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A Unified Family for Generating Probabilistic Models: Properties, Bayesian and Non-Bayesian Inference with Real-Data Applications

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Abstract

This paper introduces a new generator called the inverse-power Burr–Hatke-G (IPBH-G) family. The special models of the IPBH-G family accommodate different monotone and nonmonotone failure rates, so it turns out to be quite flexible family for analyzing non-negative real-life data. We provide three special sub-models of the family and derive its key mathematical properties. The parameters of the special IPBH-exponential model are explored from using eleven frequentist and Bayesian estimation approaches. The Bayes estimators for the unknown parameters are obtained under three different loss functions. Numerical simulations are performed to compare and rank the proposed methods based on partial and overall ranks. Furthermore, the superiority of the IPBH-exponential model over other distributions are illustrated empirically by means of three real-life data sets from applied sciences including industry, medicine and agriculture.

Key Words: Exponential distribution; order statistics; maximum likelihood; Bayesian estimation; simulation; data analysis.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Statisticians have recently become more interested in generating new generators or generalized models by adding additional parameters to the baseline distributions. These generators of generalized models offer new insights for flexible modeling of real-world data in several applied fields such as biology, economics, engineering, computer science, industry, medicine, life testing, and insurance. Due to the limitations of classical distributions, which are unable to exhibit vast flexibility and have relatively restricted qualities, the flexible generated models are therefore crucial to provide sufficient fits to real-life data in these disciplines.

Therefore, several families or generators of distributions having one or more extra shape parameters have been introduced in the literature to provide flexible models. For example, the Topp–Leone odd log-logistic-G by Brito et al. (2017), generalized transmuted-G by Nofal et al. (2017), generalized odd Lindley-G by Afify et al. (2019), flexible

Weibull-G by Alizadeh et al. (2020), extended failure rate-G by Afify et al. (2022), odd Fréchet Lehmann type II-G by Al Mutairi et al.(2022), weighted Lindley-G by Alnssyan et al. (2023).

The main purpose of this article is to introduce a new flexible family of distributions, called the IPBH-G family which has desirable properties and can model different real-life data sets in a wide range of applications. The most important feature of the new family is its ability to capture all important failure rate shapes. The motivation behind developing this family is to improve the general performance of classical distributions. The two extra parameters of the IPBH-G family make its special sub-models very flexible and capable of modeling real-life data with increasing, decreasing, J-shape, reversed-J shape, bathtub, upside-down bathtub, and modified bathtub shapes. Additionally, the densities of the sub-models of the IPBH-G class can provide right-skewed, J-shape, symmetric, left-skewed, reversed-J shape, unimodal, and bimodal shapes. The hazard rate function of this new family of distribution can capture. We consider three baseline models namely the exponential, Weibull and Burr XII to generate three special sub-models of the IPBH-G family. These special models called the IPBH-exponential (IPBHEx), IPBH-Weibull (IPBHW), and IPBH-Burr III (IPBHBIII) models. Moreover, the parameters of the IPBHEx distribution are estimated by eleven classical and Bayesian estimation approaches. The performance of these methods is evaluated by a detailed simulation study. Finally, three real-life data sets from industry, medicine and agriculture are analyzed to illustrate the flexibility of the proposed family. It is also shown that the two additional shape parameters α and η provide greater flexibility in modeling real data (See section 7). The p-values of the IPBHEx model are 0.9218, 0.9520 and 0.6797 whereas the associated p-values of the baseline exponential model are 0.4947, 0.0008 and 0.1086 for the three data sets, respectively.

The rest of the paper is organized as follows: In Section 2, the IPBH-G family is defined. In Section 3, three special distributions of the IPBH-G family are presented. Mathematical properties of the family are addressed in Section 4. The estimation of the IPBHEx parameters is explored in Section 5. The Bayesian estimators of the IPBHE parameters are discussed in Section 6. Simulations results are provided in Section 7. Three real data applications are given in Section 8. Some final remarks are presented in Section 9.

2. The IPBH-G Family

Recently, Afify et al. (2021) introduced the IPBH distribution which has the following the cumulative distribution function (cdf) and probability density function (pdf)

$$F_{IPBH}(x;\alpha,\eta)=\frac{\exp\left(-\alpha x^{-\eta}\right)}{x^{-\eta}+1},\quad \alpha,\eta>0, \tag{1}$$

and

$$f_{IPBH}(x;\alpha,\eta)=\frac{\eta\exp\left(-\alpha x^{-\eta}\right)\left[\alpha+\left(\alpha+1\right)x^{\eta}\right]}{x(x^{\eta}+1)^{2}}, \tag{2}$$

where η and α are shape parameters.

Based on the T-X generator of Alzaatreh et al. (2013), we define the new IPBH-G family by taking $W[G(x;\boldsymbol{\psi})]=\frac{G(x,\boldsymbol{\psi})}{1-G(x,\boldsymbol{\psi})}$ and $r(t)=f_{IPBH}(x;\alpha,\eta)$, where $G(x;\boldsymbol{\psi})$ is the baseline cdf which depends on a vector of parameters $\boldsymbol{\psi}$. where $G(x;\boldsymbol{\psi})$ is the baseline cdf which depends on a vector of parameters $\boldsymbol{\psi}$.

The cdf of the IPBH-G family is given (for $x\in\mathcal{R}$) by

$$F(x;\alpha,\eta,\boldsymbol{\psi})=\int_{0}^{\frac{G(x;\boldsymbol{\psi})}{1-G(x;\boldsymbol{\psi})}}r(t)\,dt=\frac{\exp\left\{-\alpha\left[\frac{G(x;\boldsymbol{\psi})}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}\right\}}{1+\left[\frac{G(x;\boldsymbol{\psi})}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}},\quad \alpha,\eta>0. \tag{3}$$

The pdf corresponding to (3) has the form

$$f\left(x;\alpha,\eta,\boldsymbol{\psi}\right)=\frac{\eta g\left(x;\boldsymbol{\psi}\right)}{\left[1-G\left(x;\boldsymbol{\psi}\right)\right]}\frac{\exp\left\{-\alpha\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}\right\}\left\{\alpha+\left(\alpha+1\right)\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{\eta}\right\}}{G\left(x,\psi\right)\left\{1+\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{\eta}\right\}^{2}}, \tag{4}$$

where $g\left(x\right)=dG(x)/dx$ is the baseline density corresponding to $G(x)$.
The cumulative hrf of the IPBH-G family is

$$H\left(x;\alpha,\eta,\boldsymbol{\psi}\right)=-\log\left(1-\frac{\exp\left\{-\alpha\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}\right\}}{1+\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}}\right). \tag{5}$$

The hazard rate function (hrf) of the IPBH-G family reduces to

$$\begin{aligned}h\left(x;\alpha,\eta,\boldsymbol{\psi}\right) &= \frac{\eta g\left(x;\boldsymbol{\psi}\right)\exp\left\{-\alpha\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}\right\}\left\{\alpha+\left(\alpha+1\right)\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{\eta}\right\}}{G\left(x;\psi\right)\left[1-G\left(x;\boldsymbol{\psi}\right)\right]\left\{1+\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{\eta}\right\}^{2}}\\ &\quad\times\frac{1+\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}}{1+\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}-\exp\left\{-\alpha\left[\frac{G\left(x;\boldsymbol{\psi}\right)}{1-G(x;\boldsymbol{\psi})}\right]^{-\eta}\right\}}\end{aligned} \tag{6}$$

Hereafter, the random variable X with pdf (4) is denoted by $X\sim$IPBH-G$(\alpha,\eta,\boldsymbol{\psi})$.

3. Special Sub-Models

Three special sub-models of the IPBH-G class, called the IPBHEx, IPBHW, and IPBHBIII models are presented. Figures 1–3 reveal that the densities of the three special models exhibit right-skewed, J-shape, symmetric, left-skewed, reversed-J shape, unimodal, and bimodal shapes. Furthermore, the plots show that the hazard functions of these models can be increasing, decreasing, J-shape, reversed-J shape, bathtub, upside-down bathtub, and modified bathtub shapes.

3.1. The IPBHEx Distribution

By taking the pdf and cdf of the exponential distribution with parameter $\lambda>0$ as a baseline model in Equation (4), the pdf of the IPBHEx distribution takes the form

$$f\left(x;\alpha,\eta,\lambda\right)=\frac{\eta\,\lambda\exp\left(\lambda x-\alpha\left(e^{\lambda x}-1\right)^{-\eta}\right)\left[\alpha+\left(\alpha+1\right)\left(e^{\lambda x}-1\right)^{\eta}\right]}{\left(e^{\lambda x}-1\right)\left[\left(e^{\lambda x}-1\right)^{\eta}+1\right]^{2}},\ x>0,\ \ \alpha,\eta,\lambda>0 \tag{7}$$

Plots the pdf and hrf of the IPBHEx distribution for selected parameter values are given in Figure 1.

3.2. The IPBHW Distribution

The pdf of the W distribution (for $x>0$) has the form $g\left(x\right)=\theta\lambda\,x^{\lambda-1}e^{-\theta x^{\lambda}},\theta,\lambda>0$. Using (4), the pdf of the IPBHW distribution follows (for $x>0$) as

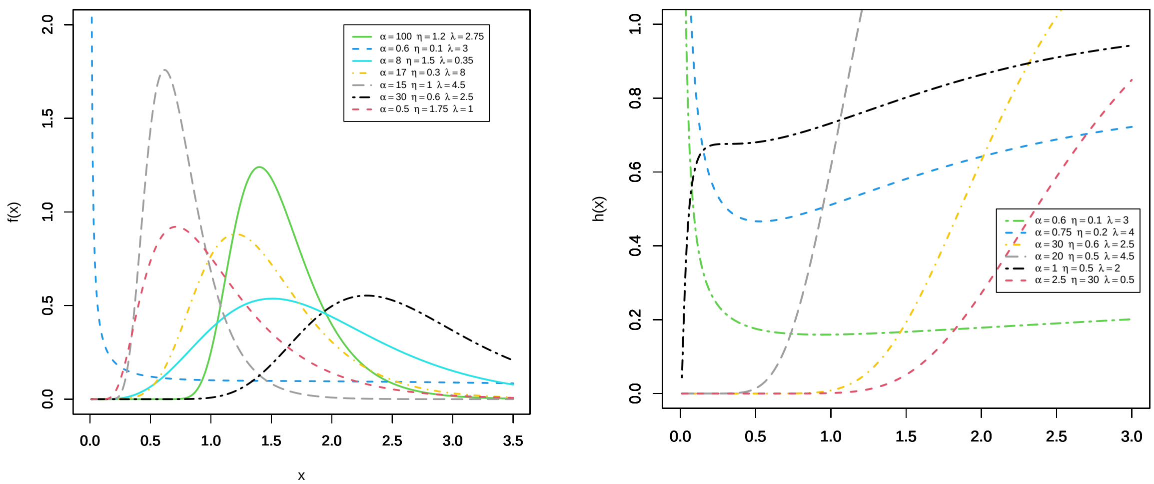

Figure 1: Plots of the pdf and hrf of the IPBHEx distribution

$$f\left(x;\alpha,\eta,\lambda,\theta\right)=\frac{\eta\theta\lambda\, x^{\lambda-1}\exp\left[-\alpha\left(e^{\theta x^{\lambda}}-1\right)^{-\eta}\right]\left[\alpha+\left(\alpha+1\right)\left(e^{\theta x^{\lambda}}-1\right)^{\eta}\right]}{1-e^{-\theta x^{\lambda}}\left[1+\left(e^{\theta x^{\lambda}}-1\right)^{\eta}\right]^{2}},\ \alpha,\eta,\lambda,\theta>0. \tag{8}$$

Plots the pdf and hrf of the IPBHW distribution for selected parameter values are displayed in Figure 2.

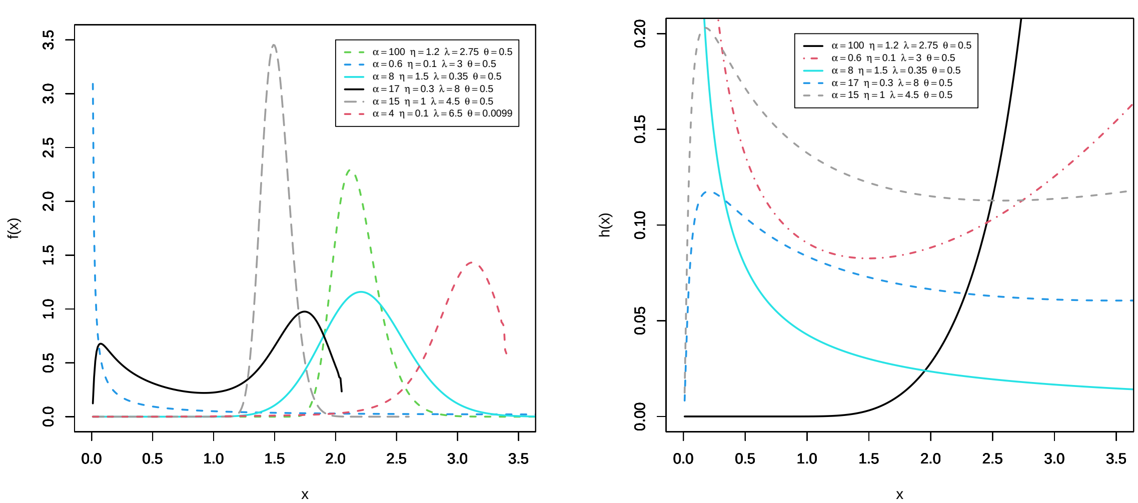

Figure 2: Plots of the pdf and hrf of the IPBHW distribution

3.3. The IPBHBIII Distribution

Using the baseline two-parameter Burr III (BIII) distribution (Burr, 1942) with pdf $g\left(x;a\ ,b\right)=abx^{-a-1}\left(1+x^{-a}\right)^{-b-1},\ x>0,a,b>0$, the cdf of the IPBHBIII distribution reduces to

$$f\left(x;\alpha,\eta,a,b\right)=\frac{\eta abx^{-a-1}\exp\left\{-\alpha\left[\left(1+x^{-a}\right)^{-b}-1\right]^{-\eta}\right\}\left\{\alpha+\left(\alpha+1\right)\left[\left(1+x^{-a}\right)^{-b}-1\right]^{\eta}\right\}}{\left(1+x^{-a}\right)\left[1-\left(1+x^{-a}\right)^{-b}\right]\left\{1+\left[\left(1+x^{-a}\right)^{-b}-1\right]^{\eta}\right\}^{2}}. \tag{9}$$

Plots the pdf and hrf of the IPBHBIII distribution for selected parameter values are displayed in Figure 3.

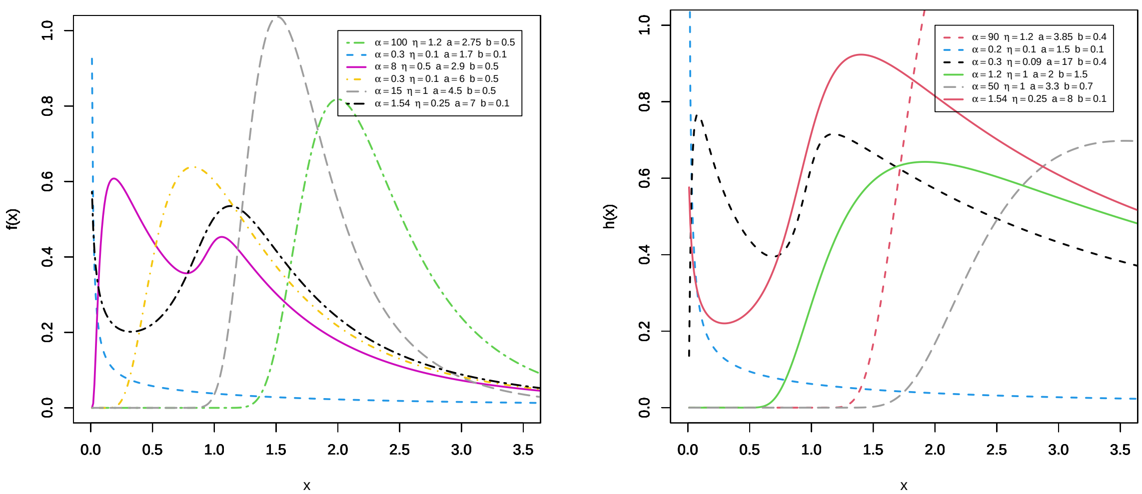

Figure 3: Plots of the pdf and hrf of the IPBHBIII distribution

4. Mathematical Properties

In this section, we present some key properties of the IPBH -G family.

4.1. Mixture Representation

In this section, we provide a useful linear representation for the cdf of the IPBH-G family. Consider the series expansion

$$e^{-Z}=\sum_{j=0}^{\infty}\frac{(-1)^{j}}{j!}z^{j}. \tag{10}$$

Applying the power series (10) to (3), we can write

$$F\left(x;\alpha,\eta,\boldsymbol{\psi}\right)=\frac{1}{1+\left[\frac{G\left(x\right)}{1-G(x)}\right]^{-\eta}}\sum_{j=0}^{\infty}\frac{(-1)^{j}}{j!}\alpha^{j}\left[\frac{G\left(x\right)}{1-G(x)}\right]^{-\eta j}. \tag{11}$$

Consider the following two power series

$$(1-z)^{q}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(q+1)}{k!\Gamma(q+1-k)}z^{k} \tag{12}$$

and

$$\frac{1}{1-(-z)} = \sum_{k=0}^{\infty} (-1)^k z^k. \tag{13}$$

Applying the power series (13) to the quantity

$$\frac{1}{1-\left[-\frac{G\,(x)}{1-G(x)}\right]^{-\eta}} = \sum_{k=0}^{\infty} (-1)^k \left[\frac{G\,(x)}{1-G(x)}\right]^{-\eta k}. \tag{14}$$

Substituting (14) in (11), we get

$$\begin{aligned} F\,(x;\alpha,\eta,\boldsymbol{\psi}) &= \sum_{k=0}^{\infty} (-1)^k \left[\frac{G\,(x)}{1-G\,(x)}\right]^{-\eta k} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\alpha^j \left[\frac{G\,(x)}{1-G\,(x)}\right]^{-\eta j} \\ &= \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}}{j!}\alpha^j G\,(x)^{-\eta(j+k)} \left[1-G\,(x)\right]^{\eta(j+k)}. \end{aligned} \tag{15}$$

After applying the power series (12), we have

$$\left[1-G(x)\right]^{\eta(j+k)} = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\eta(j+k)+1)}{l!\Gamma(\eta(j+k)+1-l)} G\,(x)^l. \tag{16}$$

Substituting (16) in (15), we get

$$F\,(x;\alpha,\eta,\boldsymbol{\psi}) = \sum_{j,k,l=0}^{\infty} \frac{(-1)^{j+k+l}}{j!l!} \frac{\Gamma(\eta(j+k)+1)}{l!\Gamma(\eta(j+k)-l+1)} \alpha^j G(x)^{l-\eta(j+k)}. \tag{17}$$

Then, the cdf of the IPBH-G family can be expressed as

$$F\,(x) = \sum_{j,k,l=0}^{\infty} a_{j,k,l}\; G(x)^{l-\eta(j+k)}, \tag{18}$$

where $a_{j,k,l} = \alpha^j \frac{(-1)^{j+k+l}}{j!l!} \frac{\Gamma(\eta(j+k)+1)}{l!\Gamma(\eta(j+k)-l+1)}$.
By differentiation of (18) the pdf of the IPBH-G family reduces to

$$f\,(x;\alpha,\eta,\boldsymbol{\psi}) = \sum_{j,k,l=0}^{\infty} b_{j,k,l}\; g(x)\; G\,(x)^{l-\eta(j+k)-1}, \tag{19}$$

where $b_{j,k,l} = a_{j,k,l}[l-\eta(j+k)]$ is the Exp-G density with power parameter $[l-\eta(j+k)]$. Thus, several mathematical properties of the IPBH-G family can be obtained simply from those of the Exp-G family.

4.2. Quantile functions

The quantile function (qf) of the IPBH-G family follows, by inverting the cdf (3), as

$$Q(p) = F^{-1}(p) = Q_G \left\{ \frac{\left[W_0\left(\frac{\alpha e^{\alpha}}{p}\right) - 1\right]^{\frac{-1}{\eta}}}{\alpha^{\frac{-1}{\eta}} + \left[W_0\left(\frac{\alpha e^{\alpha}}{p}\right) - 1\right]^{\frac{-1}{\eta}}} \right\}, \tag{20}$$

where $W_0(.)$ is Lambert function, $Q_G(.)$ denotes the baseline distribution's qf and p follows the uniform distribution $(0,\ 1)$.

4.3. Moments

The r^{th} moment of X can be obtained as

$$\mu'_r = E(X^r) = \int_{-\infty}^{+\infty} x^r f(x;\alpha,\eta,\psi)\,dx = \sum_{j,k,l=0}^{\infty} b_{j,k,l} \int_0^{+\infty} x^r g(x)\, G(x)^{l-\eta(j+k)-1}. \tag{21}$$

The sth incomplete moment of X can be obtained from (19) as

$$\varphi_S(t) = \int_{-\infty}^{t} x^S f(x)\,dx = \sum_{j,k,l=0}^{\infty} b_{j,k,l} \int_{-\infty}^{t} x^S g(x)\, G(x)^{l-\eta(j+k)-1} dx.$$

The first incomplete moment of X follows from the last equation as

$$\varphi_1(t) = \sum_{j,k,l=0}^{\infty} b_{j,k,l} \int_{-\infty}^{t} x g(x)\, G(x)^{l-\eta(j+k)-1} dx.$$

4.4. Order Statistics

Consider a random sample $X_1,\ldots,X_n$ from the IPBH-G family and its associated order statistics $X_{1:n}, X_{2:n},\ldots,X_{n:n}$. Let $X_{i:n}$ is i^{th} order statistic, then the PDF of $X_{i:n}$ is

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{j=0}^{n-j} (-1)^j \binom{n-i}{j} f(x) F(x)^{j+i-1}. \tag{22}$$

Using (3), (4), we can write

$$f(x)\,F(x)^{j+i-1} = \frac{\eta g(x) \left\{\alpha + (\alpha+1)\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\} \exp\left\{-\alpha(j+i)\left[\frac{G(x)}{1-G(x)}\right]^{-\eta}\right\}}{[1-G(\,x)]\,G(x)\left\{1+\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\}^{2}\left\{1+\left[\frac{G(x)}{1-G(x)}\right]^{-\eta}\right\}^{j+i-1}}.$$

Applying the series of expansions, we have

$$\exp\left\{-\alpha(j+i)\left[\frac{G(x)}{1-G(x)}\right]^{-\eta}\right\} = \sum_{q=0}^{\infty} (-1)^q \frac{\alpha^q (j+i)^q}{q!} \left[\frac{G(x)}{1-G(x)}\right]^{-\eta q}.$$

Following Abramowitz and Stegun (1968), we can use the power series

$$(1+z)^{-2}=\sum_{p=0}^{\infty}\binom{-2}{p}z^{p}=\sum_{p=0}^{\infty}\frac{(-1)^{p}}{2}(p+1)(p+2)z^{p}$$

$$(1+z)^{-m}=\sum_{m=0}^{\infty}\binom{-q}{m}z^{m}$$

and

$$(1-z)^{-q}=\sum_{i=0}^{\infty}\frac{\Gamma(q+i)}{i!\Gamma(q)}z^{i},$$

we can write

$$\left\{1+\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\}^{-2}=\sum_{p=0}^{\infty}\binom{-2}{p}\left[\frac{G(x)}{1-G(x)}\right]^{p\eta}$$

and

$$\left\{1+\left[\frac{G(x)}{1-G(x)}\right]^{-\eta}\right\}^{-j-i+1}=\sum_{m=0}^{\infty}\binom{-j-i+1}{m}\left[\frac{G(x)}{1-G(x)}\right]^{-\eta m}.$$

Hence, we can write

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \eta g(x)\left\{\alpha+(\alpha+1)\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\} \\ &\quad S_{q,p,m}G(x)^{\eta(p-q-m)-1}\left[1-G\ (x)\right]^{\eta(q+m-p)-1}. \end{aligned}$$

Where $\boldsymbol{S}_{q,p,m}=\sum_{q,p,m=0}^{\infty}(-1)^{q}\binom{-j-i+1}{m}\frac{\alpha^{q}(j+i)^{q}}{q!}\binom{-2}{p}$.

After applying the power series (12) to $[1-G\,(\,x)]^{\eta(q+m-p)-1}$, we have

$$[1-G(x)]^{-\eta(p+q-m)-1}=\sum_{d=0}^{\infty}\frac{\Gamma(\eta(p+q-m)+1+d)}{d!\Gamma(\eta(p-q-m))}G(x)^{d}.$$

Hence, it follows that

$$f(x)F(x)^{j+i-1}=\eta g(x)\left\{\alpha+(\alpha+1)\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\}\boldsymbol{C}_{q,p,m,d}\,G(x)^{\eta(p-q-m)+d-1},$$

where $\boldsymbol{C}_{q,p,m,d}=\sum_{d=0}^{\infty}\frac{\Gamma(\eta(q+m-p)+d)}{d!\Gamma(\eta(q+m-p))}S_{q,p,m}$.
Then, we can write

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \eta\alpha C_{q,p,m,d}\,g(x)G(x)^{\eta(p-q-m)+d-1} \\ &\quad +\eta(\alpha+1)C_{q,p,m,d}\,g(x)G(x)^{\eta(p-q-m+1)+d-1}[1-G(x)]^{-\eta}. \end{aligned}$$

After applying the power series $(1-z)^{-q}=\sum_{i=0}^{\infty}\frac{\Gamma_{(q+i)}}{i!\Gamma_{(q)}}z^{i}$, we can write

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \eta\alpha g(x)\boldsymbol{C}_{q,p,m,d}\,g(x)G(x)^{\eta(p-q-m)+d-1} \\ &\quad +\eta g(x)(\alpha+1)u_{q,p,m,d,v}\,G(x)^{\eta(p-q-m+1)+d+v-1}, \end{aligned} \tag{23}$$

where $u_{q,p,m,d,v} = \sum_{v=0}^{\infty} \frac{\Gamma_{(\eta+v)}}{v!\Gamma_{(\eta)}} \boldsymbol{C}_{q,p,m,d}.$
Substituting (23) in (22), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{q,p,m,d=0}^{\infty} \boldsymbol{\Omega}_{q,p,m,d}\, H_{\eta(q-p+m)+d}(x) + \sum_{q,p,m,d,v=0}^{\infty} \boldsymbol{\Omega}_{q,p,m,d,v}\, H_{\eta(q-p+m)+d+v}(x),$$

where $H_{\delta}(x) = \delta g(x)\, G(x)^{\delta-1}$ is the Exp-G density with power parameter δ and

$$\boldsymbol{\Omega}_{q,p,m,d} = \eta\alpha \sum_{j=0}^{n-j} \frac{(-1)^j\; n!}{(i-1)!\,(n-i)!} \binom{n-i}{j} C_{q,p,m,d},$$

and

$$\boldsymbol{\Omega}_{q,p,m,d,v} = \eta(\alpha+1) \sum_{j=0}^{n-j} \frac{(-1)^j\; n!}{(i-1)!\,(n-i)!} \binom{n-i}{j} u_{q,p,m,d,v}.$$

Then, the density function of the IPBH-G order statistics is a fourth linear combination of the Exp-G densities. Based on this equation, the properties of $X_{i:n}$ can be easily determined from those properties of the Exp-G class.

4.5. Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is given by

$$I_{\theta}(X) = \frac{1}{1-\theta} \log\left[\int_{-\infty}^{\infty} f(x)^{\theta} dx\right], \theta > 0 \text{ and } \theta \neq 1.$$

Using the pdf (4), we can write

$$f(x)^{\theta} = \frac{\eta^{\theta} g(x)^{\theta} \left\{\alpha + (\alpha+1)\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\}^{\theta}}{[1-G(x)]^{\theta}\; G(x)^{\theta}\; \left\{\left[\frac{G(x)}{1-G(x)}\right]^{\eta}+1\right\}^{2\theta}} \exp\left\{-\alpha\theta\left[\frac{G(x)}{1-G(x)}\right]^{-\eta}\right\}.$$

Applying the series expansion (10) to the last term, we obtain

$$\exp\left\{-\alpha\theta\left[\frac{G(x)}{1-G(x)}\right]^{-\eta}\right\} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \alpha^j \theta^j G(x)^{-\eta j} [1-G(x)]^{\eta j}.$$

Hence,

$$f(x)^{\theta} = \frac{\eta^{\theta} g(x)^{\theta} \left\{\alpha + (\alpha+1)\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\}^{\theta}}{\left\{\left[\frac{G(x)}{1-G(x)}\right]^{\eta}+1\right\}^{2\theta}} \sum_{j,i=0}^{\infty} \frac{(-\alpha\theta)^j}{j!} G(x)^{-\eta j-\theta} [1-G(x)]^{\eta j-\theta}. \tag{24}$$

Applying the power series

$$(1+z)^{-k} = \sum_{k=0}^{\infty} \binom{-q}{k} z^k, \tag{25}$$

gives

$$\left\{1+\left(\frac{G(x)}{1-G(x)}\right)^{\eta}\right\}^{-2\theta}=\sum_{k=0}^{\infty}\binom{-2\theta}{k}G\ (x)^{\eta k}\left[1-G\ (x)\right]^{-\eta k}. \tag{26}$$

Substituting (26) in (24), we get

$$\begin{aligned} f(x)^{\theta} &= \eta^{\theta}g(x)^{\theta}\left\{\alpha+(\alpha+1)\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\}^{\theta} \\ &\sum_{j,k=0}^{\infty}\frac{(-\alpha\theta)^{j}}{j!}\binom{-2\theta}{k}G(x)^{-\eta(j-k)-\theta}\left[1-G(x)\right]^{-\eta(k-j)-\theta}. \end{aligned} \tag{27}$$

Applying the series of expansions

$$\begin{aligned} \left\{\alpha+(\alpha+1)\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\}^{\theta} &= \alpha^{\theta}\left\{1+\frac{(\alpha+1)}{\alpha}\left[\frac{G(x)}{1-G(x)}\right]^{\eta}\right\}^{\theta} \\ &= \sum_{r=0}^{\infty}\alpha^{\theta}\binom{\theta}{r}(\alpha+1)^{r}\alpha^{-r}G\ (x)^{\eta r}\left[1-G(x)\right]^{-\eta r}. \end{aligned} \tag{28}$$

Substituting (28) in (27), we get

$$\begin{aligned} f(x)^{\theta} &= \eta^{\theta}g(x)^{\theta}\sum_{j,k,r=0}^{\infty}\frac{(-1)^{j}}{j!}\binom{-2\theta}{k}\binom{\theta}{r}\theta^{j}\alpha^{\theta+j-r}(\alpha+1)^{r} \\ &G(\ x)^{-\eta(j-k-r)-\theta}\left[1-G(x)\right]^{-\eta(k-j+r)-\theta}. \end{aligned} \tag{29}$$

Applying the binomial series

$$\left[1-G(\ x)\right]^{-\eta(k-j+r)-\theta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{-\eta(k-j+r)-\theta}{i}G(x)^{i}.$$

Hence,

$$f(x)^{\theta}=\varpi_{k}g(x)^{\theta}G(x)^{-\eta(k-j+r)-\theta+i},$$

where

$$\varpi_{k}=\eta^{\theta}\sum_{j,k,r,i=0}^{\infty}\frac{(-1)^{j+i}}{j!}\binom{-2\theta}{k}\binom{\theta}{r}\binom{-\eta(k-j+r)-\theta}{i}\theta^{j}\alpha^{\theta+j-r}(\alpha+1)^{r}.$$

The Rényi entropy of the IPBH-G family takes the form

$$I_{\theta}(X)=\frac{1}{1-\theta}\log\int_{-\infty}^{\infty}\varpi_{k}g(x)^{\theta}G(x)^{-\eta(k-j+r)-\theta+i}dx.$$

5. Estimation of the IPBHEx Parameters

In this section, we study the estimation of the IPBHEx distribution using eight estimators. Consider the random sample, say $X_1, X_2,\ ...,\ X_n$, of size n from the IPBHEx distribution and their associated order statistics, say $X_{1:n}, X_{2:n:},\ ..., X_{i:n}$. The log-likelihood function for the IPBHEx distribution with the vector of parameters $\mathcal{O}=(\alpha,\eta,\lambda)^T$ is

$$\ell(\mathcal{O}) = n\log(\eta) + n\log(\lambda) + \lambda\sum_{i=1}^{n} x_i - \alpha\sum_{i=1}^{n}\delta_i{}^{-\eta} + \sum_{i=1}^{n}\log\left[\alpha + (\alpha+1)\,(\delta_i)^{\eta}\right] - \sum_{i=1}^{n}\log(\delta_i) - 2\sum_{i=1}^{n}\log(1+\delta_i{}^{\eta}),$$

where $\delta_i = e^{\lambda x_i} - 1$.
The least squares estimators (LSE) and weighted least-squares estimators (WLSE) (Swain et al. 1988) are defined as arguments that minimize the following objective function

$$S(\mathcal{O}) = \sum_{i=1}^{n}\omega_i\left[\frac{\exp\left(-\alpha(\delta_{i:n})^{-\eta}\right)}{(\delta_{i:n})^{-\eta}+1} - \frac{i}{n+1}\right]^2,$$

where $\omega_i = 1$ for the LSE, and $\omega_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ for the WLSE method.
The percentile estimators (PCE) (Kao 1958) follow by minimizing

$$P(\mathcal{O}) = \sum_{i=1}^{n}\left(x_{i:n} - \frac{-1}{\lambda}\log\left\{\frac{\left[W\left(\frac{\alpha e^{\alpha}}{p_i}\right)-1\right]^{\frac{-1}{\eta}}}{\alpha^{\frac{-1}{\eta}} + \left[W\left(\frac{\alpha e^{\alpha}}{p_i}\right)-1\right]^{\frac{-1}{\eta}}}\right\}\right)^2,$$

where $p_i = i/(n\ +\ 1)$ be an unbiased estimator of $F\left(x_{i:n}|\alpha,\eta,\ \lambda\right)$.
Cheng and Amin (1979), (1983) introduced the maximum product of spacings (MPS) method. The MPS estimators (MPSE) are determined by maximizing the function

$$M(\mathcal{O}) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[\frac{\exp\left(-\alpha(\delta_{i:n})^{-\eta}\right)}{(\delta_{i:n})^{-\eta}+1} - \frac{\exp\left(-\alpha(\delta_{(i-1:n)})^{-\eta}\right)}{(\delta_{(i-1:n)})^{-\eta}+1}\right].$$

Cramér (1928) and MacDonald (1971) introduced the Cramér von–Mises estimators (CRVME) which follow by minimizing

$$C(\mathcal{O}) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{x_{i:n}}|\alpha,\eta,\ \lambda) - \frac{2i-1}{2n}\right]^2.$$

The Anderson–Darling (AD) test is proposed by Anderson and Darling (1952). The AD estimators (ADE) of the parameters of the IPBHEx distribution are obtained by minimizing

$$A(\mathcal{O}) = -n - \frac{1}{n} + \sum_{i=1}^{n}(2i-1)\ \left[\log F(x_{i:n}|\alpha,\eta,\ \lambda) + \log S(x_{i:n}|\alpha,\eta,\lambda)\right].$$

On the other hand, Luceño (2006) applied some modifications to the AD statistic to define the right-tail AD (RAD) statistic. Hence, the RAD estimators (RADE) of the IPBHEx parameters are specified by minimizing

$$R(\mathcal{O}) = \frac{n}{2} - 2\sum_{i=1}^{n}\ F\left(x_{(i:n)}|\alpha,\eta,\ \lambda\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{(i)}|\alpha,\eta,\lambda).$$

6. Bayesian Estimation

In this section, we explore the Bayesian estimators (BEs) of the IPBHEx parameters under three loss functions including the squared error (SE), linear exponential (LN) and general entropy (GE). The parameters α,η and λ of the IPBHEx distribution are assumed to be independent and follow gamma prior distributions. Hence, the joint prior density follows as

$$j\left(\mathcal{O}\right)=\frac{b^{a}}{\Gamma\left(a\right)}\frac{d^{c}}{\Gamma\left(c\right)}\frac{v^{h}}{\Gamma\left(h\right)}\alpha^{a-1}\eta^{c-1}\lambda^{h-1}e^{-\left(b\alpha+d\eta+v\lambda\right)},\alpha,\eta,\lambda>0, \tag{30}$$

where $\Gamma(a),\ \Gamma(c)$ and $\Gamma(v)$ denotes the standard gamma function. The associated likelihood function for $\underline{x}=x_1,x_2,\ ...,\ x_n$ is specified by

$$\ell\left(\mathcal{O}|\underline{x}\right)=\eta^{n}\ \lambda^{n}e^{\lambda\sum_{i=1}^{n}x_i-\alpha\sum_{i=1}^{n}\delta_i{}^{-\eta}}\sum_{i=1}^{n}\left[\alpha+\left(\alpha+1\right)\left(\ \delta_i\right)^{\eta}\right]\sum_{i=1}^{n}\delta_i{}^{-1}\sum_{i=1}^{n}\left(1+\delta_i{}^{\eta}\right)^{-2}. \tag{31}$$

The joint posterior density function of $\mathcal{O}$ is obtained from Equations (30) and (31) as follows

$$\pi^{*}\left(\mathcal{O}|\underline{x}\right)=\frac{\ell\left(\mathcal{O}|\underline{x}\right)j\left(\mathcal{O}\right)}{\int_{\mathcal{O}}\ell\left(\mathcal{O}|\underline{x}\right)j\left(\mathcal{O}\right)d\mathcal{O}}.$$

Thus, under the SE, LN and GE loss functions, the BEs of the parameters $\alpha,\ \eta$, and λ are given by

$$\begin{aligned}\hat{\mathcal{O}}_{SE}&=E\left(\mathcal{O}|\underline{x}\right)=\int_{\mathcal{O}}\mathcal{O}\pi^{*}\left(\mathcal{O}|\underline{x}\right)d\mathcal{O}\\&=K^{-1}\int_{\mathcal{O}}\mathcal{O}\,\pi^{*}\left(\mathcal{O}|\underline{x}\right)d\mathcal{O},\end{aligned} \tag{32}$$

$$\begin{aligned}\hat{\mathcal{O}}_{LN}&=-\frac{1}{c}\ \log\left\{E_{\mathcal{O}}\left[\exp\left(-c\ \mathcal{O}\right)\right]\right\}\\&=-\frac{1}{c}\left[K^{-1}\int_{\mathcal{O}}e^{-c\mathcal{O}}\mathcal{O}\,\pi^{*}\left(\mathcal{O}|\underline{x}\right)d\mathcal{O}\right]\end{aligned} \tag{33}$$

and

$$\begin{aligned}\hat{\mathcal{O}}_{EG}&=\left[E_{\mathcal{O}}\ \left(\mathcal{O}^{-q}\right)\right]^{\frac{-1}{q}}\\&=\left[K^{-1}\int_{\mathcal{O}}\mathcal{O}^{-q}\,\pi^{*}\left(\mathcal{O}|\underline{x}\right)d\mathcal{O}\right]^{\frac{-1}{q}},\end{aligned} \tag{34}$$

where $K^{-1}=\int_{\mathcal{O}}\ell\left(\mathcal{O}|\underline{x}\right)j\left(\mathcal{O}\right)d\left(\mathcal{O}\right)$.
Such that $E_{\mathcal{O}}\left(\mathcal{O}^{-q}\right)$ exists. It is clear that the integrals in Equations (32), (33)) and (34) are difficult to expressed in explicit forms. Therefore, we use the Monte Carlo Markov Chain approach (MCMC) technique to approximate these integrals, we will use the Metropolis Hastings algorithm as an example of the MCMC technique to obtain the Bayesian estimates.

7. Simulation Analysis

In this section, we perform a simulation study to assess the behavior of different classical and Bayesian estimators in estimating the IPBHEx parameters. We generate $N=5000$ random samples from IPBHEx distribution for different sample sizes, $n=\{30,\ 50,\ 100,\ 250,\ 500\}$ and some parametric values for $\alpha=(0.5,\ 0.6,\ 0.7,\ 0.75,\ 1,\ 1.2)$, $\eta=(0.5,\ 0.6,\ 0.8)$ and $\lambda=\ (0.25,\ 0.5,\ 0.7,\ 0.8,\ 1,\ 1.5,\ 2)$. The average of absolute biases ($|BIAS|$), mean-squared errors (MSE) and mean relative errors (MRE) are calculated all estimates. These measures are given by
$|BIAS(\hat{\Theta})|=\frac{1}{N}\sum_{i=1}^{N}|\hat{\Theta}-\Theta|,\ MSE=\frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta}-\Theta)^2$ and $MRE=\frac{\frac{1}{N}\sum_{i=1}^{N}|\hat{\Theta}-\Theta|}{\Theta}$.

The performance of the considered estimators is evaluated in terms of the above-mentioned measures. All simulations are done using R software (version 4.2.1).

Tables 1–9 report the values of $|BIAS|$, MSE and MRE of the WLSE, LSE, MLE, MPSE, CVME, ADE, RADE, PCE, SE, LN and GE. These tables also provide superscripts which indicates the rank of each estimator among all estimators and ($\sum Ranks$) which is the partial sum of ranks for each column in a certain sample size.

Table 10 gives the partial and overall rank of the considered estimators. As expected, based on our study, we can conclude that the Bayesian method outperforms all other classical methods under the three loss functions, with respective overall scores of 82.5, 88, and 107 for the SE, GE, and LN loss functions, respectively. On the other hand, we also conclude that the ML method outperforms all other classical estimation approaches with overall score of 198.5. Hence, the ML method is the best classical estimation method for estimating the IPBHEx parameters. It will be adopted in the next section to compare the fitting performance of the proposed IPBHEx model and other competing models. Finally, the Bayesian approach under the SE loss function is considered the optimal estimation method for estimating the IPBHEx parameters.

Table 1: Simulation results of the IPBHEx distribution for α=0.6, η=0.8, λ=0.7.

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
		$\hat{\alpha}$	0.62768 $^{\{5\}}$	0.67166 $^{\{7\}}$	0.64313 $^{\{6\}}$	0.70667 $^{\{10\}}$	0.60672 $^{\{4\}}$	0.69285 $^{\{9\}}$	0.68373 $^{\{8\}}$	0.71068 $^{\{11\}}$	0.09503 $^{\{3\}}$	0.08846 $^{\{2\}}$	0.06561 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.21558 $^{\{4\}}$	0.26268 $^{\{9\}}$	0.25112 $^{\{7\}}$	0.26816 $^{\{10\}}$	0.24224 $^{\{6\}}$	0.26226 $^{\{8\}}$	0.23776 $^{\{5\}}$	0.28599 $^{\{11\}}$	0.07384 $^{\{3\}}$	0.07332 $^{\{2\}}$	0.07144 $^{\{1\}}$
		$\hat{\lambda}$	0.38375 $^{\{4\}}$	0.66179 $^{\{10\}}$	0.55401 $^{\{7\}}$	0.65078 $^{\{9\}}$	0.50828 $^{\{5\}}$	0.67889 $^{\{11\}}$	0.51942 $^{\{6\}}$	0.57757 $^{\{8\}}$	0.07807 $^{\{2\}}$	0.07469 $^{\{1\}}$	0.09337 $^{\{3\}}$
		$\hat{\alpha}$	0.84810 $^{\{7\}}$	0.89497 $^{\{8\}}$	0.81609 $^{\{6\}}$	1.04108 $^{\{11\}}$	0.69618 $^{\{4\}}$	0.81402 $^{\{5\}}$	0.93856 $^{\{9\}}$	1.03235 $^{\{10\}}$	0.01622 $^{\{3\}}$	0.01288 $^{\{2\}}$	0.00617 $^{\{1\}}$
30	MSE	$\hat{\eta}$	0.06877 $^{\{4\}}$	0.10376 $^{\{8\}}$	0.09481 $^{\{7\}}$	0.10612 $^{\{10\}}$	0.08416 $^{\{6\}}$	0.10492 $^{\{9\}}$	0.08198 $^{\{5\}}$	0.12261 $^{\{11\}}$	0.00796 $^{\{3\}}$	0.00766 $^{\{2\}}$	0.00724 $^{\{1\}}$
		$\hat{\lambda}$	0.40385 $^{\{4\}}$	1.44435 $^{\{10\}}$	0.98052 $^{\{8\}}$	1.37317 $^{\{9\}}$	0.81433 $^{\{5\}}$	1.83648 $^{\{11\}}$	0.81852 $^{\{6\}}$	0.97354 $^{\{7\}}$	0.00938 $^{\{2\}}$	0.00827 $^{\{1\}}$	0.01331 $^{\{3\}}$
		$\hat{\alpha}$	1.04614 $^{\{5\}}$	1.11943 $^{\{7\}}$	1.07188 $^{\{6\}}$	1.17779 $^{\{10\}}$	1.01120 $^{\{4\}}$	1.15474 $^{\{9\}}$	1.13955 $^{\{8\}}$	1.18447 $^{\{11\}}$	0.09765 $^{\{1\}}$	0.14744 $^{\{3\}}$	0.10935 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.26947 $^{\{4\}}$	0.32834 $^{\{9\}}$	0.31390 $^{\{7\}}$	0.33520 $^{\{10\}}$	0.30280 $^{\{6\}}$	0.32782 $^{\{8\}}$	0.29720 $^{\{5\}}$	0.35749 $^{\{11\}}$	0.08694 $^{\{1\}}$	0.09165 $^{\{3\}}$	0.08930 $^{\{2\}}$
		$\hat{\lambda}$	0.54822 $^{\{4\}}$	0.94542 $^{\{10\}}$	0.79144 $^{\{7\}}$	0.92969 $^{\{9\}}$	0.72611 $^{\{5\}}$	0.96984 $^{\{11\}}$	0.74203 $^{\{6\}}$	0.82510 $^{\{8\}}$	0.08654 $^{\{1\}}$	0.10671 $^{\{2\}}$	0.13339 $^{\{3\}}$
		$\sum Ranks$	41 $^{\{4\}}$	78 $^{\{8\}}$	61 $^{\{7\}}$	88 $^{\{10.5\}}$	45 $^{\{5\}}$	81 $^{\{9\}}$	58 $^{\{6\}}$	88 $^{\{10.5\}}$	19 $^{\{3\}}$	18 $^{\{2\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.49512 $^{\{4\}}$	0.59255 $^{\{8\}}$	0.53629 $^{\{6\}}$	0.62297 $^{\{11\}}$	0.51634 $^{\{5\}}$	0.60272 $^{\{10\}}$	0.55805 $^{\{7\}}$	0.59291 $^{\{9\}}$	0.09404 $^{\{3\}}$	0.08796 $^{\{2\}}$	0.06411 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.19293 $^{\{4\}}$	0.23426 $^{\{9\}}$	0.21163 $^{\{7\}}$	0.22980 $^{\{8\}}$	0.21149 $^{\{6\}}$	0.24615 $^{\{10\}}$	0.20528 $^{\{5\}}$	0.25004 $^{\{11\}}$	0.06758 $^{\{3\}}$	0.06725 $^{\{2\}}$	0.06565 $^{\{1\}}$
		$\hat{\lambda}$	0.28775 $^{\{4\}}$	0.49214 $^{\{11\}}$	0.38061 $^{\{7\}}$	0.48487 $^{\{9\}}$	0.35674 $^{\{5\}}$	0.48639 $^{\{10\}}$	0.36860 $^{\{6\}}$	0.44260 $^{\{8\}}$	0.07396 $^{\{2\}}$	0.07041 $^{\{1\}}$	0.08841 $^{\{3\}}$
		$\hat{\alpha}$	0.45509 $^{\{4\}}$	0.65453 $^{\{10\}}$	0.53542 $^{\{6\}}$	0.75479 $^{\{11\}}$	0.45695 $^{\{5\}}$	0.54995 $^{\{7\}}$	0.57211 $^{\{8\}}$	0.61205 $^{\{9\}}$	0.01587 $^{\{3\}}$	0.01274 $^{\{2\}}$	0.00574 $^{\{1\}}$
50	MSE	$\hat{\eta}$	0.05363 $^{\{4\}}$	0.08066 $^{\{9\}}$	0.06672 $^{\{7\}}$	0.07736 $^{\{8\}}$	0.06278 $^{\{6\}}$	0.08724 $^{\{10\}}$	0.06052 $^{\{5\}}$	0.08997 $^{\{11\}}$	0.00684 $^{\{3\}}$	0.00664 $^{\{2\}}$	0.00629 $^{\{1\}}$
		$\hat{\lambda}$	0.18353 $^{\{4\}}$	0.76397 $^{\{11\}}$	0.42804 $^{\{7\}}$	0.71608 $^{\{9\}}$	0.30623 $^{\{5\}}$	0.76186 $^{\{10\}}$	0.37448 $^{\{6\}}$	0.53450 $^{\{8\}}$	0.00851 $^{\{2\}}$	0.00746 $^{\{1\}}$	0.01201 $^{\{3\}}$
		$\hat{\alpha}$	0.82520 $^{\{4\}}$	0.98758 $^{\{8\}}$	0.89382 $^{\{6\}}$	1.03829 $^{\{11\}}$	0.86057 $^{\{5\}}$	1.00454 $^{\{10\}}$	0.93009 $^{\{7\}}$	0.98818 $^{\{9\}}$	0.09930 $^{\{1\}}$	0.14660 $^{\{3\}}$	0.10685 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.24117 $^{\{4\}}$	0.29283 $^{\{9\}}$	0.26453 $^{\{7\}}$	0.28725 $^{\{8\}}$	0.26437 $^{\{6\}}$	0.30768 $^{\{10\}}$	0.25661 $^{\{5\}}$	0.31255 $^{\{11\}}$	0.07928 $^{\{1\}}$	0.08406 $^{\{3\}}$	0.08207 $^{\{2\}}$
		$\hat{\lambda}$	0.41108 $^{\{4\}}$	0.70306 $^{\{11\}}$	0.54373 $^{\{7\}}$	0.69267 $^{\{9\}}$	0.50963 $^{\{5\}}$	0.69484 $^{\{10\}}$	0.52657 $^{\{6\}}$	0.63229 $^{\{8\}}$	0.08206 $^{\{1\}}$	0.10058 $^{\{2\}}$	0.12629 $^{\{3\}}$
		$\sum Ranks$	36 $^{\{4\}}$	86 $^{\{10\}}$	60 $^{\{7\}}$	84 $^{\{8.5\}}$	48 $^{\{5\}}$	87 $^{\{11\}}$	55 $^{\{6\}}$	84 $^{\{8.5\}}$	19 $^{\{3\}}$	18 $^{\{2\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.39151 $^{\{4\}}$	0.47785 $^{\{8\}}$	0.41777 $^{\{6\}}$	0.49792 $^{\{10\}}$	0.41161 $^{\{5\}}$	0.51539 $^{\{11\}}$	0.43154 $^{\{7\}}$	0.48957 $^{\{9\}}$	0.09021 $^{\{3\}}$	0.08495 $^{\{2\}}$	0.06293 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.15896 $^{\{4\}}$	0.19356 $^{\{9\}}$	0.16572 $^{\{5\}}$	0.19157 $^{\{8\}}$	0.17983 $^{\{7\}}$	0.23594 $^{\{11\}}$	0.16922 $^{\{6\}}$	0.21319 $^{\{10\}}$	0.05647 $^{\{3\}}$	0.05629 $^{\{2\}}$	0.05495 $^{\{1\}}$
		$\hat{\lambda}$	0.21432 $^{\{4\}}$	0.32359 $^{\{9\}}$	0.24781 $^{\{5\}}$	0.33598 $^{\{11\}}$	0.25247 $^{\{6\}}$	0.33494 $^{\{10\}}$	0.25523 $^{\{7\}}$	0.32120 $^{\{8\}}$	0.06602 $^{\{2\}}$	0.06312 $^{\{1\}}$	0.07792 $^{\{3\}}$
		$\hat{\alpha}$	0.25015 $^{\{4\}}$	0.38963 $^{\{10\}}$	0.29841 $^{\{6\}}$	0.43382 $^{\{11\}}$	0.27576 $^{\{5\}}$	0.36143 $^{\{8\}}$	0.30361 $^{\{7\}}$	0.37421 $^{\{9\}}$	0.01466 $^{\{3\}}$	0.01205 $^{\{2\}}$	0.00556 $^{\{1\}}$
100	MSE	$\hat{\eta}$	0.03673 $^{\{4\}}$	0.05355 $^{\{9\}}$	0.04024 $^{\{5\}}$	0.05211 $^{\{8\}}$	0.04590 $^{\{7\}}$	0.07695 $^{\{11\}}$	0.04037 $^{\{6\}}$	0.06511 $^{\{10\}}$	0.00499 $^{\{3\}}$	0.00488 $^{\{2\}}$	0.00457 $^{\{1\}}$
		$\hat{\lambda}$	0.08238 $^{\{4\}}$	0.26746 $^{\{10\}}$	0.13499 $^{\{7\}}$	0.29314 $^{\{11\}}$	0.11975 $^{\{5\}}$	0.23549 $^{\{9\}}$	0.12827 $^{\{6\}}$	0.22890 $^{\{8\}}$	0.00679 $^{\{2\}}$	0.00598 $^{\{1\}}$	0.00927 $^{\{3\}}$
		$\hat{\alpha}$	0.65252 $^{\{4\}}$	0.79641 $^{\{8\}}$	0.69628 $^{\{6\}}$	0.82986 $^{\{10\}}$	0.68602 $^{\{5\}}$	0.85899 $^{\{11\}}$	0.71924 $^{\{7\}}$	0.81595 $^{\{9\}}$	0.09685 $^{\{1\}}$	0.14159 $^{\{3\}}$	0.10489 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.19870 $^{\{4\}}$	0.24195 $^{\{9\}}$	0.20715 $^{\{5\}}$	0.23947 $^{\{8\}}$	0.22479 $^{\{7\}}$	0.29492 $^{\{11\}}$	0.21153 $^{\{6\}}$	0.26649 $^{\{10\}}$	0.06566 $^{\{1\}}$	0.07036 $^{\{3\}}$	0.06868 $^{\{2\}}$
		$\hat{\lambda}$	0.30616 $^{\{4\}}$	0.46228 $^{\{9\}}$	0.35402 $^{\{5\}}$	0.47997 $^{\{11\}}$	0.36067 $^{\{6\}}$	0.47849 $^{\{10\}}$	0.36462 $^{\{7\}}$	0.45885 $^{\{8\}}$	0.07245 $^{\{1\}}$	0.09017 $^{\{2\}}$	0.11131 $^{\{3\}}$
		$\sum Ranks$	36 $^{\{4\}}$	81 $^{\{8.5\}}$	50 $^{\{5\}}$	88 $^{\{10\}}$	53 $^{\{6\}}$	92 $^{\{11\}}$	59 $^{\{7\}}$	81 $^{\{8.5\}}$	19 $^{\{3\}}$	18 $^{\{2\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.29684 $^{\{6\}}$	0.36231 $^{\{8\}}$	0.29134 $^{\{5\}}$	0.36425 $^{\{9\}}$	0.28732 $^{\{4\}}$	0.43032 $^{\{11\}}$	0.30370 $^{\{7\}}$	0.36950 $^{\{10\}}$	0.06160 $^{\{2\}}$	0.06145 $^{\{1\}}$	0.06249 $^{\{3\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.12604 $^{\{5\}}$	0.15049 $^{\{9\}}$	0.12340 $^{\{4\}}$	0.14828 $^{\{8\}}$	0.13122 $^{\{7\}}$	0.22555 $^{\{11\}}$	0.12812 $^{\{6\}}$	0.17124 $^{\{10\}}$	0.04053 $^{\{1\}}$	0.04070 $^{\{3\}}$	0.04054 $^{\{2\}}$
		$\hat{\lambda}$	0.15887 $^{\{4\}}$	0.21166 $^{\{8\}}$	0.16180 $^{\{5\}}$	0.21234 $^{\{9\}}$	0.16324 $^{\{6\}}$	0.24969 $^{\{11\}}$	0.16545 $^{\{7\}}$	0.22031 $^{\{10\}}$	0.04291 $^{\{1\}}$	0.04299 $^{\{2\}}$	0.04316 $^{\{3\}}$
		$\hat{\alpha}$	0.13121 $^{\{4\}}$	0.19435 $^{\{9\}}$	0.13647 $^{\{5\}}$	0.20071 $^{\{10\}}$	0.13880 $^{\{7\}}$	0.23722 $^{\{11\}}$	0.13808 $^{\{6\}}$	0.19094 $^{\{8\}}$	0.00529 $^{\{2\}}$	0.00516 $^{\{1\}}$	0.00537 $^{\{3\}}$
250	MSE	$\hat{\eta}$	0.02411 $^{\{5\}}$	0.03191 $^{\{9\}}$	0.02362 $^{\{4\}}$	0.03112 $^{\{8\}}$	0.02693 $^{\{7\}}$	0.07007 $^{\{11\}}$	0.02451 $^{\{6\}}$	0.04249 $^{\{10\}}$	0.00259 $^{\{3\}}$	0.00258 $^{\{2\}}$	0.00255 $^{\{1\}}$
		$\hat{\lambda}$	0.03945 $^{\{4\}}$	0.07947 $^{\{8\}}$	0.04726 $^{\{7\}}$	0.08189 $^{\{9\}}$	0.04575 $^{\{6\}}$	0.09389 $^{\{11\}}$	0.04424 $^{\{5\}}$	0.08220 $^{\{10\}}$	0.00282 $^{\{3\}}$	0.00280 $^{\{1\}}$	0.00281 $^{\{2\}}$
		$\hat{\alpha}$	0.49473 $^{\{6\}}$	0.60385 $^{\{8\}}$	0.48557 $^{\{5\}}$	0.60708 $^{\{9\}}$	0.47887 $^{\{4\}}$	0.71719 $^{\{11\}}$	0.50616 $^{\{7\}}$	0.61584 $^{\{10\}}$	0.09805 $^{\{1\}}$	0.10241 $^{\{2\}}$	0.10416 $^{\{3\}}$
	MRE	$\hat{\eta}$	0.15755 $^{\{5\}}$	0.18811 $^{\{9\}}$	0.15425 $^{\{4\}}$	0.18535 $^{\{8\}}$	0.16402 $^{\{7\}}$	0.28193 $^{\{11\}}$	0.16015 $^{\{6\}}$	0.21405 $^{\{10\}}$	0.04980 $^{\{1\}}$	0.05088 $^{\{3\}}$	0.05068 $^{\{2\}}$
		$\hat{\lambda}$	0.22696 $^{\{4\}}$	0.30237 $^{\{8\}}$	0.23114 $^{\{5\}}$	0.30335 $^{\{9\}}$	0.23320 $^{\{6\}}$	0.35670 $^{\{11\}}$	0.23636 $^{\{7\}}$	0.31473 $^{\{10\}}$	0.05970 $^{\{1\}}$	0.06142 $^{\{2\}}$	0.06166 $^{\{3\}}$
		$\sum Ranks$	43 $^{\{4\}}$	76 $^{\{8\}}$	44 $^{\{5\}}$	79 $^{\{9\}}$	54 $^{\{6\}}$	99 $^{\{11\}}$	57 $^{\{7\}}$	88 $^{\{10\}}$	15 $^{\{1\}}$	17 $^{\{2\}}$	22 $^{\{3\}}$
		$\hat{\alpha}$	0.23216 $^{\{6\}}$	0.29108 $^{\{9\}}$	0.22203 $^{\{5\}}$	0.29007 $^{\{8\}}$	0.19954 $^{\{4\}}$	0.37110 $^{\{11\}}$	0.24228 $^{\{7\}}$	0.29162 $^{\{10\}}$	0.06049 $^{\{1\}}$	0.06070 $^{\{2\}}$	0.06131 $^{\{3\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.10404 $^{\{6\}}$	0.12716 $^{\{9\}}$	0.09623 $^{\{5\}}$	0.12477 $^{\{8\}}$	0.09454 $^{\{4\}}$	0.20745 $^{\{11\}}$	0.10455 $^{\{7\}}$	0.13966 $^{\{10\}}$	0.03200 $^{\{1\}}$	0.03218 $^{\{3\}}$	0.03209 $^{\{2\}}$
		$\hat{\lambda}$	0.12308 $^{\{6\}}$	0.16337 $^{\{10\}}$	0.11850 $^{\{5\}}$	0.16248 $^{\{8\}}$	0.11189 $^{\{4\}}$	0.21017 $^{\{11\}}$	0.13001 $^{\{7\}}$	0.16286 $^{\{9\}}$	0.03699 $^{\{1\}}$	0.03716 $^{\{2\}}$	0.03722 $^{\{3\}}$
		$\hat{\alpha}$	0.07926 $^{\{6\}}$	0.11993 $^{\{10\}}$	0.07728 $^{\{4\}}$	0.11959 $^{\{9\}}$	0.07806 $^{\{5\}}$	0.18102 $^{\{11\}}$	0.08868 $^{\{7\}}$	0.11846 $^{\{8\}}$	0.00502 $^{\{2\}}$	0.00498 $^{\{1\}}$	0.00508 $^{\{3\}}$
500	MSE	$\backslash hat\eta$	0.01799 $^{\{7\}}$	0.02319 $^{\{9\}}$	0.01576 $^{\{4\}}$	0.02251 $^{\{8\}}$	0.01637 $^{\{5\}}$	0.06307 $^{\{11\}}$	0.01726 $^{\{6\}}$	0.03020 $^{\{10\}}$	0.00160 $^{\{2.5\}}$	0.00160 $^{\{2.5\}}$	0.00159 $^{\{1\}}$
		$\hat{\lambda}$	0.02281 $^{\{4\}}$	0.04116 $^{\{9\}}$	0.02321 $^{\{5\}}$	0.04118 $^{\{10\}}$	0.02334 $^{\{6\}}$	0.06160 $^{\{11\}}$	0.02657 $^{\{7\}}$	0.04006 $^{\{8\}}$	0.00209 $^{\{2.5\}}$	0.00208 $^{\{1\}}$	0.00209 $^{\{2.5\}}$
		$\hat{\alpha}$	0.38694 $^{\{6\}}$	0.48514 $^{\{9\}}$	0.37006 $^{\{5\}}$	0.48344 $^{\{8\}}$	0.33257 $^{\{4\}}$	0.61850 $^{\{11\}}$	0.40380 $^{\{7\}}$	0.48603 $^{\{10\}}$	0.09878 $^{\{1\}}$	0.10117 $^{\{2\}}$	0.10219 $^{\{3\}}$
	MRE	$\hat{\eta}$	0.13004 $^{\{6\}}$	0.15894 $^{\{9\}}$	0.12028 $^{\{5\}}$	0.15596 $^{\{8\}}$	0.11817 $^{\{4\}}$	0.25931 $^{\{11\}}$	0.13069 $^{\{7\}}$	0.17457 $^{\{10\}}$	0.03940 $^{\{1\}}$	0.04023 $^{\{3\}}$	0.04011 $^{\{2\}}$
		$\hat{\lambda}$	0.17582 $^{\{6\}}$	0.23339 $^{\{10\}}$	0.16928 $^{\{5\}}$	0.23212 $^{\{8\}}$	0.15984 $^{\{4\}}$	0.30024 $^{\{11\}}$	0.18573 $^{\{7\}}$	0.23265 $^{\{9\}}$	0.05227 $^{\{1\}}$	0.05309 $^{\{2\}}$	0.05317 $^{\{3\}}$
		$\sum Ranks$	53 $^{\{6\}}$	84 $^{\{9.5\}}$	43 $^{\{5\}}$	75 $^{\{8\}}$	40 $^{\{4\}}$	99 $^{\{11\}}$	62 $^{\{7\}}$	84 $^{\{9.5\}}$	13 $^{\{1\}}$	18.5 $^{\{2\}}$	22.5 $^{\{3\}}$

Table 2: Simulation results of the IPBHEx distribution for α=0.75, η=0.5, λ=1.

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
		$\hat{\alpha}$	$0.49177^{\{4\}}$	$0.52836^{\{7\}}$	$0.53003^{\{8\}}$	$0.56118^{\{9\}}$	$0.50086^{\{5\}}$	$0.59102^{\{11\}}$	$0.52258^{\{6\}}$	$0.58659^{\{10\}}$	$0.08027^{\{3\}}$	$0.07798^{\{2\}}$	$0.07553^{\{1\}}$
	$\lvert BIAS \rvert$	$\hat{\eta}$	$0.17441^{\{4\}}$	$0.22035^{\{9\}}$	$0.20127^{\{7\}}$	$0.21329^{\{8\}}$	$0.19991^{\{6\}}$	$0.25825^{\{11\}}$	$0.17788^{\{5\}}$	$0.23582^{\{10\}}$	$0.04425^{\{3\}}$	$0.04422^{\{2\}}$	$0.04380^{\{1\}}$
		$\hat{\lambda}$	$0.45883^{\{4\}}$	$0.86152^{\{10\}}$	$0.77766^{\{8\}}$	$0.82587^{\{9\}}$	$0.74955^{\{7\}}$	$1.00351^{\{11\}}$	$0.58034^{\{5\}}$	$0.69171^{\{6\}}$	$0.10317^{\{2\}}$	$0.10036^{\{1\}}$	$0.11251^{\{3\}}$
		$\hat{\alpha}$	$0.38014^{\{5\}}$	$0.40902^{\{6\}}$	$0.45502^{\{8\}}$	$0.50754^{\{10\}}$	$0.37304^{\{4\}}$	$0.47735^{\{9\}}$	$0.43079^{\{7\}}$	$0.56766^{\{11\}}$	$0.00957^{\{3\}}$	$0.00876^{\{2\}}$	$0.00802^{\{1\}}$
30	MSE	$\hat{\eta}$	$0.05014^{\{5\}}$	$0.07259^{\{9\}}$	$0.06299^{\{7\}}$	$0.06860^{\{8\}}$	$0.06128^{\{6\}}$	$0.09684^{\{11\}}$	$0.04899^{\{4\}}$	$0.08549^{\{10\}}$	$0.00283^{\{3\}}$	$0.00278^{\{2\}}$	$0.00272^{\{1\}}$
		$\hat{\lambda}$	$0.39482^{\{4\}}$	$2.76461^{\{9\}}$	$2.48869^{\{8\}}$	$2.38809^{\{7\}}$	$6.96223^{\{11\}}$	$3.17287^{\{10\}}$	$0.72034^{\{5\}}$	$0.91555^{\{6\}}$	$0.01565^{\{2\}}$	$0.01448^{\{1\}}$	$0.01862^{\{3\}}$
		$\hat{\alpha}$	$0.65569^{\{4\}}$	$0.70448^{\{7\}}$	$0.70670^{\{8\}}$	$0.74824^{\{9\}}$	$0.66781^{\{5\}}$	$0.78803^{\{11\}}$	$0.69677^{\{6\}}$	$0.78212^{\{10\}}$	$0.09527^{\{1\}}$	$0.10398^{\{3\}}$	$0.10071^{\{2\}}$
	MRE	$\hat{\eta}$	$0.34881^{\{4\}}$	$0.44069^{\{9\}}$	$0.40254^{\{7\}}$	$0.42658^{\{8\}}$	$0.39981^{\{6\}}$	$0.51651^{\{11\}}$	$0.35577^{\{5\}}$	$0.47165^{\{10\}}$	$0.08522^{\{1\}}$	$0.08843^{\{3\}}$	$0.08760^{\{2\}}$
		$\hat{\lambda}$	$0.45883^{\{4\}}$	$0.86152^{\{10\}}$	$0.77766^{\{8\}}$	$0.82587^{\{9\}}$	$0.74955^{\{7\}}$	$1.00351^{\{11\}}$	$0.58034^{\{5\}}$	$0.69171^{\{6\}}$	$0.09108^{\{1\}}$	$0.10036^{\{2\}}$	$0.11251^{\{3\}}$
		$\sum Ranks$	$38^{\{4\}}$	$76^{\{8\}}$	$69^{\{7\}}$	$77^{\{9\}}$	$57^{\{6\}}$	$96^{\{11\}}$	$48^{\{5\}}$	$79^{\{10\}}$	$19^{\{3\}}$	$18^{\{2\}}$	$17^{\{1\}}$
		$\hat{\alpha}$	$0.40223^{\{4\}}$	$0.46517^{\{9\}}$	$0.43756^{\{7\}}$	$0.46397^{\{8\}}$	$0.42189^{\{5\}}$	$0.54110^{\{11\}}$	$0.42362^{\{6\}}$	$0.48781^{\{10\}}$	$0.07673^{\{3\}}$	$0.07538^{\{2\}}$	$0.07374^{\{1\}}$
	$\lvert BIAS \rvert$	$\hat{\eta}$	$0.14801^{\{4\}}$	$0.19007^{\{9\}}$	$0.16718^{\{7\}}$	$0.17965^{\{8\}}$	$0.16619^{\{6\}}$	$0.25579^{\{11\}}$	$0.14983^{\{5\}}$	$0.21089^{\{10\}}$	$0.04107^{\{2\}}$	$0.04117^{\{3\}}$	$0.04090^{\{1\}}$
		$\hat{\lambda}$	$0.36146^{\{4\}}$	$0.59858^{\{10\}}$	$0.48839^{\{7\}}$	$0.58822^{\{9\}}$	$0.44896^{\{6\}}$	$0.74231^{\{11\}}$	$0.43350^{\{5\}}$	$0.54968^{\{8\}}$	$0.09800^{\{2\}}$	$0.09598^{\{1\}}$	$0.10410^{\{3\}}$
		$\hat{\alpha}$	$0.24487^{\{4\}}$	$0.31339^{\{8\}}$	$0.28752^{\{7\}}$	$0.32991^{\{9\}}$	$0.26289^{\{5\}}$	$0.39019^{\{11\}}$	$0.27719^{\{6\}}$	$0.35073^{\{10\}}$	$0.00866^{\{3\}}$	$0.00816^{\{2\}}$	$0.00763^{\{1\}}$
50	MSE	$\hat{\eta}$	$0.03862^{\{5\}}$	$0.05748^{\{9\}}$	$0.04633^{\{7\}}$	$0.05128^{\{8\}}$	$0.04519^{\{6\}}$	$0.09697^{\{11\}}$	$0.03686^{\{4\}}$	$0.07168^{\{10\}}$	$0.00247^{\{3\}}$	$0.00245^{\{2\}}$	$0.00241^{\{1\}}$
		$\hat{\lambda}$	$0.21888^{\{4\}}$	$1.00324^{\{10\}}$	$0.54282^{\{7\}}$	$0.98594^{\{9\}}$	$0.47924^{\{6\}}$	$1.45861^{\{11\}}$	$0.35243^{\{5\}}$	$0.55007^{\{8\}}$	$0.01404^{\{2\}}$	$0.01320^{\{1\}}$	$0.01583^{\{3\}}$
		$\hat{\alpha}$	$0.53631^{\{4\}}$	$0.62023^{\{9\}}$	$0.58341^{\{7\}}$	$0.61863^{\{8\}}$	$0.56252^{\{5\}}$	$0.72147^{\{11\}}$	$0.56483^{\{6\}}$	$0.65042^{\{10\}}$	$0.09509^{\{1\}}$	$0.10051^{\{3\}}$	$0.09832^{\{2\}}$
	MRE	$\hat{\eta}$	$0.29601^{\{4\}}$	$0.38014^{\{9\}}$	$0.33435^{\{7\}}$	$0.35929^{\{8\}}$	$0.33237^{\{6\}}$	$0.51157^{\{11\}}$	$0.29965^{\{5\}}$	$0.42177^{\{10\}}$	$0.07982^{\{1\}}$	$0.08235^{\{3\}}$	$0.08180^{\{2\}}$
		$\hat{\lambda}$	$0.36146^{\{4\}}$	$0.59858^{\{10\}}$	$0.48839^{\{7\}}$	$0.58822^{\{9\}}$	$0.44896^{\{6\}}$	$0.74231^{\{11\}}$	$0.43350^{\{5\}}$	$0.54968^{\{8\}}$	$0.08920^{\{1\}}$	$0.09598^{\{2\}}$	$0.10410^{\{3\}}$
		$\sum Ranks$	$37^{\{4\}}$	$83^{\{9\}}$	$63^{\{7\}}$	$76^{\{8\}}$	$51^{\{6\}}$	$99^{\{11\}}$	$47^{\{5\}}$	$84^{\{10\}}$	$18^{\{2\}}$	$19^{\{3\}}$	$17^{\{1\}}$
		$\hat{\alpha}$	$0.30885^{\{4\}}$	$0.35961^{\{9\}}$	$0.31241^{\{5\}}$	$0.35619^{\{8\}}$	$0.31876^{\{6\}}$	$0.46398^{\{11\}}$	$0.32469^{\{7\}}$	$0.37157^{\{10\}}$	$0.07446^{\{3\}}$	$0.07388^{\{2\}}$	$0.07271^{\{1\}}$
	$\lvert BIAS \rvert$	$\hat{\eta}$	$0.11251^{\{4\}}$	$0.14413^{\{9\}}$	$0.11634^{\{5\}}$	$0.13850^{\{8\}}$	$0.12112^{\{7\}}$	$0.24443^{\{11\}}$	$0.11677^{\{6\}}$	$0.16793^{\{10\}}$	$0.03545^{\{2\}}$	$0.03561^{\{3\}}$	$0.03536^{\{1\}}$
		$\hat{\lambda}$	$0.27109^{\{4\}}$	$0.38625^{\{9\}}$	$0.30639^{\{6\}}$	$0.36919^{\{8\}}$	$0.30727^{\{7\}}$	$0.53582^{\{11\}}$	$0.30626^{\{5\}}$	$0.39309^{\{10\}}$	$0.08836^{\{2\}}$	$0.08701^{\{1\}}$	$0.09219^{\{3\}}$
		$\hat{\alpha}$	$0.14670^{\{4\}}$	$0.19043^{\{9\}}$	$0.15247^{\{5\}}$	$0.18892^{\{8\}}$	$0.16144^{\{6\}}$	$0.29212^{\{11\}}$	$0.16163^{\{7\}}$	$0.20474^{\{10\}}$	$0.00799^{\{3\}}$	$0.00771^{\{2\}}$	$0.00733^{\{1\}}$
100	MSE	$\hat{\eta}$	$0.02464^{\{4\}}$	$0.03535^{\{9\}}$	$0.02485^{\{5\}}$	$0.03326^{\{8\}}$	$0.02826^{\{7\}}$	$0.09577^{\{11\}}$	$0.02492^{\{6\}}$	$0.05067^{\{10\}}$	$0.00191^{\{2.5\}}$	$0.00191^{\{2.5\}}$	$0.00187^{\{1\}}$
		$\hat{\lambda}$	$0.11876^{\{4\}}$	$0.27078^{\{10\}}$	$0.17173^{\{7\}}$	$0.26471^{\{9\}}$	$0.16013^{\{5\}}$	$0.51695^{\{11\}}$	$0.16058^{\{6\}}$	$0.26143^{\{8\}}$	$0.01148^{\{2\}}$	$0.01093^{\{1\}}$	$0.01244^{\{3\}}$
		$\hat{\alpha}$	$0.41179^{\{4\}}$	$0.47947^{\{9\}}$	$0.41654^{\{5\}}$	$0.47492^{\{8\}}$	$0.42501^{\{6\}}$	$0.61864^{\{11\}}$	$0.43292^{\{7\}}$	$0.49543^{\{10\}}$	$0.09398^{\{1\}}$	$0.09851^{\{3\}}$	$0.09695^{\{2\}}$
	MRE	$\hat{\eta}$	$0.22502^{\{4\}}$	$0.28825^{\{9\}}$	$0.23268^{\{5\}}$	$0.27701^{\{8\}}$	$0.24223^{\{7\}}$	$0.48887^{\{11\}}$	$0.23354^{\{6\}}$	$0.33587^{\{10\}}$	$0.06941^{\{1\}}$	$0.07121^{\{3\}}$	$0.07072^{\{2\}}$
		$\hat{\lambda}$	$0.27109^{\{4\}}$	$0.38625^{\{9\}}$	$0.30639^{\{6\}}$	$0.36919^{\{8\}}$	$0.30727^{\{7\}}$	$0.53582^{\{11\}}$	$0.30626^{\{5\}}$	$0.39309^{\{10\}}$	$0.08100^{\{1\}}$	$0.08701^{\{2\}}$	$0.09219^{\{3\}}$
		$\sum Ranks$	$36^{\{4\}}$	$82^{\{9\}}$	$49^{\{5\}}$	$73^{\{8\}}$	$58^{\{7\}}$	$99^{\{11\}}$	$55^{\{6\}}$	$88^{\{10\}}$	$18^{\{2\}}$	$20^{\{3\}}$	$17^{\{1\}}$
		$\hat{\alpha}$	$0.20022^{\{5\}}$	$0.24064^{\{8\}}$	$0.20196^{\{6\}}$	$0.24271^{\{9\}}$	$0.19735^{\{4\}}$	$0.36867^{\{11\}}$	$0.21337^{\{7\}}$	$0.24857^{\{10\}}$	$0.06986^{\{1\}}$	$0.07038^{\{2\}}$	$0.07063^{\{3\}}$
	$\lvert BIAS \rvert$	$\hat{\eta}$	$0.06905^{\{4\}}$	$0.09557^{\{9\}}$	$0.07043^{\{6\}}$	$0.09498^{\{8\}}$	$0.06955^{\{5\}}$	$0.20801^{\{11\}}$	$0.07463^{\{7\}}$	$0.10899^{\{10\}}$	$0.02935^{\{1\}}$	$0.02959^{\{3\}}$	$0.02955^{\{2\}}$
		$\hat{\lambda}$	$0.17266^{\{4\}}$	$0.24116^{\{9\}}$	$0.18615^{\{6\}}$	$0.23790^{\{8\}}$	$0.18175^{\{5\}}$	$0.38608^{\{11\}}$	$0.19707^{\{7\}}$	$0.25867^{\{10\}}$	$0.07435^{\{1\}}$	$0.07494^{\{2\}}$	$0.07496^{\{3\}}$
		$\hat{\alpha}$	$0.06553^{\{4\}}$	$0.09384^{\{8\}}$	$0.06659^{\{5\}}$	$0.09511^{\{9\}}$	$0.06668^{\{6\}}$	$0.20633^{\{11\}}$	$0.07439^{\{7\}}$	$0.10048^{\{10\}}$	$0.00680^{\{1\}}$	$0.00682^{\{2\}}$	$0.00687^{\{3\}}$
250	MSE	$\hat{\eta}$	$0.01059^{\{6\}}$	$0.01842^{\{9\}}$	$0.01039^{\{4\}}$	$0.01823^{\{8\}}$	$0.01052^{\{5\}}$	$0.08051^{\{11\}}$	$0.01127^{\{7\}}$	$0.02435^{\{10\}}$	$0.00132^{\{1.5\}}$	$0.00133^{\{3\}}$	$0.00132^{\{1.5\}}$
		$\hat{\lambda}$	$0.04864^{\{4\}}$	$0.09656^{\{9\}}$	$0.05741^{\{6\}}$	$0.09393^{\{8\}}$	$0.05640^{\{5\}}$	$0.23786^{\{11\}}$	$0.06537^{\{7\}}$	$0.11189^{\{10\}}$	$0.00816^{\{1\}}$	$0.00821^{\{2.5\}}$	$0.00821^{\{2.5\}}$
		$\hat{\alpha}$	$0.26697^{\{5\}}$	$0.32085^{\{8\}}$	$0.26927^{\{6\}}$	$0.32361^{\{9\}}$	$0.26314^{\{4\}}$	$0.49156^{\{11\}}$	$0.28450^{\{7\}}$	$0.33143^{\{10\}}$	$0.09230^{\{1\}}$	$0.09383^{\{2\}}$	$0.09417^{\{3\}}$
	MRE	$\hat{\eta}$	$0.13811^{\{4\}}$	$0.19114^{\{9\}}$	$0.14086^{\{6\}}$	$0.18996^{\{8\}}$	$0.13909^{\{5\}}$	$0.41603^{\{11\}}$	$0.14926^{\{7\}}$	$0.21798^{\{10\}}$	$0.05860^{\{1\}}$	$0.05919^{\{3\}}$	$0.05909^{\{2\}}$
		$\hat{\lambda}$	$0.17266^{\{4\}}$	$0.24116^{\{9\}}$	$0.18615^{\{6\}}$	$0.23790^{\{8\}}$	$0.18175^{\{5\}}$	$0.38608^{\{11\}}$	$0.19707^{\{7\}}$	$0.25867^{\{10\}}$	$0.07379^{\{1\}}$	$0.07494^{\{2\}}$	$0.07496^{\{3\}}$
		$\sum Ranks$	$40^{\{4\}}$	$78^{\{9\}}$	$51^{\{6\}}$	$75^{\{8\}}$	$44^{\{5\}}$	$99^{\{11\}}$	$63^{\{7\}}$	$90^{\{10\}}$	$10^{\{1\}}$	$22^{\{2\}}$	$23^{\{3\}}$
		$\hat{\alpha}$	$0.13647^{\{5\}}$	$0.17125^{\{8\}}$	$0.14590^{\{7\}}$	$0.17259^{\{9\}}$	$0.13612^{\{4\}}$	$0.28271^{\{11\}}$	$0.14273^{\{6\}}$	$0.17628^{\{10\}}$	$0.06889^{\{1\}}$	$0.06950^{\{2\}}$	$0.06961^{\{3\}}$
	$\lvert BIAS \rvert$	$\hat{\eta}$	$0.04408^{\{5\}}$	$0.06385^{\{8.5\}}$	$0.04905^{\{7\}}$	$0.06385^{\{8.5\}}$	$0.04381^{\{4\}}$	$0.16408^{\{11\}}$	$0.04666^{\{6\}}$	$0.07285^{\{10\}}$	$0.02573^{\{1\}}$	$0.02596^{\{3\}}$	$0.02594^{\{2\}}$
		$\hat{\lambda}$	$0.11875^{\{4\}}$	$0.16727^{\{8\}}$	$0.13358^{\{7\}}$	$0.16866^{\{9\}}$	$0.12223^{\{5\}}$	$0.29591^{\{11\}}$	$0.13012^{\{6\}}$	$0.17852^{\{10\}}$	$0.06855^{\{1\}}$	$0.06916^{\{2\}}$	$0.06917^{\{3\}}$
		$\hat{\alpha}$	$0.03055^{\{4\}}$	$0.04827^{\{8\}}$	$0.03432^{\{7\}}$	$0.04860^{\{9\}}$	$0.03206^{\{5\}}$	$0.13818^{\{11\}}$	$0.03316^{\{6\}}$	$0.05225^{\{10\}}$	$0.00661^{\{1\}}$	$0.00665^{\{2\}}$	$0.00667^{\{3\}}$
500	MSE	$\hat{\eta}$	$0.00387^{\{5\}}$	$0.00831^{\{9\}}$	$0.00448^{\{7\}}$	$0.00811^{\{8\}}$	$0.00380^{\{4\}}$	$0.05974^{\{11\}}$	$0.00411^{\{6\}}$	$0.01135^{\{10\}}$	$0.00102^{\{1\}}$	$0.00103^{\{2.5\}}$	$0.00103^{\{2.5\}}$
		$\hat{\lambda}$	$0.02279^{\{4\}}$	$0.04571^{\{8\}}$	$0.02892^{\{7\}}$	$0.04622^{\{9\}}$	$0.02575^{\{5\}}$	$0.14199^{\{11\}}$	$0.02809^{\{6\}}$	$0.05384^{\{10\}}$	$0.00696^{\{1\}}$	$0.00701^{\{2.5\}}$	$0.00701^{\{2.5\}}$
		$\hat{\alpha}$	$0.18195^{\{5\}}$	$0.22834^{\{8\}}$	$0.19453^{\{7\}}$	$0.23012^{\{9\}}$	$0.18149^{\{4\}}$	$0.37694^{\{11\}}$	$0.19030^{\{6\}}$	$0.23504^{\{10\}}$	$0.09140^{\{1\}}$	$0.09266^{\{2\}}$	$0.09282^{\{3\}}$
	MRE	$\hat{\eta}$	$0.08816^{\{5\}}$	$0.12769^{\{8\}}$	$0.09810^{\{7\}}$	$0.12771^{\{9\}}$	$0.08762^{\{4\}}$	$0.32816^{\{11\}}$	$0.09333^{\{6\}}$	$0.14569^{\{10\}}$	$0.05127^{\{1\}}$	$0.05192^{\{3\}}$	$0.05188^{\{2\}}$
		$\hat{\lambda}$	$0.11875^{\{4\}}$	$0.16727^{\{8\}}$	$0.13358^{\{7\}}$	$0.16866^{\{9\}}$	$0.12223^{\{5\}}$	$0.29591^{\{11\}}$	$0.13012^{\{6\}}$	$0.17852^{\{10\}}$	$0.06842^{\{1\}}$	$0.06916^{\{2\}}$	$0.06917^{\{3\}}$
		$\sum Ranks$	$41^{\{5\}}$	$73.5^{\{8\}}$	$63^{\{7\}}$	$79.5^{\{9\}}$	$40^{\{4\}}$	$99^{\{11\}}$	$54^{\{6\}}$	$90^{\{10\}}$	$9^{\{1\}}$	$21^{\{2\}}$	$24^{\{3\}}$

Table 3: Simulation results of the IPBHEx distribution for α=0.6, η=0.5, λ=0.25.

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
30	$\|BIAS\|$	$\hat{\alpha}$	$0.42568^{\{4\}}$	$0.46081^{\{8\}}$	$0.45701^{\{7\}}$	$0.48976^{\{10\}}$	$0.43860^{\{5\}}$	$0.46379^{\{9\}}$	$0.45682^{\{6\}}$	$0.50963^{\{11\}}$	$0.06723^{\{3\}}$	$0.06470^{\{2\}}$	$0.06206^{\{1\}}$
		$\hat{\eta}$	$0.16108^{\{4\}}$	$0.20112^{\{9\}}$	$0.18426^{\{6\}}$	$0.19693^{\{8\}}$	$0.18601^{\{7\}}$	$0.20959^{\{10\}}$	$0.16607^{\{5\}}$	$0.21710^{\{11\}}$	$0.04391^{\{3\}}$	$0.04387^{\{2\}}$	$0.04337^{\{1\}}$
		$\hat{\lambda}$	$0.11211^{\{4\}}$	$0.19458^{\{10\}}$	$0.17153^{\{8\}}$	$0.19011^{\{9\}}$	$0.16770^{\{7\}}$	$0.20644^{\{11\}}$	$0.13844^{\{5\}}$	$0.16739^{\{6\}}$	$0.02545^{\{2\}}$	$0.02473^{\{1\}}$	$0.02758^{\{3\}}$
	MSE	$\hat{\alpha}$	$0.29350^{\{5\}}$	$0.31878^{\{7\}}$	$0.34763^{\{9\}}$	$0.39813^{\{10\}}$	$0.29162^{\{4\}}$	$0.30789^{\{6\}}$	$0.33655^{\{8\}}$	$0.42966^{\{11\}}$	$0.00683^{\{3\}}$	$0.00612^{\{2\}}$	$0.00541^{\{1\}}$
		$\hat{\eta}$	$0.04150^{\{4\}}$	$0.05988^{\{9\}}$	$0.05227^{\{7\}}$	$0.05815^{\{8\}}$	$0.05147^{\{6\}}$	$0.06890^{\{10\}}$	$0.04195^{\{5\}}$	$0.07104^{\{11\}}$	$0.00282^{\{3\}}$	$0.00277^{\{2\}}$	$0.00270^{\{1\}}$
		$\hat{\lambda}$	$0.02455^{\{4\}}$	$0.13764^{\{9\}}$	$0.11500^{\{7\}}$	$0.12563^{\{8\}}$	$0.31576^{\{11\}}$	$0.15340^{\{10\}}$	$0.04259^{\{5\}}$	$0.05614^{\{6\}}$	$0.00096^{\{2\}}$	$0.00089^{\{1\}}$	$0.00113^{\{3\}}$
	MRE	$\hat{\alpha}$	$0.70946^{\{4\}}$	$0.76802^{\{8\}}$	$0.76168^{\{7\}}$	$0.81627^{\{10\}}$	$0.73100^{\{5\}}$	$0.77298^{\{9\}}$	$0.76137^{\{6\}}$	$0.84939^{\{11\}}$	$0.09622^{\{1\}}$	$0.10784^{\{3\}}$	$0.10343^{\{2\}}$
		$\hat{\eta}$	$0.32215^{\{4\}}$	$0.40225^{\{9\}}$	$0.36852^{\{6\}}$	$0.39387^{\{8\}}$	$0.37203^{\{7\}}$	$0.41918^{\{10\}}$	$0.33214^{\{5\}}$	$0.43419^{\{11\}}$	$0.08437^{\{1\}}$	$0.08773^{\{3\}}$	$0.08674^{\{2\}}$
		$\hat{\lambda}$	$0.44843^{\{4\}}$	$0.77834^{\{10\}}$	$0.68611^{\{8\}}$	$0.76044^{\{9\}}$	$0.67080^{\{7\}}$	$0.82576^{\{11\}}$	$0.55378^{\{5\}}$	$0.66955^{\{6\}}$	$0.09102^{\{1\}}$	$0.09892^{\{2\}}$	$0.11034^{\{3\}}$
		$\sum Ranks$	$37^{\{4\}}$	$79^{\{8\}}$	$65^{\{7\}}$	$80^{\{9\}}$	$59^{\{6\}}$	$86^{\{11\}}$	$50^{\{5\}}$	$84^{\{10\}}$	$19^{\{3\}}$	$18^{\{2\}}$	$17^{\{1\}}$
50	$\|BIAS\|$	$\hat{\alpha}$	$0.34989^{\{4\}}$	$0.40496^{\{9\}}$	$0.37783^{\{7\}}$	$0.40475^{\{8\}}$	$0.37222^{\{6\}}$	$0.42061^{\{10\}}$	$0.37092^{\{5\}}$	$0.42325^{\{11\}}$	$0.06375^{\{3\}}$	$0.06220^{\{2\}}$	$0.06015^{\{1\}}$
		$\hat{\eta}$	$0.14039^{\{4\}}$	$0.17477^{\{9\}}$	$0.15658^{\{6\}}$	$0.16597^{\{8\}}$	$0.16013^{\{7\}}$	$0.20080^{\{11\}}$	$0.14257^{\{5\}}$	$0.19343^{\{10\}}$	$0.04059^{\{2\}}$	$0.04064^{\{3\}}$	$0.04031^{\{1\}}$
		$\hat{\lambda}$	$0.08855^{\{4\}}$	$0.13696^{\{10\}}$	$0.11177^{\{7\}}$	$0.13694^{\{9\}}$	$0.10871^{\{6\}}$	$0.15214^{\{11\}}$	$0.10399^{\{5\}}$	$0.13220^{\{8\}}$	$0.02408^{\{2\}}$	$0.02353^{\{1\}}$	$0.02563^{\{3\}}$
	MSE	$\hat{\alpha}$	$0.18533^{\{4\}}$	$0.23905^{\{8\}}$	$0.21444^{\{7\}}$	$0.25505^{\{10\}}$	$0.20329^{\{5\}}$	$0.24725^{\{9\}}$	$0.21341^{\{6\}}$	$0.26506^{\{11\}}$	$0.00601^{\{3\}}$	$0.00555^{\{2\}}$	$0.00502^{\{1\}}$
		$\hat{\eta}$	$0.03337^{\{5\}}$	$0.04722^{\{9\}}$	$0.03966^{\{6\}}$	$0.04302^{\{8\}}$	$0.04001^{\{7\}}$	$0.06536^{\{11\}}$	$0.03296^{\{4\}}$	$0.05834^{\{10\}}$	$0.00244^{\{3\}}$	$0.00242^{\{2\}}$	$0.00237^{\{1\}}$
		$\hat{\lambda}$	$0.01328^{\{4\}}$	$0.04772^{\{9\}}$	$0.02510^{\{7\}}$	$0.05094^{\{10\}}$	$0.02412^{\{6\}}$	$0.06858^{\{11\}}$	$0.02055^{\{5\}}$	$0.03297^{\{8\}}$	$0.00086^{\{2\}}$	$0.00080^{\{1\}}$	$0.00097^{\{3\}}$
	MRE	$\hat{\alpha}$	$0.58315^{\{4\}}$	$0.67494^{\{9\}}$	$0.62971^{\{7\}}$	$0.67458^{\{8\}}$	$0.62036^{\{6\}}$	$0.70101^{\{10\}}$	$0.61821^{\{5\}}$	$0.70541^{\{11\}}$	$0.09584^{\{1\}}$	$0.10367^{\{3\}}$	$0.10025^{\{2\}}$
		$\hat{\eta}$	$0.28078^{\{4\}}$	$0.34955^{\{9\}}$	$0.31316^{\{6\}}$	$0.33193^{\{8\}}$	$0.32025^{\{7\}}$	$0.40161^{\{11\}}$	$0.28515^{\{5\}}$	$0.38686^{\{10\}}$	$0.07871^{\{1\}}$	$0.08128^{\{3\}}$	$0.08061^{\{2\}}$
		$\hat{\lambda}$	$0.35421^{\{4\}}$	$0.54785^{\{10\}}$	$0.44709^{\{7\}}$	$0.54774^{\{9\}}$	$0.43483^{\{6\}}$	$0.60855^{\{11\}}$	$0.41597^{\{5\}}$	$0.52881^{\{8\}}$	$0.08723^{\{1\}}$	$0.09412^{\{2\}}$	$0.10253^{\{3\}}$
		$\sum Ranks$	$37^{\{4\}}$	$82^{\{9\}}$	$60^{\{7\}}$	$78^{\{8\}}$	$56^{\{6\}}$	$95^{\{11\}}$	$45^{\{5\}}$	$87^{\{10\}}$	$18^{\{2\}}$	$19^{\{3\}}$	$17^{\{1\}}$
100	$\|BIAS\|$	$\hat{\alpha}$	$0.27335^{\{4\}}$	$0.31711^{\{9\}}$	$0.27701^{\{5\}}$	$0.31364^{\{8\}}$	$0.28654^{\{6\}}$	$0.35442^{\{11\}}$	$0.28697^{\{7\}}$	$0.32733^{\{10\}}$	$0.06055^{\{3\}}$	$0.05972^{\{2\}}$	$0.05828^{\{1\}}$
		$\hat{\eta}$	$0.11004^{\{4\}}$	$0.13640^{\{9\}}$	$0.11412^{\{6\}}$	$0.13146^{\{8\}}$	$0.12084^{\{7\}}$	$0.19069^{\{11\}}$	$0.11356^{\{5\}}$	$0.15820^{\{10\}}$	$0.03543^{\{2\}}$	$0.03560^{\{3\}}$	$0.03537^{\{1\}}$
		$\hat{\lambda}$	$0.06766^{\{4\}}$	$0.09256^{\{9\}}$	$0.07490^{\{6\}}$	$0.08863^{\{8\}}$	$0.07665^{\{7\}}$	$0.11089^{\{11\}}$	$0.07451^{\{5\}}$	$0.09492^{\{10\}}$	$0.02224^{\{2\}}$	$0.02194^{\{1\}}$	$0.02319^{\{3\}}$
	MSE	$\hat{\alpha}$	$0.11261^{\{4\}}$	$0.14554^{\{9\}}$	$0.11923^{\{5\}}$	$0.14450^{\{8\}}$	$0.12678^{\{7\}}$	$0.17886^{\{11\}}$	$0.12533^{\{6\}}$	$0.15593^{\{10\}}$	$0.00533^{\{3\}}$	$0.00505^{\{2\}}$	$0.00470^{\{1\}}$
		$\hat{\eta}$	$0.02206^{\{4\}}$	$0.03049^{\{9\}}$	$0.02349^{\{6\}}$	$0.02881^{\{8\}}$	$0.02641^{\{7\}}$	$0.06244^{\{11\}}$	$0.02288^{\{5\}}$	$0.04285^{\{10\}}$	$0.00189^{\{3\}}$	$0.00188^{\{2\}}$	$0.00186^{\{1\}}$
		$\hat{\lambda}$	$0.00727^{\{4\}}$	$0.01501^{\{9\}}$	$0.01002^{\{7\}}$	$0.01460^{\{8\}}$	$0.00964^{\{6\}}$	$0.02453^{\{11\}}$	$0.00945^{\{5\}}$	$0.01522^{\{10\}}$	$0.00073^{\{2\}}$	$0.00069^{\{1\}}$	$0.00079^{\{3\}}$
	MRE	$\hat{\alpha}$	$0.45558^{\{4\}}$	$0.52851^{\{9\}}$	$0.46169^{\{5\}}$	$0.52273^{\{8\}}$	$0.47757^{\{6\}}$	$0.59070^{\{11\}}$	$0.47828^{\{7\}}$	$0.54555^{\{10\}}$	$0.09400^{\{1\}}$	$0.09954^{\{3\}}$	$0.09713^{\{2\}}$
		$\hat{\eta}$	$0.22009^{\{4\}}$	$0.27280^{\{9\}}$	$0.22825^{\{6\}}$	$0.26292^{\{8\}}$	$0.24167^{\{7\}}$	$0.38138^{\{11\}}$	$0.22713^{\{5\}}$	$0.31641^{\{10\}}$	$0.06889^{\{1\}}$	$0.07119^{\{3\}}$	$0.07073^{\{2\}}$
		$\hat{\lambda}$	$0.27064^{\{4\}}$	$0.37022^{\{9\}}$	$0.29960^{\{6\}}$	$0.35454^{\{8\}}$	$0.30658^{\{7\}}$	$0.44354^{\{11\}}$	$0.29804^{\{5\}}$	$0.37968^{\{10\}}$	$0.08210^{\{1\}}$	$0.08774^{\{2\}}$	$0.09276^{\{3\}}$
		$\sum Ranks$	$36^{\{4\}}$	$81^{\{9\}}$	$52^{\{6\}}$	$72^{\{8\}}$	$60^{\{7\}}$	$99^{\{11\}}$	$50^{\{5\}}$	$90^{\{10\}}$	$18^{\{2\}}$	$19^{\{3\}}$	$17^{\{1\}}$
250	$\|BIAS\|$	$\hat{\alpha}$	$0.18271^{\{4\}}$	$0.21617^{\{8\}}$	$0.18419^{\{6\}}$	$0.21824^{\{9\}}$	$0.18360^{\{5\}}$	$0.29045^{\{11\}}$	$0.19135^{\{7\}}$	$0.22430^{\{10\}}$	$0.05673^{\{1\}}$	$0.05709^{\{2\}}$	$0.05727^{\{3\}}$
		$\hat{\eta}$	$0.07155^{\{4\}}$	$0.09390^{\{9\}}$	$0.07236^{\{5\}}$	$0.09347^{\{8\}}$	$0.07424^{\{6\}}$	$0.17316^{\{11\}}$	$0.07458^{\{7\}}$	$0.10776^{\{10\}}$	$0.02733^{\{1\}}$	$0.02755^{\{3\}}$	$0.02751^{\{2\}}$
		$\hat{\lambda}$	$0.04422^{\{4\}}$	$0.05952^{\{9\}}$	$0.04716^{\{6\}}$	$0.05884^{\{8\}}$	$0.04709^{\{5\}}$	$0.08306^{\{11\}}$	$0.04867^{\{7\}}$	$0.06328^{\{10\}}$	$0.01806^{\{1\}}$	$0.01820^{\{2\}}$	$0.01821^{\{3\}}$
	MSE	$\hat{\alpha}$	$0.05330^{\{4\}}$	$0.07376^{\{8\}}$	$0.05503^{\{5\}}$	$0.07496^{\{9\}}$	$0.05691^{\{6\}}$	$0.13122^{\{11\}}$	$0.05945^{\{7\}}$	$0.08034^{\{10\}}$	$0.00450^{\{1.5\}}$	$0.00450^{\{1.5\}}$	$0.00453^{\{3\}}$
		$\hat{\eta}$	$0.01098^{\{5\}}$	$0.01684^{\{9\}}$	$0.01091^{\{4\}}$	$0.01679^{\{8\}}$	$0.01172^{\{7\}}$	$0.05911^{\{11\}}$	$0.01106^{\{6\}}$	$0.02323^{\{10\}}$	$0.00117^{\{2\}}$	$0.00117^{\{2\}}$	$0.00117^{\{2\}}$
		$\hat{\lambda}$	$0.00315^{\{4\}}$	$0.00574^{\{9\}}$	$0.00367^{\{5\}}$	$0.00560^{\{8\}}$	$0.00370^{\{6\}}$	$0.01171^{\{11\}}$	$0.00395^{\{7\}}$	$0.00653^{\{10\}}$	$0.00049^{\{2\}}$	$0.00049^{\{2\}}$	$0.00049^{\{2\}}$
	MRE	$\hat{\alpha}$	$0.30452^{\{4\}}$	$0.36029^{\{8\}}$	$0.30698^{\{6\}}$	$0.36373^{\{9\}}$	$0.30600^{\{5\}}$	$0.48409^{\{11\}}$	$0.31892^{\{7\}}$	$0.37383^{\{10\}}$	$0.09337^{\{1\}}$	$0.09515^{\{2\}}$	$0.09546^{\{3\}}$
		$\hat{\eta}$	$0.14310^{\{4\}}$	$0.18779^{\{9\}}$	$0.14473^{\{5\}}$	$0.18694^{\{8\}}$	$0.14848^{\{6\}}$	$0.34631^{\{11\}}$	$0.14915^{\{7\}}$	$0.21553^{\{10\}}$	$0.05455^{\{1\}}$	$0.05510^{\{3\}}$	$0.05502^{\{2\}}$
		$\hat{\lambda}$	$0.17690^{\{4\}}$	$0.23809^{\{9\}}$	$0.18863^{\{6\}}$	$0.23536^{\{8\}}$	$0.18838^{\{5\}}$	$0.33223^{\{11\}}$	$0.19468^{\{7\}}$	$0.25311^{\{10\}}$	$0.07174^{\{1\}}$	$0.07280^{\{2\}}$	$0.07285^{\{3\}}$
		$\sum Ranks$	$37^{\{4\}}$	$78^{\{9\}}$	$48^{\{5\}}$	$75^{\{8\}}$	$51^{\{6\}}$	$99^{\{11\}}$	$62^{\{7\}}$	$90^{\{10\}}$	$11.5^{\{1\}}$	$19.5^{\{2\}}$	$23^{\{3\}}$
500	$\|BIAS\|$	$\hat{\alpha}$	$0.12707^{\{5\}}$	$0.15834^{\{9\}}$	$0.13450^{\{7\}}$	$0.15809^{\{8\}}$	$0.12683^{\{4\}}$	$0.22368^{\{11\}}$	$0.13121^{\{6\}}$	$0.16053^{\{10\}}$	$0.05635^{\{1\}}$	$0.05682^{\{2\}}$	$0.05694^{\{3\}}$
		$\hat{\eta}$	$0.04710^{\{5\}}$	$0.06583^{\{9\}}$	$0.05122^{\{7\}}$	$0.06495^{\{8\}}$	$0.04670^{\{4\}}$	$0.13969^{\{11\}}$	$0.04844^{\{6\}}$	$0.07294^{\{10\}}$	$0.02384^{\{1\}}$	$0.02406^{\{3\}}$	$0.02403^{\{2\}}$
		$\hat{\lambda}$	$0.03087^{\{4\}}$	$0.04250^{\{9\}}$	$0.03418^{\{7\}}$	$0.04248^{\{8\}}$	$0.03219^{\{5\}}$	$0.06375^{\{11\}}$	$0.03315^{\{6\}}$	$0.04426^{\{10\}}$	$0.01594^{\{1\}}$	$0.01607^{\{2\}}$	$0.01608^{\{3\}}$
	MSE	$\hat{\alpha}$	$0.02697^{\{4\}}$	$0.04119^{\{9\}}$	$0.02990^{\{7\}}$	$0.04063^{\{8\}}$	$0.02828^{\{5\}}$	$0.08948^{\{11\}}$	$0.02863^{\{6\}}$	$0.04323^{\{10\}}$	$0.00441^{\{1\}}$	$0.00443^{\{2\}}$	$0.00445^{\{3\}}$
		$\hat{\eta}$	$0.00476^{\{5\}}$	$0.00904^{\{9\}}$	$0.00537^{\{7\}}$	$0.00846^{\{8\}}$	$0.00464^{\{4\}}$	$0.04572^{\{11\}}$	$0.00477^{\{6\}}$	$0.01141^{\{10\}}$	$0.00087^{\{2\}}$	$0.00087^{\{2\}}$	$0.00087^{\{2\}}$
		$\hat{\lambda}$	$0.00157^{\{4\}}$	$0.00294^{\{9\}}$	$0.00192^{\{7\}}$	$0.00291^{\{8\}}$	$0.00175^{\{5\}}$	$0.00718^{\{11\}}$	$0.00185^{\{6\}}$	$0.00329^{\{10\}}$	$0.00038^{\{2\}}$	$0.00038^{\{2\}}$	$0.00038^{\{2\}}$
	MRE	$\hat{\alpha}$	$0.21179^{\{5\}}$	$0.26389^{\{9\}}$	$0.22417^{\{7\}}$	$0.26349^{\{8\}}$	$0.21139^{\{4\}}$	$0.37280^{\{11\}}$	$0.21868^{\{6\}}$	$0.26755^{\{10\}}$	$0.09343^{\{1\}}$	$0.09469^{\{2\}}$	$0.09490^{\{3\}}$
		$\hat{\eta}$	$0.09419^{\{5\}}$	$0.13166^{\{9\}}$	$0.10244^{\{7\}}$	$0.12990^{\{8\}}$	$0.09340^{\{4\}}$	$0.27938^{\{11\}}$	$0.09688^{\{6\}}$	$0.14588^{\{10\}}$	$0.04753^{\{1\}}$	$0.04811^{\{3\}}$	$0.04807^{\{2\}}$
		$\hat{\lambda}$	$0.12348^{\{4\}}$	$0.16999^{\{9\}}$	$0.13670^{\{7\}}$	$0.16991^{\{8\}}$	$0.12876^{\{5\}}$	$0.25502^{\{11\}}$	$0.13262^{\{6\}}$	$0.17703^{\{10\}}$	$0.06368^{\{1\}}$	$0.06429^{\{2\}}$	$0.06431^{\{3\}}$
		$\sum Ranks$	$41^{\{5\}}$	$81^{\{9\}}$	$63^{\{7\}}$	$72^{\{8\}}$	$40^{\{4\}}$	$99^{\{11\}}$	$54^{\{6\}}$	$90^{\{10\}}$	$11^{\{1\}}$	$20^{\{2\}}$	$23^{\{3\}}$

Table 4: Simulation results of the IPBHEx distribution for α=0.5, η=0.6, λ=0.5.

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
		$\hat{\alpha}$	0.43860 {4}	0.47374 {8}	0.44981 {6}	0.49165 {9}	0.43924 {5}	0.50744 {11}	0.46638 {7}	0.50162 {10}	0.06415 {3}	0.06024 {2}	0.05311 {1}
	$\|BIAS\|$	$\hat{\eta}$	0.17163 {4}	0.20382 {8}	0.19438 {6}	0.20937 {9}	0.19650 {7}	0.22670 {10}	0.18323 {5}	0.23195 {11}	0.05317 {3}	0.05289 {2}	0.05211 {1}
		$\hat{\lambda}$	0.23824 {4}	0.37320 {10}	0.31392 {7}	0.36052 {9}	0.29882 {6}	0.45663 {11}	0.29343 {5}	0.33795 {8}	0.05186 {2}	0.05004 {1}	0.05787 {3}
		$\hat{\alpha}$	0.36124 {6}	0.37842 {8}	0.34083 {5}	0.42350 {11}	0.31363 {4}	0.38689 {9}	0.37509 {7}	0.42308 {10}	0.00661 {3}	0.00554 {2}	0.00404 {1}
30	MSE	$\hat{\eta}$	0.04405 {4}	0.05987 {8}	0.05469 {7}	0.06430 {9}	0.05293 {6}	0.07483 {10}	0.04786 {5}	0.08136 {11}	0.00416 {3}	0.00405 {2}	0.00389 {1}
		$\hat{\lambda}$	0.13050 {4}	0.39484 {10}	0.25981 {7}	0.36290 {9}	0.21453 {6}	0.77692 {11}	0.21040 {5}	0.28221 {8}	0.00403 {2}	0.00366 {1}	0.00505 {3}
		$\hat{\alpha}$	0.87720 {4}	0.94747 {8}	0.89963 {6}	0.98331 {9}	0.87848 {5}	1.01488 {11}	0.93275 {7}	1.00324 {10}	0.09685 {1}	0.12049 {3}	0.10622 {2}
	MRE	$\hat{\eta}$	0.28605 {4}	0.33971 {8}	0.32397 {6}	0.34896 {9}	0.32751 {7}	0.37783 {10}	0.30538 {5}	0.38658 {11}	0.08376 {1}	0.08815 {3}	0.08685 {2}
		$\hat{\lambda}$	0.47647 {4}	0.74639 {10}	0.62784 {7}	0.72103 {9}	0.59764 {6}	0.91326 {11}	0.58687 {5}	0.67591 {8}	0.08923 {1}	0.10008 {2}	0.11573 {3}
		$\sum Ranks$	38 {4}	78 {8}	57 {7}	83 {9}	52 {6}	94 {11}	51 {5}	87 {10}	19 {3}	18 {2}	17 {1}
		$\hat{\alpha}$	0.35350 {4}	0.41398 {8}	0.38015 {6}	0.42471 {10}	0.37700 {5}	0.44262 {11}	0.38063 {7}	0.42074 {9}	0.06201 {3}	0.05914 {2}	0.05218 {1}
	$\|BIAS\|$	$\hat{\eta}$	0.15548 {4}	0.18425 {9}	0.16824 {6}	0.17881 {8}	0.17424 {7}	0.21278 {11}	0.15886 {5}	0.20188 {10}	0.04862 {3}	0.04858 {2}	0.04784 {1}
		$\hat{\lambda}$	0.18562 {4}	0.28015 {10}	0.22661 {7}	0.27492 {9}	0.22493 {6}	0.32780 {11}	0.21309 {5}	0.26493 {8}	0.04842 {2}	0.04693 {1}	0.05335 {3}
		$\hat{\alpha}$	0.20208 {4}	0.27185 {9}	0.22615 {6}	0.29897 {11}	0.20973 {5}	0.27578 {10}	0.23179 {7}	0.27094 {8}	0.00601 {3}	0.00523 {2}	0.00379 {1}
50	MSE	$\hat{\eta}$	0.03616 {4}	0.04870 {9}	0.04183 {6}	0.04645 {8}	0.04253 {7}	0.06568 {11}	0.03682 {5}	0.05962 {10}	0.00355 {3}	0.00348 {2}	0.00336 {1}
		$\hat{\lambda}$	0.06283 {4}	0.19718 {10}	0.11294 {7}	0.18969 {9}	0.09368 {5}	0.33085 {11}	0.09780 {6}	0.14808 {8}	0.00353 {2}	0.00323 {1}	0.00429 {3}
		$\hat{\alpha}$	0.70699 {4}	0.82796 {8}	0.76030 {6}	0.84943 {10}	0.75401 {5}	0.88525 {11}	0.76126 {7}	0.84149 {9}	0.09678 {1}	0.11828 {3}	0.10437 {2}
	MRE	$\hat{\eta}$	0.25913 {4}	0.30708 {9}	0.28040 {6}	0.29801 {8}	0.29039 {7}	0.35464 {11}	0.26477 {5}	0.33646 {10}	0.07785 {1}	0.08096 {3}	0.07974 {2}
		$\hat{\lambda}$	0.37125 {4}	0.56031 {10}	0.45322 {7}	0.54984 {9}	0.44986 {6}	0.65560 {11}	0.42618 {5}	0.52986 {8}	0.08377 {1}	0.09387 {2}	0.10671 {3}
		$\sum Ranks$	36 {4}	82 {9.5}	57 {7}	82 {9.5}	53 {6}	98 {11}	52 {5}	80 {8}	19 {3}	18 {2}	17 {1}
		$\hat{\alpha}$	0.28018 {4}	0.33061 {8}	0.29712 {6}	0.33871 {9}	0.30472 {7}	0.38088 {11}	0.29438 {5}	0.34091 {10}	0.05735 {3}	0.05501 {2}	0.05042 {1}
	$\|BIAS\|$	$\hat{\eta}$	0.12587 {4}	0.15218 {9}	0.13229 {6}	0.15005 {8}	0.14936 {7}	0.21006 {11}	0.12979 {5}	0.17352 {10}	0.04126 {2}	0.04132 {3}	0.04079 {1}
		$\hat{\lambda}$	0.14033 {4}	0.19338 {8}	0.15858 {6}	0.19834 {9}	0.16747 {7}	0.22969 {11}	0.15642 {5}	0.20068 {10}	0.04457 {2}	0.04348 {1}	0.04810 {3}
		$\hat{\alpha}$	0.11759 {4}	0.16270 {8}	0.13180 {6}	0.17401 {10}	0.13348 {7}	0.18756 {11}	0.12784 {5}	0.16720 {9}	0.00496 {3}	0.00442 {2}	0.00349 {1}
100	MSE	$\hat{\eta}$	0.02522 {4}	0.03373 {9}	0.02696 {6}	0.03260 {7}	0.03367 {8}	0.06384 {11}	0.02552 {5}	0.04530 {10}	0.00263 {3}	0.00260 {2}	0.00252 {1}
		$\hat{\lambda}$	0.03141 {4}	0.07361 {8}	0.04403 {7}	0.07915 {10}	0.04399 {6}	0.09575 {11}	0.04095 {5}	0.07414 {9}	0.00297 {2}	0.00277 {1}	0.00343 {3}
		$\hat{\alpha}$	0.56036 {4}	0.66123 {8}	0.59424 {6}	0.67743 {9}	0.60943 {7}	0.76176 {11}	0.58876 {5}	0.68182 {10}	0.09638 {1}	0.11001 {3}	0.10084 {2}
	MRE	$\hat{\eta}$	0.20978 {4}	0.25363 {9}	0.22048 {6}	0.25009 {8}	0.24894 {7}	0.35010 {11}	0.21631 {5}	0.28920 {10}	0.06585 {1}	0.06886 {3}	0.06799 {2}
		$\hat{\lambda}$	0.28066 {4}	0.38676 {8}	0.31717 {6}	0.39667 {9}	0.33494 {7}	0.45938 {11}	0.31284 {5}	0.40136 {10}	0.07841 {1}	0.08697 {2}	0.09621 {3}
		$\sum Ranks$	36 {4}	75 {8}	55 {6}	79 {9}	63 {7}	99 {11}	45 {5}	88 {10}	18 {2}	19 {3}	17 {1}
		$\hat{\alpha}$	0.20962 {6}	0.24308 {8}	0.20622 {4}	0.24391 {9}	0.21744 {7}	0.31287 {11}	0.20771 {5}	0.24884 {10}	0.04889 {1}	0.04905 {2}	0.04951 {3}
	$\|BIAS\|$	$\hat{\eta}$	0.09553 {6}	0.11401 {9}	0.09436 {4}	0.11209 {8}	0.10675 {7}	0.19072 {11}	0.09468 {5}	0.12968 {10}	0.03094 {1}	0.03116 {3}	0.03107 {2}
		$\hat{\lambda}$	0.10395 {4}	0.13050 {8}	0.10615 {6}	0.13058 {9}	0.11221 {7}	0.17614 {11}	0.10496 {5}	0.13694 {10}	0.03344 {1}	0.03364 {2}	0.03368 {3}
		$\hat{\alpha}$	0.06541 {6}	0.08470 {8}	0.06443 {4}	0.08591 {9}	0.07214 {7}	0.13275 {11}	0.06467 {5}	0.08967 {10}	0.00333 {2}	0.00330 {1}	0.00336 {3}
250	MSE	$\hat{\eta}$	0.01598 {6}	0.02079 {9}	0.01543 {5}	0.01999 {7}	0.02018 {8}	0.05646 {11}	0.01519 {4}	0.02817 {10}	0.00150 {2.5}	0.00150 {2.5}	0.00149 {1}
		$\hat{\lambda}$	0.01650 {4}	0.02682 {8}	0.01807 {6}	0.02728 {9}	0.01945 {7}	0.04781 {11}	0.01704 {5}	0.02983 {10}	0.00169 {2}	0.00169 {2}	0.00169 {2}
		$\hat{\alpha}$	0.41924 {6}	0.48616 {8}	0.41244 {4}	0.48781 {9}	0.43489 {7}	0.62574 {11}	0.41543 {5}	0.49768 {10}	0.09618 {1}	0.09810 {2}	0.09902 {3}
	MRE	$\hat{\eta}$	0.15922 {6}	0.19002 {9}	0.15726 {4}	0.18681 {8}	0.17792 {7}	0.31787 {11}	0.15781 {5}	0.21613 {10}	0.05114 {1}	0.05193 {3}	0.05178 {2}
		$\hat{\lambda}$	0.20791 {4}	0.26101 {8}	0.21230 {6}	0.26117 {9}	0.22442 {7}	0.35228 {11}	0.20992 {5}	0.27387 {10}	0.06599 {1}	0.06728 {2}	0.06736 {3}
		$\sum Ranks$	48 {6}	75 {8}	43 {4}	77 {9}	64 {7}	99 {11}	44 {5}	90 {10}	12.5 {1}	19.5 {2}	22 {3}
		$\hat{\alpha}$	0.15694 {6}	0.18972 {9}	0.15104 {4}	0.18619 {8}	0.15564 {5}	0.25859 {11}	0.16275 {7}	0.19070 {10}	0.04859 {1}	0.04892 {2}	0.04913 {3}
	$\|BIAS\|$	$\hat{\eta}$	0.07239 {5}	0.09156 {9}	0.06768 {4}	0.08778 {8}	0.07541 {7}	0.16592 {11}	0.07484 {6}	0.10045 {10}	0.02513 {1}	0.02532 {3}	0.02528 {2}
		$\hat{\lambda}$	0.07705 {5}	0.09962 {9}	0.07594 {4}	0.09763 {8}	0.07933 {6}	0.14140 {11}	0.08161 {7}	0.10107 {10}	0.02891 {1}	0.02912 {2}	0.02914 {3}
		$\hat{\alpha}$	0.03865 {5}	0.05384 {9}	0.03609 {4}	0.05176 {8}	0.04196 {7}	0.09857 {11}	0.04190 {6}	0.05602 {10}	0.00325 {1.5}	0.00325 {1.5}	0.00328 {3}
500	MSE	$\hat{\eta}$	0.01035 {5}	0.01461 {9}	0.00866 {4}	0.01328 {8}	0.01172 {7}	0.04834 {11}	0.01047 {6}	0.01904 {10}	0.00099 {2}	0.00099 {2}	0.00099 {2}
		$\hat{\lambda}$	0.00945 {5}	0.01515 {9}	0.00924 {4}	0.01473 {8}	0.01055 {6}	0.02983 {11}	0.01061 {7}	0.01585 {10}	0.00126 {1}	0.00127 {2.5}	0.00127 {2.5}
		$\hat{\alpha}$	0.31387 {6}	0.37944 {9}	0.30209 {4}	0.37238 {8}	0.31129 {5}	0.51719 {11}	0.32549 {7}	0.38140 {10}	0.09611 {1}	0.09785 {2}	0.09826 {3}
	MRE	$\hat{\eta}$	0.12065 {5}	0.15260 {9}	0.11280 {4}	0.14630 {8}	0.12568 {7}	0.27654 {11}	0.12473 {6}	0.16742 {10}	0.04151 {1}	0.04221 {3}	0.04214 {2}
		$\hat{\lambda}$	0.15411 {5}	0.19924 {9}	0.15187 {4}	0.19527 {8}	0.15866 {6}	0.28280 {11}	0.16322 {7}	0.20213 {10}	0.05724 {1}	0.05824 {2}	0.05828 {3}
		$\sum Ranks$	47 {5}	81 {9}	36 {4}	72 {8}	56 {6}	99 {11}	59 {7}	90 {10}	10.5 {1}	20 {2}	23.5 {3}

Table 5: Simulation results of the IPBHEx distribution for α=1.2, η=0.8, λ=1.5.

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
		$\hat{\alpha}$	1.00927 $^{\{5\}}$	1.04347 $^{\{7\}}$	1.01459 $^{\{6\}}$	1.10377 $^{\{10\}}$	0.94249 $^{\{4\}}$	1.10207 $^{\{9\}}$	1.07148 $^{\{8\}}$	1.10450 $^{\{11\}}$	0.15801 $^{\{3\}}$	0.14910 $^{\{2\}}$	0.12498 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.27219 $^{\{4\}}$	0.32545 $^{\{8\}}$	0.31860 $^{\{7\}}$	0.33431 $^{\{10\}}$	0.30918 $^{\{6\}}$	0.33021 $^{\{9\}}$	0.29583 $^{\{5\}}$	0.35724 $^{\{11\}}$	0.07175 $^{\{3\}}$	0.07120 $^{\{2\}}$	0.07081 $^{\{1\}}$
		$\hat{\lambda}$	0.88837 $^{\{4\}}$	1.54920 $^{\{10\}}$	1.38835 $^{\{8\}}$	1.51905 $^{\{9\}}$	1.24502 $^{\{5\}}$	1.71336 $^{\{11\}}$	1.26144 $^{\{6\}}$	1.32762 $^{\{7\}}$	0.17241 $^{\{2\}}$	0.16264 $^{\{1\}}$	0.21307 $^{\{3\}}$
		$\hat{\alpha}$	1.81437 $^{\{6\}}$	1.81631 $^{\{7\}}$	1.67694 $^{\{5\}}$	2.13519 $^{\{11\}}$	1.37392 $^{\{4\}}$	1.81939 $^{\{8\}}$	1.98387 $^{\{9\}}$	2.07665 $^{\{10\}}$	0.04108 $^{\{3\}}$	0.03436 $^{\{2\}}$	0.02208 $^{\{1\}}$
30	MSE	$\hat{\eta}$	0.10839 $^{\{4\}}$	0.14284 $^{\{8\}}$	0.14063 $^{\{7\}}$	0.15011 $^{\{9\}}$	0.12842 $^{\{6\}}$	0.15060 $^{\{10\}}$	0.11884 $^{\{5\}}$	0.18232 $^{\{11\}}$	0.00751 $^{\{3\}}$	0.00726 $^{\{2\}}$	0.00719 $^{\{1\}}$
		$\hat{\lambda}$	1.91073 $^{\{4\}}$	6.36401 $^{\{10\}}$	5.04768 $^{\{8\}}$	6.10489 $^{\{9\}}$	4.32386 $^{\{7\}}$	10.37738 $^{\{11\}}$	4.13435 $^{\{5\}}$	4.23468 $^{\{6\}}$	0.04536 $^{\{2\}}$	0.03932 $^{\{1\}}$	0.06635 $^{\{3\}}$
		$\hat{\alpha}$	0.84106 $^{\{5\}}$	0.86956 $^{\{7\}}$	0.84549 $^{\{6\}}$	0.91981 $^{\{10\}}$	0.78541 $^{\{4\}}$	0.91839 $^{\{9\}}$	0.89290 $^{\{8\}}$	0.92041 $^{\{11\}}$	0.09550 $^{\{1\}}$	0.12425 $^{\{3\}}$	0.10415 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.34024 $^{\{4\}}$	0.40681 $^{\{8\}}$	0.39825 $^{\{7\}}$	0.41789 $^{\{10\}}$	0.38647 $^{\{6\}}$	0.41277 $^{\{9\}}$	0.36979 $^{\{5\}}$	0.44655 $^{\{11\}}$	0.08498 $^{\{1\}}$	0.08900 $^{\{3\}}$	0.08851 $^{\{2\}}$
		$\hat{\lambda}$	0.59225 $^{\{4\}}$	1.03280 $^{\{10\}}$	0.92556 $^{\{8\}}$	1.01270 $^{\{9\}}$	0.83001 $^{\{5\}}$	1.14224 $^{\{11\}}$	0.84096 $^{\{6\}}$	0.88508 $^{\{7\}}$	0.08537 $^{\{1\}}$	0.10842 $^{\{2\}}$	0.14204 $^{\{3\}}$
		$\sum Ranks$	40 $^{\{4\}}$	75 $^{\{8\}}$	62 $^{\{7\}}$	87 $^{\{10.5\}}$	47 $^{\{5\}}$	87 $^{\{10.5\}}$	57 $^{\{6\}}$	85 $^{\{9\}}$	19 $^{\{3\}}$	18 $^{\{2\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.81484 $^{\{4\}}$	0.93943 $^{\{8\}}$	0.88002 $^{\{6\}}$	0.98402 $^{\{11\}}$	0.83453 $^{\{5\}}$	0.96731 $^{\{10\}}$	0.88188 $^{\{7\}}$	0.94536 $^{\{9\}}$	0.14536 $^{\{3\}}$	0.13795 $^{\{2\}}$	0.12043 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.24945 $^{\{4\}}$	0.29739 $^{\{9\}}$	0.27372 $^{\{7\}}$	0.28980 $^{\{8\}}$	0.27050 $^{\{6\}}$	0.31274 $^{\{10\}}$	0.25611 $^{\{5\}}$	0.32049 $^{\{11\}}$	0.06717 $^{\{3\}}$	0.06705 $^{\{2\}}$	0.06657 $^{\{1\}}$
		$\hat{\lambda}$	0.68111 $^{\{4\}}$	1.21782 $^{\{11\}}$	0.99545 $^{\{7\}}$	1.19575 $^{\{10\}}$	0.89063 $^{\{5\}}$	1.16437 $^{\{9\}}$	0.89962 $^{\{6\}}$	1.07799 $^{\{8\}}$	0.15828 $^{\{2\}}$	0.14978 $^{\{1\}}$	0.18987 $^{\{3\}}$
		$\hat{\alpha}$	1.03291 $^{\{4\}}$	1.36635 $^{\{9\}}$	1.20250 $^{\{6\}}$	1.58059 $^{\{11\}}$	1.04219 $^{\{5\}}$	1.28686 $^{\{8\}}$	1.22694 $^{\{7\}}$	1.37438 $^{\{10\}}$	0.03341 $^{\{3\}}$	0.02881 $^{\{2\}}$	0.02024 $^{\{1\}}$
50	MSE	$\hat{\eta}$	0.09314 $^{\{5\}}$	0.11943 $^{\{9\}}$	0.10669 $^{\{7\}}$	0.11484 $^{\{8\}}$	0.10250 $^{\{6\}}$	0.13562 $^{\{10\}}$	0.09179 $^{\{4\}}$	0.14479 $^{\{11\}}$	0.00658 $^{\{3\}}$	0.00644 $^{\{2\}}$	0.00635 $^{\{1\}}$
		$\hat{\lambda}$	0.90854 $^{\{4\}}$	3.90866 $^{\{10\}}$	2.58152 $^{\{7\}}$	3.70982 $^{\{9\}}$	1.87929 $^{\{5\}}$	4.34424 $^{\{11\}}$	2.00250 $^{\{6\}}$	2.65428 $^{\{8\}}$	0.03824 $^{\{2\}}$	0.03355 $^{\{1\}}$	0.05284 $^{\{3\}}$
		$\hat{\alpha}$	0.67904 $^{\{4\}}$	0.78286 $^{\{8\}}$	0.73335 $^{\{6\}}$	0.82002 $^{\{11\}}$	0.69545 $^{\{5\}}$	0.80609 $^{\{10\}}$	0.73490 $^{\{7\}}$	0.78780 $^{\{9\}}$	0.09570 $^{\{1\}}$	0.11496 $^{\{3\}}$	0.10036 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.31181 $^{\{4\}}$	0.37173 $^{\{9\}}$	0.34216 $^{\{7\}}$	0.36226 $^{\{8\}}$	0.33813 $^{\{6\}}$	0.39093 $^{\{10\}}$	0.32013 $^{\{5\}}$	0.40061 $^{\{11\}}$	0.08126 $^{\{1\}}$	0.08382 $^{\{3\}}$	0.08321 $^{\{2\}}$
		$\hat{\lambda}$	0.45407 $^{\{4\}}$	0.81188 $^{\{11\}}$	0.66363 $^{\{7\}}$	0.79717 $^{\{10\}}$	0.59375 $^{\{5\}}$	0.77624 $^{\{9\}}$	0.59975 $^{\{6\}}$	0.71866 $^{\{8\}}$	0.08195 $^{\{1\}}$	0.09985 $^{\{2\}}$	0.12658 $^{\{3\}}$
		$\sum Ranks$	37 $^{\{4\}}$	84 $^{\{8\}}$	60 $^{\{7\}}$	86 $^{\{10\}}$	48 $^{\{5\}}$	87 $^{\{11\}}$	53 $^{\{6\}}$	85 $^{\{9\}}$	19 $^{\{3\}}$	18 $^{\{2\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.64028 $^{\{4\}}$	0.77330 $^{\{8\}}$	0.69688 $^{\{6\}}$	0.79979 $^{\{10\}}$	0.65166 $^{\{5\}}$	0.85571 $^{\{11\}}$	0.70028 $^{\{7\}}$	0.78258 $^{\{9\}}$	0.13659 $^{\{3\}}$	0.13070 $^{\{2\}}$	0.11938 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.19471 $^{\{4\}}$	0.24564 $^{\{9\}}$	0.21016 $^{\{6\}}$	0.24457 $^{\{8\}}$	0.21735 $^{\{7\}}$	0.31444 $^{\{11\}}$	0.20832 $^{\{5\}}$	0.27277 $^{\{10\}}$	0.05917 $^{\{2\}}$	0.05927 $^{\{3\}}$	0.05891 $^{\{1\}}$
		$\hat{\lambda}$	0.50102 $^{\{4\}}$	0.82559 $^{\{10\}}$	0.63789 $^{\{7\}}$	0.84832 $^{\{11\}}$	0.59766 $^{\{5\}}$	0.80576 $^{\{9\}}$	0.63240 $^{\{6\}}$	0.79772 $^{\{8\}}$	0.14346 $^{\{2\}}$	0.13789 $^{\{1\}}$	0.16352 $^{\{3\}}$
		$\hat{\alpha}$	0.61264 $^{\{4\}}$	0.87738 $^{\{9\}}$	0.74778 $^{\{7\}}$	0.96331 $^{\{11\}}$	0.66695 $^{\{5\}}$	0.95070 $^{\{10\}}$	0.73192 $^{\{6\}}$	0.86860 $^{\{8\}}$	0.02852 $^{\{3\}}$	0.02543 $^{\{2\}}$	0.01981 $^{\{1\}}$
100	MSE	$\hat{\eta}$	0.06154 $^{\{4\}}$	0.08402 $^{\{9\}}$	0.06649 $^{\{6\}}$	0.08321 $^{\{8\}}$	0.07360 $^{\{7\}}$	0.13664 $^{\{11\}}$	0.06359 $^{\{5\}}$	0.10850 $^{\{10\}}$	0.00525 $^{\{3\}}$	0.00519 $^{\{2\}}$	0.00512 $^{\{1\}}$
		$\hat{\lambda}$	0.42076 $^{\{4\}}$	1.64635 $^{\{10\}}$	0.86110 $^{\{7\}}$	1.76085 $^{\{11\}}$	0.70767 $^{\{5\}}$	1.26481 $^{\{8\}}$	0.78989 $^{\{6\}}$	1.31927 $^{\{9\}}$	0.03092 $^{\{2\}}$	0.02782 $^{\{1\}}$	0.03930 $^{\{3\}}$
		$\hat{\alpha}$	0.53356 $^{\{4\}}$	0.64442 $^{\{8\}}$	0.58073 $^{\{6\}}$	0.66649 $^{\{10\}}$	0.54305 $^{\{5\}}$	0.71309 $^{\{11\}}$	0.58356 $^{\{7\}}$	0.65215 $^{\{9\}}$	0.09463 $^{\{1\}}$	0.10892 $^{\{3\}}$	0.09948 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.24338 $^{\{4\}}$	0.30704 $^{\{9\}}$	0.26270 $^{\{6\}}$	0.30571 $^{\{8\}}$	0.27169 $^{\{7\}}$	0.39306 $^{\{11\}}$	0.26040 $^{\{5\}}$	0.34097 $^{\{10\}}$	0.07236 $^{\{1\}}$	0.07409 $^{\{3\}}$	0.07364 $^{\{2\}}$
		$\hat{\lambda}$	0.33402 $^{\{4\}}$	0.55039 $^{\{10\}}$	0.42526 $^{\{7\}}$	0.56555 $^{\{11\}}$	0.39844 $^{\{5\}}$	0.53717 $^{\{9\}}$	0.42160 $^{\{6\}}$	0.53182 $^{\{8\}}$	0.07704 $^{\{1\}}$	0.09192 $^{\{2\}}$	0.10901 $^{\{3\}}$
		$\sum Ranks$	36 $^{\{4\}}$	82 $^{\{9\}}$	58 $^{\{7\}}$	88 $^{\{10\}}$	51 $^{\{5\}}$	91 $^{\{11\}}$	53 $^{\{6\}}$	81 $^{\{8\}}$	18 $^{\{2\}}$	19 $^{\{3\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.46710 $^{\{5\}}$	0.57968 $^{\{8\}}$	0.47555 $^{\{6\}}$	0.58406 $^{\{9\}}$	0.43277 $^{\{4\}}$	0.73176 $^{\{11\}}$	0.48541 $^{\{7\}}$	0.58465 $^{\{10\}}$	0.11578 $^{\{1\}}$	0.11635 $^{\{2\}}$	0.11714 $^{\{3\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.14086 $^{\{4\}}$	0.18123 $^{\{9\}}$	0.14489 $^{\{6\}}$	0.17898 $^{\{8\}}$	0.14422 $^{\{5\}}$	0.30342 $^{\{11\}}$	0.14639 $^{\{7\}}$	0.20507 $^{\{10\}}$	0.04598 $^{\{1\}}$	0.04629 $^{\{3\}}$	0.04619 $^{\{2\}}$
		$\hat{\lambda}$	0.35381 $^{\{5\}}$	0.50657 $^{\{8\}}$	0.38593 $^{\{6\}}$	0.51036 $^{\{9\}}$	0.35375 $^{\{4\}}$	0.60576 $^{\{11\}}$	0.38708 $^{\{7\}}$	0.52849 $^{\{10\}}$	0.10346 $^{\{1\}}$	0.10402 $^{\{2\}}$	0.10419 $^{\{3\}}$
		$\hat{\alpha}$	0.33268 $^{\{4\}}$	0.48249 $^{\{8\}}$	0.35721 $^{\{7\}}$	0.49286 $^{\{10\}}$	0.34276 $^{\{5\}}$	0.70672 $^{\{11\}}$	0.35689 $^{\{6\}}$	0.49021 $^{\{9\}}$	0.01857 $^{\{2\}}$	0.01845 $^{\{1\}}$	0.01872 $^{\{3\}}$
250	MSE	$\hat{\eta}$	0.03646 $^{\{5\}}$	0.04999 $^{\{9\}}$	0.03671 $^{\{6\}}$	0.04825 $^{\{8\}}$	0.04070 $^{\{7\}}$	0.14011 $^{\{11\}}$	0.03564 $^{\{4\}}$	0.06794 $^{\{10\}}$	0.00330 $^{\{2\}}$	0.00331 $^{\{3\}}$	0.00329 $^{\{1\}}$
		$\hat{\lambda}$	0.19906 $^{\{4\}}$	0.46234 $^{\{8\}}$	0.26334 $^{\{7\}}$	0.47318 $^{\{9\}}$	0.23594 $^{\{5\}}$	0.53762 $^{\{11\}}$	0.24528 $^{\{6\}}$	0.49711 $^{\{10\}}$	0.01600 $^{\{2\}}$	0.01599 $^{\{1\}}$	0.01601 $^{\{3\}}$
		$\hat{\alpha}$	0.38925 $^{\{5\}}$	0.48307 $^{\{8\}}$	0.39629 $^{\{6\}}$	0.48671 $^{\{9\}}$	0.36064 $^{\{4\}}$	0.60980 $^{\{11\}}$	0.40451 $^{\{7\}}$	0.48720 $^{\{10\}}$	0.09554 $^{\{1\}}$	0.09695 $^{\{2\}}$	0.09762 $^{\{3\}}$
	MRE	$\hat{\eta}$	0.17607 $^{\{4\}}$	0.22654 $^{\{9\}}$	0.18112 $^{\{6\}}$	0.22373 $^{\{8\}}$	0.18027 $^{\{5\}}$	0.37928 $^{\{11\}}$	0.18299 $^{\{7\}}$	0.25634 $^{\{10\}}$	0.05696 $^{\{1\}}$	0.05786 $^{\{3\}}$	0.05774 $^{\{2\}}$
		$\hat{\lambda}$	0.23588 $^{\{5\}}$	0.33771 $^{\{8\}}$	0.25728 $^{\{6\}}$	0.34024 $^{\{9\}}$	0.23583 $^{\{4\}}$	0.40384 $^{\{11\}}$	0.25806 $^{\{7\}}$	0.35233 $^{\{10\}}$	0.06807 $^{\{1\}}$	0.06935 $^{\{2\}}$	0.06946 $^{\{3\}}$
		$\sum Ranks$	41 $^{\{4\}}$	75 $^{\{8\}}$	56 $^{\{6\}}$	79 $^{\{9\}}$	43 $^{\{5\}}$	99 $^{\{11\}}$	58 $^{\{7\}}$	89 $^{\{10\}}$	12 $^{\{1\}}$	19 $^{\{2\}}$	23 $^{\{3\}}$
		$\hat{\alpha}$	0.33020 $^{\{5\}}$	0.45328 $^{\{10\}}$	0.33379 $^{\{6\}}$	0.44365 $^{\{8\}}$	0.27562 $^{\{4\}}$	0.61658 $^{\{11\}}$	0.37233 $^{\{7\}}$	0.44533 $^{\{9\}}$	0.11522 $^{\{1\}}$	0.11608 $^{\{2\}}$	0.11656 $^{\{3\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.09777 $^{\{5\}}$	0.14323 $^{\{9\}}$	0.09788 $^{\{6\}}$	0.13756 $^{\{8\}}$	0.09457 $^{\{4\}}$	0.26646 $^{\{11\}}$	0.11133 $^{\{7\}}$	0.15735 $^{\{10\}}$	0.03869 $^{\{1\}}$	0.03899 $^{\{3\}}$	0.03894 $^{\{2\}}$
		$\hat{\lambda}$	0.24800 $^{\{5\}}$	0.37184 $^{\{10\}}$	0.25867 $^{\{6\}}$	0.36489 $^{\{8\}}$	0.21951 $^{\{4\}}$	0.49372 $^{\{11\}}$	0.29026 $^{\{7\}}$	0.36920 $^{\{9\}}$	0.09496 $^{\{1\}}$	0.09567 $^{\{2\}}$	0.09571 $^{\{3\}}$
		$\hat{\alpha}$	0.17793 $^{\{4\}}$	0.30176 $^{\{9\}}$	0.18403 $^{\{6\}}$	0.29166 $^{\{8\}}$	0.18141 $^{\{5\}}$	0.54636 $^{\{11\}}$	0.21991 $^{\{7\}}$	0.30303 $^{\{10\}}$	0.01826 $^{\{1\}}$	0.01829 $^{\{2\}}$	0.01844 $^{\{3\}}$
500	MSE	$\hat{\eta}$	0.01966 $^{\{5\}}$	0.03363 $^{\{9\}}$	0.01839 $^{\{4\}}$	0.03090 $^{\{8\}}$	0.02248 $^{\{7\}}$	0.12255 $^{\{11\}}$	0.02232 $^{\{6\}}$	0.04563 $^{\{10\}}$	0.00233 $^{\{1.5\}}$	0.00234 $^{\{3\}}$	0.00233 $^{\{1.5\}}$
		$\hat{\lambda}$	0.10113 $^{\{4\}}$	0.21914 $^{\{9\}}$	0.11431 $^{\{6\}}$	0.21511 $^{\{8\}}$	0.10946 $^{\{5\}}$	0.34901 $^{\{11\}}$	0.13742 $^{\{7\}}$	0.22025 $^{\{10\}}$	0.01325 $^{\{1\}}$	0.01330 $^{\{2.5\}}$	0.01330 $^{\{2.5\}}$
		$\hat{\alpha}$	0.27516 $^{\{5\}}$	0.37773 $^{\{10\}}$	0.27816 $^{\{6\}}$	0.36971 $^{\{8\}}$	0.22968 $^{\{4\}}$	0.51381 $^{\{11\}}$	0.31028 $^{\{7\}}$	0.37111 $^{\{9\}}$	0.09491 $^{\{1\}}$	0.09673 $^{\{2\}}$	0.09713 $^{\{3\}}$
	MRE	$\hat{\eta}$	0.12221 $^{\{5\}}$	0.17904 $^{\{9\}}$	0.12235 $^{\{6\}}$	0.17195 $^{\{8\}}$	0.11821 $^{\{4\}}$	0.33308 $^{\{11\}}$	0.13916 $^{\{7\}}$	0.19668 $^{\{10\}}$	0.04810 $^{\{1\}}$	0.04873 $^{\{3\}}$	0.04867 $^{\{2\}}$
		$\hat{\lambda}$	0.16534 $^{\{5\}}$	0.24789 $^{\{10\}}$	0.17245 $^{\{6\}}$	0.24326 $^{\{8\}}$	0.14634 $^{\{4\}}$	0.32914 $^{\{11\}}$	0.19351 $^{\{7\}}$	0.24613 $^{\{9\}}$	0.06299 $^{\{1\}}$	0.06378 $^{\{2\}}$	0.06381 $^{\{3\}}$
		$\sum Ranks$	43 $^{\{5\}}$	85 $^{\{9\}}$	52 $^{\{6\}}$	72 $^{\{8\}}$	41 $^{\{4\}}$	99 $^{\{11\}}$	62 $^{\{7\}}$	86 $^{\{10\}}$	9.5 $^{\{1\}}$	21.5 $^{\{2\}}$	23 $^{\{3\}}$

Table 6: Simulation results of the IPBHEx distribution for α=1, η=0.5, λ=0.8.

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
		$\hat{\alpha}$	0.59152 $^{\{4\}}$	0.63212 $^{\{7\}}$	0.63604 $^{\{8\}}$	0.67939 $^{\{9\}}$	0.60151 $^{\{5\}}$	0.70736 $^{\{10\}}$	0.62439 $^{\{6\}}$	0.71149 $^{\{11\}}$	0.10528 $^{\{3\}}$	0.10326 $^{\{2\}}$	0.10134 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.19071 $^{\{4\}}$	0.24415 $^{\{9\}}$	0.22549 $^{\{7\}}$	0.23913 $^{\{8\}}$	0.22171 $^{\{6\}}$	0.27937 $^{\{11\}}$	0.19423 $^{\{5\}}$	0.26142 $^{\{10\}}$	0.04512 $^{\{3\}}$	0.04511 $^{\{2\}}$	0.04484 $^{\{1\}}$
		$\hat{\lambda}$	0.37657 $^{\{4\}}$	0.78113 $^{\{10\}}$	0.72449 $^{\{7\}}$	0.73307 $^{\{9\}}$	0.72760 $^{\{8\}}$	0.79816 $^{\{11\}}$	0.49034 $^{\{5\}}$	0.57134 $^{\{6\}}$	0.08260 $^{\{2\}}$	0.08002 $^{\{1\}}$	0.08998 $^{\{3\}}$
		$\hat{\alpha}$	0.54283 $^{\{5\}}$	0.57694 $^{\{6\}}$	0.64187 $^{\{8\}}$	0.72464 $^{\{10\}}$	0.52928 $^{\{4\}}$	0.68326 $^{\{9\}}$	0.60501 $^{\{7\}}$	0.85297 $^{\{11\}}$	0.01622 $^{\{3\}}$	0.01517 $^{\{2\}}$	0.01420 $^{\{1\}}$
30	MSE	$\hat{\eta}$	0.06200 $^{\{5\}}$	0.09016 $^{\{9\}}$	0.08058 $^{\{7\}}$	0.08727 $^{\{8\}}$	0.07813 $^{\{6\}}$	0.11614 $^{\{11\}}$	0.05940 $^{\{4\}}$	0.10932 $^{\{10\}}$	0.00292 $^{\{3\}}$	0.00287 $^{\{2\}}$	0.00283 $^{\{1\}}$
		$\hat{\lambda}$	0.26335 $^{\{4\}}$	2.18611 $^{\{9\}}$	2.22527 $^{\{10\}}$	1.79306 $^{\{7\}}$	6.77111 $^{\{11\}}$	1.89920 $^{\{8\}}$	0.49495 $^{\{5\}}$	0.58673 $^{\{6\}}$	0.01005 $^{\{2\}}$	0.00925 $^{\{1\}}$	0.01201 $^{\{3\}}$
		$\hat{\alpha}$	0.59152 $^{\{4\}}$	0.63212 $^{\{7\}}$	0.63604 $^{\{8\}}$	0.67939 $^{\{9\}}$	0.60151 $^{\{5\}}$	0.70736 $^{\{10\}}$	0.62439 $^{\{6\}}$	0.71149 $^{\{11\}}$	0.09589 $^{\{1\}}$	0.10326 $^{\{3\}}$	0.10134 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.38142 $^{\{4\}}$	0.48830 $^{\{9\}}$	0.45098 $^{\{7\}}$	0.47826 $^{\{8\}}$	0.44342 $^{\{6\}}$	0.55873 $^{\{11\}}$	0.38846 $^{\{5\}}$	0.52285 $^{\{10\}}$	0.08726 $^{\{1\}}$	0.09022 $^{\{3\}}$	0.08968 $^{\{2\}}$
		$\hat{\lambda}$	0.47071 $^{\{4\}}$	0.97641 $^{\{10\}}$	0.90562 $^{\{7\}}$	0.91634 $^{\{9\}}$	0.90951 $^{\{8\}}$	0.99770 $^{\{11\}}$	0.61293 $^{\{5\}}$	0.71418 $^{\{6\}}$	0.09118 $^{\{1\}}$	0.10003 $^{\{2\}}$	0.11248 $^{\{3\}}$
		$\sum Ranks$	38 $^{\{4\}}$	76 $^{\{8\}}$	69 $^{\{7\}}$	77 $^{\{9\}}$	59 $^{\{6\}}$	92 $^{\{11\}}$	48 $^{\{5\}}$	81 $^{\{10\}}$	19 $^{\{3\}}$	18 $^{\{2\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.48502 $^{\{4\}}$	0.55545 $^{\{8\}}$	0.52664 $^{\{7\}}$	0.55790 $^{\{9\}}$	0.50223 $^{\{6\}}$	0.64387 $^{\{11\}}$	0.49951 $^{\{5\}}$	0.59026 $^{\{10\}}$	0.10025 $^{\{3\}}$	0.09925 $^{\{2\}}$	0.09837 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.15909 $^{\{5\}}$	0.20818 $^{\{9\}}$	0.18371 $^{\{7\}}$	0.19914 $^{\{8\}}$	0.17713 $^{\{6\}}$	0.27502 $^{\{11\}}$	0.15720 $^{\{4\}}$	0.23418 $^{\{10\}}$	0.04167 $^{\{2\}}$	0.04177 $^{\{3\}}$	0.04150 $^{\{1\}}$
		$\hat{\lambda}$	0.29751 $^{\{4\}}$	0.54279 $^{\{10\}}$	0.43931 $^{\{7\}}$	0.52660 $^{\{9\}}$	0.38584 $^{\{6\}}$	0.58888 $^{\{11\}}$	0.36171 $^{\{5\}}$	0.46149 $^{\{8\}}$	0.07853 $^{\{2\}}$	0.07649 $^{\{1\}}$	0.08407 $^{\{3\}}$
		$\hat{\alpha}$	0.35699 $^{\{4\}}$	0.44787 $^{\{8\}}$	0.41763 $^{\{7\}}$	0.47264 $^{\{9\}}$	0.37814 $^{\{5\}}$	0.56600 $^{\{11\}}$	0.38768 $^{\{6\}}$	0.51851 $^{\{10\}}$	0.01456 $^{\{3\}}$	0.01392 $^{\{2\}}$	0.01336 $^{\{1\}}$
50	MSE	$\hat{\eta}$	0.04739 $^{\{5\}}$	0.07059 $^{\{9\}}$	0.05827 $^{\{7\}}$	0.06455 $^{\{8\}}$	0.05497 $^{\{6\}}$	0.11830 $^{\{11\}}$	0.04149 $^{\{4\}}$	0.09210 $^{\{10\}}$	0.00254 $^{\{3\}}$	0.00252 $^{\{2\}}$	0.00249 $^{\{1\}}$
		$\hat{\lambda}$	0.14801 $^{\{4\}}$	0.90927 $^{\{10\}}$	0.49537 $^{\{7\}}$	0.82987 $^{\{9\}}$	0.52042 $^{\{8\}}$	0.91414 $^{\{11\}}$	0.24762 $^{\{5\}}$	0.37075 $^{\{6\}}$	0.00907 $^{\{2\}}$	0.00846 $^{\{1\}}$	0.01038 $^{\{3\}}$
		$\hat{\alpha}$	0.48502 $^{\{4\}}$	0.55545 $^{\{8\}}$	0.52664 $^{\{7\}}$	0.55790 $^{\{9\}}$	0.50223 $^{\{6\}}$	0.64387 $^{\{11\}}$	0.49951 $^{\{5\}}$	0.59026 $^{\{10\}}$	0.09503 $^{\{1\}}$	0.09925 $^{\{3\}}$	0.09837 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.31818 $^{\{5\}}$	0.41637 $^{\{9\}}$	0.36741 $^{\{7\}}$	0.39828 $^{\{8\}}$	0.35427 $^{\{6\}}$	0.55004 $^{\{11\}}$	0.31440 $^{\{4\}}$	0.46836 $^{\{10\}}$	0.08221 $^{\{1\}}$	0.08353 $^{\{3\}}$	0.08300 $^{\{2\}}$
		$\hat{\lambda}$	0.37188 $^{\{4\}}$	0.67848 $^{\{10\}}$	0.54913 $^{\{7\}}$	0.65825 $^{\{9\}}$	0.48230 $^{\{6\}}$	0.73610 $^{\{11\}}$	0.45213 $^{\{5\}}$	0.57687 $^{\{8\}}$	0.08799 $^{\{1\}}$	0.09561 $^{\{2\}}$	0.10509 $^{\{3\}}$
		$\sum Ranks$	39 $^{\{4\}}$	81 $^{\{9\}}$	63 $^{\{7\}}$	78 $^{\{8\}}$	55 $^{\{6\}}$	99 $^{\{11\}}$	43 $^{\{5\}}$	82 $^{\{10\}}$	18 $^{\{2\}}$	19 $^{\{3\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.36200 $^{\{4\}}$	0.42750 $^{\{9\}}$	0.36927 $^{\{5\}}$	0.42381 $^{\{8\}}$	0.36931 $^{\{6\}}$	0.54998 $^{\{11\}}$	0.38190 $^{\{7\}}$	0.44378 $^{\{10\}}$	0.09591 $^{\{3\}}$	0.09550 $^{\{2\}}$	0.09471 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.11475 $^{\{4\}}$	0.15591 $^{\{9\}}$	0.12292 $^{\{7\}}$	0.14827 $^{\{8\}}$	0.12230 $^{\{6\}}$	0.25961 $^{\{11\}}$	0.12144 $^{\{5\}}$	0.18301 $^{\{10\}}$	0.03713 $^{\{1\}}$	0.03733 $^{\{3\}}$	0.03715 $^{\{2\}}$
		$\hat{\lambda}$	0.21780 $^{\{4\}}$	0.33428 $^{\{10\}}$	0.25772 $^{\{7\}}$	0.31829 $^{\{8\}}$	0.25168 $^{\{5\}}$	0.42387 $^{\{11\}}$	0.25495 $^{\{6\}}$	0.33128 $^{\{9\}}$	0.07284 $^{\{2\}}$	0.07164 $^{\{1\}}$	0.07617 $^{\{3\}}$
		$\hat{\alpha}$	0.20621 $^{\{4\}}$	0.27517 $^{\{9\}}$	0.21678 $^{\{5\}}$	0.27160 $^{\{8\}}$	0.22385 $^{\{6\}}$	0.42863 $^{\{11\}}$	0.22842 $^{\{7\}}$	0.29951 $^{\{10\}}$	0.01307 $^{\{3\}}$	0.01273 $^{\{2\}}$	0.01236 $^{\{1\}}$
100	MSE	$\hat{\eta}$	0.02706 $^{\{4\}}$	0.04304 $^{\{9\}}$	0.02911 $^{\{6\}}$	0.03899 $^{\{8\}}$	0.03068 $^{\{7\}}$	0.11704 $^{\{11\}}$	0.02796 $^{\{5\}}$	0.06361 $^{\{10\}}$	0.00206 $^{\{2.5\}}$	0.00206 $^{\{2.5\}}$	0.00204 $^{\{1\}}$
		$\hat{\lambda}$	0.07838 $^{\{4\}}$	0.22147 $^{\{10\}}$	0.13058 $^{\{7\}}$	0.21934 $^{\{9\}}$	0.11389 $^{\{5\}}$	0.32758 $^{\{11\}}$	0.11577 $^{\{6\}}$	0.18422 $^{\{8\}}$	0.00776 $^{\{2\}}$	0.00738 $^{\{1\}}$	0.00844 $^{\{3\}}$
		$\hat{\alpha}$	0.36200 $^{\{4\}}$	0.42750 $^{\{9\}}$	0.36927 $^{\{5\}}$	0.42381 $^{\{8\}}$	0.36931 $^{\{6\}}$	0.54998 $^{\{11\}}$	0.38190 $^{\{7\}}$	0.44378 $^{\{10\}}$	0.09240 $^{\{1\}}$	0.09550 $^{\{3\}}$	0.09471 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.22950 $^{\{4\}}$	0.31182 $^{\{9\}}$	0.24585 $^{\{7\}}$	0.29654 $^{\{8\}}$	0.24460 $^{\{6\}}$	0.51922 $^{\{11\}}$	0.24289 $^{\{5\}}$	0.36601 $^{\{10\}}$	0.07303 $^{\{1\}}$	0.07467 $^{\{3\}}$	0.07430 $^{\{2\}}$
		$\hat{\lambda}$	0.27225 $^{\{4\}}$	0.41785 $^{\{10\}}$	0.32215 $^{\{7\}}$	0.39787 $^{\{8\}}$	0.31460 $^{\{5\}}$	0.52983 $^{\{11\}}$	0.31869 $^{\{6\}}$	0.41410 $^{\{9\}}$	0.08351 $^{\{1\}}$	0.08955 $^{\{2\}}$	0.09522 $^{\{3\}}$
		$\sum Ranks$	36 $^{\{4\}}$	84 $^{\{9\}}$	56 $^{\{7\}}$	73 $^{\{8\}}$	52 $^{\{5\}}$	99 $^{\{11\}}$	54 $^{\{6\}}$	86 $^{\{10\}}$	16.5 $^{\{1\}}$	19.5 $^{\{3\}}$	18 $^{\{2\}}$
		$\hat{\alpha}$	0.22690 $^{\{5\}}$	0.27714 $^{\{8\}}$	0.23059 $^{\{6\}}$	0.28122 $^{\{9\}}$	0.22241 $^{\{4\}}$	0.42427 $^{\{11\}}$	0.24214 $^{\{7\}}$	0.28915 $^{\{10\}}$	0.09110 $^{\{1\}}$	0.09183 $^{\{2\}}$	0.09204 $^{\{3\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.06599 $^{\{4\}}$	0.09680 $^{\{8\}}$	0.06973 $^{\{6\}}$	0.09710 $^{\{9\}}$	0.06660 $^{\{5\}}$	0.21289 $^{\{11\}}$	0.07273 $^{\{7\}}$	0.11379 $^{\{10\}}$	0.03111 $^{\{1\}}$	0.03138 $^{\{3\}}$	0.03134 $^{\{2\}}$
		$\hat{\lambda}$	0.13514 $^{\{4\}}$	0.19876 $^{\{9\}}$	0.14878 $^{\{6\}}$	0.19618 $^{\{8\}}$	0.14452 $^{\{5\}}$	0.29568 $^{\{11\}}$	0.15802 $^{\{7\}}$	0.21638 $^{\{10\}}$	0.06225 $^{\{1\}}$	0.06277 $^{\{2\}}$	0.06281 $^{\{3\}}$
		$\hat{\alpha}$	0.08422 $^{\{4\}}$	0.12544 $^{\{8\}}$	0.08723 $^{\{6\}}$	0.12924 $^{\{9\}}$	0.08458 $^{\{5\}}$	0.29488 $^{\{11\}}$	0.09509 $^{\{7\}}$	0.13857 $^{\{10\}}$	0.01163 $^{\{1\}}$	0.01168 $^{\{2\}}$	0.01174 $^{\{3\}}$
250	MSE	$\hat{\eta}$	0.00915 $^{\{5\}}$	0.01853 $^{\{8\}}$	0.01005 $^{\{7\}}$	0.01880 $^{\{9\}}$	0.00907 $^{\{4\}}$	0.09651 $^{\{11\}}$	0.00983 $^{\{6\}}$	0.02729 $^{\{10\}}$	0.00148 $^{\{1\}}$	0.00149 $^{\{2.5\}}$	0.00149 $^{\{2.5\}}$
		$\hat{\lambda}$	0.02972 $^{\{4\}}$	0.06769 $^{\{9\}}$	0.03714 $^{\{6\}}$	0.06610 $^{\{8\}}$	0.03575 $^{\{5\}}$	0.14637 $^{\{11\}}$	0.04284 $^{\{7\}}$	0.07969 $^{\{10\}}$	0.00565 $^{\{1\}}$	0.00568 $^{\{2.5\}}$	0.00568 $^{\{2.5\}}$
		$\hat{\alpha}$	0.22690 $^{\{5\}}$	0.27714 $^{\{8\}}$	0.23059 $^{\{6\}}$	0.28122 $^{\{9\}}$	0.22241 $^{\{4\}}$	0.42427 $^{\{11\}}$	0.24214 $^{\{7\}}$	0.28915 $^{\{10\}}$	0.09070 $^{\{1\}}$	0.09183 $^{\{2\}}$	0.09204 $^{\{3\}}$
	MRE	$\hat{\eta}$	0.13199 $^{\{4\}}$	0.19361 $^{\{8\}}$	0.13946 $^{\{6\}}$	0.19420 $^{\{9\}}$	0.13320 $^{\{5\}}$	0.42579 $^{\{11\}}$	0.14545 $^{\{7\}}$	0.22757 $^{\{10\}}$	0.06181 $^{\{1\}}$	0.06275 $^{\{3\}}$	0.06268 $^{\{2\}}$
		$\hat{\lambda}$	0.16893 $^{\{4\}}$	0.24845 $^{\{9\}}$	0.18597 $^{\{6\}}$	0.24522 $^{\{8\}}$	0.18064 $^{\{5\}}$	0.36960 $^{\{11\}}$	0.19752 $^{\{7\}}$	0.27048 $^{\{10\}}$	0.07752 $^{\{1\}}$	0.07847 $^{\{2\}}$	0.07851 $^{\{3\}}$
		$\sum Ranks$	39 $^{\{4\}}$	75 $^{\{8\}}$	55 $^{\{6\}}$	78 $^{\{9\}}$	42 $^{\{5\}}$	99 $^{\{11\}}$	62 $^{\{7\}}$	90 $^{\{10\}}$	9 $^{\{1\}}$	21 $^{\{2\}}$	24 $^{\{3\}}$
		$\hat{\alpha}$	0.15461 $^{\{5\}}$	0.19726 $^{\{8\}}$	0.16707 $^{\{7\}}$	0.19985 $^{\{9\}}$	0.15311 $^{\{4\}}$	0.31384 $^{\{11\}}$	0.16412 $^{\{6\}}$	0.20375 $^{\{10\}}$	0.08789 $^{\{1\}}$	0.08867 $^{\{2\}}$	0.08875 $^{\{3\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.04278 $^{\{4\}}$	0.06492 $^{\{8\}}$	0.04916 $^{\{7\}}$	0.06530 $^{\{9\}}$	0.04287 $^{\{5\}}$	0.15791 $^{\{11\}}$	0.04681 $^{\{6\}}$	0.07516 $^{\{10\}}$	0.02758 $^{\{1\}}$	0.02784 $^{\{3\}}$	0.02782 $^{\{2\}}$
		$\hat{\lambda}$	0.09317 $^{\{4\}}$	0.13711 $^{\{8\}}$	0.10735 $^{\{7\}}$	0.13841 $^{\{9\}}$	0.09762 $^{\{5\}}$	0.22183 $^{\{11\}}$	0.10523 $^{\{6\}}$	0.14837 $^{\{10\}}$	0.05785 $^{\{1\}}$	0.05837 $^{\{2\}}$	0.05838 $^{\{3\}}$
		$\hat{\alpha}$	0.03811 $^{\{4\}}$	0.06295 $^{\{8\}}$	0.04434 $^{\{7\}}$	0.06425 $^{\{9\}}$	0.04023 $^{\{5\}}$	0.18450 $^{\{11\}}$	0.04268 $^{\{6\}}$	0.06923 $^{\{10\}}$	0.01097 $^{\{1\}}$	0.01105 $^{\{2\}}$	0.01107 $^{\{3\}}$
500	MSE	$\hat{\eta}$	0.00312 $^{\{4\}}$	0.00784 $^{\{8\}}$	0.00409 $^{\{7\}}$	0.00796 $^{\{9\}}$	0.00318 $^{\{5\}}$	0.06420 $^{\{11\}}$	0.00361 $^{\{6\}}$	0.01149 $^{\{10\}}$	0.00116 $^{\{2\}}$	0.00116 $^{\{2\}}$	0.00116 $^{\{2\}}$
		$\hat{\lambda}$	0.01374 $^{\{4\}}$	0.03065 $^{\{8\}}$	0.01858 $^{\{7\}}$	0.03121 $^{\{9\}}$	0.01604 $^{\{5\}}$	0.08387 $^{\{11\}}$	0.01820 $^{\{6\}}$	0.03748 $^{\{10\}}$	0.00493 $^{\{1\}}$	0.00497 $^{\{2.5\}}$	0.00497 $^{\{2.5\}}$
		$\hat{\alpha}$	0.15461 $^{\{5\}}$	0.19726 $^{\{8\}}$	0.16707 $^{\{7\}}$	0.19985 $^{\{9\}}$	0.15311 $^{\{4\}}$	0.31384 $^{\{11\}}$	0.16412 $^{\{6\}}$	0.20375 $^{\{10\}}$	0.08764 $^{\{1\}}$	0.08867 $^{\{2\}}$	0.08875 $^{\{3\}}$
	MRE	$\hat{\eta}$	0.08556 $^{\{4\}}$	0.12984 $^{\{8\}}$	0.09832 $^{\{7\}}$	0.13060 $^{\{9\}}$	0.08573 $^{\{5\}}$	0.31582 $^{\{11\}}$	0.09362 $^{\{6\}}$	0.15033 $^{\{10\}}$	0.05494 $^{\{1\}}$	0.05567 $^{\{3\}}$	0.05564 $^{\{2\}}$
		$\hat{\lambda}$	0.11646 $^{\{4\}}$	0.17138 $^{\{8\}}$	0.13419 $^{\{7\}}$	0.17302 $^{\{9\}}$	0.12202 $^{\{5\}}$	0.27729 $^{\{11\}}$	0.13154 $^{\{6\}}$	0.18546 $^{\{10\}}$	0.07203 $^{\{1\}}$	0.07296 $^{\{2\}}$	0.07298 $^{\{3\}}$
		$\sum Ranks$	38 $^{\{4\}}$	72 $^{\{8\}}$	63 $^{\{7\}}$	81 $^{\{9\}}$	43 $^{\{5\}}$	99 $^{\{11\}}$	54 $^{\{6\}}$	90 $^{\{10\}}$	10 $^{\{1\}}$	20.5 $^{\{2\}}$	23.5 $^{\{3\}}$

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Table 7: Simulation results of the IPBHEx distribution for α=0.7, η=0.5, λ=2.

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
		$\hat{\alpha}$	0.47107 {4}	0.50549 {8}	0.50278 {7}	0.53783 {9}	0.47967 {5}	0.56850 {11}	0.50043 {6}	0.56180 {10}	0.07550 {3}	0.07313 {2}	0.07136 {1}
	$\|BIAS\|$	$\hat{\eta}$	0.17089 {4}	0.21345 {9}	0.19321 {6}	0.20844 {8}	0.19454 {7}	0.25686 {11}	0.17422 {5}	0.23066 {10}	0.04471 {3}	0.04468 {2}	0.04425 {1}
		$\hat{\lambda}$	0.90969 {4}	1.67241 {10}	1.49964 {8}	1.61723 {9}	1.44654 {7}	2.02287 {11}	1.14101 {5}	1.37020 {6}	0.20637 {2}	0.20121 {1}	0.22359 {3}
		$\hat{\alpha}$	0.35055 {5}	0.37715 {6}	0.41879 {8}	0.46945 {10}	0.34408 {4}	0.44075 {9}	0.39832 {7}	0.52038 {11}	0.00847 {3}	0.00769 {2}	0.00710 {1}
30	MSE	$\hat{\eta}$	0.04776 {5}	0.06777 {9}	0.05851 {7}	0.06533 {8}	0.05757 {6}	0.09503 {11}	0.04673 {4}	0.08133 {10}	0.00289 {3}	0.00284 {2}	0.00278 {1}
		$\hat{\lambda}$	1.56437 {4}	10.53459 {9}	9.68263 {8}	9.36686 {7}	28.39775 {11}	13.34791 {10}	2.82590 {5}	3.63127 {6}	0.06268 {2}	0.05820 {1}	0.07404 {3}
		$\hat{\alpha}$	0.67296 {4}	0.72213 {8}	0.71826 {7}	0.76833 {9}	0.68524 {5}	0.81215 {11}	0.71490 {6}	0.80258 {10}	0.09650 {1}	0.10447 {3}	0.10194 {2}
	MRE	$\hat{\eta}$	0.34178 {4}	0.42691 {9}	0.38642 {6}	0.41688 {8}	0.38907 {7}	0.51371 {11}	0.34844 {5}	0.46132 {10}	0.08594 {1}	0.08936 {3}	0.08849 {2}
		$\hat{\lambda}$	0.45484 {4}	0.83620 {10}	0.74982 {8}	0.80861 {9}	0.72327 {7}	1.01143 {11}	0.57051 {5}	0.68510 {6}	0.09212 {1}	0.10061 {2}	0.11180 {3}
		$\sum Ranks$	38 {4}	78 {9}	65 {7}	77 {8}	59 {6}	96 {11}	48 {5}	79 {10}	19 {3}	18 {2}	17 {1}
		$\hat{\alpha}$	0.38545 {4}	0.44559 {9}	0.41493 {7}	0.44407 {8}	0.40452 {5}	0.52679 {11}	0.40646 {6}	0.46708 {10}	0.07218 {3}	0.07071 {2}	0.06942 {1}
	$\|BIAS\|$	$\hat{\eta}$	0.14562 {4}	0.18517 {9}	0.16076 {6}	0.17487 {8}	0.16333 {7}	0.25872 {11}	0.14713 {5}	0.20534 {10}	0.04111 {2}	0.04119 {3}	0.04088 {1}
		$\hat{\lambda}$	0.71833 {4}	1.16359 {10}	0.95284 {7}	1.15062 {9}	0.88078 {6}	1.52777 {11}	0.85658 {5}	1.08768 {8}	0.19084 {2}	0.18636 {1}	0.20228 {3}
		$\hat{\alpha}$	0.22497 {4}	0.28789 {8}	0.26417 {7}	0.30360 {9}	0.24115 {5}	0.36519 {11}	0.25585 {6}	0.32124 {10}	0.00766 {3}	0.00717 {2}	0.00667 {1}
50	MSE	$\hat{\eta}$	0.03682 {5}	0.05387 {9}	0.04334 {7}	0.04806 {8}	0.04308 {6}	0.09638 {11}	0.03533 {4}	0.06728 {10}	0.00247 {3}	0.00244 {2}	0.00240 {1}
		$\hat{\lambda}$	0.86395 {4}	3.68282 {9}	2.06705 {7}	3.75686 {10}	1.72827 {6}	6.26089 {11}	1.38536 {5}	2.17452 {8}	0.05364 {2}	0.05035 {1}	0.06029 {3}
		$\hat{\alpha}$	0.55065 {4}	0.63656 {9}	0.59275 {7}	0.63439 {8}	0.57789 {5}	0.75256 {11}	0.58066 {6}	0.66726 {10}	0.09544 {1}	0.10102 {3}	0.09917 {2}
	MRE	$\hat{\eta}$	0.29123 {4}	0.37034 {9}	0.32151 {6}	0.34973 {8}	0.32666 {7}	0.51743 {11}	0.29426 {5}	0.41067 {10}	0.08008 {1}	0.08237 {3}	0.08177 {2}
		$\hat{\lambda}$	0.35917 {4}	0.58179 {10}	0.47642 {7}	0.57531 {9}	0.44039 {6}	0.76388 {11}	0.42829 {5}	0.54384 {8}	0.08748 {1}	0.09318 {2}	0.10114 {3}
		$\sum Ranks$	37 {4}	82 {9}	61 {7}	77 {8}	53 {6}	99 {11}	47 {5}	84 {10}	18 {2}	19 {3}	17 {1}
		$\hat{\alpha}$	0.29553 {5}	0.34646 {9}	0.29424 {4}	0.34201 {8}	0.30777 {6}	0.45937 {11}	0.31199 {7}	0.35778 {10}	0.06928 {3}	0.06862 {2}	0.06755 {1}
	$\|BIAS\|$	$\hat{\eta}$	0.11040 {4}	0.14213 {9}	0.11160 {5}	0.13604 {8}	0.11986 {7}	0.25384 {11}	0.11529 {6}	0.16540 {10}	0.03540 {2}	0.03556 {3}	0.03535 {1}
		$\hat{\lambda}$	0.53854 {4}	0.76297 {9}	0.59540 {5}	0.72826 {8}	0.60393 {6}	1.11264 {11}	0.60643 {7}	0.78169 {10}	0.17638 {2}	0.17372 {1}	0.18371 {3}
		$\hat{\alpha}$	0.13354 {4}	0.17556 {9}	0.13775 {5}	0.17339 {8}	0.14880 {6}	0.28049 {11}	0.14925 {7}	0.18884 {10}	0.00690 {3}	0.00663 {2}	0.00630 {1}
100	MSE	$\hat{\eta}$	0.02316 {5}	0.03403 {9}	0.02310 {4}	0.03162 {8}	0.02722 {7}	0.09795 {11}	0.02421 {6}	0.04835 {10}	0.00189 {3}	0.00188 {2}	0.00185 {1}
		$\hat{\lambda}$	0.46645 {4}	1.04095 {10}	0.65957 {7}	1.01304 {8}	0.62814 {5}	2.19818 {11}	0.63239 {6}	1.03980 {9}	0.04587 {2}	0.04373 {1}	0.04961 {3}
		$\hat{\alpha}$	0.42219 {5}	0.49494 {9}	0.42035 {4}	0.48859 {8}	0.43967 {6}	0.65624 {11}	0.44569 {7}	0.51112 {10}	0.09397 {1}	0.09802 {3}	0.09650 {2}
	MRE	$\hat{\eta}$	0.22080 {4}	0.28427 {9}	0.22319 {5}	0.27209 {8}	0.23973 {7}	0.50768 {11}	0.23058 {6}	0.33080 {10}	0.06932 {1}	0.07113 {3}	0.07070 {2}
		$\hat{\lambda}$	0.26927 {4}	0.38148 {9}	0.29770 {5}	0.36413 {8}	0.30197 {6}	0.55632 {11}	0.30322 {7}	0.39084 {10}	0.08123 {1}	0.08686 {2}	0.09185 {3}
		$\sum Ranks$	39 {4}	82 {9}	44 {5}	72 {8}	56 {6}	99 {11}	59 {7}	89 {10}	18 {2}	19 {3}	17 {1}
		$\hat{\alpha}$	0.19288 {6}	0.23286 {8}	0.18589 {4}	0.23468 {9}	0.19146 {5}	0.37806 {11}	0.20274 {7}	0.24122 {10}	0.06571 {1}	0.06618 {2}	0.06635 {3}
	$\|BIAS\|$	$\hat{\eta}$	0.06860 {5}	0.09516 {9}	0.06561 {4}	0.09442 {8}	0.07025 {6}	0.22684 {11}	0.07283 {7}	0.10904 {10}	0.02892 {1}	0.02917 {3}	0.02913 {2}
		$\hat{\lambda}$	0.34478 {4}	0.48020 {9}	0.35169 {5}	0.47400 {8}	0.35244 {6}	0.82250 {11}	0.38526 {7}	0.51378 {10}	0.14638 {1}	0.14757 {2}	0.14763 {3}
		$\hat{\alpha}$	0.06021 {5}	0.08723 {8}	0.05713 {4}	0.08824 {9}	0.06302 {6}	0.20856 {11}	0.06745 {7}	0.09428 {10}	0.00600 {1}	0.00601 {2}	0.00605 {3}
250	MSE	$\hat{\eta}$	0.01020 {5}	0.01802 {9}	0.00902 {4}	0.01775 {8}	0.01098 {7}	0.08887 {11}	0.01071 {6}	0.02447 {10}	0.00130 {1.5}	0.00131 {3}	0.00130 {1.5}
		$\hat{\lambda}$	0.19296 {4}	0.38004 {9}	0.21123 {5}	0.36983 {8}	0.22689 {6}	1.03577 {11}	0.25232 {7}	0.43756 {10}	0.03169 {1}	0.03185 {2}	0.03188 {3}
		$\hat{\alpha}$	0.27555 {6}	0.33266 {8}	0.26555 {4}	0.33526 {9}	0.27351 {5}	0.54009 {11}	0.28963 {7}	0.34460 {10}	0.09269 {1}	0.09455 {2}	0.09479 {3}
	MRE	$\hat{\eta}$	0.13721 {5}	0.19032 {9}	0.13122 {4}	0.18884 {8}	0.14050 {6}	0.45368 {11}	0.14566 {7}	0.21808 {10}	0.05764 {1}	0.05834 {3}	0.05825 {2}
		$\hat{\lambda}$	0.17239 {4}	0.24010 {9}	0.17585 {5}	0.23700 {8}	0.17622 {6}	0.41125 {11}	0.19263 {7}	0.25689 {10}	0.07267 {1}	0.07379 {2}	0.07382 {3}
		$\sum Ranks$	44 {5}	78 {9}	39 {4}	75 {8}	53 {6}	99 {11}	62 {7}	90 {10}	9.5 {1}	21 {2}	23.5 {3}
		$\hat{\alpha}$	0.13255 {5}	0.16724 {8}	0.13524 {6}	0.16819 {9}	0.13077 {4}	0.29695 {11}	0.13636 {7}	0.17218 {10}	0.06420 {1}	0.06475 {2}	0.06484 {3}
	$\|BIAS\|$	$\hat{\eta}$	0.04456 {5}	0.06435 {8}	0.04659 {7}	0.06443 {9}	0.04300 {4}	0.18475 {11}	0.04602 {6}	0.07359 {10}	0.02492 {1}	0.02514 {3}	0.02512 {2}
		$\hat{\lambda}$	0.23860 {5}	0.33644 {8}	0.25352 {6}	0.33853 {9}	0.22808 {4}	0.63972 {11}	0.25590 {7}	0.35792 {10}	0.13307 {1}	0.13424 {3}	0.13423 {2}
		$\hat{\alpha}$	0.02907 {4}	0.04588 {8}	0.03019 {6}	0.04637 {9}	0.02994 {5}	0.14699 {11}	0.03087 {7}	0.05002 {10}	0.00576 {1}	0.00580 {2}	0.00581 {3}
500	MSE	$\hat{\eta}$	0.00406 {5}	0.00847 {9}	0.00412 {6}	0.00841 {8}	0.00385 {4}	0.06967 {11}	0.00415 {7}	0.01179 {10}	0.00096 {1}	0.00097 {2.5}	0.00097 {2.5}
		$\hat{\lambda}$	0.09285 {4}	0.18404 {8}	0.10793 {6}	0.18661 {9}	0.10215 {5}	0.63045 {11}	0.11099 {7}	0.21617 {10}	0.02671 {1}	0.02690 {3}	0.02689 {2}
		$\hat{\alpha}$	0.18936 {5}	0.23891 {8}	0.19319 {6}	0.24028 {9}	0.18681 {4}	0.42422 {11}	0.19480 {7}	0.24597 {10}	0.09129 {1}	0.09251 {2}	0.09264 {3}
	MRE	$\hat{\eta}$	0.08911 {5}	0.12871 {8}	0.09318 {7}	0.12887 {9}	0.08599 {4}	0.36950 {11}	0.09203 {6}	0.14718 {10}	0.04959 {1}	0.05028 {3}	0.05025 {2}
		$\hat{\lambda}$	0.11930 {5}	0.16822 {8}	0.12676 {6}	0.16926 {9}	0.11404 {4}	0.31986 {11}	0.12795 {7}	0.17896 {10}	0.06630 {1}	0.06712 {2.5}	0.06712 {2.5}
		$\sum Ranks$	43 {5}	73 {8}	56 {6}	80 {9}	38 {4}	99 {11}	61 {7}	90 {10}	9 {1}	23 {3}	22 {2}

Table 8: Simulation results of the IPBHEx distribution for α=0.75, η=0.5, λ=2

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
		$\hat{\alpha}$	0.49898 $^{\{5\}}$	0.53313 $^{\{8\}}$	0.50756 $^{\{6\}}$	0.55331 $^{\{9\}}$	0.49707 $^{\{4\}}$	0.60641 $^{\{11\}}$	0.52110 $^{\{7\}}$	0.58799 $^{\{10\}}$	0.08030 $^{\{3\}}$	0.07802 $^{\{2\}}$	0.07554 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.17214 $^{\{4\}}$	0.21211 $^{\{8\}}$	0.19644 $^{\{6\}}$	0.21782 $^{\{9\}}$	0.19793 $^{\{7\}}$	0.26679 $^{\{11\}}$	0.18343 $^{\{5\}}$	0.24329 $^{\{10\}}$	0.04422 $^{\{3\}}$	0.04419 $^{\{2\}}$	0.04377 $^{\{1\}}$
		$\hat{\lambda}$	0.91580 $^{\{4\}}$	1.44464 $^{\{10\}}$	1.26134 $^{\{7\}}$	1.38332 $^{\{8\}}$	1.16492 $^{\{6\}}$	2.01771 $^{\{11\}}$	1.15477 $^{\{5\}}$	1.38807 $^{\{9\}}$	0.20601 $^{\{2\}}$	0.20036 $^{\{1\}}$	0.22467 $^{\{3\}}$
		$\hat{\alpha}$	0.40411 $^{\{6\}}$	0.41870 $^{\{7\}}$	0.39136 $^{\{5\}}$	0.46084 $^{\{9\}}$	0.36019 $^{\{4\}}$	0.51145 $^{\{10\}}$	0.42124 $^{\{8\}}$	0.55642 $^{\{11\}}$	0.00958 $^{\{3\}}$	0.00878 $^{\{2\}}$	0.00803 $^{\{1\}}$
30	MSE	$\hat{\eta}$	0.04845 $^{\{4\}}$	0.06670 $^{\{8\}}$	0.05897 $^{\{7\}}$	0.07341 $^{\{9\}}$	0.05819 $^{\{6\}}$	0.10325 $^{\{11\}}$	0.05169 $^{\{5\}}$	0.09091 $^{\{10\}}$	0.00283 $^{\{3\}}$	0.00278 $^{\{2\}}$	0.00272 $^{\{1\}}$
		$\hat{\lambda}$	1.62105 $^{\{4\}}$	4.73269 $^{\{10\}}$	3.49619 $^{\{7\}}$	4.22778 $^{\{9\}}$	2.76324 $^{\{6\}}$	13.59373 $^{\{11\}}$	2.75557 $^{\{5\}}$	3.69800 $^{\{8\}}$	0.06242 $^{\{2\}}$	0.05776 $^{\{1\}}$	0.07431 $^{\{3\}}$
		$\hat{\alpha}$	0.66531 $^{\{5\}}$	0.71084 $^{\{8\}}$	0.67674 $^{\{6\}}$	0.73775 $^{\{9\}}$	0.66276 $^{\{4\}}$	0.80854 $^{\{11\}}$	0.69480 $^{\{7\}}$	0.78399 $^{\{10\}}$	0.09522 $^{\{1\}}$	0.10403 $^{\{3\}}$	0.10071 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.34428 $^{\{4\}}$	0.42421 $^{\{8\}}$	0.39288 $^{\{6\}}$	0.43563 $^{\{9\}}$	0.39586 $^{\{7\}}$	0.53357 $^{\{11\}}$	0.36686 $^{\{5\}}$	0.48658 $^{\{10\}}$	0.08517 $^{\{1\}}$	0.08838 $^{\{3\}}$	0.08755 $^{\{2\}}$
		$\hat{\lambda}$	0.45790 $^{\{4\}}$	0.72232 $^{\{10\}}$	0.63067 $^{\{7\}}$	0.69166 $^{\{8\}}$	0.58246 $^{\{6\}}$	1.00885 $^{\{11\}}$	0.57739 $^{\{5\}}$	0.69403 $^{\{9\}}$	0.09098 $^{\{1\}}$	0.10018 $^{\{2\}}$	0.11233 $^{\{3\}}$
		$\sum Ranks$	40 $^{\{4\}}$	77 $^{\{8\}}$	57 $^{\{7\}}$	79 $^{\{9\}}$	50 $^{\{5\}}$	98 $^{\{11\}}$	52 $^{\{6\}}$	87 $^{\{10\}}$	19 $^{\{3\}}$	18 $^{\{2\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.39820 $^{\{4\}}$	0.46154 $^{\{8\}}$	0.42232 $^{\{6\}}$	0.47422 $^{\{9\}}$	0.41727 $^{\{5\}}$	0.54774 $^{\{11\}}$	0.42632 $^{\{7\}}$	0.48697 $^{\{10\}}$	0.07672 $^{\{3\}}$	0.07539 $^{\{2\}}$	0.07379 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.15022 $^{\{4\}}$	0.18628 $^{\{9\}}$	0.16243 $^{\{6\}}$	0.18160 $^{\{8\}}$	0.16344 $^{\{7\}}$	0.26376 $^{\{11\}}$	0.15519 $^{\{5\}}$	0.21002 $^{\{10\}}$	0.04098 $^{\{2\}}$	0.04108 $^{\{3\}}$	0.04081 $^{\{1\}}$
		$\hat{\lambda}$	0.72251 $^{\{4\}}$	1.12071 $^{\{10\}}$	0.91077 $^{\{7\}}$	1.08717 $^{\{9\}}$	0.87057 $^{\{6\}}$	1.47595 $^{\{11\}}$	0.86332 $^{\{5\}}$	1.07079 $^{\{8\}}$	0.19576 $^{\{2\}}$	0.19165 $^{\{1\}}$	0.20813 $^{\{3\}}$
		$\hat{\alpha}$	0.23907 $^{\{4\}}$	0.31390 $^{\{8\}}$	0.26856 $^{\{6\}}$	0.33468 $^{\{9\}}$	0.25405 $^{\{5\}}$	0.39760 $^{\{11\}}$	0.27770 $^{\{7\}}$	0.34445 $^{\{10\}}$	0.00867 $^{\{3\}}$	0.00817 $^{\{2\}}$	0.00764 $^{\{1\}}$
50	MSE	$\hat{\eta}$	0.03997 $^{\{5\}}$	0.05429 $^{\{9\}}$	0.04427 $^{\{7\}}$	0.05254 $^{\{8\}}$	0.04409 $^{\{6\}}$	0.09980 $^{\{11\}}$	0.03922 $^{\{4\}}$	0.07275 $^{\{10\}}$	0.00246 $^{\{3\}}$	0.00244 $^{\{2\}}$	0.00240 $^{\{1\}}$
		$\hat{\lambda}$	0.85711 $^{\{4\}}$	2.65434 $^{\{10\}}$	1.64546 $^{\{7\}}$	2.44526 $^{\{9\}}$	1.34217 $^{\{5\}}$	5.21754 $^{\{11\}}$	1.44663 $^{\{6\}}$	2.03238 $^{\{8\}}$	0.05600 $^{\{2\}}$	0.05263 $^{\{1\}}$	0.06325 $^{\{3\}}$
		$\hat{\alpha}$	0.53094 $^{\{4\}}$	0.61539 $^{\{8\}}$	0.56309 $^{\{6\}}$	0.63229 $^{\{9\}}$	0.55636 $^{\{5\}}$	0.73032 $^{\{11\}}$	0.56842 $^{\{7\}}$	0.64930 $^{\{10\}}$	0.09501 $^{\{1\}}$	0.10051 $^{\{3\}}$	0.09839 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.30044 $^{\{4\}}$	0.37257 $^{\{9\}}$	0.32486 $^{\{6\}}$	0.36320 $^{\{8\}}$	0.32687 $^{\{7\}}$	0.52752 $^{\{11\}}$	0.31038 $^{\{5\}}$	0.42003 $^{\{10\}}$	0.07956 $^{\{1\}}$	0.08215 $^{\{3\}}$	0.08162 $^{\{2\}}$
		$\hat{\lambda}$	0.36125 $^{\{4\}}$	0.56036 $^{\{10\}}$	0.45539 $^{\{7\}}$	0.54359 $^{\{9\}}$	0.43528 $^{\{6\}}$	0.73798 $^{\{11\}}$	0.43166 $^{\{5\}}$	0.53539 $^{\{8\}}$	0.08871 $^{\{1\}}$	0.09583 $^{\{2\}}$	0.10406 $^{\{3\}}$
		$\sum Ranks$	37 $^{\{4\}}$	81 $^{\{9\}}$	58 $^{\{7\}}$	78 $^{\{8\}}$	52 $^{\{6\}}$	99 $^{\{11\}}$	51 $^{\{5\}}$	84 $^{\{10\}}$	18 $^{\{2\}}$	19 $^{\{3\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.30093 $^{\{4\}}$	0.36389 $^{\{8\}}$	0.31424 $^{\{7\}}$	0.36696 $^{\{10\}}$	0.31127 $^{\{5\}}$	0.47864 $^{\{11\}}$	0.31313 $^{\{6\}}$	0.36435 $^{\{9\}}$	0.07418 $^{\{3\}}$	0.07363 $^{\{2\}}$	0.07264 $^{\{1\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.10875 $^{\{4\}}$	0.14702 $^{\{9\}}$	0.11407 $^{\{6\}}$	0.14356 $^{\{8\}}$	0.11826 $^{\{7\}}$	0.26236 $^{\{11\}}$	0.11337 $^{\{5\}}$	0.16171 $^{\{10\}}$	0.03577 $^{\{2\}}$	0.03593 $^{\{3\}}$	0.03569 $^{\{1\}}$
		$\hat{\lambda}$	0.52520 $^{\{4\}}$	0.77680 $^{\{8\}}$	0.61244 $^{\{7\}}$	0.78316 $^{\{10\}}$	0.60049 $^{\{5\}}$	1.12900 $^{\{11\}}$	0.60446 $^{\{6\}}$	0.78043 $^{\{9\}}$	0.17811 $^{\{2\}}$	0.17539 $^{\{1\}}$	0.18616 $^{\{3\}}$
		$\hat{\alpha}$	0.14008 $^{\{4\}}$	0.19679 $^{\{8\}}$	0.15468 $^{\{7\}}$	0.20185 $^{\{10\}}$	0.15077 $^{\{5\}}$	0.30913 $^{\{11\}}$	0.15190 $^{\{6\}}$	0.19804 $^{\{9\}}$	0.00795 $^{\{3\}}$	0.00766 $^{\{2\}}$	0.00730 $^{\{1\}}$
100	MSE	$\hat{\eta}$	0.02327 $^{\{4\}}$	0.03663 $^{\{9\}}$	0.02412 $^{\{6\}}$	0.03435 $^{\{8\}}$	0.02617 $^{\{7\}}$	0.10554 $^{\{11\}}$	0.02361 $^{\{5\}}$	0.04727 $^{\{10\}}$	0.00195 $^{\{3\}}$	0.00194 $^{\{2\}}$	0.00191 $^{\{1\}}$
		$\hat{\lambda}$	0.43899 $^{\{4\}}$	1.09509 $^{\{9\}}$	0.66782 $^{\{7\}}$	1.14573 $^{\{10\}}$	0.61024 $^{\{5\}}$	2.28287 $^{\{11\}}$	0.62479 $^{\{6\}}$	1.03759 $^{\{8\}}$	0.04664 $^{\{2\}}$	0.04446 $^{\{1\}}$	0.05078 $^{\{3\}}$
		$\hat{\alpha}$	0.40124 $^{\{4\}}$	0.48519 $^{\{8\}}$	0.41898 $^{\{7\}}$	0.48928 $^{\{10\}}$	0.41502 $^{\{5\}}$	0.63818 $^{\{11\}}$	0.41751 $^{\{6\}}$	0.48581 $^{\{9\}}$	0.09418 $^{\{1\}}$	0.09818 $^{\{3\}}$	0.09685 $^{\{2\}}$
	MRE	$\hat{\eta}$	0.21750 $^{\{4\}}$	0.29403 $^{\{9\}}$	0.22814 $^{\{6\}}$	0.28712 $^{\{8\}}$	0.23652 $^{\{7\}}$	0.52472 $^{\{11\}}$	0.22674 $^{\{5\}}$	0.32342 $^{\{10\}}$	0.07009 $^{\{1\}}$	0.07186 $^{\{3\}}$	0.07138 $^{\{2\}}$
		$\hat{\lambda}$	0.26260 $^{\{4\}}$	0.38840 $^{\{8\}}$	0.30622 $^{\{7\}}$	0.39158 $^{\{10\}}$	0.30024 $^{\{5\}}$	0.56450 $^{\{11\}}$	0.30223 $^{\{6\}}$	0.39022 $^{\{9\}}$	0.08148 $^{\{1\}}$	0.08770 $^{\{2\}}$	0.09308 $^{\{3\}}$
		$\sum Ranks$	36 $^{\{4\}}$	76 $^{\{8\}}$	60 $^{\{7\}}$	84 $^{\{10\}}$	51 $^{\{5.5\}}$	99 $^{\{11\}}$	51 $^{\{5.5\}}$	83 $^{\{9\}}$	18 $^{\{2\}}$	19 $^{\{3\}}$	17 $^{\{1\}}$
		$\hat{\alpha}$	0.20368 $^{\{7\}}$	0.23970 $^{\{8\}}$	0.19633 $^{\{4\}}$	0.24751 $^{\{9\}}$	0.20203 $^{\{5\}}$	0.37557 $^{\{11\}}$	0.20296 $^{\{6\}}$	0.25700 $^{\{10\}}$	0.06986 $^{\{1\}}$	0.07039 $^{\{2\}}$	0.07055 $^{\{3\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.06811 $^{\{5\}}$	0.09168 $^{\{8\}}$	0.06791 $^{\{4\}}$	0.09445 $^{\{9\}}$	0.07037 $^{\{6\}}$	0.21977 $^{\{11\}}$	0.07090 $^{\{7\}}$	0.11283 $^{\{10\}}$	0.02984 $^{\{1\}}$	0.03009 $^{\{3\}}$	0.03005 $^{\{2\}}$
		$\hat{\lambda}$	0.35491 $^{\{4\}}$	0.47659 $^{\{8\}}$	0.36601 $^{\{6\}}$	0.48813 $^{\{9\}}$	0.35741 $^{\{5\}}$	0.79891 $^{\{11\}}$	0.37084 $^{\{7\}}$	0.52878 $^{\{10\}}$	0.14947 $^{\{1\}}$	0.15066 $^{\{2\}}$	0.15071 $^{\{3\}}$
		$\hat{\alpha}$	0.06702 $^{\{5\}}$	0.09119 $^{\{8\}}$	0.06404 $^{\{4\}}$	0.09696 $^{\{9\}}$	0.06843 $^{\{6\}}$	0.21442 $^{\{11\}}$	0.06850 $^{\{7\}}$	0.10779 $^{\{10\}}$	0.00678 $^{\{1\}}$	0.00680 $^{\{2\}}$	0.00683 $^{\{3\}}$
250	MSE	$\hat{\eta}$	0.00963 $^{\{5\}}$	0.01641 $^{\{8\}}$	0.00934 $^{\{4\}}$	0.01763 $^{\{9\}}$	0.01133 $^{\{7\}}$	0.08773 $^{\{11\}}$	0.01029 $^{\{6\}}$	0.02640 $^{\{10\}}$	0.00137 $^{\{1.5\}}$	0.00138 $^{\{3\}}$	0.00137 $^{\{1.5\}}$
		$\hat{\lambda}$	0.20276 $^{\{4\}}$	0.37630 $^{\{8\}}$	0.23323 $^{\{7\}}$	0.39453 $^{\{9\}}$	0.23014 $^{\{6\}}$	0.99990 $^{\{11\}}$	0.22916 $^{\{5\}}$	0.47510 $^{\{10\}}$	0.03331 $^{\{1\}}$	0.03349 $^{\{2\}}$	0.03352 $^{\{3\}}$
		$\hat{\alpha}$	0.27157 $^{\{7\}}$	0.31961 $^{\{8\}}$	0.26178 $^{\{4\}}$	0.33001 $^{\{9\}}$	0.26937 $^{\{5\}}$	0.50076 $^{\{11\}}$	0.27061 $^{\{6\}}$	0.34267 $^{\{10\}}$	0.09228 $^{\{1\}}$	0.09385 $^{\{2\}}$	0.09407 $^{\{3\}}$
	MRE	$\hat{\eta}$	0.13622 $^{\{5\}}$	0.18336 $^{\{8\}}$	0.13581 $^{\{4\}}$	0.18890 $^{\{9\}}$	0.14075 $^{\{6\}}$	0.43954 $^{\{11\}}$	0.14180 $^{\{7\}}$	0.22567 $^{\{10\}}$	0.05947 $^{\{1\}}$	0.06019 $^{\{3\}}$	0.06010 $^{\{2\}}$
		$\hat{\lambda}$	0.17746 $^{\{4\}}$	0.23830 $^{\{8\}}$	0.18301 $^{\{6\}}$	0.24407 $^{\{9\}}$	0.17870 $^{\{5\}}$	0.39945 $^{\{11\}}$	0.18542 $^{\{7\}}$	0.26439 $^{\{10\}}$	0.07431 $^{\{1\}}$	0.07533 $^{\{2\}}$	0.07536 $^{\{3\}}$
		$\sum Ranks$	46 $^{\{5\}}$	72 $^{\{8\}}$	43 $^{\{4\}}$	81 $^{\{9\}}$	51 $^{\{6\}}$	99 $^{\{11\}}$	58 $^{\{7\}}$	90 $^{\{10\}}$	9.5 $^{\{1\}}$	21 $^{\{2\}}$	23.5 $^{\{3\}}$
		$\hat{\alpha}$	0.13832 $^{\{6\}}$	0.17124 $^{\{9\}}$	0.13665 $^{\{5\}}$	0.17005 $^{\{8\}}$	0.13147 $^{\{4\}}$	0.29920 $^{\{11\}}$	0.14514 $^{\{7\}}$	0.17244 $^{\{10\}}$	0.06862 $^{\{1\}}$	0.06922 $^{\{2\}}$	0.06934 $^{\{3\}}$
	$\|BIAS\|$	$\hat{\eta}$	0.04405 $^{\{5\}}$	0.06371 $^{\{9\}}$	0.04495 $^{\{6\}}$	0.06253 $^{\{8\}}$	0.04195 $^{\{4\}}$	0.17777 $^{\{11\}}$	0.04825 $^{\{7\}}$	0.07138 $^{\{10\}}$	0.02579 $^{\{1\}}$	0.02602 $^{\{3\}}$	0.02600 $^{\{2\}}$
		$\hat{\lambda}$	0.24135 $^{\{5\}}$	0.33707 $^{\{9\}}$	0.24810 $^{\{6\}}$	0.33294 $^{\{8\}}$	0.22352 $^{\{4\}}$	0.61911 $^{\{11\}}$	0.26539 $^{\{7\}}$	0.35102 $^{\{10\}}$	0.13693 $^{\{1\}}$	0.13816 $^{\{2.5\}}$	0.13816 $^{\{2.5\}}$
		$\hat{\alpha}$	0.03076 $^{\{5\}}$	0.04756 $^{\{9\}}$	0.03086 $^{\{6\}}$	0.04692 $^{\{8\}}$	0.02971 $^{\{4\}}$	0.15087 $^{\{11\}}$	0.03454 $^{\{7\}}$	0.05100 $^{\{10\}}$	0.00657 $^{\{1\}}$	0.00661 $^{\{2\}}$	0.00663 $^{\{3\}}$
500	MSE	$\hat{\eta}$	0.00367 $^{\{5\}}$	0.00773 $^{\{9\}}$	0.00377 $^{\{6\}}$	0.00749 $^{\{8\}}$	0.00335 $^{\{4\}}$	0.06687 $^{\{11\}}$	0.00421 $^{\{7\}}$	0.01108 $^{\{10\}}$	0.00102 $^{\{1\}}$	0.00103 $^{\{2.5\}}$	0.00103 $^{\{2.5\}}$
		$\hat{\lambda}$	0.09461 $^{\{4\}}$	0.18394 $^{\{9\}}$	0.10222 $^{\{6\}}$	0.18100 $^{\{8\}}$	0.09696 $^{\{5\}}$	0.59253 $^{\{11\}}$	0.11669 $^{\{7\}}$	0.21263 $^{\{10\}}$	0.02774 $^{\{1\}}$	0.02795 $^{\{3\}}$	0.02794 $^{\{2\}}$
		$\hat{\alpha}$	0.18443 $^{\{6\}}$	0.22832 $^{\{9\}}$	0.18220 $^{\{5\}}$	0.22674 $^{\{8\}}$	0.17529 $^{\{4\}}$	0.39894 $^{\{11\}}$	0.19352 $^{\{7\}}$	0.22992 $^{\{10\}}$	0.09108 $^{\{1\}}$	0.09230 $^{\{2\}}$	0.09246 $^{\{3\}}$
	MRE	$\hat{\eta}$	0.08811 $^{\{5\}}$	0.12741 $^{\{9\}}$	0.08991 $^{\{6\}}$	0.12506 $^{\{8\}}$	0.08389 $^{\{4\}}$	0.35553 $^{\{11\}}$	0.09650 $^{\{7\}}$	0.14276 $^{\{10\}}$	0.05145 $^{\{1\}}$	0.05204 $^{\{3\}}$	0.05200 $^{\{2\}}$
		$\hat{\lambda}$	0.12067 $^{\{5\}}$	0.16854 $^{\{9\}}$	0.12405 $^{\{6\}}$	0.16647 $^{\{8\}}$	0.11176 $^{\{4\}}$	0.30955 $^{\{11\}}$	0.13270 $^{\{7\}}$	0.17551 $^{\{10\}}$	0.06835 $^{\{1\}}$	0.06908 $^{\{2.5\}}$	0.06908 $^{\{2.5\}}$
		$\sum Ranks$	46 $^{\{5\}}$	81 $^{\{9\}}$	52 $^{\{6\}}$	72 $^{\{8\}}$	37 $^{\{4\}}$	99 $^{\{11\}}$	63 $^{\{7\}}$	90 $^{\{10\}}$	9 $^{\{1\}}$	22.5 $^{\{2.5\}}$	22.5 $^{\{2.5\}}$

Table 9: Simulation results of the IPBHEx distribution for α=3, η=0.5, λ=2.5.

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
		$\hat{\alpha}$	0.28452 {1}	0.31633 {8}	0.29889 {6}	0.33703 {9}	0.29332 {3}	0.34010 {10}	0.31341 {7}	0.35287 {11}	0.29808 {5}	0.29631 {4}	0.29330 {2}
	$\|BIAS\|$	$\hat{\eta}$	0.13271 {4}	0.16001 {9}	0.13992 {5}	0.15948 {8}	0.14855 {7}	0.19852 {11}	0.14010 {6}	0.17328 {10}	0.04645 {2}	0.04585 {1}	0.04708 {3}
		$\hat{\lambda}$	0.90674 {4}	1.37971 {9}	1.17204 {7}	1.43628 {10}	1.12409 {6}	2.27326 {11}	1.05806 {5}	1.35967 {8}	0.29498 {2}	0.27310 {1}	0.39699 {3}
		$\hat{\alpha}$	0.15670 {5}	0.17141 {6}	0.18275 {9}	0.21835 {10}	0.15273 {4}	0.18106 {8}	0.17941 {7}	0.22313 {11}	0.12726 {3}	0.12303 {2}	0.11894 {1}
30	MSE	$\hat{\eta}$	0.02732 {4}	0.03800 {8}	0.03180 {6}	0.03844 {9}	0.03232 {7}	0.05759 {11}	0.02994 {5}	0.04434 {10}	0.00314 {2}	0.00300 {1}	0.00319 {3}
		$\hat{\lambda}$	1.76558 {4}	6.20379 {7}	7.36103 {8}	7.46013 {9}	8.44893 {10}	20.19542 {11}	2.79282 {5}	4.20474 {6}	0.12751 {2}	0.10710 {1}	0.20530 {3}
		$\hat{\alpha}$	0.94841 {4}	1.05444 {8}	0.99630 {6}	1.12342 {9}	0.97774 {5}	1.13366 {10}	1.04471 {7}	1.17624 {11}	0.09490 {1}	0.09877 {3}	0.09777 {2}
	MRE	$\hat{\eta}$	0.26541 {4}	0.32003 {9}	0.27983 {5}	0.31897 {8}	0.29711 {7}	0.39704 {11}	0.28020 {6}	0.34657 {10}	0.08766 {1}	0.09170 {2}	0.09416 {3}
		$\hat{\lambda}$	0.40299 {4}	0.61320 {9}	0.52091 {7}	0.63835 {10}	0.49960 {6}	1.01034 {11}	0.47025 {5}	0.60430 {8}	0.08740 {1}	0.12138 {2}	0.17644 {3}
		$\sum Ranks$	34 {4}	73 {8}	59 {7}	82 {9}	55 {6}	94 {11}	53 {5}	85 {10}	19 {2}	17 {1}	23 {3}
		$\hat{\alpha}$	0.22985 {1}	0.27576 {5}	0.24059 {2}	0.27896 {6}	0.25025 {4}	0.31042 {11}	0.24898 {3}	0.28969 {10}	0.28965 {9}	0.28936 {8}	0.28643 {7}
	$\|BIAS\|$	$\hat{\eta}$	0.11449 {5}	0.13845 {9}	0.11438 {4}	0.13512 {8}	0.12987 {7}	0.18257 {11}	0.11648 {6}	0.15146 {10}	0.04444 {2}	0.04415 {1}	0.04501 {3}
		$\hat{\lambda}$	0.71670 {4}	1.03357 {8}	0.77629 {5}	1.05268 {9}	0.82913 {7}	1.63301 {11}	0.79883 {6}	1.06764 {10}	0.26765 {2}	0.24848 {1}	0.33962 {3}
		$\hat{\alpha}$	0.09046 {1}	0.12005 {8}	0.10655 {3}	0.13535 {10}	0.09949 {2}	0.13625 {11}	0.10704 {4}	0.13282 {9}	0.11909 {7}	0.11659 {6}	0.11299 {5}
50	MSE	$\hat{\eta}$	0.02018 {4}	0.02844 {9}	0.02058 {5}	0.02744 {8}	0.02458 {7}	0.04815 {11}	0.02096 {6}	0.03390 {10}	0.00287 {2}	0.00279 {1}	0.00292 {3}
		$\hat{\lambda}$	0.94879 {4}	2.59697 {9}	1.58050 {7}	2.83058 {10}	1.35750 {5}	8.53649 {11}	1.39916 {6}	2.45007 {8}	0.10471 {2}	0.08929 {1}	0.15431 {3}
		$\hat{\alpha}$	0.76618 {4}	0.91919 {8}	0.80195 {5}	0.92985 {9}	0.83416 {7}	1.03473 {11}	0.82992 {6}	0.96563 {10}	0.09300 {1}	0.09645 {3}	0.09548 {2}
	MRE	$\hat{\eta}$	0.22897 {5}	0.27690 {9}	0.22876 {4}	0.27023 {8}	0.25973 {7}	0.36515 {11}	0.23296 {6}	0.30291 {10}	0.08578 {1}	0.08830 {2}	0.09002 {3}
		$\hat{\lambda}$	0.31853 {4}	0.45937 {8}	0.34502 {5}	0.46786 {9}	0.36850 {7}	0.72578 {11}	0.35503 {6}	0.47451 {10}	0.08718 {1}	0.11044 {2}	0.15094 {3}
		$\sum Ranks$	32 {3.5}	73 {8}	40 {5}	77 {9}	53 {7}	99 {11}	49 {6}	87 {10}	27 {2}	25 {1}	32 {3.5}
		$\hat{\alpha}$	0.18352 {3}	0.22052 {6}	0.17584 {1}	0.21525 {5}	0.20036 {4}	0.26690 {11}	0.18146 {2}	0.22699 {7}	0.26637 {9}	0.26684 {10}	0.26594 {8}
	$\|BIAS\|$	$\hat{\eta}$	0.09462 {6}	0.11348 {9}	0.08701 {4}	0.10871 {8}	0.10675 {7}	0.17441 {11}	0.08754 {5}	0.12878 {10}	0.04167 {2}	0.04163 {1}	0.04201 {3}
		$\hat{\lambda}$	0.56755 {6}	0.74328 {9}	0.52792 {4}	0.70649 {8}	0.62858 {7}	1.16758 {11}	0.53389 {5}	0.77037 {10}	0.24046 {2}	0.22756 {1}	0.27904 {3}
		$\hat{\alpha}$	0.05230 {1}	0.07098 {6}	0.05313 {2}	0.07026 {5}	0.06076 {4}	0.09185 {8}	0.05493 {3}	0.07373 {7}	0.10203 {11}	0.10087 {10}	0.09951 {9}
100	MSE	$\hat{\eta}$	0.01408 {6}	0.01913 {9}	0.01236 {4}	0.01820 {8}	0.01746 {7}	0.04306 {11}	0.01266 {5}	0.02479 {10}	0.00251 {2}	0.00247 {1}	0.00252 {3}
		$\hat{\lambda}$	0.53242 {4}	1.02240 {9}	0.61545 {6}	0.97975 {8}	0.70115 {7}	2.97210 {11}	0.57331 {5}	1.04956 {10}	0.08457 {2}	0.07537 {1}	0.10920 {3}
		$\hat{\alpha}$	0.61172 {6}	0.73508 {9}	0.58613 {4}	0.71752 {8}	0.66787 {7}	0.88967 {11}	0.60486 {5}	0.75665 {10}	0.08757 {1}	0.08895 {3}	0.08865 {2}
	MRE	$\hat{\eta}$	0.18924 {6}	0.22695 {9}	0.17402 {4}	0.21741 {8}	0.21349 {7}	0.34883 {11}	0.17509 {5}	0.25755 {10}	0.08169 {1}	0.08326 {2}	0.08402 {3}
		$\hat{\lambda}$	0.25224 {6}	0.33035 {9}	0.23463 {4}	0.31400 {8}	0.27937 {7}	0.51893 {11}	0.23729 {5}	0.34239 {10}	0.08569 {1}	0.10114 {2}	0.12402 {3}
		$\sum Ranks$	44 {6}	75 {9}	33 {3}	66 {8}	57 {7}	96 {11}	40 {5}	84 {10}	31 {1.5}	31 {1.5}	37 {4}
		$\hat{\alpha}$	0.13144 {3}	0.15124 {6}	0.11772 {2}	0.14950 {5}	0.13496 {4}	0.22494 {8}	0.11593 {1}	0.16482 {7}	0.24207 {9}	0.24415 {10}	0.24430 {11}
	$\|BIAS\|$	$\hat{\eta}$	0.07107 {6}	0.08253 {9}	0.05877 {5}	0.07942 {8}	0.07313 {7}	0.16323 {11}	0.05697 {4}	0.09769 {10}	0.03943 {1}	0.03976 {3}	0.03971 {2}
		$\hat{\lambda}$	0.40196 {7}	0.48650 {9}	0.33804 {5}	0.47354 {8}	0.39425 {6}	0.86255 {11}	0.32591 {4}	0.54467 {10}	0.19866 {1}	0.20023 {3}	0.20008 {2}
		$\hat{\alpha}$	0.02539 {3}	0.03437 {6}	0.02282 {1}	0.03415 {5}	0.02862 {4}	0.06500 {8}	0.02354 {2}	0.03958 {7}	0.08545 {9}	0.08599 {10}	0.08608 {11}
250	MSE	$\hat{\eta}$	0.00873 {6}	0.01139 {9}	0.00618 {5}	0.01061 {8}	0.00942 {7}	0.03739 {11}	0.00594 {4}	0.01582 {10}	0.00225 {1.5}	0.00226 {3}	0.00225 {1.5}
		$\hat{\lambda}$	0.24563 {6}	0.39732 {9}	0.21612 {5}	0.38055 {8}	0.29547 {7}	1.19487 {11}	0.21318 {4}	0.46392 {10}	0.05519 {1}	0.05542 {3}	0.05532 {2}
		$\hat{\alpha}$	0.43814 {6}	0.50413 {9}	0.39242 {5}	0.49834 {8}	0.44988 {7}	0.74981 {11}	0.38644 {4}	0.54941 {10}	0.08024 {1}	0.08138 {2}	0.08143 {3}
	MRE	$\hat{\eta}$	0.14214 {6}	0.16506 {9}	0.11754 {5}	0.15883 {8}	0.14626 {7}	0.32645 {11}	0.11394 {4}	0.19537 {10}	0.07859 {1}	0.07952 {3}	0.07942 {2}
		$\hat{\lambda}$	0.17865 {7}	0.21622 {9}	0.15024 {5}	0.21046 {8}	0.17522 {6}	0.38336 {11}	0.14485 {4}	0.24208 {10}	0.08795 {1}	0.08899 {3}	0.08892 {2}
		$\sum Ranks$	50 {6}	75 {9}	38 {4}	66 {8}	55 {7}	93 {11}	31 {2}	84 {10}	25.5 {1}	40 {5}	36.5 {3}
		$\hat{\alpha}$	0.10131 {4}	0.10943 {5}	0.09039 {2}	0.10951 {6}	0.09757 {3}	0.18960 {8}	0.07834 {1}	0.12503 {7}	0.22059 {9}	0.22260 {10}	0.22273 {11}
	$\|BIAS\|$	$\hat{\eta}$	0.05603 {7}	0.05865 {8}	0.04522 {5}	0.05913 {9}	0.05232 {6}	0.14964 {11}	0.03805 {4}	0.07375 {10}	0.03766 {1}	0.03801 {3}	0.03798 {2}
		$\hat{\lambda}$	0.30952 {7}	0.33972 {8}	0.25965 {5}	0.34154 {9}	0.26986 {6}	0.70923 {11}	0.21694 {4}	0.40322 {10}	0.19708 {1}	0.19882 {3}	0.19876 {2}
		$\hat{\alpha}$	0.01606 {3}	0.01905 {5}	0.01343 {2}	0.01937 {6}	0.01697 {4}	0.04889 {8}	0.01119 {1}	0.02397 {7}	0.07217 {9}	0.07274 {10}	0.07282 {11}
500	MSE	$\hat{\eta}$	0.00602 {7}	0.00640 {8}	0.00380 {5}	0.00660 {9}	0.00567 {6}	0.03350 {11}	0.00298 {4}	0.01008 {10}	0.00204 {1}	0.00206 {3}	0.00205 {2}
		$\hat{\lambda}$	0.15340 {6}	0.19973 {8}	0.12334 {5}	0.20303 {9}	0.16585 {7}	0.73597 {11}	0.09646 {4}	0.25960 {10}	0.05430 {1}	0.05468 {3}	0.05461 {2}
		$\hat{\alpha}$	0.33770 {7}	0.36478 {8}	0.30130 {5}	0.36505 {9}	0.32524 {6}	0.63200 {11}	0.26114 {4}	0.41675 {10}	0.07325 {1}	0.07420 {2}	0.07424 {3}
	MRE	$\hat{\eta}$	0.11205 {7}	0.11730 {8}	0.09044 {5}	0.11827 {9}	0.10465 {6}	0.29927 {11}	0.07610 {4}	0.14750 {10}	0.07508 {1}	0.07601 {3}	0.07596 {2}
		$\hat{\lambda}$	0.13756 {7}	0.15099 {8}	0.11540 {5}	0.15179 {9}	0.11994 {6}	0.31521 {11}	0.09642 {4}	0.17921 {10}	0.08721 {1}	0.08837 {3}	0.08834 {2}
		$\sum Ranks$	55 {7}	66 {8}	39 {4}	75 {9}	50 {6}	93 {11}	30 {2}	84 {10}	25 {1}	40 {5}	37 {3}

Table 10: Ranks findings for all estimation methods and for different combinations of α, η and λ.

Parameters	n	MLE	LSE	WLSE	CRVME	MPSE	PCE	ADE	RADE	SE	LN	GE
	30	4	8	7	10.5	5	9	6	10.5	3	2	1
$\hat{\alpha}=0.6$	50	4	10	7	8.5	5	11	6	8.5	3	2	1
$\hat{\eta}=0.8$	100	4	8.5	5	10	6	11	7	8.5	3	2	1
$\hat{\lambda}=0.7$	250	4	8	5	9	6	11	7	10	1	2	3
	500	6	9.5	5	8	4	11	7	9.5	1	2	3
	30	4	8	7	9	6	11	5	10	3	2	1
$\hat{\alpha}=0.75$	50	4	9	7	8	6	11	5	10	2	3	1
$\hat{\eta}=0.5$	100	4	9	5	8	7	11	6	10	2	3	1
$\hat{\lambda}=1$	250	4	9	6	8	5	11	7	10	1	2	3
	500	5	8	7	9	4	11	6	10	1	2	3
	30	4	8	7	9	6	11	5	10	3	2	1
$\hat{\alpha}=0.6$	50	4	9	7	8	6	11	5	10	2	3	1
$\hat{\eta}=0.5$	100	4	9	6	8	7	11	5	10	2	3	1
$\hat{\lambda}=0.25$	250	4	9	5	8	6	11	7	10	1	2	3
	500	5	9	7	8	4	11	6	10	1	2	3
	30	4	8	7	9	6	11	5	10	3	2	1
$\hat{\alpha}=0.5$	50	4	9.5	7	9.5	6	11	5	8	3	2	1
$\hat{\eta}=0.6$	100	4	8	6	9	7	11	5	10	2	3	1
$\hat{\lambda}=0.5$	250	6	8	4	9	7	11	5	10	1	2	3
	500	5	9	4	8	6	11	7	10	1	2	3
	30	4	8	7	1.5	5	10.5	6	9	3	2	1
$\hat{\alpha}=1.2$	50	4	8	7	10	5	11	6	9	3	2	1
$\hat{\eta}=0.8$	100	4	9	7	10	5	11	6	8	2	3	1
$\hat{\lambda}=1.5$	250	4	8	6	9	5	11	7	10	1	2	3
	500	5	9	6	8	4	11	7	10	1	2	3
	30	4	8	7	9	6	11	5	10	3	2	1
$\hat{\alpha}=1$	50	4	9	7	8	6	11	5	10	2	3	1
$\hat{\eta}=0.5$	100	4	9	7	8	5	11	6	10	1	3	2
$\hat{\lambda}=0.8$	250	4	8	6	9	5	11	7	10	1	2	3
	500	4	8	7	9	5	11	6	10	1	2	3
	30	4	9	7	8	6	11	5	10	3	2	1
$\hat{\alpha}=0.7$	50	4	9	7	8	6	11	5	10	2	3	1
$\hat{\eta}=0.5$	100	4	9	5	8	6	11	7	10	2	3	1
$\hat{\lambda}=2$	250	5	9	4	8	6	11	7	10	1	2	3
	500	5	8	6	9	4	11	7	10	1	3	2
	30	4	8	7	9	5	11	6	10	3	2	1
$\hat{\alpha}=0.75$	50	4	9	7	8	6	11	5	10	2	3	1
$\hat{\eta}=0.5$	100	4	8	7	10	5.5	11	5.5	9	2	3	1
$\hat{\lambda}=2$	250	5	8	4	9	6	11	7	10	1	2	3
	500	5	9	6	8	4	11	7	10	1	2.5	2.5
	30	4	8	7	9	6	11	5	10	2	1	3
$\hat{\alpha}=3$	50	3.5	8	5	9	7	11	6	10	2	1	3.5
$\hat{\eta}=0.5$	100	6	9	3	8	7	11	5	10	1.5	1.5	4
$\hat{\lambda}=2.5$	250	6	9	4	8	7	11	2	10	1	5	3
	500	7	8	4	9	6	11	2	10	1	5	3
$\sum Ranks$		198.5	385.5	271	383	253.5	492.5	259.5	440	82.5	107	88
Overall rank		4	9	7	8	5	11	6	10	1	3	2

8. Applications to Real-Life Data

In this section, we illustrate the importance of the IPBHEx distribution using three real-life data sets from applied fields such as industry, medicine and agriculture. The first data set refers to failure times of product reliability during the design phase for 50 items can be found in Lai et al. (2006). The data values are: 0.12, 0.43, 0.92, 1.14, 1.24, 1.61, 1.93, 2.38, 4.51, 5.09, 6.79, 7.64, 8.45, 11.9, 11.94, 13.01, 13.25, 14.32, 17.47, 18.1, 18.66, 19.23, 24.39, 25.01, 26.41, 26.8, 27.75, 29.69, 29.84, 31.65, 32.64, 35, 40.7, 42.34, 43.05, 43.4, 44.36, 45.4, 48.14, 49.1, 49.44, 51.17, 58.62, 60.29, 72.13, 72.22, 72.25, 72.29, 85.2, 89.52.

The second data set represents the estimated time since growth hormone medication until the children reached the target age. This data set is studied by Alizadeh et al. (2018). The data values are: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51,4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

The third data set represents the exceedances of 72 flood peaks in m^3/s of the Wheaton river near Carcross in Yukon Territory. This data set is analyzed by Agu et al. (2020). The data values are: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 27.0, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5.

The IPBHEx distribution is fitted and compared with some competitive models namely: the exponential (Ex), Marshall–Olkin exponential (MOEx) (Marshall and Olkin 1997), exponentiated-exponential (EEx) (Gupta et al. 1998), beta generalized exponential (BGEx) (Barreto-Souza et al. 2010), Weibull-exponential (WEx) (Oguntunde et al. 2015), alpha-power exponential (APEx) (Mahdavi and Kundu 2017), modified exponential (MEx) (Rasekhi et at. 2017), transmuted generalized–exponential (TGEx) (Khan et al. 2017), Marshall–Olkin alpha-power exponential (MOAPEx) (Nassar et al. 2019), alpha power exponentiated-exponential (APEEx) (Afify et al. 2020), and modified weighted exponential (MWEx) (Chesneau et al. 2022).
The fitting performance of the above models is checked based on some standard performance information criteria (IC) such as $-\hat{\ell}$ (where $\hat{\ell}$ is the maximized log-likelihood), Akaike IC (AIC), consistent AIC (CAIC), Hannan–Quinn IC (HQIC), Bayesian IC (BIC), Cramér–von Mises (W^*), Anderson–Darling (A^*), and Kolmogorov–Smirnov (KS) statistics with the KS p-value.

The ML estimates and their standard errors (SEs) (in parentheses) for all fitted distributions are reported in Tables 11–13 for the three data sets, respectively.

Moreover, the goodness-of-fit measures for the three data sets are listed in Tables 14–16. The results in these tables provide evidence that the proposed IPBHEx distribution provides close fit for the three analyzed data as compared to eleven competing extensions of the Ex model.

Furthermore, the plots of fitted functions including of the IPBHEx distribution are displayed in Figures 4–6 for the three data sets. These plots support the numerical findings in Tables 14–16 and illustrate that the developed IPBHEx distribution fits the three studied data sets better than other competing Ex models.

Table 11: The ML estimates and SEs of the considered models for first data.

Model	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$
IPBHEx	1.2195(0.4228)	0.2595(0.0780)	0.1759(0.0658)	
MEx	0.3126(0.3973)	0.5382(1.1590)	0.5255(0.3228)	0.1542(0.2000)
BGEx	0.0495(0.2562)	102.9888(335.8098)	0.0150(0.0534)	15.4643(89.6664)
APEx	2.1200(1.6608)		0.03933(0.0080)	
APEEx	7.5580(10.6534)		0.0387(0.0069)	0.61840(0.2644)
MOAPEx	1.0016(1.5041)		0.0425(0.0010)	1.7284(1.5510)
MWEx	0.4068(0.3765)		0.0400(0.0056)	
TGEx	0.7588(0.2160)		0.0345(0.0062)	0.4647(0.3560)
EEx	0.9082(0.1622)		0.0312(0.0059)	
MOEx	1.7311(0.8545)		0.0424(0.0099)	
WEx	1164.9293(3069.0000)		0.0332(0.0047)	
Ex			0.0331(0.0047)	

Table 12: The ML estimates and SEs of the considered models for second data.

Model	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$
IPBHEx	1.0111(1.7443)	1.6069(0.5876)	0.2276(0.1188)	
MEx	0.2522(0.0448)	83.9418(847.2020)	46.4534(35.5325)	2.2098(0.0817)
BGEx	14.1438(6.5898)	0.1549(0.0282)	2.2524(0.0475)	12.3616(0.3227)
APEx	5029.3959(11094.7100)		0.5249(0.0670)	
APEEx	414466.9000(16779.9800)		0.4957(0.0817)	.05580(.02045)
MOAPEx	18156.6000(10628.14203)		.03883(0.1251)	.02316(0.2842)
MWEx	2.5506(1.4284)		0.2210(0.0351)	
TGEx	6.4059(2.2167)		0.4379(0.1050)	0.3499(0.4651)
EEx	6.5130(2.2693)		0.4820(0.0798)	
MOEx	15.4318(9.2241)		0.5781(0.1054)	
WEx	0.0001(0.6427)		0.3769(0.1292)	
Ex			0.1885(0.0319)	

Table 13: The ML estimates and SEs of the considered models for third data.

Model	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$
IPBHEx	0.4444(0.3172)	0.4839(0.1251)	0.1820(0.0660)	
MEx	1.3504(3.7228)	0.4930(1.8734)	0.7964(0.1811)	0.0656(0.1892)
BGEx	0.0726(0.1013)	2.4922(2.3648)	0.0520(0.0270)	10.3780(15.2950)
APEx	0.5867(0.4647)		0.0720(0.0175)	
APEEx	1.1414(1.1524)		0.0740(0.0160)	0.8114(0.1808)
MOAPEx	0.9999(1.4215)		0.0687(0.0180)	0.6972(0.5810)
MWEx	0.0050(0.8771)		0.0824(0.0711)	
TGEx	0.8184(0.1598)		0.0733(0.0142)	0.0384(0.3809)
EEx	0.8284(0.1231)		0.0724(0.0117)	
MOEx	0.6969(0.3032)		0.0686(0.0180)	
WEx	291.4096(665.6528)		0.0822(0.0097)	
Ex				0.0819(0.0097)

Table 14: Findings of goodness-of-fit statistics for first data.

Model	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC	W*	A*	KS	p-value
IPBHEx	217.0428	440.085	440.607	445.821	442.269	0.035314	0.2548	0.07491	0.921826
MEx	218.601	445.202	446.090	452.85	448.114	0.062485	0.4299	0.08745	0.807143
BGEx	216.4137	440.827	441.716	448.475	443.739	0.037672	0.2511	0.08038	0.877443
APEx	219.9456	443.891	444.146	447.715	445.347	0.096463	0.6232	0.09876	0.684217
APEEx	218.8173	443.634	444.156	449.370	445.819	0.070333	0.4754	0.08708	0.811151
MOAPEx	219.7785	445.557	446.078	451.293	447.741	0.084249	0.5517	0.10208	0.637316
MWEx	219.9044	443.808	444.064	447.632	445.265	0.101980	0.6539	0.09541	0.716928
TGEx	219.477	444.954	445.475	450.690	447.138	0.095970	0.6222	0.10570	0.594175
EEx	220.2089	444.417	444.673	448.241	445.874	0.123140	0.7767	0.12429	0.390481
MOEx	219.7758	443.557	443.812	447.381	445.012	0.084235	0.5516	0.10209	0.637241
WEx	220.3235	444.647	444.902	448.471	446.103	0.275115	2.0422	0.82899	0.008882
Ex	220.3567	442.713	442.796	444.625	443.441	0.124177	0.7831	0.11433	0.494731

Table 15: Findings of goodness-of-fit statistics for second data.

Model	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC	W*	A*	KS	p-value
IPBHEx	77.74903	161.498	162.272	166.164	163.108	0.040574	0.2880	0.08737	0.9520714
MEx	77.84811	163.696	165.029	169.917	165.843	0.046042	0.3161	0.09882	0.8839317
BGEx	76.78941	161.578	192.912	167.800	163.726	0.407909	2.6106	0.11428	0.7505653
APEx	79.69246	163.920	164.299	167.035	164.998	0.086090	0.5711	0.10569	0.8291043
APEEx	79.60361	165.207	165.981	169.877	166.817	0.081315	0.5415	0.10359	0.8468233
MOAPEx	79.10673	164.213	164.987	168.879	165.824	0.055259	0.3989	0.10635	0.8234132
MWEx	88.21471	180.429	180.804	183.540	181.503	0.097343	0.5991	0.25522	0.0209303
TGEx	78.85662	163.713	164.487	168.379	165.324	0.061890	0.4264	0.91537	0.9310549
EEx	79.09753	162.195	162.570	165.305	163.268	0.070356	0.4747	0.10124	0.8656559
MOEx	83.49900	170.998	171.373	174.108	172.091	0.182326	1.1103	0.13776	0.5196911
WEx	84.24941	192.498	172.873	175.609	173.572	0.111047	0.7146	0.66569	0.0006750
Ex	93.40745	188.814	188.936	190.370	189.531	0.100374	0.6518	0.33317	0.0008441

Table 16: Findings of goodness-of-fit statistics for third data.

Model	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC	W*	A*	KS	p-value
IPBHEx	248.8127	503.625	503.978	510.455	506.344	0.0734071	0.4233	0.08492	0.676715
MEx	251.1836	510.367	510.964	519.473	513.992	0.1218414	0.7124	0.10454	0.410740
BGEx	250.6413	509.282	509.879	518.389	512.907	0.0947009	0.5808	0.10449	0.411413
APEx	251.8650	507.730	507.904	512.283	509.543	0.1415468	0.8011	0.12293	0.226563
APEEx	251.2840	508.569	508.922	515.399	511.288	0.1266564	0.7348	0.10327	0.426297
MOAPEx	251.7600	509.521	509.874	516.351	512.240	0.1447345	0.8141	0.11361	0.307576
MWEx	252.1280	508.256	508.429	512.809	510.068	0.1305420	0.7522	0.14221	0.108686
TGEx	251.2880	508.577	508.930	515.407	511.296	0.1274820	0.7378	0.10259	0.434620
EEx	251.2930	506.587	507.761	511.140	508.399	0.1286180	0.7420	0.10166	0.446286
MOEx	251.7606	507.5212	507.6951	512.074	509.333	0.1447539	0.81422	0.11389	0.307740
WEx	251.9780	507.956	508.130	512.509	509.769	0.1195960	0.8590	0.63985	0.000000
Ex	252.1280	506.255	506.313	508.532	507.162	0.130550	0.7523	0.14220	0.108698

9. Conclusions

In this paper, we proposed and investigated a new flexible class called the inverse-power Burr-Hatke-G (IPBH-G) family. The special sub-models of the family provide monotone and nonmonotone failure rates, as well as unimodal, left-skewed, bimodal, right-skewed, J-shape, and reversed-J shape densities. A useful mixture representation of the IPBH-G family was derived in terms of exponentiated-G family. Its mathematical properties are addressed. Three special sub-models of the family are studied. Eleven classical and Bayesian estimation approaches are adopted to estimate the IPBH-exponential (IPBHEx) parameters. Simulation results are presented to explore the performance

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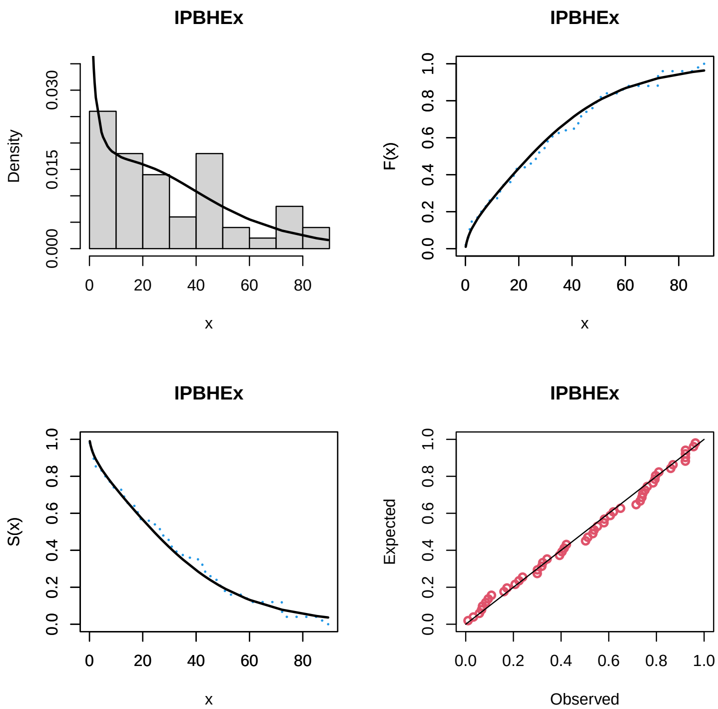

Figure 4: The fitted pdf, cdf, sf, and probability-probability (pp) plots of the IPBHEx for first data.

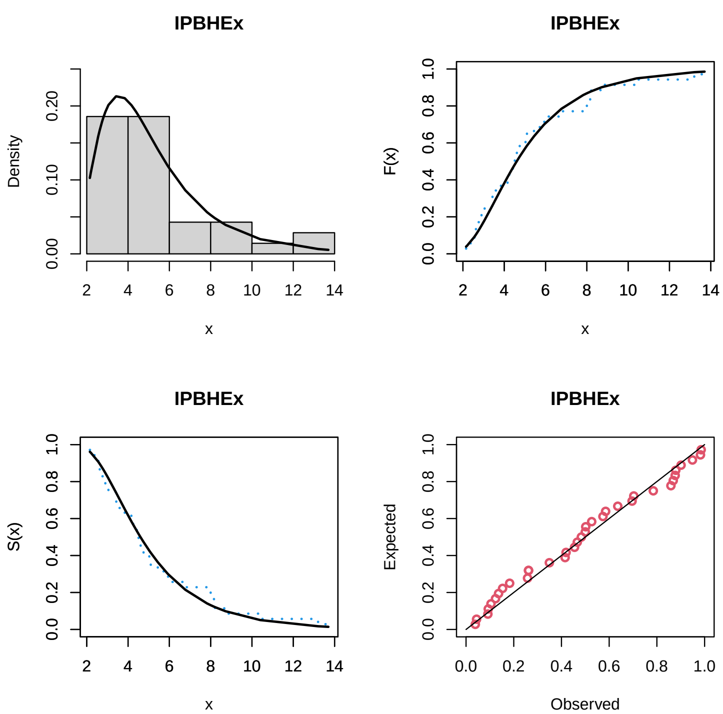

Figure 5: The fitted pdf, cdf, sf, and pp plots of the IPBHEx for second data.

of the these estimators. We also determined the best estimation approach using partial and overall ranks for all estimators, showing that the Bayesian method is the best estimation approach for estimating the IPBHEx parameters under different losses functions. Three real-life data sets are fitted to explore the flexibility of the IPBHEx distribution. The IPBHEx model outperforms many competing exponential models including the Marshall–Olkin exponential,

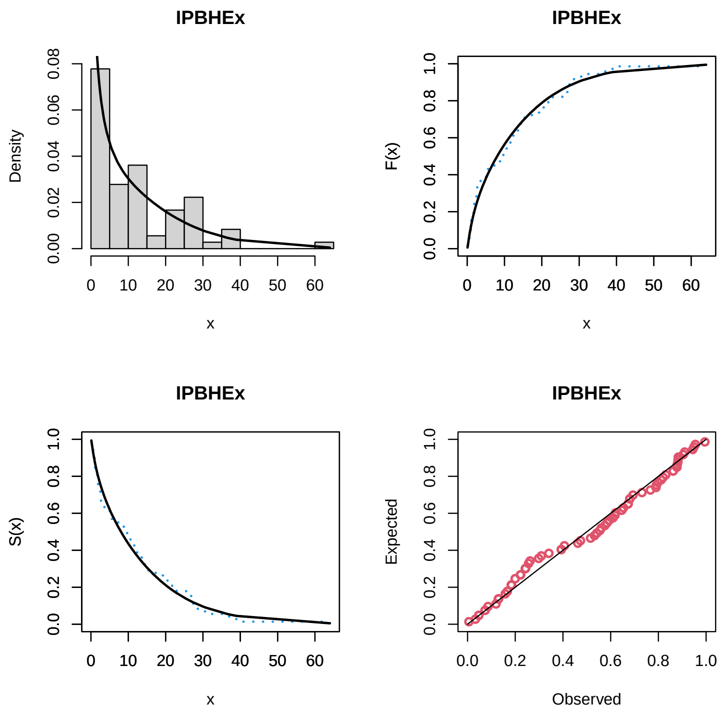

Figure 6: The fitted pdf, cdf, sf, and pp plots of the IPBHEx for third data.

exponentiated-exponential, beta generalized exponential, Weibull-exponential, modified exponential, and alpha-power exponentiated-exponential distributions.

Data Availability

The data are fully available in the article and the mentioned references.

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