias(\widehat{\gamma})} = \overline{\widehat{\gamma}} - \widehat{\gamma},$$

and

$$\widehat{SE(\widehat{\gamma})} = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}\left(\widehat{\gamma}^{(i)} - \overline{\widehat{\gamma}}\right)^2}$$

where

$$\overline{\widehat{\gamma}} = \frac{1}{B}\sum_{i=1}^{B}\widehat{\gamma}^{(i)},$$

is the average of the bootstrap estimates. The estimator $\overline{\widehat{\gamma}}$ is known as the BP point estimate of γ, which we denote it by $\widehat{\gamma}^{PBE}$.

Note that in the non-parametric bootstrap method, there is no assumption about the type of population distribution. Additionally, bootstrap samples are generated by resampling with replacement from the original sample. So, the steps for the non-parametric bootstrap procedure are similar to the parametric bootstrap. The only difference is that in non-parametric bootstrap , the bootstrap samples $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(B)}$ are generated by random sampling with replacement from the original data $\mathbf{x} = (x_1, x_2, ..., x_m)$. In this paper, since the population distribution is known, we used the parametric bootstrap.

5. Bayesian method

In a Bayesian approach, a prior distribution is considered as an initial beliefs about an unknown parameter before seeing any data. Once we have the data, we update this prior knowledge by incorporating the new information to form the posterior distribution. This posterior distribution is then used for estimation and making other inferences. Here we consider the GAM(α_1, α_2) distribution as a prior distribution for γ, where α_1 and α_2 are the prior parameters (or hyperparameters), i.e., we consider the prior pdf as $\pi_{\alpha_1,\alpha_2}(\gamma) = f_{GAM}(\gamma, \alpha_1, \alpha_2)$. Combining the likelihood function in Eq. (3) and the prior pdf, we can derive the posterior pdf of γ given the data $\mathbf{x}$ as

$$\begin{aligned}\pi(\gamma \mid \mathbf{x}) &\propto \pi_{\alpha_1,\alpha_2}(\gamma)\; L(\gamma, \mathbf{x}) \\ &\propto \frac{\gamma^{4m+\alpha_1-1}}{(1+\gamma)^{2n}}\; e^{-\gamma\left(\sum_{i=1}^{m} x_i + (n-m)x_m + \alpha_2\right)} \left[\gamma^3 a(x_m) + \gamma^2 b(x_m) + \gamma c(x_m) + 6\right]^{n-m}.\end{aligned} \tag{12}$$

Here we compute the Bayesian estimator (BE) of γ under the quadratic loss (or squared error loss)

$$L(\gamma, \widehat{\gamma}) = (\widehat{\gamma} - \gamma)^2,$$

which is the most common and popular loss function in Bayesian estimation. The BE of γ under this loss function, denoted by $\widehat{\gamma}^{BE}$ is the posterior mean, i.e., $\widehat{\gamma}^{BE} = E(\gamma \mid \mathbf{x})$. Because we can not find the posterior mean in a closed form, the BE of γ can not be calculated directly. Instead, we use numerical methods to approximate the BE. Here, two approximate methods are employed to obtain the BE: Lindley's method and importance sampling method. First, note that we can write the BE of γ of as the ratio of two integrals in the following form:

$$\widehat{\gamma}^{BE} = \frac{\int_0^\infty \gamma\pi(\gamma \mid \mathbf{x})d\gamma}{\int_0^\infty \pi(\gamma \mid \mathbf{x})d\gamma},$$

since $\int_0^\infty \pi(\gamma \mid \mathbf{x})d\gamma = 1$.

5.1. Lindley's approximation

Lindley's method is an approximate method for calculating the BEs, which was first proposed by Lindley in 1958. In this approximate method, one must first write the BEs as the ratio of two integrals and then use the approximation. In the literature, it is common to use the Lindley's approximation method to calculate the BEs when the posterior distribution has a complicated form, see e.g., Sharma et al.(2017) and Asgharzadeh et al.(2017b). For the case when the statistical model includes only one parameter, such as the 2S-Lindley model under study, the Lindley's approximation for any function $U(\gamma)$ of γ is as follows:

$$E\left(U(\gamma) \mid \mathbf{X} = \mathbf{x}\right) = U(\gamma) + \frac{1}{2}[U_2(\gamma) + 2U_1(\gamma)\rho_1(\gamma)]\,\sigma_\gamma^2 + \frac{1}{2}l_3(\gamma)U_1(\gamma)\sigma_\gamma^4, \tag{13}$$

where for $i = 1, 2, 3$ and $j = 1, 2$

$$l_i(\gamma) = \frac{\partial^i}{\partial\gamma^i}l(\gamma, \mathbf{x}),$$

$$U_j(\gamma) = \frac{\partial^j}{\partial\gamma^j}U(\gamma),$$

and

$$\rho_1(\gamma) = \frac{\partial}{\partial\gamma}\rho(\gamma), \quad \rho(\gamma) = \ln(\pi(\gamma)).$$

Now, from the gamma prior pdf $\pi_{\alpha_1,\alpha_2}(\gamma)$, we have $\rho_1(\gamma) = \frac{\alpha_1-1}{\gamma} - \alpha_2$. To get the BE of γ, we consider $U(\gamma)$ as $U(\gamma) = \gamma$, from which we obviously obtain $U_1(\gamma) = 1$ and $U_2(\gamma) = 0$. Also, the third-order derivative of $l(\gamma; \mathbf{x})$ can be found as follows

$$l_3(\gamma) = \frac{8m}{\gamma^3} - \frac{4n}{(1+\gamma)^3} + (n-m)A(m,\gamma,x_m) + (n-m)B(m,\gamma,x_m),$$

where

$$A(m,\gamma,x_m) = \frac{6a_m^3\gamma^6 + 12a_m^2b_m\gamma^5 + (12a_mb_m^2 - 6a_m^2c_m)\gamma^4 + (10a_m^2c_m + 4b_m^2 - 4a_mb_mc_m - 252a_m^2)\gamma^3}{(a_m\gamma^3 + b_m\gamma^2 + c_m\gamma + 6)^3}$$

and $B(m,\gamma,x_m)$ is equal to the expression:

$$\frac{(6a_mb_mc_m + 6a_mc_m^2 - 252a_mb_m)\gamma^2 + (2a_mc_m^2 + 4b_mc_m^2 - 36a_mc_m - 72b_m^2)\gamma + (216a_m + 2c_m^3 - 12a_mc_m - 24b_mc_m)}{(a_m\gamma^3 + b_m\gamma^2 + c_m\gamma + 6)^3},$$

with $a_m = a(x_m),\ b_m = b(x_m)$ and $c_m = c(x_m)$.
Now, by replacing the above expressions in Eq. (13), the BE of γ using Lindley's approximation, denoted by $\widehat{\gamma}^{BEL}$, is

$$\widehat{\gamma}^{BEL} = \widehat{\gamma} + \left(\frac{\alpha_1 - 1}{\widehat{\gamma}} - \alpha_2\right)\,\sigma_{\widehat{\gamma}}^2 + \frac{1}{2}l_3(\widehat{\gamma})\,\sigma_{\widehat{\gamma}}^4, \tag{14}$$

where $\widehat{\gamma}$ is the MLE of γ.

5.2. Importance sampling method

Here, the importance sampling technique is employed to generate samples from the posterior distribution and then approximate the BE of the parameter γ. First, note that the posterior pdf of γ in Eq. (12) can be expressed in an alternative form:

$$\pi(\gamma \mid \mathbf{X}) \propto \pi_{\alpha_1',\alpha_2'}(\gamma)\ g(\mathbf{x},\gamma), \tag{15}$$

where $\pi_{\alpha_1',\alpha_2'}(\gamma)$ is a gamma pdf with shape parameter $\alpha_1' = 4m + \alpha_1$ and scale parameter $\alpha_2' = \sum_{i=1}^m x_i + (n-m)x_m + \alpha_2$, and $g(\mathbf{x};\gamma)$ is given by

$$g(\mathbf{x},\gamma) = \frac{\left[\gamma^3 a(x_m) + \gamma^2 b(x_m) + \gamma c(x_m) + 6\right]^{n-m}}{(1+\gamma)^{2n}}.$$

Now, using (15), we can rewrite the BE of γ as

$$\begin{aligned}\widehat{\gamma}^{BE} = E(\gamma \mid \mathbf{x}) &= \frac{\int_0^\infty \gamma\, \pi(\gamma \mid \mathbf{x})d\gamma}{\int_0^\infty \pi(\gamma \mid \mathbf{x})d\gamma},\\ &= \frac{\int_0^\infty \gamma\, \pi_{\alpha_1',\alpha_2'}(\gamma)\, g(\mathbf{x},\gamma)d\gamma}{\int_0^\infty \pi_{\alpha_1',\alpha_2'}(\gamma)\, g(\mathbf{x},\gamma)d\gamma},\\ &= \frac{E_{\pi'}\left[\gamma\, g(\mathbf{x},\gamma)\right]}{E\left[g(\mathbf{x},\gamma)\right]},\end{aligned}$$

where $E_{\pi'}(.)$ denotes the expectation with respect to the gamma pdf $\pi'(\gamma) = \pi_{\alpha_1',\alpha_2'}(\gamma)$. So, we can approximate the BE $\widehat{\gamma}^{BE}$ as

$$\widehat{\gamma}^{BE} \approx \frac{\frac{1}{M}\sum_{i=1}^M \gamma_i\, g(\mathbf{x},\gamma_i)}{\frac{1}{M}\sum_{i=1}^M g(\mathbf{x},\gamma_i)},$$

based on a random sample $\gamma_1, ..., \gamma_M$ generated from the gamma distribution with parameters α_1' and α_2', i.e., GAM(α_1', α_2'). Consequently, we can compute the BE of γ using the following two-step algorithm:

step1. Generate $\gamma_1, \ldots, \gamma_M$ from the GAM(α_1', α_2') distribution,
step 2. Choose

$$\widehat{\gamma}^{BEI} = \frac{\sum_{i=1}^M \gamma_i\, g(\mathbf{x} \mid \gamma_i)}{\sum_{i=1}^M g(\mathbf{x} \mid \gamma_i)} \tag{16}$$

as the BE of γ using the importance sampling technique.
Now, we find the Bayesian credible interval (BCI) for γ using the technique given in Chen and Shao(1999). For this purpose, suppose $\Pi(\gamma \mid \mathbf{x})$ be the posterior cdf corresponding to the posterior pdf of $\pi(\gamma \mid \mathbf{X})$, and $\gamma^{(\delta)}$ (for $0 < \delta < 1$) is the δ-th quantile of γ defined as:

$$\gamma^{(\delta)} = \inf\left\{\gamma : \Pi(\gamma \mid \mathbf{x}) \geq \delta\right\}.$$

For a given value of γ^*, we clearly have

$$\Pi(\gamma^* \mid \mathbf{x}) = E\{I_{\gamma \leq \gamma^*}(\gamma) \mid \mathbf{x}\} = \frac{\int_0^\infty I_{\gamma \leq \gamma^*}(\gamma)\ \pi(\gamma \mid \mathbf{x})d\gamma}{\int_0^\infty \pi(\gamma \mid \mathbf{x})d\gamma},$$

where $I_{\gamma \leq \gamma^*}(\gamma)$ is the indicator function defined by

$$I_{\gamma \leq \gamma^*}(\gamma) = \begin{cases} 1 & \gamma \leq \gamma^*, \\ 0 & \gamma > \gamma^*. \end{cases}$$

Therefore, as described before, we can approximate $\Pi(\gamma^* \mid \mathbf{x})$ as

$$\widehat{\Pi}(\gamma^* \mid \mathbf{x}) \approx \frac{\frac{1}{M}\sum_{i=1}^M I_{\gamma_i \leq \gamma^*}(\gamma_i) g(\mathbf{x},\gamma_i)}{\frac{1}{M}\sum_{i=1}^M g(\mathbf{x};\gamma_i)},$$

based on the random sample $\gamma_1, ..., \gamma_M$ generated from the GAM(α_1', α_2') distribution.

Now, let us sort the generated sample $\gamma_1, ..., \gamma_M$ as $\gamma_{1:M}, ..., \gamma_{M:M}$, and define

$$w_i = \frac{g(\mathbf{x}, \gamma_{i:M})}{\sum_{i=1}^{M} g(\mathbf{x}, \gamma_{i:M})}.$$

Then, we can approximately rewrite $\widehat{\Pi}(\gamma^* \mid \mathbf{x})$ as

$$\widehat{\Pi}(\gamma^* \mid \mathbf{x}) = \begin{cases} 0 & \gamma^* < \gamma_{(1)}, \\ \sum_{j=1}^{i} w_j & \gamma_{i:M} < \gamma^* < \gamma_{i+1:M}, \\ 1 & \gamma^* \geq \gamma_{M:M}. \end{cases}$$

Consequently, an approximate value for $\gamma^{(\delta)}$ can be given as:

$$\widehat{\gamma}^{(\delta)} \approx \begin{cases} \gamma_{(1)} & \delta = 0, \\ \gamma_{(i)} & \sum_{j=1}^{i-1} w_j < \delta \leq \sum_{j=1}^{i} w_j. \end{cases}$$

Finally, to construct a $100(1-\delta)\%$ highest posterior density (HPD) BCI for γ, we consider all the BCIs given as:

$$\left(\widehat{\gamma}^{\left[\frac{j}{M}\right]}, \widehat{\gamma}^{\left[\frac{j+[(1-\delta)M]}{M}\right]}\right), \quad j = 1, 2, ..., M - [(1-\delta)M],$$

where $[k]$ is the largest integer function that represents the greatest integer less than or equal to k. Among all these BCIs, the interval with the shortest length will be a $100(1-\delta)\%$ HPD BCI for γ.

6. Real data analysis

Here we consider a real data set and discuss the various estimation techniques described in this paper using this data set. This data set is taken from Gross and Clark(1976) and shows the relief times (in minutes) of 20 patient(s) receiving analgesics. Recently, Chesneau et al.(2020), using various goodness-of-fit tests, showed that the 2S-Lindley distribution fits these data very well and can be a suitable distribution for modeling this data set. The data in ascending order are given in Table 1 below.

Table 1: Relief times data.

1.1354	1.2872	1.3124	1.4550	1.4794	1.5112	1.6012	1.6879	1.7463	1.7128
1.7755	1.8946	1.8124	1.9965	2.0023	2.2187	2.3283	2.7128	3.0235	4.1846

To achieve the Type-II censored sample, suppose that in this study, as soon as the 16th relief time was observed, the experiment was stopped and the relief times of the remaining 4 patient(s) were not observed. Therefore, we are faced with a Type-II censored sample with $n = 20$ and $m = 16$. All the computations were performed using the statistical software R.
Based on these data, we computed point and interval estimates of γ from the 2S-Lindley distribution using the various estimation techniques described in Sections 2 to 5. The results for point estimation are summarized in Table 2 and the results for interval estimation are summarized in Table 3. To calculate the interval estimates (or CIs), we considered the confidence level to be $1 - \alpha = 0.95$. Also, in the Bayesian technique, the GAM(a_1, a_2) prior distribution with $a_1 = a_2 = 0.001$ was used to calculate the Bayesian point and interval estimates. Although this prior distribution is a proper prior distribution, it is approximately a non-informative prior distribution.

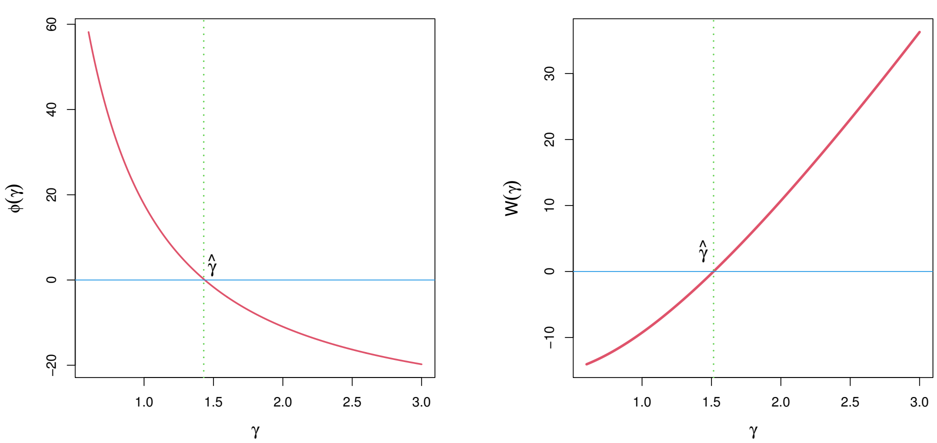
Since the MBE (and its corresponding CI) and MLE involve solving some nonlinear equations, the function "`uniroot`" in R was used to solve the corresponding nonlinear equations and calculate these estimates. We can also use the graphical method to compute the MBE and MLE. In Figure 1, we provided the plots of $\phi(\gamma)$ and $w(\gamma)$. As we can see, $\phi(\gamma)$ and $w(\gamma)$ are strictly decreasing and strictly increasing in terms of γ, respectively (as shown theoretically in Section(s) 2 and 3). The intersection of the graph of these functions with the horizontal axis, i.e. the roots of the equations $\phi(\gamma) = 0$ and $w(\gamma) = 0$ respectively represent the MLE and MBE. Theses plots again show the existence and uniqueness of the MLE and MBE.

Table 2: Point estimates of γ for the relief times data.

Technique	MBE	MLE	PBE	BEL	BEI
Point estimate	1.5127	1.4374	1.4621	1.4386	1.4561

Table 3: Interval estimates of γ for the relief times data.

Technique	Interval estimate
ECI	(1.1502, 1.9154)
ACI	(1.0648, 1.8101)
ACI*	(1.1092, 1.8628)
PB-CI	(1.0968, 1.9133)
BCI	(1.2466, 1.7767)



(a) mle (b) mbe

Figure 1: Plots of $\phi(\gamma)$ and $w(\gamma)$ for the relief times data.

7. Simulation and numerical comparisons

Given that many of the point estimators discussed in this paper are computed numerically and lack explicit formulas, assessing their performance requires a simulation study. For this, we conduct a Monte Carlo simulation study designed to evaluate and compare the effectiveness of these estimators, both point and interval. Our simulation is performed for different Type-II censoring schemes, two default values for the parameter γ, and two distinct prior distributions. In this context, let us consider the following Type-II censoring schemes:

$$(n, m) = (7, 4), (10, 7), (15, 11), (15, 12), (20, 16), (20, 17), (25, 20), (25, 21), \text{ and } (25, 22).$$

We use two default values for γ: $\gamma = 0.75$ and $\gamma = 1.5$. For the prior distributions of γ, we consider GAM$(0.001, 0.001)$ and GAM$(3, 3)$. While the first prior is nearly non-informative, the later is informative.
Based on different censoring scheme, two default values for the parameter γ and two priors considered above, we randomly generated samples of the 2S-L(γ) distribution and for each generated sample, we calculated the point estimates MBE, MLE, PBE, BEL and BEI. We also calculated different 95% CIs of γ for each generated sample. We compared the performance of point estimates in terms of mean squared error MSE, based on $N = 3000$ simulations. The MSE is the most popular and fundamental measure for evaluating different estimators, which includes two components, bias and variance. The estimated MSE in the simulation is calculated as

$$\widehat{MSE} = N^{-1} \sum_{i=1}^{N} (\widehat{\gamma}_i - \gamma)^2,$$

where $\widehat{\gamma}_i$ is the estimate of γ obtained in the ith simulation (for $i = 1, 2, ..., N$).
The performance of different CIs including ECI, ACI, ACI*, bootstrap and Bayesian CIs are evaluated based on their average interval lengths (AILs) and simulated coverage probabilities (CPs). If (L_i, U_i) represents a CI for γ obtained in the ith iteration of the simulation, then the simulated CP (SCP) is computed as SCP $= N^{-1} \sum_{i=1}^{N} I_{(L_i, U_i)}(\gamma)$, where $I(.)$ is the indicator function.

The simulation results are given in Tables 4-7. The estimated MSE (EMSE) of the different point estimates of γ are presented in Tables 4 and 5. The AIL and SCP of the different CIs are shown in Tables 6 and 7. From Tables 4 and 5, we see that the Bayesian point estimates perform better than the other point estimates, even based on the prior GAM$(0.001, 0.001)$. The MLE works better than the MBE. The bootstrap estimation does not perform well and gives the largest EMSE in most cases considered. Also, the Bayes estimates with the prior GAM$(3, 3)$ perform better than the Bayes estimates with the prior GAM$(0.001, 0.001)$, which is reasonable. In comparing the Bayes estimates based on the Lindley method and the importance sampling method, Lindley's Bayes estimates have a better performance.

For the interval estimation, it is evident from Tables 6 and 7 that for most cases, the SCP of different intervals are close to the true 95% confidence level. Bayesian intervals using the prior GAM$(3, 3)$ are shorter than those using the prior GAM$(0.001, 0.001)$. The approximate CIs based on the asymptotic normality of the MLE are shorter than the exact CIs. The bootstrap method generally performs poorly, producing the longest intervals in most cases considered. Additionally, approximate CIs based on the asymptotic normality of $\log$ MLE are larger than those based on the MLE. Finally, as shown in Tables 4-7, as m increases (with n fixed), the EMSE of the point estimates and the AIL of the CIs both decrease. This can be explained by the fact that as m increases, the amount of information from the data grows, leading to improved performance of both point estimates and CIs.

Table 4: The EMSE of the different point estimates when $\gamma = 0.75$.

					GAM$(0.001, 0.001)$		GAM$(3, 3)$	
n	m	MBE	MLE	PBE	BEI	BEL	BEI	BEL
7	4	0.0525	0.0489	0.0518	0.0493	0.0487	0.0400	0.0373
10	7	0.0238	0.0224	0.0267	0.0239	0.0237	0.0220	0.0218
15	11	0.0154	0.0147	0.0168	0.0143	0.0141	0.0135	0.0137
	12	0.0141	0.0136	0.0151	0.0142	0.0141	0.0134	0.0134
20	16	0.0097	0.0094	0.0101	0.0095	0.0094	0.0093	0.0093
	17	0.0095	0.0093	0.0101	0.0094	0.0093	0.0091	0.0091
25	20	0.0079	0.0076	0.0081	0.0077	0.0077	0.0076	0.0075
	21	0.0078	0.0075	0.0080	0.0071	0.0073	0.0072	0.0072
	22	0.0075	0.0072	0.0077	0.0073	0.0071	0.0071	0.0070

Table 5: The EMSE of the different point estimates when $\gamma = 1.5$.

					GAM$(0.001, 0.001)$		GAM$(3, 3)$	
n	m	MBE	MLE	PBE	BEI	BEL	BEI	BEL
7	4	0.2498	0.2358	0.3424	0.2386	0.2355	0.0856	0.0547
10	7	0.1178	0.1118	0.14212	0.1165	0.1163	0.0641	0.0540
15	11	0.0718	0.0687	0.0793	0.0646	0.0641	0.0449	0.0419
	12	0.0652	0.0622	0.0710	0.0622	0.0603	0.0444	0.1253
20	16	0.0458	0.0445	0.0496	0.0507	0.0470	0.0375	0.0351
	17	0.0447	0.0432	0.0481	0.0485	0.0431	0.0367	0.0329
25	20	0.0357	0.0341	0.0372	0.0380	0.0346	0.0312	0.0280
	21	0.0350	0.0337	0.0368	0.0402	0.0336	0.0319	0.0275
	22	0.0331	0.0320	0.0346	0.0403	0.0308	0.0314	0.0252

Table 6: The AIL and SCP for the different CIs when $\gamma = 0.75$.

							GAM$(0.001, 0.001)$	GAM$(3, 3)$
n	m		ECI	ACI	ACI*	PB-CI	BCI	BCI
7	4	AIL	0.759	0.758	0.786	0.880	0.731	0.691
		SCP	0.940	0.956	0.930	0.914	0.951	0.969
10	7	AIL	0.567	0.564	0.576	0.608	0.556	0.539
		SCP	0.941	0.940	0.933	0.930	0.954	0.960
15	11	AIL	0.455	0.451	0.458	0.470	0.439	0.432
		SCP	0.953	0.952	0.955	0.942	0.951	0.955
	12	AIL	0.435	0.431	0.437	0.448	0.426	0.419
		SCP	0.955	0.955	0.955	0.946	0.945	0.947
20	16	AIL	0.375	0.371	0.374	0.382	0.363	0.359
		SCP	0.957	0.952	0.958	0.949	0.944	0.949
	17	AIL	0.370	0.366	0.369	0.376	0.353	0.349
		SCP	0.943	0.948	0.941	0.940	0.940	0.938
25	20	AIL	0.339	0.335	0.338	0.341	0.324	0.321
		SCP	0.954	0.950	0.944	0.940	0.942	0.947
	21	AIL	0.333	0.329	0.332	0.336	0.311	0.309
		SCP	0.949	0.947	0.951	0.944	0.929	0.931
	22	AIL	0.328	0.324	0.326	0.330	0.295	0.293
		SCP	0.958	0.962	0.957	0.953	0.913	0.919

Table 7: The AIL and SCP for the different CIs when $\gamma = 1.5$.

							GAM$(0.001, 0.001)$	GAM$(3, 3)$
n	m		ECI	ACI	ACI*	PB-CI	BCI	BCI
7	4	AIL	1.635	1.639	1.708	1.949	1.590	1.296
		SCP	0.949	0.964	0.943	0.922	0.945	0.953
10	7	AIL	1.215	1.210	1.241	1.326	1.169	1.042
		SCP	0.946	0.945	0.935	0.935	0.942	0.949
15	11	AIL	0.951	0.945	0.960	0.999	0.895	0.837
		SCP	0.947	0.950	0.949	0.945	0.925	0.940
	12	AIL	0.922	0.916	0.929	0.953	0.822	0.779
		SCP	0.954	0.949	0.952	0.936	0.891	0.923
20	16	AIL	0.805	0.799	0.808	0.824	0.664	0.642
		SCP	0.950	0.963	0.952	0.947	0.869	0.897
	17	AIL	0.786	0.780	0.788	0.807	0.604	0.587
		SCP	0.949	0.951	0.955	0.941	0.818	0.863
25	20	AIL	0.708	0.703	0.709	0.720	0.554	0.542
		SCP	0.957	0.954	0.959	0.949	0.833	0.867
	21	AIL	0.705	0.699	0.705	0.717	0.509	0.498
		SCP	0.948	0.952	0.952	0.936	0.791	0.820
	22	AIL	0.685	0.679	0.685	0.696	0.464	0.457
		SCP	0.933	0.938	0.939	0.937	0.737	0.780

8. Concluding remarks

In this study, we examined various estimation techniques for Type-II censored samples from the 2S-L(γ) distribution. We provided point estimates using methods including MLE, MBE, bootstrap, and Bayesian approaches, and constructed confidence intervals for the model parameters. Our simulation results indicate that Bayesian methods show superior point and interval estimatess compared to the others. While this paper focused on estimation for the 2S-Lindley distribution with Type-II censored data, future research could tackle the prediction of future failures in reliability and survival analysis, assuming a 2S-Lindley lifetime distribution. Another area for future work could involve studying stress-strength reliability with Type-II censored data. Additionally, the estimation methods discussed can be adapted for other data types, such as progressively censoring or record data. We are currently working on these extensions.

References

1. Abdi, M., Asgharzadeh, A., Bakouch, H. S., and Alipour, Z. (2019). A new compound gamma and Lindley distribution with application to failure data. *Austrian Journal of Statistics*, 48(3):54–75.
2. Al-Mutairi, D., Ghitany, M., and Kundu, D. (2013). Inferences on stress-strength reliability from Lindley distributions. *Communications in Statistics-Theory and Methods*, 42(8):1443–1463.
3. Asgharzadeh, A., Fallah, A., Raqab, M., and Valiollahi, R. (2018). Statistical inference based on Lindley record data. *Statistical Papers*, 59:759–779.
4. Asgharzadeh, A., Nadarajah, S., and Sharafi, F. (2017a). Generalized inverse Lindley distribution with application to danish fire insurance data. *Communications in Statistics-Theory and Methods*, 46(10):5001–5021.
5. Asgharzadeh, A., Ng, H. K., Valiollahi, R., and Azizpour, M. (2017b). Statistical inference for Lindley model based on type ii censored data. *Journal of Statistical Theory and Applications*, 16(2):178–197.
6. Bakouch, H. S., Al-Zahrani, B. M., Al-Shomrani, A. A., Marchi, V. A., and Louzada, F. (2012). An extended Lindley distribution. *Journal of the Korean Statistical Society*, 41:75–85.
7. Bartlett, M. (1953). Approximate confidence intervals. *Biometrika*, 40(1/2):12–19.
8. Casella, G. and Berger, R. (2024). *Statistical inference(Reprint ed.).* CRC press.
9. Chen, M.-H. and Shao, Q.-M. (1999). Monte carlo estimation of Bayesian credible and HPD intervals. *Journal of computational and Graphical Statistics*, 8(1):69–92.
10. Chesneau, C., D'cruz, V., Maya, R., and Irshad, M. (2023). A novel discrete distribution based on the mixture of poisson and sum of two Lindley random variables: Accepted-september 2023. *REVSTAT-Statistical Journal*.
11. Chesneau, C., Tomy, L., and Gillariose, J. (2020). On a sum and difference of two Lindley distributions: Theory and application. *REVSTAT-Statistical Journal*, 18(5):673–695.
12. Efron, B. and Hinkley, D. V. (1978). Assessing the accuracy of the maximum likelihood estimator: Observed versus expected Fisher information. *Biometrika*, 65(3):457–483.
13. Efron, B. and Tibshirani, R. (1993). *An Introduction to the Bootstrap*. Boca Raton, Florida, Tibshirani, R.J;Chapman and Hall/CRC;Springer, 1 edition.
14. Ghitany, M. E., Al-Mutairi, D. K., Balakrishnan, N., and Al-Enezi, L. (2013). Power Lindley distribution and associated inference. *Computational Statistics & Data Analysis*, 64:20–33.
15. Goel, R. and Krishna, H. (2022). Statistical inference for two Lindley populations under balanced joint progressive type-ii censoring scheme. *Computational Statistics*, 37(1):263–286.
16. Gross, A. J. and Clark, V. (1976). Review of survival distributions: Reliability applications in the biomedical sciences. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 3:303–303.
17. Gupta, P. K. and Singh, B. (2013). Parameter estimation of Lindley distribution with hybrid censored data. *International Journal of System Assurance Engineering and Management*, 4:378–385.
18. Hamedani, G. and Najaf, M. (2021). Remarks on and characterizations of 2S-Lindley and 2D-Lindley distributions introduced by Chesneau et al.(2020). *Pakistan Journal of Statistics and Operation Research*, pages 227–234.
19. Sharma, V. K., Singh, S. K., Singh, U., and Ul-Farhat, K. (2017). Bayesian estimation on interval censored Lindley distribution using Lindley's approximation. *International Journal of System Assurance Engineering and Management*, 8:799–810.
20. Sukhatme, P. V. (1937). Tests of significance for samples of the x2-population with two degrees of freedom. *Annals of Eugenics*, 8(1):52–56.
21. Valiollahi, R., Asgharzadeh, A., and Ng, H. K. (2017). Prediction for Lindley distribution based on type-ii right censored samples. *Journal of the Iranian Statistical Society*, 16(2):1–19.