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# E-Bayesian estimation and prediction of insurance premium in Poisson model

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**Abstract**

Premium estimation and prediction are widely applied in insurance, healthcare, and finance to improve risk management, pricing accuracy, and customer personalization. They help insurers balance profitability with fairness, while giving customers more transparent and tailored options. In this paper, the E-Bayesian estimation of premium and predicting the number of claims is considered when the number of claims follows a Poisson distribution. The Escher premium principle is used to obtain the estimators and predictors. The Bayesian and E-Bayesian estimators of premium are derived under three densities for hyperparameters of prior distribution and compared by using a simulation study. A real data analysis is given to illustrate the results. The method of E-Bayesian estimation is extended to E-Bayesian predicting the number of claims. Performance of the proposed predictors are evaluated conducting a prequential analysis within a simulation.

**Key Words:** E-Bayesian estimation and prediction; Esscher premium; Poisson model; Prequential analysis.

**Mathematical Subject Classification: 62F15; 62P05.**

## 1. Introduction

Calculating insurance premium in actuarial sciences has different methods and principles. As mentioned in Gerber (1979), a fundamental concept in insurance is the premium calculation principle, which is a mathematical function designed to allocate a standard premium to various distributions of claims. Let $L:\Theta\times\mathrm{P}\rightarrow R$ be a loss function that assigns to any $(\theta,P)\in\Theta\times\mathrm{P}$ the loss incurred by a decision maker who takes the action P and is confronted with the realization $x$ resulting from a stochastic event. The premium denoted $P(\theta)$ which is called the risk premium, could be obtained by minimizing the expected loss $E_f[L(X,P)]$ with respect to $\theta$. L is usually taken as the weighted squared-error loss function, i.e. $L(a,x) = h(x)(x-a)^2$. Using different functional forms for $h(x)$, we have different premium principles. For instance, we obtain the net, Esscher and variance premium principles, respectively, when $h(x) =1$, $h(x) = e^{cx}$, $c > 0$ and $h(x) = x$, for more detail, see Gómez-Déniz et. al. (1999) and Gómez-Déniz (2009).

The Bayesian analysis approach has attracted the attention of many researchers in recent years due to its consistency and stability in calculations and also its application in risk theory. To model the insurance loss using Bayesian analysis, it is often necessary to choose a prior distribution. But choosing a suitable prior distribution is a difficult task in practice. To overcome the uncertainty caused by choosing a prior distribution, researchers have proposed the robust Bayesian analysis approach, see Berger (1985), Lavine (1991) and Berger (1994).

E-Bayesian or expected Bayesian estimation is introduced and developed by Han (1997). This method is based on the expected value of the Bayesian estimator over the density of hyperparameters of prior distribution. In recent decades, many researchers have used the E-Bayesian method in some estimation and prediction problems, see for example, Han (2011), Jaheen and Okasha (2011), Han (2017), Yaghoobzadeh and Makhdom (2021) and more recently, Alhamaideh et. al. (2022,2023), Hendi and Naghizadeh Qomi (2024), Hendi et al. (2024), Sanei et al. (2024) and Ghasabani et al. (2025).

Kiapour (2018) used this method for estimation of net premium and extend it to predict of claim size. In this paper, we used the Esscher premium principal for providing E-Bayesian estimation of premium and E-Bayesian prediction for the number of claims. The corresponding loss function is as

$$L(x,P)=e^{\alpha x}(x-P)^2,\quad \alpha>0, \tag{1}$$

where $P$ is the corresponding premium of the number of claims $x$. It is assumed that $X$ to be the number of claims and follows a Poisson distribution with density function

$$f(x\,|\,\theta)=e^{-\theta}\frac{\theta^x}{x!};\quad \theta>0, \tag{2}$$

where the unknown risk parameter $\theta$ is the expected number of claims per period.

The rest of the paper is organized as follows. In Section 2, estimation of the premium is presented by using Bayesian and E-Bayesian methods. E-Bayesian premium estimators are obtained using three densities for the hyperparameters of the prior distribution and a comparison study is conducted. A real data example is provided to illustrate the results. In Section 3, The E-Bayesian predictors of next number of claims are computed and assessed using a prequential analysis. Conclusions and discussions are offered in Section 4.

## 2. Bayesian and E-Bayesian premium estimation

Let $X_i$ be the claim size in year $i=1,2,3,\ldots$ and $X_1,X_2,\ldots$ be a sequence of random claim size. Suppose $\underline{X}=(X_1,X_2,\ldots,X_n)$ be the vector of the claim sizes up to year $n$ and $\underline{x}=(x_1,x_2,\ldots,x_n)$ be the vector of corresponding observed values. Suppose that given $\theta,X_1,X_2,\ldots$ are stochastically iid. Under the Escher loss (1), the Escher premium can be obtained as

$$P(\theta)=\frac{E(Xe^{\alpha X}|\theta)}{E(e^{\alpha X}|\theta)}.$$

Assuming the Poisson model (2), the Escher premium is $P(\theta)=\theta e^{\alpha}$.

### 2.1 Bayesian premium estimation

Considering a Gamma prior for $\theta$ with the following density function

$$\pi(\theta\,|\,a,b)=\frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta},\ \theta>0,\ a>0,\ b>0, \tag{3}$$

the posterior distribution is Gamma with density

$$\pi(\theta\,|\,\underline{x})=\frac{(n+b)^{a+T}}{\Gamma(a+T)}\theta^{a+T-1}e^{-\theta(n+b)},\ \theta>0, \tag{3}$$

where $\Gamma(z)=\int_0^\infty t^z e^{-z}dt$ is the complete gamma function and $T=\sum_{i=1}^{n}X_i$. To obtain the Bayesian estimator of $P(\theta)$ under the loss function

$$L(P, P(\theta)) = e^{\alpha P(\theta)}(P - P(\theta))^2,$$

we minimize the corresponding posterior risk

$$\rho(\pi, P) = E[e^{\alpha P(\theta)}(P - P(\theta))^2|\underline{X}] = P^2 E[e^{\alpha P(\theta)}|\underline{X}] - 2PE[P(\theta)e^{\alpha P(\theta)}|\underline{X}] + E[P(\theta)^2 e^{\alpha P(\theta)}|\underline{X}].$$

Afetr some simple calculations, we obtain the Bayesian premium as

$$P^B(\underline{X}) = \frac{E[P(\theta)e^{\alpha P(\theta)}|\underline{X}]}{E[e^{\alpha P(\theta)}|\underline{X}]}. \tag{5}$$

Thus, the Bayesian premium under Poisson model is given by

$$P^B = \frac{(a+T)e^{\alpha}}{b+n-\alpha e^{\alpha}}. \tag{6}$$

### 2.2. E-Bayesian premium estimation

In this subsection, we compute the E-Bayesian estimator of premium.

**Definition 2.1.** According to Han (2009), the E-Bayesian (expected Bayesian) estimator of premium is defined as

$$P^{EB}(\underline{X}) = \iint_D P^B(\underline{X})\pi(a,b)\,da\,db, \tag{7}$$

where $D$ is the domain of $a$ and $b$, $P^B$ is the Bayesian estimator of parameter θ, and $\pi(a,b)$ is the density function of $a$ and $b$ over $D$.

Consider prior $\pi(\theta|a,b)$ for $\theta$ with hyperparameters $a$ and $b$. According to Han (1997), the prior parameters $a$ and $b$ should be selected to guarantee that $\pi(\theta|a,b)$ is a decreasing function of $\theta$ which implies that hyperparameters $a$ and $b$ should be in the ranges $0 < a < 1$ and $b > 0$, respectively. According to Berger (1985), the prior distribution with thinner tail would make worse robustness of Bayesian distribution. Therefore, $b$ should not be too big while $0 < a < 1$. This implies to choose $0 < a < 1$ and $0 < b < c$, where $c$ is a positive constant value. This approach robustifies an estimation rule's dependency to prior hyperparameters.

Assuming that $a$ and $b$ are independent with bivariate density function $\pi(a,b)=\pi(a)\pi(b)$, we consider three densities for hyperparameters $a$ and $b$ as follows:

$$\pi_1(a,b) = \frac{2(c-b)}{c^2} \times \frac{a^{u-1}(1-a)^{v-1}}{B(u,v)},\ 0<a<1,\ 0<b<c, \tag{8}$$

$$\pi_2(a,b) = \frac{1}{c} \times \frac{a^{u-1}(1-a)^{v-1}}{B(u,v)},\ 0<a<1,\ 0<b<c, \tag{9}$$

$$\pi_3(a,b) = \frac{2b}{c^2} \times \frac{a^{u-1}(1-a)^{v-1}}{B(u,v)},\ 0<a<1,\ 0<b<c, \tag{10}$$

where $B(u,v) = \int_0^1 s^{u-1}(1-s)^{v-1}\,ds$ is the complete beta function.

These distributions are used to investigate the influence of the different prior distributions on the E-Bayesian estimation, see Jaheen and Okasha (2011).

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**Theorem 2.1.** The E-Bayesian premium estimators corresponding to the prior densities (8)-(10) are given respectively by

$$P^{EB_1} = [\frac{2(n+c-\alpha e^{\alpha})}{c^2} \times \ln \frac{n+c-\alpha e^{\alpha}}{n-\alpha e^{\alpha}} - \frac{2}{c}] \times e^{\alpha} (\frac{u}{u+v} + T),$$

$$P^{EB_2} = \frac{1}{c} \times \ln \frac{n+c-\alpha e^{\alpha}}{n-\alpha e^{\alpha}} \times e^{\alpha} (\frac{u}{u+v} + T),$$

$$P^{EB_3} = [\frac{2(\alpha e^{\alpha} - n)}{c^2} \times \ln \frac{n+c-\alpha e^{\alpha}}{n-\alpha e^{\alpha}} + \frac{2}{c}] \times e^{\alpha} (\frac{u}{u+v} + T) \cdot$$

**Proof.** Using the Definition 2.1, under density (8), the E-Bayesian premium estimate is obtained as

$$\begin{aligned} P^{EB_1} &= \iint_D P^B(\underline{x}) \pi_1(a,b) dadb \\ &= \int_0^c \int_0^1 \frac{(a+T)e^{\alpha}}{b+n-\alpha e^{\alpha}} \times \frac{2(c-b)}{c^2} \times \frac{a^{u-1}(1-a)^{v-1}}{B(u,v)} dadb \\ &= \frac{2}{c^2} \int_0^c \frac{c-b}{b+n-\alpha e^{\alpha}} \int_0^1 \frac{(a+T)e^{\alpha} \times a^{u-1}(1-a)^{v-1}}{B(u,v)} dadb \\ &= \left[ \frac{2(n+c-\alpha e^{\alpha})}{c^2} \times \ln \frac{n+c-\alpha e^{\alpha}}{n-\alpha e^{\alpha}} - \frac{2}{c} \right] \times e^{\alpha} (\frac{u}{u+v} + T). \end{aligned}$$

Using density (9), the E-Bayesian premium estimate would be

$$\begin{aligned} P^{EB_2}(\underline{x}) &= \iint_D P^B(\underline{x}) \pi_2(a,b) dadb \\ &= \int_0^c \int_0^1 \frac{(a+T)e^{\alpha}}{b+n-\alpha e^{\alpha}} \times \frac{1}{c} \times \frac{a^{u-1}(1-a)^{v-1}}{B(u,v)} dadb \\ &= \frac{1}{c} \int_0^c \frac{1}{b+n-\alpha e^{\alpha}} \int_0^1 \frac{(a+T)e^{\alpha} \times a^{u-1}(1-a)^{v-1}}{B(u,v)} dadb \\ &= \frac{1}{c} \times \ln \frac{n+c-\alpha e^{\alpha}}{n-\alpha e^{\alpha}} \times e^{\alpha} \left( \frac{u}{u+v} + T \right). \end{aligned}$$

Similarly, the E-Bayesian premium estimator according to the density (10), is given by

$$\begin{aligned} P^{EB_3}(\underline{x}) &= \iint_D P^B(\underline{x}) \pi_3(a,b) dadb \\ &= \int_0^c \int_0^1 \frac{(a+T)e^{\alpha}}{b+n-\alpha e^{\alpha}} \times \frac{2b}{c^2} \times \frac{a^{u-1}(1-a)^{v-1}}{B(u,v)} dadb \\ &= \frac{2}{c^2} \int_0^c \frac{b}{b+n-\alpha e^{\alpha}} \int_0^1 \frac{(a+T)e^{\alpha} \times a^{u-1}(1-a)^{v-1}}{B(u,v)} dadb \end{aligned}$$

$$=[\frac{2(\alpha e^{\alpha}-n)}{c^{2}}\times\ln\frac{n+c-\alpha e^{\alpha}}{n-\alpha e^{\alpha}}+\frac{2}{c}]\times e^{\alpha}(\frac{u}{u+v}+T)\cdot$$

### 2.3. Comparison of the premium estimators

To make a numerical comparison between the Bayesian and E-Bayesian premium estimators, a simulation study is conducted. To do, sequences $n$ of independent random samples from Poisson distribution with true value of parameter $\theta=2$ is generated. It is also assumed that $u=2$, $v=3$, $\alpha=1$, $a=0.4$, b=0.2, $c=1,3,5$ and $n=10,30,50,80$. The estimators and the corresponding risks are calculated by repeated 10000 times simulation runs and summarized in Table 1. In this Table, $B$ denotes the Bayesian premium estimate and $EB_i, i=1,2,3$ stands for E-Bayesian estimates.

From Table 1, it is observed that the E-Bayesian estimators have lower risk compared to the Bayesian estimator. Moreover, the risk Escher of estimators decreases when the sample size increases. Furthermore, the risks tend to decrease as the value of *c* increases. For a comparison purpose of proposed estimators, we can state the following relation between the proposed estimators: $EB_3 < EB_2 < EB_1 < B$.

Choosing a specific prior or hyperparameter can be quite critical, but for better interpretation about the simulation results, we explained some reasons for choosing the specified hyperparameters. the hyperparameters a and b can be selected according to the equation $E(\theta|a,b)=a/b$. For the priors given in (8)-(10), we assumed that $\pi_i$(a,b) = $\pi_i$(a) $\pi_i$(b) , i=1,2,3 and therefore $a \sim Beta(u,v)$ which implies that $E(a|u,v)=u/(u+v)$. The hyperparameter $b$ can be selected using the fact $b=a/E(\theta|a,b)$. For instance, by choosing $u=2$ and $v=3$, we have $E(a|u,v)=0.4$. If we set $\theta=2$, we get $b=0.4/2=0.2$. Thus, the hyperparameters $a$ and $b$ have been chosen as 0.4 and 0.2, respectively. This method for choosing hyperparameters is rich enough to contain all prior means achieved by the priors $\pi_i, i=1,2,3$.

**Table 1.** Results of estimated values and corresponding risks for Bayesian and E-Bayesian premium estimators.

| | | Estimate | | | | Risk | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | *B* | $EB_1$ | $EB_2$ | $EB_3$ | *B* | $EB_1$ | $EB_2$ | $EB_3$ |
| **n=10** | c=1 | 7.5935 | 7.4675 | 7.3108 | **7.1540** | 34.12544 | 32.43930 | 30.44562 | **28.59161** |
| | c=3 | 7.5935 | 6.9083 | 6.5334 | **6.1585** | 34.12544 | 25.72355 | 22.00240 | **19.06726** |
| | c=5 | 7.5935 | 6.4548 | 5.9396 | **5.4245** | 34.12544 | 21.37429 | 17.80427 | **15.98350** |
| **n=30** | c=1 | 6.0505 | 6.0217 | 5.9858 | **5.9499** | 12.47222 | 12.16996 | 11.83009 | **11.53174** |
| | c=3 | 6.0505 | 5.8829 | 5.7824 | **5.6819** | 12.47222 | 10.97581 | 10.16298 | **9.54248** |
| | c=5 | 6.0505 | 5.7533 | 5.5964 | **5.4395** | 12.47222 | 9.97876 | 9.10766 | **8.81955** |
| **n=50** | c=1 | 5.7707 | 5.7547 | 5.7347 | **5.7147** | 8.46816 | 8.33945 | 8.18976 | **8.05199** |
| | c=3 | 5.7707 | 5.6763 | 5.6187 | **5.5611** | 8.46816 | 7.79738 | 7.47639 | **7.22587** |
| | c=5 | 5.7707 | 5.6010 | 5.5087 | **5.4164** | 8.46816 | 7.39578 | 7.06738 | **6.87643** |
| **n=80** | c=1 | 5.6768 | 5.6671 | 5.6549 | **5.6428** | 6.3943 | 6.33270 | 6.26087 | **6.19449** |
| | c=3 | 5.6768 | 5.6192 | 5.5838 | **5.5483** | 6.3943 | 6.06984 | 5.90987 | **5.76934** |
| | c=5 | 5.6768 | 5.5725 | 5.5149 | **5.4573** | 6.3943 | 5.86485 | 5.64875 | **5.51896** |

### 2.4. A real data analysis

In order to illustrate the application of the estimation results, in this section, calculations of estimators were performed using a real data set. This data set, which is given in Table 2, is related to the declaration of car insurance claims in the second month of summer of 2021, which was registered in the online agencies of Iran Insurance Company in Rasht city. First, a chi-square goodness-of-fit test with test statistic 1.2049 and p-value 0.5447 ensure that the data follows a Poisson distribution with parameter $\hat{\theta}=1.35$. Then, the Escher premium can be obtained as $P(\theta)=$

3.6828. The Bayesian and E-Bayesian estimators of insurance premium are calculated for $\alpha = 1, u = 2, v = 3, a = 0.4, b = 0.3$ and $c = 5$ and reported in Table 3. It is observed that $EB_3$ is close to the estimated premium.

**Table 2.** Car insurance claims data.

| Number of claims | Frequencies |
|---|---|
| 0 | 8 |
| 1 | 12 |
| 2 | 5 |
| 3 | 4 |
| 4 | 2 |
| 5 | 0 |

**Table 3.** Results of Bayesian and E-Bayesian premium estimates

| B | $EB_1$ | $EB_2$ | $EB_3$ |
|---|---|---|---|
| 4.0325 | 3.8543 | 3.7525 | 3.6508 |

**3. Bayesian and E-Bayesian Prediction of claim size**

Since insurance compensation constitute the great liabilities carried by insurers, one of the main activities of a practicing actuary is required reserves estimating. Ruin would be happened, when a large claim is occurred and the insurer company fails to meet its commitments. So, to avoid ruin, it is vital to predict the future claim and the required reserves. To overcome the uncertainties involved in the payment of losses, Bayesian prediction is an optimal approach.

**3.1. Bayesian prediction**

In this section, we predict the future number of claims $y = x_{n+1}$ using Bayesian method under the prediction loss

$$L(y, P_{n+1}) = e^{\alpha y}(y - P_{n+1})^2. \quad (11)$$

**Theorem 3.1.** Let $\underline{x} = (x_1, x_2, ..., x_n)$ be the $n$ occurred number of claims. The Bayesian predictor of the future number of claims, $y = x_{n+1}$, under the prediction loss (11) is given by

$$\hat{P}_{n+1}^{B}(\underline{x}) = \frac{E(ye^{\alpha y} \mid \underline{x})}{E(e^{\alpha y} \mid \underline{x})} = \frac{e^{\alpha}(a+T)}{n+b-(e^{\alpha}-1)}.$$

**Proof.** The posterior risk is given by

$$E(e^{\alpha y}(y - P_{n+1})^2 \mid \underline{x}) = E(y^2 e^{\alpha y} \mid \underline{x}) + P_{n+1}^2 E(e^{\alpha y} \mid \underline{x}) - 2P_{n+1}E(ye^{\alpha y} \mid \underline{x}),$$

The Bayesian predictor is computed by minimizing the posterior risk as

$$\hat{P}_{n+1}^{B}(\underline{x}) = \frac{E(ye^{\alpha y} \mid \underline{x})}{E(e^{\alpha y} \mid \underline{x})}. \quad (12)$$

We get

$$E(e^{\alpha y} \mid \underline{x}) = E(E(e^{\alpha y} \mid \theta) \mid \underline{x}) = E(e^{\theta(e^{\alpha}-1)} \mid \underline{x})$$

$$= \int e^{\theta(e^{\alpha}-1)} \frac{1}{\Gamma(a+T)}(n+b)^{(a+T)} e^{-(n+b)\theta} \theta^{(a+T)-1} d\theta = \left(\frac{n+b}{n+b-e^{\alpha}+1}\right)^{(a+T)},$$

Moreover, we can obtain

$$E(ye^{\alpha y} \mid \underline{x}) = E(E(ye^{\alpha y} \mid \theta) \mid \underline{x})$$

$$E(ye^{\alpha y} \mid \theta) = \sum ye^{\alpha y} \frac{e^{-\theta}\theta^{y}}{y!} = \sum e^{\alpha y} \frac{e^{-\theta}\theta^{y}}{(y-1)!} \xrightarrow{y-1=t}$$

$$E(ye^{\alpha y} \mid \theta) = \sum e^{\alpha(t+1)} \frac{e^{-\theta}\theta^{(t+1)}}{t!} = \theta e^{\alpha} \sum \frac{e^{\alpha t} e^{-\theta}\theta^{t}}{t!} = \theta e^{\alpha} e^{\theta(e^{\alpha}-1)}$$

$$\Rightarrow E(ye^{\alpha y} \mid \underline{x}) = E(E(ye^{\alpha y} \mid \theta) \mid \underline{x}) = E(\theta e^{\alpha} e^{\theta(e^{\alpha}-1)} \mid \underline{x})$$

$$= e^{\alpha} \int \frac{\theta e^{\theta(e^{\alpha}-1)}}{\Gamma(a+T)} (n+b)^{(a+T)} e^{-(n+b)\theta} \theta^{(a+T)-1} d\theta = (a+T)e^{\alpha} \frac{(n+b)^{(a+T)}}{(n+b-e^{\alpha}+1)^{(a+T+1)}},$$

Therefore, the Bayesian predictor (12) can be expressed as

$$\hat{P}_{n+1}^{B}(\underline{x}) = \frac{E(ye^{\alpha y} \mid \underline{x})}{E(e^{\alpha y} \mid \underline{x})} = \frac{e^{\alpha}(a+T)}{n+b-(e^{\alpha}-1)}.$$

### 3.2. E-Bayesian Prediction of next number of claims

Here, we extend the E-Bayesian estimation method to the E-Bayesian prediction and employ it for obtaining the E-Bayesian prediction of future number of claims.

**Theorem 3.2.** Let $\underline{x} = (x_1, x_2, ..., x_n)$ be the sample observations from the Poisson distribution (1). Then, the E-Bayesian Predictions of the Esscher premium $y = x_{n+1}$ corresponding to the prior densities (8), (9) and (10) are

$$\hat{P}^{EB_1}(\underline{x}) = [\frac{2(n+c-e^{\alpha}+1)}{c^2} \times \ln \frac{n+c-e^{\alpha}+1}{n-e^{\alpha}+1} - \frac{2}{c}] \times e^{\alpha}(\frac{u}{u+v}+T), \quad (13)$$

$$\hat{P}^{EB_2}(\underline{x}) = \frac{1}{c} \times \ln \frac{n+c-e^{\alpha}+1}{n-e^{\alpha}+1} \times e^{\alpha}(\frac{u}{u+v}+T), \quad (14)$$

$$\hat{P}^{EB_3}(\underline{x}) = [\frac{2(e^{\alpha}-n-1)}{c^2} \times \ln \frac{n+c-e^{\alpha}+1}{n-e^{\alpha}+1} + \frac{2}{c}] \times e^{\alpha}(\frac{u}{u+v}+T) \cdot \quad (15)$$

**Proof.** Using the Definition 2.1, under density (8), the E-Bayesian premium prediction under the loss function (11) is obtained as

$$\hat{P}_{n+1}^{EB_1}(\underline{x}) = \iint_D \hat{P}_{n+1}^{B}(\underline{x}) \pi_1(a,b) da db$$

$$= \int_0^c \int_0^1 \frac{(a+T)e^{\alpha}}{b+n-e^{\alpha}+1} \times \frac{2(c-b)}{c^2} \times \frac{a^{u-1}(1-a)^{v-1}}{B(u,v)} da db$$

$$= -\frac{2}{c^2} \int_0^c \frac{b-c}{b+n-e^{\alpha}+1} \int_0^1 \frac{(a+T)e^{\alpha} \times a^{u-1}(1-a)^{v-1}}{B(u,v)} da db$$

$$= [\frac{2(n+c-e^{\alpha}+1)}{c^2} \times \ln \frac{n+c-e^{\alpha}+1}{n-e^{\alpha}+1} - \frac{2}{c}] \times e^{\alpha}(\frac{u}{u+v}+T),$$

Using density (9), the E-Bayesian premium prediction would be

$$\hat{P}_{n+1}^{EB_2}(\underline{x}) = \iint_D \hat{P}_{n+1}^{B}(\underline{x}) \pi_2(a,b) da db$$

$$= \int_0^c \int_0^1 \frac{(a+T)e^{\alpha}}{b+n-e^{\alpha}+1} \times \frac{1}{c} \times \frac{a^{u-1}(1-a)^{v-1}}{B(u,v)} da db$$

$$=\frac{1}{c}\int_0^c \frac{1}{b+n-e^{\alpha}+1}\int_0^1 \frac{(a+T)e^{\alpha}\times a^{u-1}(1-a)^{\nu-1}}{B(u,\nu)}dadb$$
$$=\frac{1}{c}\times \ln\frac{n+c-e^{\alpha}+1}{n-e^{\alpha}+1}\times e^{\alpha}(\frac{u}{u+\nu}+T),$$

and according to the density (10), the corresponding E-Bayesian premium prediction is given by

$$\hat{P}_{n+1}^{EB_3}(\underline{x})=\iint_D \hat{P}_{n+1}^{B}(\underline{x})\pi_3(a,b)dadb$$
$$=\int_0^c\int_0^1 \frac{(a+T)e^{\alpha}}{b+n-e^{\alpha}+1}\times\frac{2b}{c^2}\times\frac{a^{u-1}(1-a)^{\nu-1}}{B(u,\nu)}dadb$$
$$=\frac{2}{c^2}\int_0^c \frac{b}{b+n-e^{\alpha}+1}\int_0^1 \frac{(a+T)e^{\alpha}\times a^{u-1}(1-a)^{\nu-1}}{B(u,\nu)}dadb$$
$$=[\frac{2(e^{\alpha}-n-1)}{c^2}\times \ln\frac{n+c-e^{\alpha}+1}{n-e^{\alpha}+1}+\frac{2}{c}]\times e^{\alpha}(\frac{u}{u+\nu}+T)\cdot$$

### 3.3. Numerical Comparison of predictors

To compare the proposed predictors of $y = x_{n+1}$, a simulation study is conducted using the prequential analysis due to Kiapour and Nematollahi (2011). To do, we assume $u = 2$, $v = 3$, $\alpha = 0.5$, $a = 0.4$ and $b = 0.2$.
The steps of the simulation are as follows:

1. Generate the sample $x_1, x_2, ..., x_n$ from the Poisson model with $\theta = 2, i.e., Po(2)$.
2. Generate $y$ from $Po(2)$ and compute a predictor $\hat{P}_{n+1}^{B}(\underline{x})$ utilizing data for $y$.
3. Compute the prediction error for $y$ as $e^{\alpha y}\left(y-\hat{P}_{n+1}\right)^2$.
4. Increase $n$ by 1 and repeat Steps 2 and 3 until $n$=$m$, when $m$ = 5, 10, 30, 50, 100.
5. Compute the average prediction error (APE) given by
$$APE=\frac{1}{m}\sum_{n=1}^{m}\left[e^{\alpha y}\left(y-\hat{P}_{n+1}\right)^2\right].$$
6. Compute the simulated APE as the mean of the APEs over $10^4$ repetitions.

The results of simulation ae summarized in Table 4 for selected values of $c = 0.5,1,5$. From the Table 4, we conclude that the third E-Bayesian predictor, $\hat{P}_{n+1}^{EB_3}$, has a good performance when $c = 0.5,1$ while the first E-Bayesian predictor, $\hat{P}_{n+1}^{EB_1}$ , is better than other predictors when $c = 5$. The simulated APEs tend to decrease when the sample size increases.
For better presentation of the results, Figure 1 displays the APEs versus sample size, $m$, for selected values of $c = 0.5$ and $c = 1.0$. In view of Figure 1, it is observed that the E-Bayesian predictor $\hat{P}_{n+1}^{EB_3}$ has smaller APE than other predictors when $c = 0.5,1$.

**Table 4.** Results of prediction analysis of estimators.

| c | $m$ | $\hat{P}_{n+1}^{B}$ | $\hat{P}_{n+1}^{EB_1}$ | $\hat{P}_{n+1}^{EB_2}$ | $\hat{P}_{n+1}^{EB_3}$ |
|---|---|---|---|---|---|
| 0.5 | 5 | 3.267421 | 3.325750 | 3.225987 | **3.146560** |
| | 10 | 2.967498 | 2.996809 | 2.945444 | **2.904702** |
| | 30 | 2.633290 | 2.642598 | 2.626309 | **2.613577** |
| | 50 | 2.523825 | 2.529989 | 2.519011 | **2.510288** |
| | 100 | 2.442978 | 2.445877 | 2.440689 | **2.436596** |
| 1 | 5 | 3.267421 | 3.164995 | 3.074901 | **3.032179** |
| | 10 | 2.967498 | 2.913234 | 2.866246 | **2.844702** |
| | 30 | 2.633290 | 2.616220 | 2.602061 | **2.596632** |
| | 50 | 2.523825 | 2.511994 | 2.501790 | **2.497134** |
| | 100 | 2.442978 | 2.437373 | 2.432630 | **2.430602** |
| 5 | 5 | 3.267421 | **3.136589** | 3.373592 | 3.744610 |
| | 10 | 2.967498 | **2.911359** | 3.066104 | 3.309615 |

| | | | | |
|---|---|---|---|---|
| 30 | 2.633290 | **2.631437** | 2.701198 | 2.808457 |
| 50 | 2.523825 | **2.514698** | 2.556582 | 2.623022 |
| 100 | 2.442978 | **2.440502** | 2.462842 | 2.497915 |

## 5. Discussion and Concluding remarks

The paper considered developing Bayesian and E-Bayesian estimation of Escher premium and Bayesian and E-Bayesian prediction for number of claims in a Poisson model. Two simulation studies are conducted to compare the proposed estimators and predictors. Findings of this work show that the third E-Bayesian estimator has smaller risk than other estimators and can be offered for estimation of premium. Moreover, the third E-Bayesian predictor of claim size has a good performance for small values of *c*.

The E-Bayesian estimators and predictors in this paper can be compared using new criteria called E-posterior risk of estimators and predictors, respectively. This work is under consideration and construction following the new paper due to Alaa Alhamaidah et al. (2023). We may have some information about $\theta$ as a prior point guess value $\theta_0$. Following Naghizadeh Qomi (2017), we can obtain the improved estimator of $\theta$ by constructing a linear shrinkage estimator.

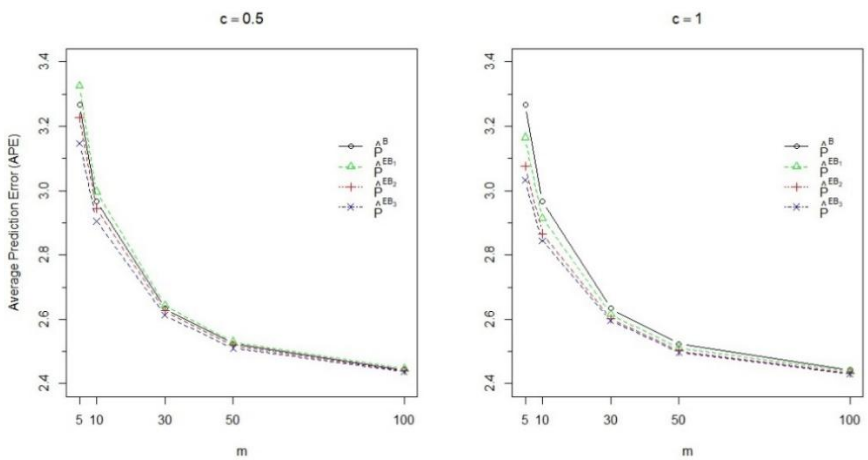


**Figure 1.** The APEs versus sample size, $m$, when $c = 0.5$ $(left)$ and $c = 1.0$ $(right)$.

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