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Modeling Anemia Dynamics Among Women of Reproductive Age Using Topp-Leone Exponentiated Generalized Exponential (TLEG-E) Distribution

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Abstract
Anemia continues to be a significant public health issue, particularly impacting women aged 15 to 49. To improve the modeling of anemia prevalence, this study introduces the proposed distribution, offering enhanced flexibility for capturing skewed and heavy-tailed data structures. The model is applied to country-level data from Pakistan, with global trends from World Bank data serving as a comparative backdrop. The TLEG-E distribution demonstrates superior fit and interpretability compared to traditional models, effectively highlighting a declining trend in anemia among Pakistani women, potentially reflecting the impact of health policy reforms and improved nutritional access. While global prevalence varies widely across regions, the emphasis here lies in the methodological advancement and its utility for health data modeling. The proposed framework provides a robust statistical foundation for tracking anemia trends and can support more targeted policy interventions. Its adaptability makes it suitable for broader applications in epidemiological research, enabling more precise assessments of public health initiatives across diverse populations.

Key Words: Anemia, Reproductive age, moments, Survival rate.

1. Introduction:

Anemia continues to be a pressing global health concern, especially among women aged 15 to 49, due to its substantial clinical and societal impacts. Anemia plays a role in maternal health problems, including premature births, low birth weights, and a higher likelihood of infant mortality. Effectively tackling anemia is crucial for achieving Sustainable Development Goal 3 (SDG 3) (General, 2015). The third Sustainable Development Goal (SDG) encompasses several health-related goals. The goals include reducing maternal mortality, averting infant and child fatalities, combating infectious diseases, and attaining universal access to healthcare (Rahman et al., 2016). Women who do not receive treatment for anemia face a higher likelihood of experiencing complications, including premature delivery, maternal death, and developmental problems in their newborns. (Morón-Duarte, Ramirez Varela, Segura, & Freitas da Silveira, 2019).

Given these health implications, there is an urgent need for robust statistical models that can effectively assess and predict anemia trends. Existing models often fail to capture the complexities of anemia prevalence, which is influenced by diverse factors, including menstruation, pregnancy, and childbirth (Alaarg, Schiffelers, van Solinge, & Van Wijk, 2013). Furthermore, such models typically overlook regional variations and the socio-economic factors that influence anemia prevalence. This gap underscores the necessity for more advanced and adaptable statistical methods to accurately analyze anemia patterns. Biological differences, such as menstrual blood loss and heightened iron requirements during pregnancy, make women especially vulnerable to developing anemia. (Bryant, 2010),(Vlassoff, 2007). This statistical innovation allows for more precise modeling of anemia trends, providing valuable insights into the underlying factors contributing to the condition.

The model is also compared with global anemia prevalence data, offering a comparative perspective on regional and socio-economic differences. The results suggest a declining trend in anemia among Pakistani women, potentially reflecting improvements in health policies and food accessibility. This demonstrates how advanced modeling techniques can not only provide more accurate representations of health data but also inform more targeted public health interventions.

By introducing the TLEG-E model early, this study aims to fill a critical gap in the current statistical toolkit, offering a more flexible and accurate method for analyzing anemia prevalence. The implications of this work extend beyond data analysis, as the insights gained can guide evidence-based policy decisions, particularly in regions where anemia poses a significant burden.

Anemia studies often focus on high-income countries, overlooking the unique challenges in low-income regions like Pakistan. Reliable statistical models are essential for evidence-based policymaking, helping to evaluate interventions and design focused health policies. In Pakistan, where anemia is prevalent among women and children, developing specialized models is crucial to understanding the scope and impact of the issue. This study introduces the TLEG-E distribution, which captures the complex characteristics of anemia data, such as skewness and heavy tails, offering a more accurate representation than traditional models.

The TLEG-E model enables better analysis of anemia prevalence and informs healthcare planning by providing insights into the factors driving the condition. By improving access to healthcare, nutrition, and targeted interventions, the model supports policies that address anemia effectively. Moreover, this study contributes to global health research by facilitating cross-country comparisons and promoting international collaboration on anemia prevention.

1.1. Research Gap:

Despite extensive studies on anemia prevalence, limited research has explored its trends using advanced statistical modeling techniques. Most existing studies rely on conventional statistical methods, which may not adequately capture the complex patterns of anemia prevalence over time. Furthermore, while global databases, such as the World Bank, provide valuable comparative insights, there is a lack of model-based studies specifically tailored to Pakistan's context. The application of the TLEG-E distribution remains unexplored in this domain, leaving a gap in understanding the survival characteristics and trend analysis of anemia prevalence in Pakistan.

1.2. Objective and Study Overview:

This study investigates the prevalence and survival patterns of anemia among women aged 15 to 49 in Pakistan using the newly proposed Topp-Leone Exponentiated Generalized Exponential (TLEG-E) distribution. While global anemia data from the World Bank are used for comparative context, the model is specifically developed and applied to Pakistan's dataset to offer localized insights. By employing a flexible probabilistic framework, the study aims to provide a more accurate representation of anemia trends, supporting evidence-based policymaking and future research in public health modeling.

The main objective is to construct a robust statistical model that effectively captures anemia prevalence patterns in Pakistani women, offering an alternative to traditional parametric models. Though the study does not directly propose interventions, the adaptability of the TLEG-E model creates opportunities for targeted health strategy development. Key findings are emphasized in the discussion, while technical derivations are provided separately to maintain clarity. The structure of the study proceeds from model development to global and national data application, concluding with implications and recommendations.

The remainder of the paper is structured as follows: Section 2 introduces the TLEG-E distribution and its mathematical properties. Section 3 presents the data and methodology. Section 4 illustrates global anemia patterns through data visualization. Section 5 discusses the model's application to Pakistan and key findings. Section 6 concludes the study by summarizing its implications and potential extensions.

2. Methods:

Accurately modeling anemia prevalence requires a statistical approach that captures the variability and trends in the data. Traditional parametric models, such as the Weibull and Exponential distributions, have been widely used in epidemiological studies. Conventional models such as the Exponential, Weibull, and Gamma distributions are frequently employed in survival analysis and epidemiological research. However, a key limitation of these models lies in their assumption of a monotonic hazard function, which may not adequately reflect the complex and varying patterns observed in conditions like anemia, where prevalence is influenced by a combination of socio-economic and

healthcare-related factors. However, these models often assume a monotonic hazard function (either increasing or decreasing over time), which may not adequately capture the complex non-monotonic trends observed in anemia prevalence. Anemia prevalence does not follow a simple increasing or decreasing pattern but exhibits variations influenced by factors such as dietary improvements, healthcare access, and regional disparities.

The Topp-Leone family of distributions was first introduced by Topp and Leone (1955) to model bounded data with increasing or decreasing hazard rates. It gained attention due to its flexibility and closed-form expressions for distributional properties. As a generator, the Topp-Leone distribution has been widely used to create new families of distributions by compounding it with baseline models, allowing for enhanced modeling of real-life data, particularly in reliability analysis and survival studies. Its ability to accommodate skewness and peakedness makes it a suitable base for developing extended models, such as the proposed Topp-Leone Exponentiated Generalized Exponential (TLEG-E) distribution.

The TLEG-E distribution provides a more flexible framework for modeling anemia prevalence, accommodating different shapes of hazard rates, including upside-down umbrella patterns observed in the survival characteristics of anemia among women in Pakistan. This flexibility enables the model to represent the varying risk dynamics associated with anemia over time accurately, making it a more suitable choice compared to conventional parametric models. Furthermore, while simpler models require additional assumptions or multiple distributions to capture complex trends, the TLEG-E model inherently incorporates multiple hazard rate shapes within a single distribution, reducing the need for model switching. This makes it an efficient and robust choice for analyzing trends in anemia prevalence while maintaining interpretability. To support our choice of the TLEG-E model, an empirical comparison using model selection criteria Model comparison metrics, including the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), were utilized. The findings indicate that the TLEG-E distribution offers a notably superior fit compared to other competing models, reinforcing its effectiveness in capturing anemia prevalence patterns.

Rasheed (2020) proposed class of Generalized G-class distribution with mixture of Dagum distribution. The PDF and CDF of TLEG- class distribution, respectively, are as follows:

$$f(\mathrm{x};\mathrm{a},\mathrm{b},\alpha) = 2\alpha\mathrm{abg}(\mathrm{x},,\xi)[1-\mathrm{G}(\mathrm{x},\xi)]^{\mathrm{a}-1}(1-[1-\mathrm{G}(\mathrm{x},\xi)]^{\mathrm{a}})^{\mathrm{b}-1}\Big[1 - \Big\{1-\big(1-\mathrm{G}(\mathrm{x},\xi)\big)^{\mathrm{a}}\Big\}^{\mathrm{b}}\Big]\Big[1-\Big[1-\Big\{1-\big(1-\mathrm{G}(\mathrm{x},\xi)\big)^{\mathrm{a}}\Big\}^{\mathrm{b}}\Big]^{2}\Big]^{(\alpha-1)}, \quad (1)$$

and

$$F(\mathrm{x};\mathrm{a},\mathrm{b},\alpha) = \Big[1-\Big\{1-\{1-\{1-\mathrm{G}(\mathrm{x},\xi)\}^{\mathrm{a}}\}^{\mathrm{b}}\Big\}^{2}\Big]^{\alpha}, \quad (2)$$

where $\mathrm{G}(\mathrm{x},\xi)$ is Cumulative Distribution Function (CDF) and $\mathrm{g}(\mathrm{x},,\xi)$ is the Probability Distribution Function (PDF) of baseline distribution with its parametric space denoted by ξ.

In this research note, the exponential distribution was selected as the baseline due to its wide application in real-life phenomena, mathematical simplicity and closed-form expressions, allowing for easier derivation of properties and tractable model extensions. While it assumes a constant hazard rate, the introduction of exponentiation and Topp-Leone compounding significantly enhances its flexibility. Compared to baseline alternatives like Weibull or Gamma, this approach retains computational simplicity while introducing skewness and non-monotonic hazard behavior. An exponential pattern may emerge when anemia prevalence among women in a community steadily declines over time. (Martial, Ngaba, Julius, Khan, & Yu, 2023). Because of better diet or medical interventions, anemia rates are falling (Bhadra & Deb, 2020), (Mata Lorenzo, Ali, Mealing, & Moss, 2023). Pregnant women's anemia treatment advances the larger goal of lowering infant mortality (Sundararajan & Rabe, 2021). In probability theory, the exponential distribution models the time between successive occurrences of rare events, which are assumed to follow a Poisson process. (Daley & Vere-Jones, 2003). Only exponential distribution holds the unique property of being memoryless Nelsen (1987). The “PDF” and “CDF” of the exponential distribution, respectively, are as follows Birnbaum and Saunders (1958):

$$\mathrm{f}(\mathrm{x};\lambda) = \lambda\mathrm{e}^{-\lambda\mathrm{x}} \quad , \ \mathrm{x} > 0, \quad (3)$$

and

$$\mathrm{F}(\mathrm{x};\lambda) = 1-\mathrm{e}^{-\lambda\mathrm{x}}, \quad \mathrm{x} > 0 \quad (4)$$

where, $\lambda > 0$ is the only parameter, which is also called the rate parameter of exponential distribution(Guo, Chen, Meng, & Dong, 2014). The exponential distribution is characterized by having equal mean and standard deviation values, with both determined by the inverse of the rate parameter., i.e., λ (Gupta & Kundu, 1999).

The data used in this study were sourced from the World Development Indicators, drawing on records from the World Health Organization's Global Health Observatory and World Health Statistics. The indicator—prevalence of anemia among women aged 15 to 49—reflects the proportion of pregnant women with hemoglobin levels below 11 g/dL and non-pregnant women with levels below 12 g/dL. The use of survival analysis-inspired metrics enables flexible modeling of dynamic trends in anemia prevalence over time, even when only aggregated population-level data are available.

3. Topp-Leone Exponentiated Generalized Exponential Distribution:

In this section, the PDF and CDF for TLEG-E distribution are derived. Let $'\mathbf{x}'$ be a random variable with PDF and the CDF, then by substituting PDF and CDF in equation (2), the four-parameter TLEG-E distribution's CDF is given by:

$$\boldsymbol{F(x,a,b,\alpha,\lambda)} = \left[\mathbf{1} - \left\{\mathbf{1} - \left\{\mathbf{1} - \mathbf{e}^{-\lambda \mathbf{ax}}\right\}^{\mathbf{b}}\right\}^{\mathbf{2}}\right]^{\boldsymbol{\alpha}} \tag{5}$$

Figure 1 shows different shapes of CDF of TLEG-E distribution for different parametric values.

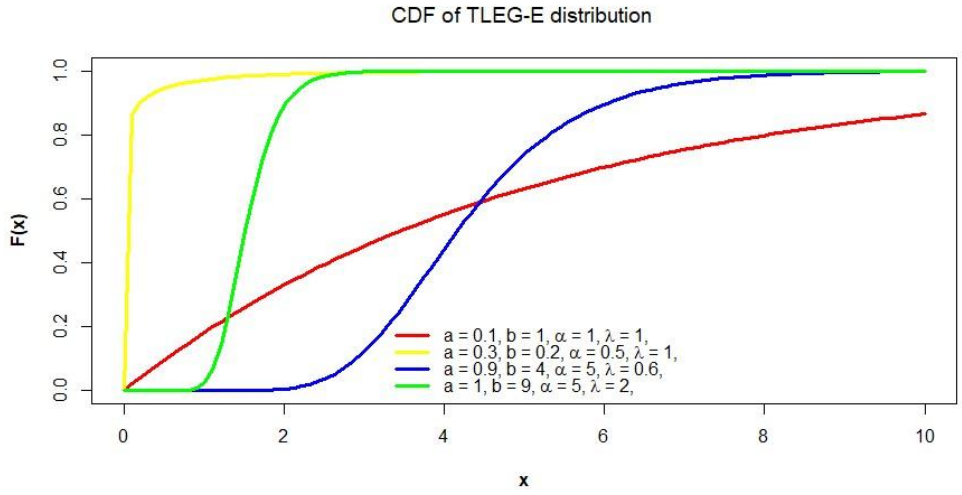

Figure 1: Behavior of CDF of TLEG-E Distribution.

After substituting equation PDF and CDF of exponential distribution in equation (1), the PDF of TLEG-E is as follows:

$$f(\mathrm{x};\mathrm{a},\mathrm{b},\alpha,\lambda) = 2\alpha\mathrm{ab}\lambda \mathrm{e}^{-\lambda \mathrm{ax}}\left(1-\mathrm{e}^{-\lambda \mathrm{ax}}\right)^{\mathrm{b}-1}\left[1 - \left\{1-\mathrm{e}^{-\lambda \mathrm{ax}}\right\}^{\mathrm{b}}\right]\left[1-\left[1-\left\{1-\mathrm{e}^{-\lambda \mathrm{ax}}\right\}^{\mathrm{b}}\right]^{2}\right]^{(\alpha-1)} \quad \mathrm{a},\mathrm{b},\alpha,\lambda,\mathrm{x} > 0 \tag{6}$$

where $'\mathrm{a}'$ is the location parameter, $'\mathrm{b}'$ and $'\lambda'$ are scale parameters, and $'\alpha'$ is the shape parameter each contributing uniquely to the model's flexibility in capturing the dynamics of anemia prevalence among women of reproductive age.

- **Shape Parameter (α)**: This governs the peakedness and skewness of the distribution. Higher values of α yield a sharper peak, indicating that anemia may be concentrated during specific life stages such as pregnancy or adolescence. Smaller α\alphaα values suggest a flatter distribution, reflecting more evenly distributed risk across the age span or population segments.
- **Scale Parameters (λ and b)**: These two parameters control the spread and steepness of the distribution. In the context of anemia, increasing λ or b results in a faster decline in the hazard or prevalence rate, potentially signifying effective public health interventions or improved access to nutrition and care. Lower values

correspond to more persistent or gradually declining trends, possibly due to slow policy response or structural barriers in healthcare delivery.

- **Location Parameter (a)**: This shifts the distribution along the time or age axis. It identifies the point at which anemia prevalence begins to rise meaningfully. For instance, a higher aaa could imply delayed onset of anemia prevalence in a population, whereas a lower a might indicate earlier exposure to anemia risks, such as during adolescence or early adulthood.

Figure 2 shows the shapes of the PDF for different parametric values of the TLEG-E distribution.

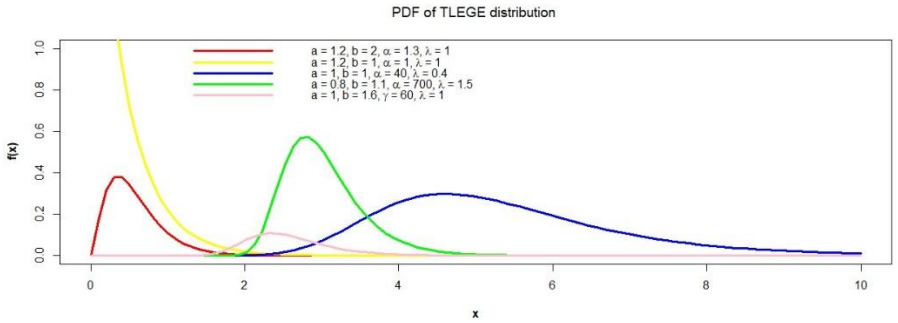

Figure 2. Graph of PDF of TLEG-E Distribution.

The following table shows the family of TLEG-E distribution. These models can be used to model different climate data sets

Table 1: Family of TLEG-E distribution

Parametric Values	Family	Model Names
$a = 1$	$F(x) = \left[\left\{\left(1 - e^{-\lambda x}\right)^{b}\right\}^{2}\right]^{\alpha}$	TLEEx
$b = 1$	$F(x) = \left(1 - e^{-2\lambda a x}\right)^{\alpha}$	TLEx(a)
$a = b = 1$	$F(x) = \left(1 - e^{-2\lambda x}\right)^{\alpha}$	TLEx
$b = \alpha = a = 1$	$F(x) = 1 - e^{-2\lambda x}$	Weibull ($\beta = 2$)
$a = b = 1$	$F(x) = \left(1 - e^{-2\lambda x}\right)^{\alpha}$	EW ($\beta = 2$)

3.1. The Properties of TLEG-E Distribution

Important characteristics in the reliability theory of TLEG-E distribution are derived.

- **Survival Rate:**

If $X \sim TLEG - E(a, b, \alpha, \lambda)$, the survival rate of TLEG-E can be derived by using the subsequent expression:

$$S(x, \emptyset) = \bar{F}(x, \emptyset) = 1 - F(x, \emptyset), \tag{7}$$

The survival function of TLEG-E distribution is stated as:

$$S(x) = 1 - \left[1 - \left\{1 - \left\{1 - e^{-\lambda a x}\right\}^{b}\right\}^{2}\right]^{\alpha}, \tag{8}$$

This expression describes the probability that the time to event exceeds a given value , $'x'$making it a crucial component in reliability and medical survival studies.

- **Hazard Rate:**

The hazard rate of a random variable $'X'$ is rate:(Khan & Javid, 2025).

$$\mathrm{h(x, \emptyset)} = \frac{\mathrm{f(x, \emptyset)}}{\mathrm{s(x, \emptyset)}} \qquad \mathrm{s(x, \emptyset)} > 0$$

Where

$$S(x, \emptyset) = 1 - F(x, \emptyset)$$

So,

$$h(x) = \frac{f(x,\emptyset)}{1 - F(x,\emptyset)} \tag{9}$$

Substituting equation PDF Eq. (6) and Survival Function Eq. (7) of TLEG-E general formulae of hazard rate, the hazard rate for the TLEGE distribution become:

$$h(x) = \frac{2\alpha ab\left[\lambda e^{-\lambda x}\right]\left[e^{-\lambda x}\right]^{a-1}\left(1 - e^{-\lambda ax}\right)^{b-1}\left[1 - \left\{1 - e^{-\lambda ax}\right\}^{b}\right]\left[1 - \left[1 - \left\{1 - e^{-\lambda ax}\right\}^{b}\right]^{2}\right]^{(\alpha-1)}}{1 - [1 - \{1 - \{1 - e^{-\lambda ax}\}^{b}\}^{2}]^{\alpha}} \tag{10}$$

The derived hazard rate function helps describe the instantaneous failure rate and is useful in identifying the risk dynamics over time.

- **Reverse Hazard Function:**

The reverse hazard rate general formula with relationship of survival rate and PDF is:

$$\hat{R}(x) = \frac{f(x)}{1 - S(x)} \tag{11}$$

By using PDF and Survival rate of TLEG-E in the aforementioned formula, the reverse hazard rate of TLEG-E is:

$$\hat{R}(x) = \frac{2\alpha ab\left[\lambda e^{-\lambda x}\right]\left[e^{-\lambda x}\right]^{a-1}\left(1 - e^{-\lambda ax}\right)^{b-1}\left[1 - \left\{1 - e^{-\lambda ax}\right\}^{b}\right]\left[1 - \left[1 - \left\{1 - e^{-\lambda ax}\right\}^{b}\right]^{2}\right]^{(\alpha-1)}}{[1 - \{1 - \{1 - e^{-\lambda ax}\}^{b}\}^{2}]^{\alpha}}, \tag{12}$$

The reverse hazard rate function provides insight into the instantaneous rate of occurrence given survival beyond a certain point and is particularly useful in characterizing lifetime distributions from a backward-looking perspective.

- **Cumulative Hazard Rate:**

The cumulative hazard rate is expressed by using the general expression as follows:

$$\omega(x) = -ln\big(s(x)\big) \tag{13}$$

By using Survival rate of TLEG-E in equation (13), the cumulative hazard rate of TLEG-E is:

$$\omega(x) = -ln\left(1 - \left[1 - \left\{1 - \left\{1 - e^{-\lambda ax}\right\}^{b}\right\}^{2}\right]^{\alpha}\right)$$

The cumulative hazard function summarizes the accumulated risk over time and plays a key role in survival analysis and model interpretation.

- **Odds Function:**

The odds function is as follows:

$$O_{TLEG-E} = \frac{F(x)}{S(x)} \tag{14}$$

By putting CDF and Survival rate in an aforementioned formulae of the odds function, the odds of TLEG-E is:

$$O_{TLEGE} = \frac{\left[1 - \left\{1 - \left\{1 - e^{-\lambda ax}\right\}^{b}\right\}^{2}\right]^{\alpha}}{1 - \left[1 - \left\{1 - \left\{1 - e^{-\lambda ax}\right\}^{b}\right\}^{2}\right]^{\alpha}}$$

The odds function offers a measure of the relative likelihood of failure versus survival at any given time, enhancing interpretation in reliability and medical contexts.

- **Quantiles of TLEG-E Distribution:**

The quantile function is a key characteristic of any probability distribution, as it facilitates data generation. The quantile function of the TLEG-E distribution when $F(x) = u$ is as follows Gilchrist (2000):

$$x = -\frac{ln\left[1 - \left\{1 - \sqrt{1 - u^{\frac{1}{\alpha}}}\right\}^{\frac{1}{b}}\right]}{a\lambda}, \tag{15}$$

The quantile function is essential for generating random samples and facilitates practical applications such as simulation and probabilistic forecasting.

- **Quartile Deviation:**

The quartile deviation for TLEG-E distribution is:

$$Q.D = ln\frac{\left[1-\left\{1-\sqrt{1-(0.25)^{\frac{1}{\alpha}}}\right\}^{\frac{1}{b}}\right]}{a\lambda\left[1-\left\{1-\sqrt{1-(0.75)^{\frac{1}{\alpha}}}\right\}^{\frac{1}{b}}\right]}, \tag{16}$$

The quartile deviation provides a robust measure of dispersion, highlighting the spread of the central 50% of the distribution and offering insights into variability unaffected by extreme values.

- **Quantile Hazard Function**:

The general expression for quantile hazard function is:

$$H(u) = [(1-u)q(u)]^{-1} \tag{17}$$

Using the TLEG-E distribution in equation (17) is:

$$H(u) = \frac{-b\sqrt{1-u^{1/\alpha}\alpha x\left(1-\sqrt{1-u^{1/\alpha}}\right)\left(1-\left(1-\sqrt{1-u^{1/\alpha}}\right)^{1/b}\right)a\lambda}}{(1-u)\left(1-\sqrt{1-u^{1/\alpha}}\right)^{1/b}u^{1/\alpha}}$$

The quantile hazard function characterizes the risk associated with specific quantiles, offering a useful perspective for analyzing survival behavior across different portions of the distribution.

- **Reverse Hazard Quantile rate:**

The general expression is:

$$A(u) = \left(uq(u)\right)^{-1} \tag{18}$$

the reverse hazard quantile rate is:

$$A(u) = \frac{-b\sqrt{1-u^{1/\alpha}\alpha x\left(1-\sqrt{1-u^{1/\alpha}}\right)\left(1-\left(1-\sqrt{1-u^{1/\alpha}}\right)^{1/b}\right)a\lambda}}{u\left(1-\sqrt{1-u^{1/\alpha}}\right)^{1/b}u^{1/\alpha}} \tag{19}$$

The reverse quantile hazard rate helps assess the instantaneous risk from a quantile-based perspective, particularly useful in retrospective survival analysis and reliability assessments.

- **Inter-p Range (Qausi Range):**

Inter-p range is a spread function which is also known as quasi-range. The general formula for quasi range is:

$$ipr(p) = \tilde{Q}(q) - \tilde{Q}(p) = ld(p) + ud(p)\frac{r-1}{n} < p \leq \frac{r}{n} \quad and \quad p = (1-q) \tag{20}$$

Using TLEG-E distribution the inter-p range or qausi range is:

$$ipr(p) = -\frac{1}{a\lambda}\left[ln\left\{\left[1-\left\{1-\sqrt{1-(0.75)^{\frac{1}{\alpha}}}\right\}^{\frac{1}{b}}\right]\left[1-\left\{1-\sqrt{1-(0.25)^{\frac{1}{\alpha}}}\right\}^{\frac{1}{b}}\right]\right\}\right] \tag{16}$$

The interquartile range captures the spread of the middle 50% of the data, serving as a robust indicator of variability less influenced by outliers.

- **Mode of TLEG-E Distribution:**

The mode of TLEG-E can be derived using its PDF:

$$f(x;a,b,\alpha,\lambda) = 2\alpha ab\lambda e^{-\lambda ax}\left(1-e^{-\lambda ax}\right)^{b-1}\Big[1 - \left\{1-e^{-\lambda ax}\right\}^{b}\Big]\Big[1-\Big[1-\left\{1-e^{-\lambda ax}\right\}^{b}\Big]^{2}\Big]^{(\alpha-1)}, x>0$$

By applying the natural logarithm to both sides of the equation and then differentiating with respect to xxx, we obtain, put $f'(x)=0$. then the mode is:

$$= -\lambda a + \frac{(b-1)\lambda a e^{-\lambda ax}}{(1-e^{-\lambda ax})} - \frac{b\left\{1-e^{-\lambda ax}\right\}^{b-1}\lambda a e^{-\lambda ax}}{[1-\{1-e^{-\lambda ax}\}^{b}]} + \frac{2(\alpha-1)\lambda ab e^{-\lambda ax}\left(1-\left\{1-e^{-\lambda ax}\right\}^{b}\right)\left\{1-e^{-\lambda ax}\right\}^{b-1}}{[1-[1-\{1-e^{-\lambda ax}\}^{b}]^{2}]} = 0.$$

The mode identifies the most probable value of the distribution, offering insight into the peak location and central tendency of anemia prevalence within the modeled population.

- **Moments about Origin of TLEG-E Distribution:**

The general preceedure to derived the moment formula for specific distribution is as follows:

$$E(x^r) = \int_0^{\infty} x^r f(\mathrm{x};\ \mathrm{a},\mathrm{b},\alpha,\lambda)dx$$

Using the TLEG-E density, the moment function is:

$$E(x^r) = \frac{2\alpha ab\lambda\sum_{p,q,r=0}^{\infty}(-1)^{p+q+r}\binom{\alpha-1}{p}\binom{2p+1}{q}\binom{b(q+1)-1}{r}}{\left(-\lambda a(r+1)\right)^{r+1}}\lceil r+1.$$

The moments about the origin provide essential measures of the distribution's shape characteristics, including mean, skewness, and kurtosis, which are fundamental for understanding the behavior of anemia prevalence data.

- **Moment Generating Function:**

The general procedure to derive the MGF is:

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}.f(x;a,b,\alpha,\lambda)\,dx$$

The MGF of TLEG-E of $E(x^r)$ is:

$$M_x(t) = \sum_{s=0}^{\infty}\frac{(t)^s}{s!}\frac{2\alpha ab\lambda\sum_{p,q,r=0}^{\infty}(-1)^{p+q+r}\binom{\alpha-1}{p}\binom{2p+1}{q}\binom{b(q+1)-1}{r}}{\left(-\lambda a(r+1)\right)^{r+1}}\lceil r+1.$$

The moment generating function uniquely characterizes the distribution and facilitates the computation of all moments, making it a vital tool for theoretical analysis and model validation.

- **Renyi Entropy:**

Renyi entropy measures the stability of any function developed and to check the randomness of the data. The general expression is:

$$I_R(v) = \frac{1}{1-v}log\left[\int_0^{\infty}[f(x)]^v dx\right]$$

If $X\sim TLEG-E(x;a,b,\alpha,\lambda)$ then $I_R(v)$ is:

$$\frac{1}{1-v}log\left[\int_0^{\infty}[f(x)]^v dx\right] = \frac{1}{1-v}log\left[\frac{(2\alpha ab\lambda)^v}{\lambda a(r+v)}\right]\sum_{p,q,r=0}^{\infty}(-1)^{p+q+r}\binom{v(\alpha-1)}{p}\binom{2p+v}{q}\binom{bq+(b-1)}{r}$$

The Rényi entropy also quantifies the uncertainty and diversity within the distribution, providing a generalized measure of information that enhances understanding of the complexity in anemia prevalence patterns.

4. Maximum Likelihood Estimation of the Parameters

Suppose $x_1, x_2, x_3, \ldots, x_n$be a random representative sample from TLEG-E probability model then the following model is expressed as:

$$L = \prod_{i=1}^{n} f(x)$$

$$L = \prod_{i=1}^{n} 2\alpha ab\lambda e^{-\lambda ax}\left(1-e^{-\lambda ax}\right)^{b-1}\left[1-\left\{1-e^{-\lambda ax}\right\}^{b}\right]\left[1-\left[1-\left\{1-e^{-\lambda ax}\right\}^{b}\right]^{2}\right]^{(\alpha-1)}$$

The normal equations for $\hat{a}, \hat{b}, \hat{\lambda}\ and\ \hat{\alpha}$ are:

$$\frac{\partial lnL}{\partial(\hat{\alpha})} = \frac{n}{\alpha} + \sum ln\left[1-\left\{1-\left(1-e^{-\lambda ax}\right)^{b}\right\}^{2}\right] = 0$$

$$\frac{\partial lnL}{\partial(\hat{b})} = \frac{n}{b} + \sum ln\left(1-e^{-\lambda ax}\right) + \sum ln\left[\frac{\left(1-e^{-\lambda ax}\right)}{1-(1-e^{-\lambda ax})^{b}}\right] + (\alpha-1)\frac{\sum ln\left[2\left\{-\left(1-e^{-\lambda ax}\right)^{b}\right\}\right]}{[1-1-\{(1-e^{-\lambda ax})^{b}\}^{2}]} = 0$$

$$\frac{\partial lnL}{\partial(\hat{a})} = \frac{n}{a} - \sum(\lambda x) + (b-1)\frac{\sum ln\left[-e^{-\lambda ax}(-\lambda x)\right]}{(1-e^{-\lambda ax})} + \frac{\sum ln\left[-\left(1-e^{-\lambda ax}\right)^{b}\lambda x e^{-\lambda ax}\right]}{\{(1-e^{-\lambda ax})^{b}\}} + (\alpha-1)\frac{\sum ln\left[-\left\{1-\left(1-e^{-\lambda ax}\right)\right\}^{2}\left\{\lambda x\left(1-e^{-\lambda ax}\right)\left(-e^{-\lambda ax}\right)\right\}\right]}{[1-\{1-(1-e^{-\lambda ax})^{b}\}^{2}]}$$

$$\frac{\partial lnL}{\partial(\hat{\lambda})} = \frac{n}{\lambda} + \sum(ax) + (b-1)\frac{\sum ln\left(axe^{-\lambda ax}\right)}{(1-e^{-\lambda ax})} + \frac{\sum ln\left[-\left(1-e^{-\lambda ax}\right)axe^{-\lambda ax}\right]}{\{1-(1-e^{-\lambda ax})^{b}\}} + (\alpha-1)\frac{\sum\left[\left\{1-\left(1-e^{-\lambda ax}\right)^{b}\right\}^{2}\left(1-e^{-\lambda ax}\right)^{b}axe^{-\lambda ax}\right]}{[1-\{1-(1-e^{-\lambda ax})^{b}\}^{2}]} = 0$$

The above equation can be solved simultaneously, and the Newton-Raphson method can be applied within the standard algorithm to obtain the final form of these non-linear equations for parameter estimation. As this involves a straightforward two-dimensional optimization problem, obtaining initial estimates is relatively uncomplicated.

5. Results

This section highlights the prevalence of anemia worldwide and presents model-based numerical measures to effectively address the problem.

It is important to note that the survival and hazard functions derived from the TLEG-E distribution are used here as theoretical constructs to study the dynamic behavior and shape of anemia prevalence trends over time. These functions do not represent actual individual-level survival probabilities or risks but rather offer a mathematical framework for examining how the modeled risk evolves across the time domain. As such, the values produced by these functions are not directly comparable to observed prevalence percentages but are useful for identifying patterns, inflection points, and the model's sensitivity to parameter changes. The survival and hazard functions derived from the TLEG-E model are not interpreted in the traditional biomedical or clinical sense. Rather, they serve as theoretical tools to assess the shape, variation, and trend dynamics of anemia prevalence over time. These constructs allow for understanding how the modeled prevalence evolves, peaks, or declines in response to socio-economic and policy-driven factors, rather than reflecting individual-level time-to-event data.

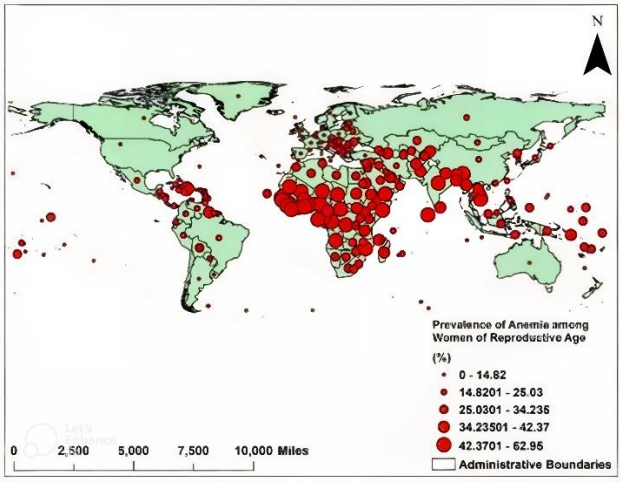

Figure 3. Prevalence of Anemia percentages

Figure 3 shows the percentage of women suffering from anemia in different regions of the world. Even when considering this health issue at a continental level, it becomes evident that 45% of women of reproductive age in Africa are afflicted by anemia. In West Africa the situation is severe, where nearly 60% of women experienced anemia. Although, South Europe is also affected, the percentage of anemic women appears to be relatively low, around 20%. South Asian women seem to suffer a slightly higher degree, with a percentage of around 35%. In Southeast Asia, the percentage of anemic women falls within 25%-35%. North America exhibits the lowest percentage (0%-15%) of anemic women, whereas the upper regions of South America indicate high percentages (14%-40%) of women suffering from anemia.

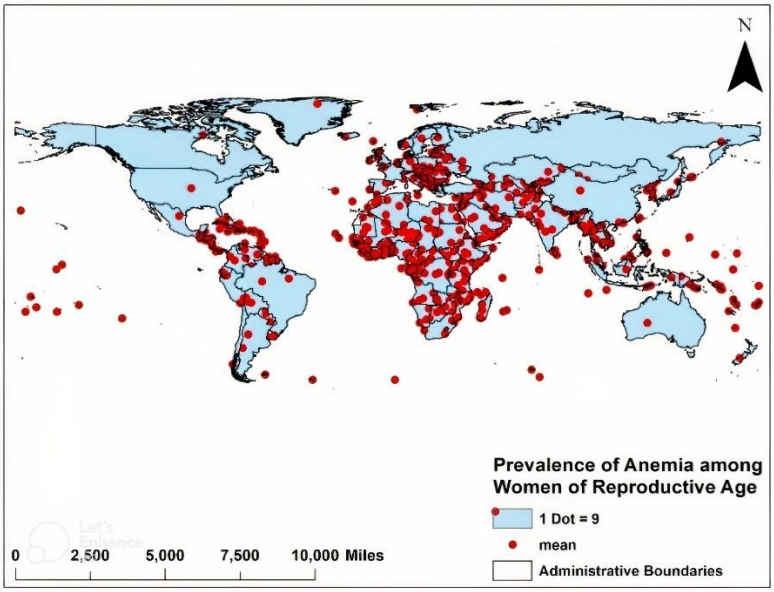

Figure 4. Mean Prevalence of Anemia.

Figure 4 illustrates the average number of women who suffer from anemia, providing a basis for countrywide comparisons of anemic women worldwide. Anemia is prevalent throughout Africa with particularly high frequencies observed in certain countries in the southwest and southeast regions. In South West Africa, countries exhibit on average, higher prevalence of deficiency of iron among women. Similarly, the southeast region, which includes Somalia, Kenya, Tanzania, Madagascar, Mozambique, and Zimbabwe, also reports a notable percentage of anemia

among women. Moving to South Europe, countries like Germany, Poland, Belarus, Ukraine, Romania, Serbia, Hungary, Italy, and Greece demonstrate, on average high frequency of women suffering from anemia in reproductive age. In South Asia, countries including Yemen, Oman, Afghanistan, Pakistan, Nepal, India, Sri Lanka, and Bangladesh, exhibits a higher prevalence of anemia. In Americas, certain North American countries including Cuba, Jamaica, Haiti, and the Dominican Republic, as well as Central American countries including Belize, Honduras, Guatemala, Costa Rica, and Panama have a higher proportion of anemic women of childbearing age as compared to others in the region. There are countries such as Russia, Canada, the US, and Mongolia that report no incidence of anemia.

- **Model Based Measure of Anemic Women of Pakistan**

A model has been developed for anemic women, and important characteristics have been discussed in this section to describe the behavior of anemia in women of Pakistan. Based on these results, one can take measures to help improve healthcare facilities.

Table 2: Goodness of Fit on Anemia among Women during Child Bearing Age (15-49) years

Distributions	Parameters (MLE)	W	A	AIC	CAIC	BIC	HQIC	-LL
TLEG-E	$\hat{a} = 0.0066$ $\hat{b} = 3.36$ $\hat{\alpha} = 7.81$ $\hat{\lambda} = 3.58$	0.052	0.406	235.94	238.6	239.92	236.72	113.972
TLE-Ex	$\hat{b} = 3.36$ $\hat{\alpha} = 7.81$ $\hat{\lambda} = 3.58$	0.049	0.385	449.46	450.9	452.4	450.046	221.73

Table 2 contains the goodness of fit measures using the Anemia data set, which shows that the TLEG-E distribution explaining the behavior of Anemia than TLE-Ex distribution according to the subsequent quality of fit standards.

Although standard parametric models such as the Exponential, Weibull, Gamma, and Lognormal distributions are widely used in epidemiological studies, they were not adopted in this study due to their structural limitations in capturing the complexity of anemia prevalence trends. The Exponential model assumes a constant hazard rate, which is unsuitable for anemia dynamics that often vary due to socioeconomic interventions, nutritional access, and healthcare disparities. While the Weibull and Gamma distributions allow for monotonic hazard behavior, they are inadequate for representing non-monotonic or peaked hazard patterns observed in real-world data, such as the sharp changes linked to pregnancy or policy shifts. The Lognormal distribution provides some asymmetry but lacks intuitive parameter interpretability in the public health context. In contrast, the proposed TLEG-E distribution offers greater modeling flexibility through its four parameters, capturing a broader range of trend behaviors, including variable hazard shapes. This adaptability makes TLEG-E more suitable for analyzing complex and fluctuating health data, such as anemia prevalence among women of reproductive age in Pakistan.

There is no unequivocal solution for four normal equations of ML estimates so they need to be explained numerically. 1000 different samples are simulated for a given known parametric space from TLEG-E distribution with different sizes. Behaviors of ML estimates are observed. From Table 4 it is observed that the standard error for the estimates of parametric space is decreasing when the sample size n increases.

Table 3: Simulation study

Actual values				Sample size	Estimated values				Standard Errors			
b	**a**	**α**	**λ**	**n**	$\hat{b}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{b}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\lambda}$
5.5	0.5	100.1	5.2	n=25	0.144	0.410	114.91	8.891	0.275	19.897	390.77	431.175
0.2	0.3	0.4	0.2	n=50	0.427	0.338	8.030	7.732	1.141	7.4409	31.742	170.095
2.2	1.3	2.4	5.2	n=100	6.679	0.756	0.353	5.2776	5.767	18.767	0.300	130.878
2.2	1.3	2.4	5.2	n=200	2.691	1.126	1.070	3.411	3.130	24.813	1.383	75.184

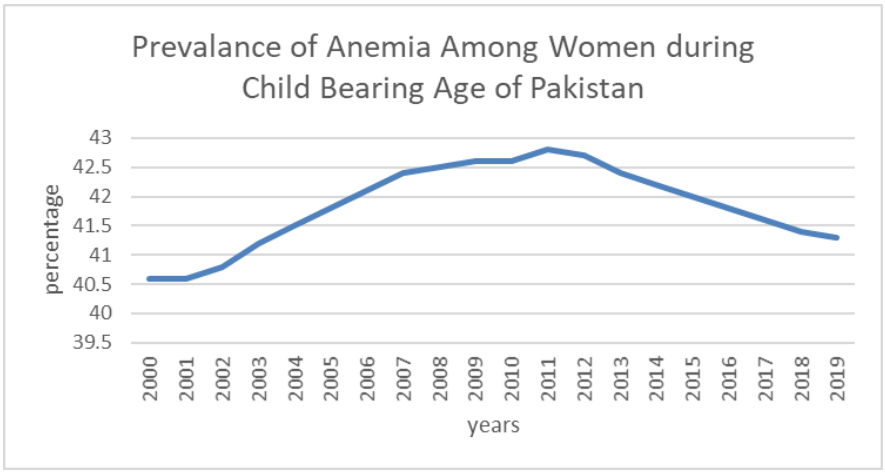

Figure 5. prevalence of Anemia in Pakistan

Figure 5 presents the observed prevalence of anemia among women of reproductive age in Pakistan from 2000 to 2019. The percentage increased from approximately 40% in 2000 to 43% in 2011, before declining to 41% by 2019. This pattern reflects national-level variations in healthcare access, nutrition, and interventions over time.

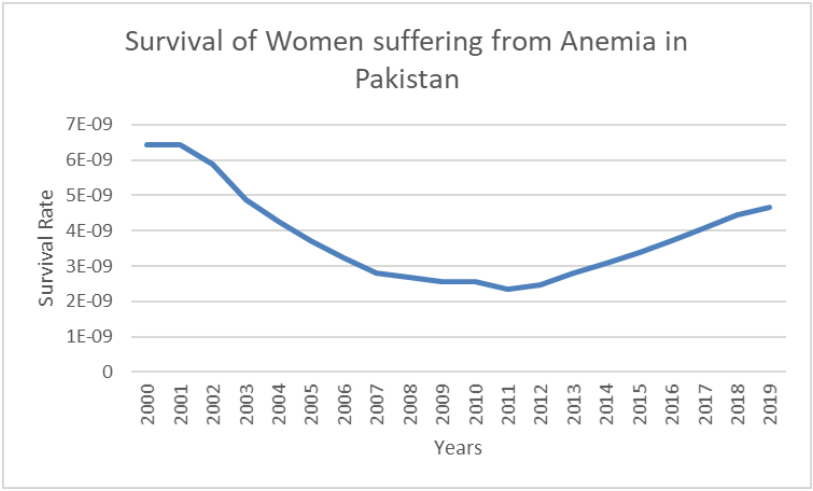

Figure 6. Model-Based Survival Function of Anemia Prevalence in Pakistan.

Figure 6 exhibits the survival behavior of anemic women in Pakistan. The inverted umbrella shape of the survival rate of anemia in Pakistani women indicates a decline from 2000 to 2011, reaching a low of $2x10^{-9}$. However, from 2011 to 2019 prevalence has been inclined with the survival rate near $5x10^{-9}$ by 2019. This suggests an improvement in the survival rate over the time. The increased survival rate of anemia in Pakistan may be attributed to the efforts of policy-making health organizations and improvements in food facilities.

Although survival analysis metrics are traditionally applied to individual-level time-to-event data, this study adapts such functions to country-level annual prevalence data as a structural modeling tool. Here, the proportion of women not experiencing anemia is treated analogously to a survival function, where the "event" is the onset of anemia and the time scale represents yearly intervals. This approach is intended for capturing temporal trends in population-level health status, not for individual-level inference.

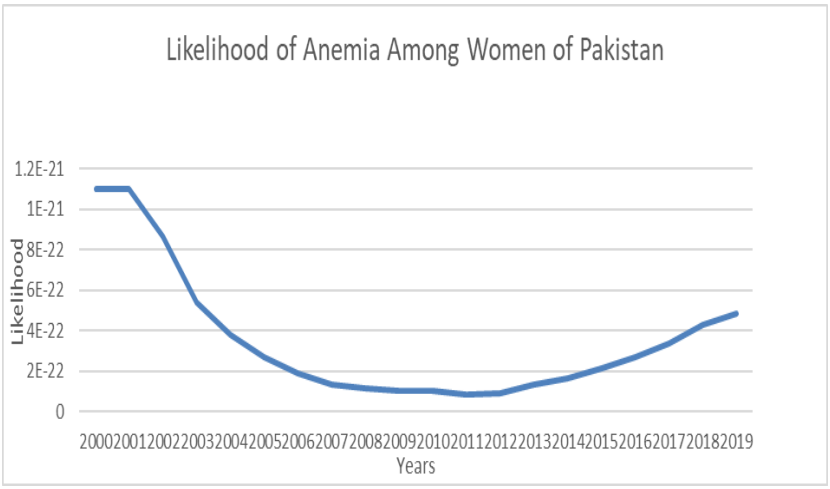

Figure 7. Hazard Function of Anemia under the TLEG-E Model.

Figure 7 illustrates the estimated hazard (or modeled likelihood) of anemia based on the fitted TLEG-E distribution. While the values appear extremely small (e.g., in the range of 10^{-22} by 2019), this result should be interpreted within the model's probabilistic framework. The curve reflects the modeled instantaneous risk under specific parameter assumptions rather than actual prevalence. The decline in the hazard values aligns with the observed downward trend in anemia prevalence (Figure 5), indicating the model's ability to capture reduced risk dynamics over time. However, the scale difference emphasizes the theoretical nature of the model output, not its direct comparability to observed percentages. It is important to clarify that the modeled hazard function values (as in Figure 7) are based on the theoretical hazard or survival functions from the TLEG-E model and are not directly comparable to observed prevalence percentages (Figure 5). These outputs help capture shape and rate of change, rather than raw proportions.

6. Conclusion

This research is based on a new continuous probability model called TLEG-E with four parameters. This statistical innovation advances our understanding of food facilities, health care, and child health after birth as critical factors associated with anemia. The World Bank Directory also observed anemia among women with worldwide data from 2000 to 2019; the behavior of the data shows that the density of anemic women is notably higher in Africa. However, there are countries such as Russia, Canada, the US, and Mongolia where the prevalence of anemia among women of childbearing age is either minimal or non-existent. In contrast, certain countries in North and Central America--including Cuba, Jamaica, Haiti, the Dominican Republic, Belize, Honduras, Guatemala, Costa Rica, and Panama--exhibit higher percentages of anemic women as compared to other countries in this region. In Pakistan, the prevalence of anemia was approximately 40% in 2000, rising to 43% by 2011. The application of the TLEG-E model reveals an upside-down umbrella-shaped survival function, indicating an overall improvement in women's health during their reproductive years. This pattern suggests a decline in anemia prevalence in recent years, highlighting the effectiveness of health interventions and nutritional improvements.

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