Pak.j.stat.oper.res. Vol.21 No. 1 2025 pp 1-13 DOI: http://dx.doi.org/10.18187/pjsor.v21i1.4703

Pakistan Journal of Statistics and Operation Research

Analysis of Two-Dimensional State Markovian Queuing Model with Multiple Vacation, Correlated Servers, Feedback and Catastrophes

Sharvan Kumar[1*], Indra[2]

* Corresponding author

1. Department of Statistics & Operational Research, Kurukshetra University, Kurukshetra, India, kumarsharvan067@gmail
2. Department of Statistics & Operational Research, Kurukshetra University, Kurukshetra, India, indra@kuk.ac.in

Abstract
This paper investigates the queuing system with multiple vacation, correlated servers, feedback and catastrophes. Inter arrival times follow an exponential distribution with parameters λ and service times follow Bivariate exponential distribution BVE (μ, μ, v) where μ is the service time parameter and v is the correlation parameter. Both the servers go on vacation with probability one when there are no units in the system. Laplace transform approach has been used to find the time-dependent solution. The model estimates the total expected cost, total expected profit and obtained the optimal values by varying time for cost and profit. The best optimal value at t=5 when service rate=2.75 and t=2 when feedback probability=0.55 for minimum cost and maximum profit respectively. These important key measures give a greater understanding of the model behaviour. Numerical analysis and graphical representations have been done by using Maple software.

Key Words: Two-Dimensional State Model; Multiple Vacation; Correlated Servers; Feedback; Catastrophes

Mathematical Subject Classification: 90B22, 60K25, 68M20

1. Introduction

This present research aims to study the use of correlated servers in day to day life. From the literature, we found that very little work has been done on correlated servers. Most previous studies focused on correlated servers Markovian queues with finite waiting space capacity. Our study takes a different approach by considering correlated servers with infinite waiting space capacity and additional parameters. This makes the model more realistic and better suited to practical applications.

Various studies have been conducted to evaluate different performance measures and to verify the robustness of the system in which a server takes a break for a random period of time called vacation. When the server returns from a vacation and finds the empty queue, it immediately goes on another vacation and continuing likewise until it finds at least one unit waiting in the system for receiving service. if server finds at least one waiting unit, then it will commence service according to the prevailing service policy *i.e.* multiple vacation policy. Different queuing systems with multiple vacation have been extensively investigated and effectively used in several fields including industries, computer & communication systems, telecommunication systems *etc.* Different types of vacation policies are available in literature such as single vacation, multiple vacation and working vacations. Researches on multiple vacation systems have grown tremendously in the last several years. Cooper(1970) was the first to study the vacation model and obtained the mean waiting time for a unit arriving at a queue served in cyclic order for the exhaustive service. Sharda and Indra (1996) obtained explicit time-dependent probabilities for a queuing system where the server takes multiple vacation and also serves the units intermittently. The vacation and the intermittently time are having the general distribution where as service time is exponentially distributed. Tian et al., 1999) considered the conditional stochastic decompositions of the stationary queue length and waiting time in multi-server queuing models with server vacations by using the matrix geometric approach. Ke and Pearn(2004) developed closed-form solutions for analysing the management policyof an M/M/1 queuing system with server breakdowns and multiple vacation by using the probability generating function approach. Ke et al., (2009) obtained the optimal values of the number of spares and the number of servers while maintaining a minimum specified level of system availability by using the Markov process and the matrix geometric approach. Indra and Bansal(2010) used the supplementary variable and Laplace transform techniques to derive explicit probabilities of the exact number of arrivals and departures by a given time as well as reliability and availability of the server. Ammar(2015) analysed transient solution of an M/M/1 queue with impatient behavior and multiple vacations and obtained time-dependent probabilities, mean and variance of the system size in terms of the modified Bessel function by using the probability-generating function along with continued fractions. Niranjan et al., (2019) analysed bulk arrival and batch service retrial queuing system with server failure and multiple vacation and also obtained optimum cost by using the Supplementary variable technique. Panta et al., (2021) carried out a brief survey of the research on vacation queuing systems with different techniques.

A system of queues in series or in parallel should ordinarily be studied taking into account the interdependence of servers, but this leads to very complicated mathematics even in very simple case of systems. So to reduce such complications of analysis the servers are considered to be independent. But this independence of servers cause impact in time bound operations such as vehicle inspection counters, toll booths, large bars and cafeterias *etc*. where for efficient system functioning the correlation between the servers contributes significantly. Nishida et al., (1974) investigated a two-server Markovian queue assuming the correlation between the servers and obtained steady-state results for a limited waiting space capacity of two units. Sharma (1990) investigated the transient solution to this problem again using only two units waiting spaces capacity. Sharma and Maheswar(1994) developed a computable matrix approach to study a correlated two-server Markovian queue with finite waiting space. They also derived waiting time distribution for steady-state and obtained the transient probabilities through steady-state by using a matrix approach and Laplace transform approach. Kumar and Indra(2023) obtained cost and profit of two-dimensional state M/M/2 queuing model with correlated servers, multiple vacation, balking and catastrophes with infinite waiting space.

Feedback in queuing literature represents customer’s dissatisfaction because of inappropriate service. In the case of feedback, after receiving the service customers either leave the system or rejoin the queue with a certain probability, called queues with feedback. For example, we order a pair of shoes from the Amazon store, but upon receiving the package, we find out that the shoes are of wrong size. We immediately contacted Amazon's customer service and explained the situation. They offered to either refund our money or send us a replacement pair in the correct size. We chose to receive a replacement pair, which arrived within a few days and we are satisfied with the service. Many researchers have been attracted to the study of queues with feedback as large number of applications have been found in many areas including production systems, post offices, supermarkets, hospital management, financial sectors, ticket offices, grocery stores, ATMs and so forth. The concept of feedback was first introduced by Finch(1959) in his paper “Cyclic queues”. Takacs(1963) determined the distribution of the queue size and the first two moments of the distribution for a queue with feedback. D’Avignon and Disney(1976) studied the non-Markovian single server queue with state-dependent feedback. Sharda et al., (1986) considered a continuous time M/M/1 queuing system with feedback and obtained explicit probabilities and the marginal probabilities of exactly i arrivals and j departures.

Queuing systems with catastrophes are also getting a lot of attention nowadays and may be used to solve a wide range of real-world problems. Catastrophes may occur at any time, resulting in the loss of units and the deactivation of the service centre, because they are totally unpredictable in nature. Such type of queues with catastrophes plays an important role in computer programs, telecommunication and ticket counters *etc.* For example, virus or hacker attacking a computer system or program causing the system fail or become idle. Kumar and Arivudainambi(2000) derived transient solution of an M/M/1 queuing model with catastrophes by using the Laplace transform approach. Kumar and Madheswari(2005) analysed transient solution of an M/M/1 queuing system with the possibility of catastrophes and server failures by using the Laplace transform approach. Tarabia(2011) carried out an analysis of infinite-buffer queuing system with single server, balking and catastrophes. Giorno et al., (2014) considered non-stationary queuing system with catastrophes and analysed the transient probabilities, the related moments and the first visit time density to zero state. Kumar (2017) considered Markovian multi-server queuing system with balking and catastrophes and obtained transient solution by using the probability generating function along with Bessel function properties. Sampath (2020) considered an M/M/1 queue with balking, catastrophes, server failure and repairs and obtained an explicit expression for the time-dependent system size probabilities in terms of the modified Bessel function of first kind. Ammar et al.,(2022) investigated a stationary fluid queue operated by a state-dependent birth-death process with catastrophes by using the Laplace transform approach.

With above concepts in mind, we analyse a two-dimensional state M/M/2 queuing model with correlated servers, multiple vacation, feedback and catastrophes.

A two-dimensional state model has been used to deal with complicated transient analysis of some queuing problem. This model is used to examine the queuing system for exact number of arrivals and departures by given time ***t***. In case of a one-dimensional state model, it is difficult to determine how many units have entered, left or waiting units in the system, while the two-dimensional state model exactly identifies the numbers of units that have entered, left, or waiting in the system. The idea of two-dimensional state model for the M/M/1 queue was first given by Pegden andRosenshine(1982). After that, the two-dimensional state model has attracted the attention of a lot of researchers. Sharda and Indra(1999) studied a two-state queuing model by utilizing the intermittently available time and the vacation time. Indra and Sharda(2004) analysed a first-come, first-served, single channel queuing system in which probabilities of arrivals in batches of different sizes at a transition mark depend upon the latest arrival run and obtained the Laplace transforms of probabilities of (i) number of units in the system (ii) exact number of arrivals and (iii) exact number of departures. Gahlawat et al., (2021) studied a two state time-dependent bulk queue model with intermittently available server. They obtained transient probabilities for an exact number of bulk arrivals and bulk departures. Sharma & Indra (2022) looked at how the impatient behaviour of the customer affects the time-dependent probabilities of a two-dimensional state queueing model with multiple vacations.

Let's consider correlated servers in the context of library book processing. There are two counters: one for book issuance and the other for book return process; both are related to each other. When there is no student in the queue for book issue or book return, the server goes on vacation, *i.e*., vacation and repeats until there is at least one student find in the queue, which can be considered multiple vacation. Some students are not satisfied with the service due to various reasons like wrong book issue, wrong author book issue, wrong edition book issue *etc*., they can join again at the end of the queue to get satisfactory service, which is called feedback. Due to any failure in the system, such as technical failure, power cut and system breakdown *etc*. all the students in the queue for book issues and book returns immediately leave the system, which is called catastrophes.

The present work has been organised in the following manner. In section 1 introduction and in section 2 presents the model assumptions, notations and description. Section 3 contains recursive solution by using the differential-difference equations to find out the time-dependent solution and section 4 describes important performance measures. Section 5 investigates the total expected cost function and total expected profit function for the given queuing system. In section 6, we present the numerical results in the form of tables and graphs to illustrate the impact of various factors on performance measures. The last section contains discussion on the findings and suggestions for further work.

2. Model Assumptions, Notations and Description

- Arrivals follow Poisson distribution with parameter $\boldsymbol{\lambda}$.
- There are two servers and the service times follow Bivariate exponential distribution BVE* ($\boldsymbol{\mu}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$) where $\boldsymbol{\mu}$ is the service time parameter and $\boldsymbol{\nu}$ is the correlation parameter.
- The vacation time of the server follow an exponential distribution with parameter $\boldsymbol{w}$.
- After completion of the service, the units rejoin at the early end of the queue to receive service with probability $\boldsymbol{q}$.
- Occurrence of catastrophes follows Poisson distribution with parameter $\boldsymbol{\xi}$.
- Various stochastic processes involved in the system are statistically independent of each other.

*introduced by Marshall and Olkin(1967)

Initially, the system starts with zero units and the server is on vacation, that is.

$$P_{0,0,V}(0) = 1; \;\; P_{0,0,B}(0) = 0 \tag{1}$$

$$\delta_{i,j} = \begin{cases} 1 & ; for \quad i = j \\ 0 & ; for \quad i \neq j \end{cases} \quad ; \qquad \sum_{i}^{j} = 0 \quad for \;\; j < i \tag{2}$$

The Two-Dimensional StateModel

$P_{i,j,V}(t)$= "The probability that there are exactly ***i*** arrivals and ***j*** departures by time ***t*** and the server is on vacation".

$P_{i,j,B}(t)$= "The probability that there are exactly ***i*** arrivals and ***j*** departures by time ***t*** and the server is busy in relation to the queue".

$P_{i,j}(t)$= "The probability that there are exactly ***i*** arrivals and ***j*** departures by time ***t***".

3. The Differential-Difference Equations for the Queuing Model under Study

$$\frac{d}{dt}P_{i,i,V}(t)=-\lambda P_{i,i,V}(t)+(q\mu+\nu)P_{i,i-1,B}(t)(1-\delta_{i,0})+\nu P_{i,i-2,B}(t)(1-\delta_{i,0}-\delta_{i,1})+\xi(1-P_{i,i,V}(t)) \quad i\geq 0 \quad (3)$$

$$\frac{d}{dt}P_{i+1,i,B}(t)=-(\lambda+q\mu+\nu+\xi)P_{i+1,i,B}(t)+2q\mu P_{i+1,i-1,B}(t)(1-\delta_{i,0})+\nu P_{i+1,i-2,B}(t)(1-\delta_{i,0}-\delta_{i,1})+wP_{i+1,i,V}(t) \quad i\geq 0 \quad (4)$$

$$\frac{d}{dt}P_{i,j,V}(t)=-(\lambda+w+\xi)P_{i,j,V}(t)+\lambda P_{i-1,j,V}(t) \quad i>j\geq 0 \quad (5)$$

$$\frac{d}{dt}P_{i,j,B}(t)=-(\lambda+2q\mu+\nu+\xi)P_{i,j,B}(t)+\lambda P_{i-1,j,B}(t)(1-\delta_{i-1,j})+2q\mu P_{i,j-1,B}(t)(1-\delta_{j,0})$$
$$+\nu P_{i,j-2,B}(t)+wP_{i,j,V}(t) \quad i>j+1 \quad (6)$$

Clearly,

$$P_{i,j}(t)=P_{i,j,V}(t)+P_{i,j,B}(t)(1-\delta_{i,j}) \quad i\geq j\geq 0 \quad (7)$$

The preceding equations (3) to (6) are solved by taking the Laplace transforms together with initial condition:

$$\overline{P}_{0,0,V}(s)=\frac{(\xi+s)}{s(s+\lambda+\xi)} \quad (8)$$

$$\overline{P}_{i,0,V}(s)=\frac{(\lambda)^i(\xi+s)}{s(s+\lambda+\xi)(s+\lambda+w+\xi)^i} \quad i>0 \quad (9)$$

$$\overline{P}_{i,i,V}(s)=\left(\frac{q\mu+\nu}{s+\lambda+\xi}\right)P_{j,j-1,B}(s)+\left(\frac{\nu}{s+\lambda+\xi}\right)P_{j,j-2,B}(s) \quad i>0 \quad (10)$$

$$\overline{P}_{i,0,B}(s)=\frac{w(\lambda)^i(\xi+s)}{s(s+\lambda+\xi)(s+\lambda+w+\xi)(s+\lambda+q\mu+\nu+\xi)(s+\lambda+2q\mu+\nu+\xi)^{i-1}}$$
$$+w(\lambda)^i\sum_{m=1}^{i-1}\frac{(\xi+s)}{s(s+\lambda+\xi)(s+\lambda+w+\xi)^{m+1}(s+\lambda+2q\mu+\nu+\xi)^{i-m}} \quad i\geq 1 \quad (11)$$

$$\overline{P}_{i+1,i,B}(s)=\left(\frac{2q\mu}{s+\lambda+q\mu+\nu+\xi}\right)P_{i+1,i-1,B}(s)+\left(\frac{\nu}{s+\lambda+q\mu+\nu+\xi}\right)P_{i+1,i-2,B}(s)+$$
$$\left(\frac{(q\mu+\nu)w\lambda}{(s+\lambda+\xi)(s+\lambda+w+\xi)(s+\lambda+q\mu+\nu+\xi)}\right)P_{i-1,i-1,B}(s) \quad i>0 \quad (12)$$

$$\overline{P}_{i,j,V}(s)=\left(\frac{q\mu+\nu}{s+\lambda+\xi}\right)\left(\frac{\lambda}{s+\lambda+w+\xi}\right)^{i-j}P_{j,j-1,B}(s)+\left(\frac{\lambda}{s+\lambda+w+\xi}\right)^{i-j}\left(\frac{\nu}{s+\lambda+\xi}\right)P_{j,j-2,B}(s) \quad i>j\geq 0 \quad (13)$$

$$\overline{P}_{i,j,B}(s)=\left(\frac{\lambda}{s+\lambda+2q\mu+\nu+\xi}\right)P_{i-1,j,B}(s)+\left(\frac{2q\mu}{s+\lambda+2q\mu+\nu+\xi}\right)P_{i,j-1,B}(s)+\left(\frac{\nu}{s+\lambda+2q\mu+\nu+\xi}\right)P_{i,j-2,B}(s)$$
$$+\left(\frac{q\mu+\nu}{s+\lambda+\xi}\right)\left(\frac{w}{s+\lambda+2q\mu+\nu+\xi}\right)\left(\frac{\lambda}{s+\lambda+w+\xi}\right)^{i-j}\left[P_{j,j-1,B}(s)+P_{j,j-2,B}(s)\right]$$

$i>j+1,\ j>0$ (14)

It is seen that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left[\overline{P}_{i,j,V}(s)+\overline{P}_{i,j,B}(s)(1-\delta_{i,j})\right]=\frac{1}{s} \quad \text{and hence} \tag{15}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left[P_{i,j,V}(t)+P_{i,j,B}(t)(1-\delta_{i,j})\right]=1 \quad \text{a verification} \tag{16}$$

4. Performance Measures

(1) The Laplace transform of $P_{i.}(t)$ of the probability that exactly ***i*** units arrive by time ***t***; when initially there are no unit in the system is given by

$$\overline{P}_{i.}(s)=\sum_{j=0}^{i}\left[\overline{P}_{i,j,V}(s)+\overline{P}_{i,j,B}(s)(1-\delta_{i,j})\right]=\sum_{j=0}^{i}\overline{P}_{i,j}(s)=\frac{(\lambda)^{i}}{(s+\lambda)^{i+1}} \tag{17}$$

And its inverse Laplace transform is $P_{i.}(t)=\frac{e^{-\lambda t}(\lambda t)^{i}}{i!}$ (18)

The arrivals follow a Poisson distribution as the probability of the total number of arrivals is not affected by vacation time of the server.

(2) $P_{.j}(t)$ is the probability that exactly ***j*** units have been served by time ***t.*** In terms of $P_{i,j}(t)$ we have

$$P_{.j}(t)=\sum_{i=j}^{\infty}P_{i,j}(t) \tag{19}$$

(3) The probability of exactly ***n*** units in the system at time ***t***, denoted by $P_n(t)$, can be expressed in terms $P_{ij}(t)$ as per

$$P_n(t)=\sum_{j=0}^{\infty}P_{j+n,j}(t) \tag{20}$$

(4) The Laplace transform of mean number of arrivals by time ***t*** is

$$\sum_{i=0}^{\infty}i\overline{P}_{i.}(s)=\frac{\lambda}{(s^2)} \tag{21}$$

And inverse of the mean number of arrivals by time ***t***is

$$\sum_{i=0}^{\infty}iP_{i.}(t)=\lambda t \tag{22}$$

(5) The mean number of units in the queue is calculated as follows

$$Q_L(t)=\sum_{n=0}^{\infty}nP_V(t)+\sum_{n=2}^{\infty}(n-2)P_B(t) \tag{23}$$

Where $n = i–j$, $P_V(t)=\sum_{j=0}^{i}P_{i,j,V}(t)$ and $P_B(t)=\sum_{j=0}^{i-1}P_{i,j,B}(t)$

5. Cost Function and Profit Function

For the given queuing system, the following notations have been used to represent various costs to find out the total expected cost and total expected profit per unit time

Let

C_H: Cost per unit time for unit in the queue.
C_B: Cost per unit time for a busy server.
C_μ: Cost per service per unit time.
C_V: Cost per unit time when the server is on vacation.
$C_{\mu-q}$: Cost per unit time when a unit rejoins at the early end of the queue as a feedback unit.

If I is the total expected amount of income generated by delivering a service per unit time then

(1) Total expected cost per unit at time ***t*** is given by

$$TC(t)=C_H*Q_L(t)+C_B*P_B(t)+C_V*P_V(t)+\mu*(C_\mu+C_{\mu-q}) \tag{24}$$

(2) Total expected income per unit at time ***t*** is given by

$$TE_I(t) = I * \mu * (1 - P_V(t)) = I * \mu * P_B(t) \tag{25}$$

(3) Total expected profit per unit at time t is given by

$$TE_P(t) = TE_I(t) - TC(t) \tag{26}$$

6. Numerical Results (Numerical Validity Check)

(1) For the state when the server is on vacation *i.e.*, $P_V(t)$

(2) For the state when the server is busy in relation to the queue *i.e.*, $P_B(t)$

(3) The probability $P_{i.}(t)$ that exactly i units arrive by time t is

$$P_{i.}(t) = \sum_{j=0}^{i} P_{i,j}(t) \tag{27}$$

(4) A numerical validity check of inversion of $\overline{P}_{i,j}(s)$ is based on the relationship

$$\Pr\{i \text{ arrivals in } (0, t)\} = \frac{e^{-\lambda t}(\lambda t)^i}{i!} = \sum_{j=0}^{\infty} P_{i,j}(t) = P_{i.}(t) \tag{28}$$

The probabilities of this model shown in last column of Table 1 given below are consistent to the last column of Pegden and Rosenshine(1982) by keeping constant values of w=1, q=0.7, ξ=0 and v=0.25 shown in table

Table 1: Numerical validity check of inversion of $\overline{P}_{i,j}(s)$

λ	μ	t	i	$\frac{e^{-\lambda t} * (\lambda t)^i}{i!}$	$\sum_{j=0}^{i} P_{i,j,V}(t)$	$\sum_{j=1}^{i-1} P_{i,j,B}(t)$	$\sum_{j=0}^{i} P_{i,j}(t)$
1	2	3	1	0.149361	0.129196	0.020165	0.149361
1	2	3	3	0.224043	0.158077	0.065966	0.224043
1	2	3	5	0.100819	0.057803	0.043016	0.100819
2	2	3	1	0.014873	0.012865	0.002008	0.014873
2	2	3	3	0.089235	0.062961	0.026274	0.089235
2	2	3	5	0.160623	0.092090	0.068533	0.160623
1	2	4	1	0.073263	0.065390	0.007873	0.073263
1	2	4	3	0.195367	0.148001	0.047366	0.195367
1	2	4	5	0.156294	0.100998	0.055296	0.156294
2	2	4	1	0.002684	0.002395	0.000289	0.002684
2	2	4	3	0.028626	0.021686	0.006940	0.028626
2	2	4	5	0.091604	0.059195	0.032409	0.091604
2	4	4	5	0.091604	0.073396	0.018208	0.091604
1	2	4	4	0.195366	0.136809	0.058557	0.195366
1	2	3	6	0.050409	0.025824	0.024585	0.050409

Table 2: Probabilities of exactly j departures by time t

λ=2, μ=4, w=1, v=0.25, q=1, ξ=0.0001					
j	t=1	t=3	t=5	t=7	t=10
0	0.67646638	0.09903097	0.007014800	0.000419236	0.0001612628
1	0.17630720	0.08098461	0.009064655	0.000579955	0.0000501230
2	0.09471300	0.11769094	0.018081886	0.001387817	0.0000594874
3	0.03728300	0.14047426	0.031349826	0.003003293	0.0000837339
4	0.01147400	0.14077174	0.047312824	0.005789865	0.0001424326
5	0.00284300	0.11936402	0.061673383	0.009784188	0.0002632795
6	0.00057450	0.08499410	0.068129303	0.014139328	0.0004652902

Table 3: Probabilities of exactly n units in the system at time t

λ=2, μ=3, w=2, v=0.25, q=1, ξ=0.0001					
n	t=1	t=2	t=3	t=4	t=5
0	0.3188454	0.2900015	0.2288654	0.1358815	0.0632150
1	0.3464042	0.3056086	0.2150051	0.1110816	0.0450873
2	0.1894659	0.1582662	0.0971954	0.0435875	0.0154702

3	0.0895022	0.0778208	0.0409972	0.0157945	0.0048669
4	0.0358991	0.0362664	0.0162732	0.0052550	0.0013771
5	0.0120816	0.0155898	0.0060336	0.0015651	0.0003341
6	0.0032706	0.0057902	0.0020221	0.0004001	0.0004001

7. Sensitivity Analysis

This part focuses on the impact of the arrival rate (λ), service rate (μ), vacation rate (w), correlation parameter (v), feedback probability (q) and catastrophes rate (ξ) on the probability when the server is on vacation ($P_V(t)$), probability when the server is busy ($P_B(t)$), expected queue length ($Q_L(t)$), total expected cost ($TC(t)$), total expected income ($TE_I(t)$) and total expected profit ($TE_P(t)$) at time t. To determine the numerical results for the sensitivity of the queuing system one parameter varied while keeping all the other parameters fixed and taking cost per unit time for unit in the queue=10, cost per unit time for a busy server=8, cost per unit time when the server is on vacation=5, cost per service per unit time=4, cost per unit time when a unit rejoins at the early end of the queue=2, total expected amount of income=100 and number of units in the system=8.

Impact of Arrival Rate (λ): We examine the behaviour of the queuing system by evaluating the effectiveness measures along with cost and profit analysis by varying $\lambda$with time, while keeping all other parameters fixed; μ=5, w=2, v=0.25, ξ=0.0001 and q=0.7. In Table 4, we observe that as the value of λ increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 4: Effectiveness Measures, Cost and Profit versus λ

t	λ	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	1.00	0.8347313	0.1652675	0.1616589	37.1123855	82.63375	45.521364
2		0.8000355	0.1997270	0.2189383	37.7873765	99.86350	62.076123
3		0.7955690	0.2006289	0.2218137	37.8010132	100.3144	62.513436
4		0.7845911	0.1940526	0.2088204	37.5635803	97.02630	59.462719
5		0.7528634	0.1790715	0.1836370	37.0332590	89.53575	52.502491
1	1.50	0.7690664	0.2309058	0.3268278	38.9608564	115.4529	76.492043
2		0.7225512	0.2736464	0.4194996	39.9969232	136.8232	96.826276
3		0.7006089	0.2591439	0.3835761	39.4119567	129.5719	90.159993
4		0.6316455	0.2156423	0.2945631	37.8289969	107.8211	69.992153
5		0.5063340	0.1557671	0.1929856	35.7076628	77.88355	42.175887
1	2.00	0.7109100	0.2888525	0.5269526	41.1348960	144.4262	103.29135
2		0.6480270	0.3306131	0.6246558	42.1315978	165.3065	123.17495
3		0.5775704	0.2697048	0.4658952	39.7044424	134.8524	95.147957
4		0.4226231	0.1700517	0.2594028	36.0675571	85.02585	48.958292
5		0.2475614	0.0855021	0.1148547	33.0703708	42.75105	9.6806792

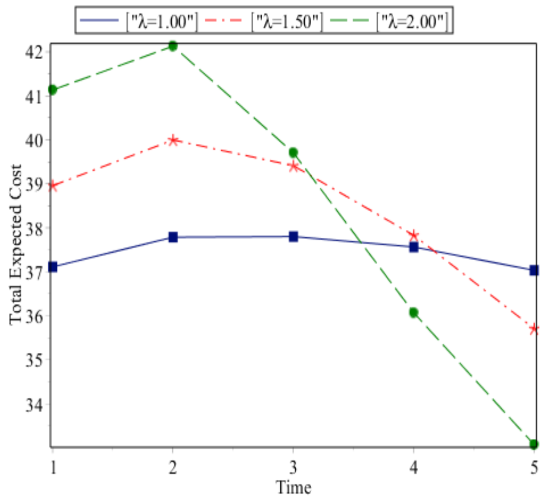

Figure 1

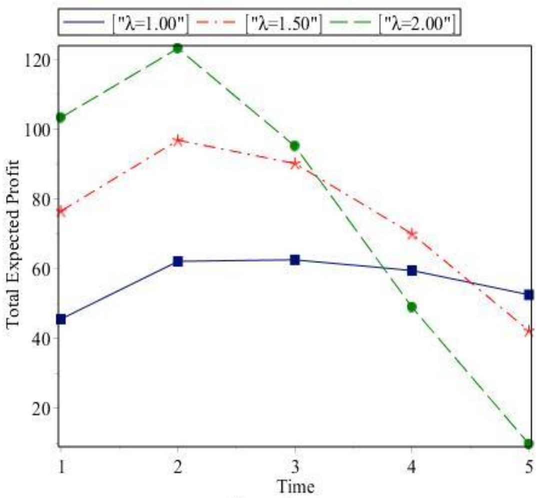

Figure 2

Figure 1: shows the variation of cost with time by varying arrival rate while keeping the other parameters fixed. (As per Table 4)

Figure 2: shows the variation ofprofitwith time by varying arrival rate while keeping the other parameters fixed. (As per Table 4)

Impact of Service Rate(μ): The behaviour of the queuing system by evaluating the effectiveness measures along with cost and profit analysis by varying $\boldsymbol{\mu}$with time $\boldsymbol{t}$, while keeping all other parameters fixed; $\boldsymbol{\lambda}$=2, $\boldsymbol{w}$=2, $\boldsymbol{\nu}$=0.25, ξ=0.0001 and $\boldsymbol{q}$=0.7. In Table 5, we observe that as the value of $\boldsymbol{\mu}$ increases with time $\boldsymbol{t}$, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 5: Effectiveness Measures, Cost and Profit versus μ

t	μ	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	2.75	0.633823	0.365839	0.655763	29.153461	100.6057	71.452318
2		0.478135	0.500454	0.890493	31.799249	137.6249	105.82571
3		0.421990	0.425284	0.678277	28.795002	116.9533	88.158317
4		0.321445	0.271229	0.370668	23.983750	74.58816	50.604416
5		0.195946	0.137116	0.161348	20.190154	37.70712	17.516965
1	3.75	0.660181	0.339581	0.585514	34.372699	127.3429	92.970250
2		0.567704	0.410935	0.728436	35.910375	154.1008	118.19051
3		0.507801	0.339473	0.541753	33.172326	127.3024	94.130161
4		0.378094	0.214580	0.298746	29.094579	80.46757	51.372995
5		0.224942	0.108121	0.131500	25.804685	40.54548	14.740801
1	4.75	0.701664	0.298097	0.536690	39.760009	141.5964	101.83639
2		0.634294	0.344345	0.640039	40.826633	163.5641	122.73747
3		0.565979	0.281295	0.476727	38.347538	133.6154	95.267871
4		0.415259	0.177415	0.265063	34.646253	84.27212	49.625872
5		0.243826	0.089237	0.117271	31.605747	42.38762	10.78187

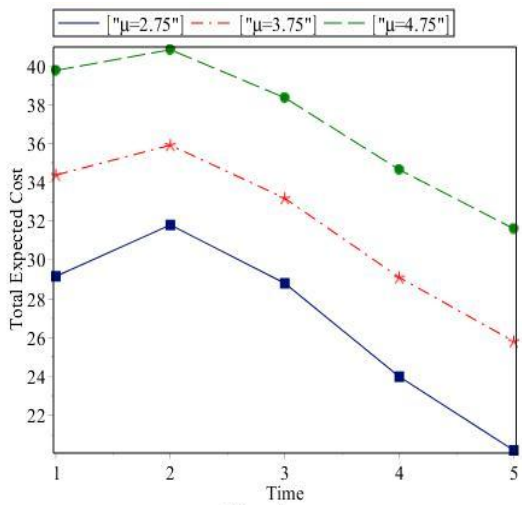

Figure 3

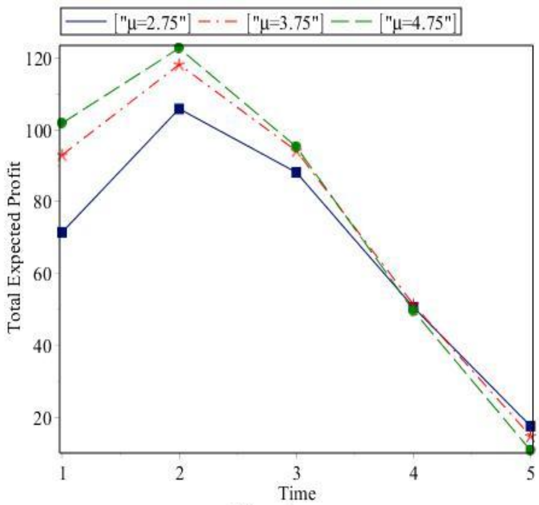

Figure 4

Figure3:shows the variation of cost with time by varying service rate while keeping the other parameters fixed**.** (As per Table 5)

Figure 4:shows the variation of profit with time by varying service rate while keeping the other parameters fixed. (As per Table 5)

Impact of Vacation Rate (w): We observe that the behaviour of the queuing system by evaluating the effectiveness measures along with cost and profit by varying $\boldsymbol{w}$ with time $\boldsymbol{t}$, while keeping all other parameters fixed; λ=2, μ=5, ν=0.25, ξ=0.0001 and $\boldsymbol{q}$=0.7. Table 6, we observe that as the value of w increases with time $\boldsymbol{t}$, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 6: Effectiveness Measures, Cost and Profit versus *w*

t	*w*	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	1.50	0.747197	0.252565	0.618685	41.943364	126.2826	84.339235
2		0.667490	0.311149	0.821146	44.038116	155.5749	111.53683
3		0.588268	0.259006	0.642533	41.438729	129.5034	88.064670
4		0.428717	0.163957	0.365994	37.115190	81.97865	44.863459
5		0.250554	0.082509	0.164417	33.557016	41.25450	7.6974840
1	1.75	0.727340	0.272422	0.569572	41.511807	136.2111	94.699292
2		0.656302	0.322337	0.710825	42.968469	161.1686	118.20018
3		0.582253	0.265022	0.540910	40.440542	132.5111	92.070557
4		0.425372	0.167302	0.304010	36.505382	83.65120	47.145817
5		0.248921	0.084141	0.135451	33.272253	42.07090	8.7986466
1	2.00	0.710910	0.288852	0.526952	41.134896	144.4262	103.29135
2		0.648027	0.330613	0.624655	42.131597	165.3065	123.17495
3		0.577570	0.269704	0.465895	39.704442	134.8524	95.147957
4		0.422623	0.170051	0.259402	36.067557	85.02585	48.958292
5		0.247561	0.085502	0.114854	33.070370	42.75105	9.6806792

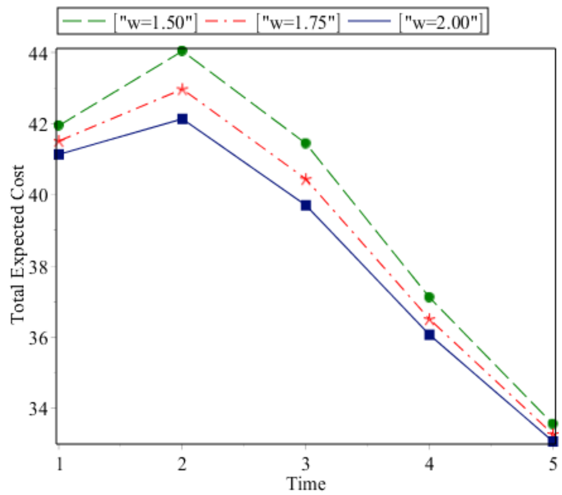

Figure 5

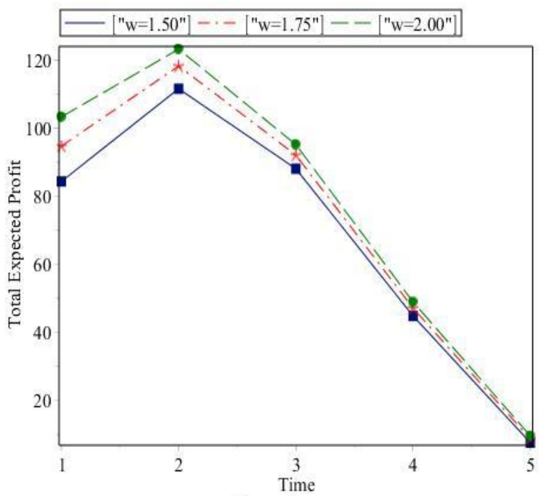

Figure 6

Figure 5:shows the variation of cost with time by varying vacation rate while keeping the other parameters fixed. (As per Table 6)
Figure 6:shows the variation of profit with time by varying vacation rate while keeping the other parameters fixed. (As per Table 6)

Impact of Correlation Parameter (v): We see that the behaviour of the queuing system by evaluating the effectiveness measures along with cost and profit analysis by varying v with time$t$, while keeping all other parameters fixed; λ=2, μ=5, w=2, ξ=0.0001 and q=0.7. InTable 7, we observe that as the value of v increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 7: Effectiveness Measures, Cost and Profit versus *v*

t	*v*	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	0.25	0.7109100	0.2888525	0.5269526	41.1348960	144.42625	103.291354
2		0.6480270	0.3306131	0.6246558	42.1315978	165.30655	123.174952
3		0.5775704	0.2697048	0.4658952	39.7044424	134.85240	95.1479576
4		0.4226231	0.1700517	0.2594028	36.0675571	85.025850	48.9582929
5		0.2475614	0.0855021	0.1148547	33.0703708	42.751050	9.68067920
1	0.50	0.7258735	0.2738889	0.5178700	40.9991787	136.94445	95.9452713
2		0.6684990	0.3101411	0.6123460	41.9470838	155.07055	113.123466

3		0.5944386	0.2528366	0.4579073	39.5739588	126.41830	86.8443412
4		0.4332878	0.1593869	0.2552740	35.9942742	79.693450	43.6991758
5		0.2529434	0.0801201	0.1130798	33.0364758	40.060050	7.02357420
1	0.75	0.7395276	0.2602349	0.5096593	40.8761102	130.11745	89.2413398
2		0.6866056	0.2920345	0.6016307	41.7856110	146.01725	104.231639
3		0.6093087	0.2379665	0.4508847	39.4591225	118.98325	79.5241275
4		0.4426921	0.1499827	0.2516100	35.9294221	74.991350	39.0619279
5		0.2576882	0.0753753	0.1114999	33.0064424	37.687650	4.68120760

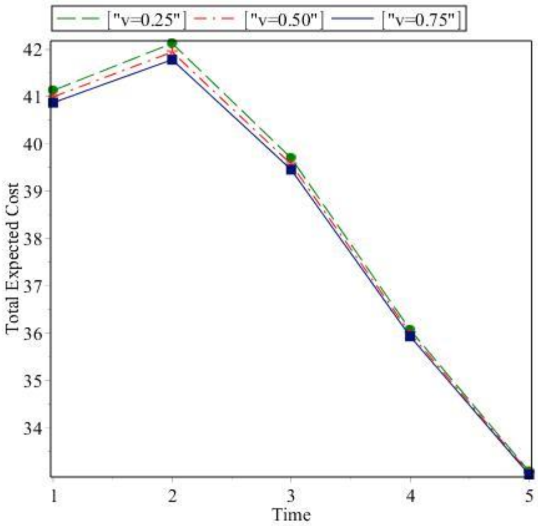

Figure 7

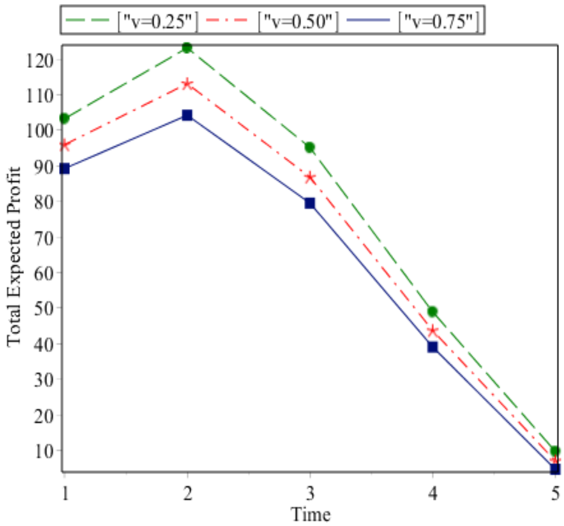

Figure 8

Figure 7:shows the variation of cost with time by varying correlation parameter while keeping the other parameters fixed**.** (As per Table 7)
Figure 8:shows the variation of profit with time by varying correlation parameter while keeping the other parameters fixed. (As per Table 7)

Impact of Feedback Probability (*q*): We see that the behaviour of the queuing system by evaluating the effectiveness measures along with cost and profit analysis by varying ***q*** with time ***t***. While keeping all other parameters fixed; λ=2, μ=5, w=2, v=0.25and ξ=0.0001. InTable 8, we observe that as the value of β increases with time ***t***, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 8: Effectiveness Measures, Cost and Profit versus *q*

t	*q*	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	0.55	0.668139	0.331623	0.575463	41.748312	165.8116	124.06333
2		0.581127	0.397513	0.708652	43.172260	198.7565	155.58423
3		0.519844	0.327430	0.526709	40.485765	163.7153	123.22958
4		0.385831	0.206843	0.290973	36.493637	103.4216	66.927962
5		0.206843	0.104185	0.128238	33.150081	52.09270	18.942618
1	0.65	0.697569	0.302192	0.541130	41.316696	151.0963	109.77960
2		0.628087	0.350552	0.647293	42.417794	175.2763	132.85855
3		0.560703	0.286572	0.481874	39.914839	143.2860	103.37121
4		0.411905	0.180769	0.267745	36.183136	90.38475	54.201614
5		0.242124	0.090939	0.118414	33.122277	45.46970	12.347422
1	0.75	0.723392	0.276370	0.514419	40.972114	138.1850	97.212930
2		0.665968	0.312671	0.605884	41.890055	156.3356	114.44554
3		0.592567	0.254707	0.452802	39.528526	127.3538	87.825323
4		0.432141	0.160533	0.252526	35.970240	80.26660	44.296359
5		0.252386	0.080677	0.111907	33.026419	40.33855	7.3121302

 DOI: http://dx.doi.org/10.18187/pjsor.v16i1.1111

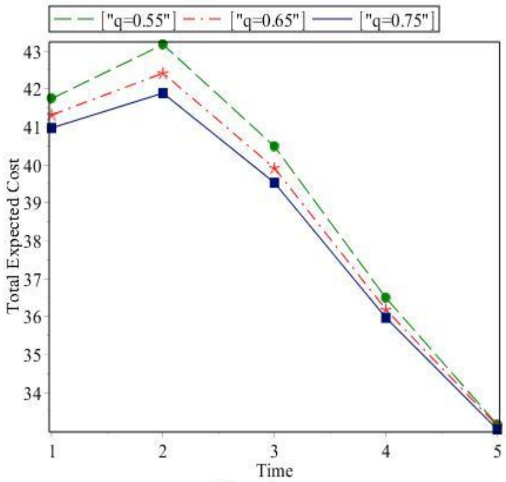

Figure 9

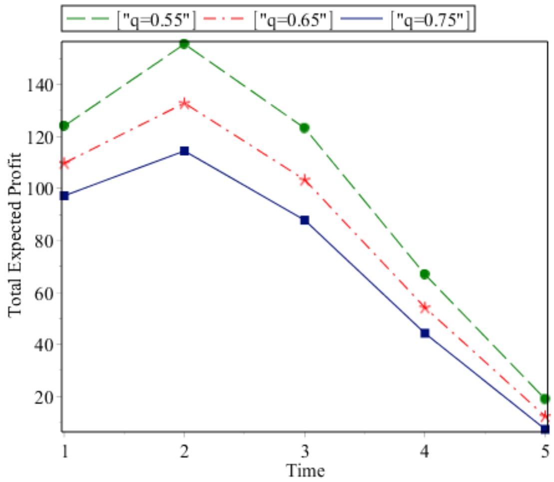

Figure 10

Figure 9: shows the variation of cost with time by varying feedback probability while keeping the other parameters fixed. (As per Table 8)

Figure 10: shows the variation of profit with time by varying feedback probability while keeping the other parameters fixed. (As per Table 8)

Impact of Catastrophes Rate (ξ): We see that the behaviour of the queuing system by evaluating the effectiveness measures along with cost and profit analysis by varying ξ with time$t$, while keeping all other parameters fixed; λ=2, μ=5, w=2, v=0.25 and q=0.7. InTable 9, we observe that as the value of ξ increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 9: Effectiveness Measures, Cost and Profit versus ξ

t	ξ	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	0.0001	0.710910	0.288852	0.526952	41.134896	144.4262	103.29135
2		0.648027	0.330613	0.624655	42.131597	165.3065	123.17495
3		0.577570	0.269704	0.465895	39.704442	134.8524	95.147957
4		0.422623	0.170051	0.259402	36.067557	85.02585	48.958292
5		0.247561	0.085502	0.114854	33.070370	42.75105	9.6806792
1	0.0002	0.710923	0.288839	0.526925	41.134589	144.4196	103.28501
2		0.648049	0.330594	0.624617	42.131181	165.2974	123.16621
3		0.577610	0.269702	0.465898	39.704651	134.8511	95.146498
4		0.422718	0.170084	0.259477	36.069039	85.04205	48.973010
5		0.247735	0.085572	0.114993	33.073192	42.78615	9.7129576
1	0.0003	0.710936	0.288825	0.526899	41.134280	144.4129	103.27861
2		0.648070	0.330576	0.624579	42.130764	165.2882	123.15748
3		0.576509	0.269699	0.465900	39.699153	134.8499	95.150746
4		0.422813	0.170116	0.259552	36.070521	85.05825	48.987728
5		0.247908	0.085642	0.115133	33.076011	42.82120	9.7451883

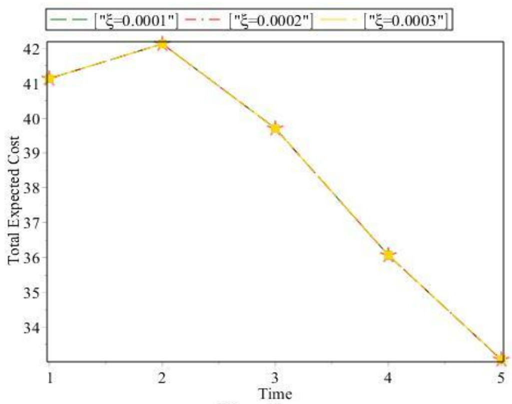

Figure 11

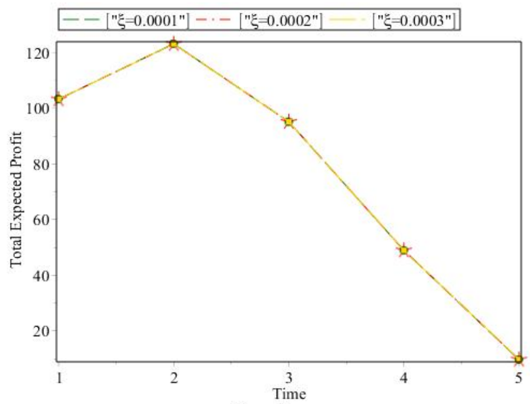

Figure 12

Figure 11:shows the variation of cost with time by varying catastrophes rate while keeping the other parameters fixed. (As per Table 9)
Figure 12:shows the variation of profit with time by varying catastrophes rate while keeping the other parameters fixed. (As per Table 9)

7.Discussion

Figure 1& Figure 2 show the variation of cost and profit with time t by varying λ(=1.00, 1.50, 2.00). The value of both cost and profit increases with increase in t upto t(=3.00) when λ=1.00 and t(=2.00) when λ=1.50 and 2.00 respectively then decreases. Hence we get the optimal value at t=5 when λ=2.00 and t=2 when λ=2.00 for minimum cost and maximum profit respectively.
Figure 3& Figure 4 show the variation of cost and profit with time t by varying μ(=2.75, 3.75, 4.75). The value of both cost and profit increases with increase in t upto t(=2.00) when μ=2.75, 3.75 and 4.75 respectively then decreases. Hence we get the optimal value at t=5 when μ=2.75 and t=2 when μ=4.75 for minimum cost and maximum profit respectively.
Figure 5& Figure 6 show the variation of cost and profit with time t by varying w(=1.50, 1.75, 2.00). The value of both cost and profit increases with increase in t upto t(=2.00) when w=1.50, 1.75 and 2.00 respectively then decreases. Hence we get the optimal value at t=5 when w=2.00 and t=2 when w=2.00 for minimum cost and maximum profit respectively.
Figure 7& Figure 8 show the variation of cost and profit with time t by varying ν(=0.25, 0.50, 0.75). The value of both cost and profit increases with increase in t upto t(=2.00) when ν=0.25, 0.50 and 0.75 respectively then decreases. Hence we get the optimal value at t=5 when ν=0.75 and t=2 when ν=0.25 for minimum cost and maximum profit respectively.
Figure 9& Figure 10 show the variation of cost and profit with time t by varying q(=0.55, 0.65, 0.75). The value of both cost and profit increases with increase in t upto t(=2.00) when q=0.55, 0.65 and 0.75 respectively then decreases. Hence we get the optimal value at t=5 when q=0.75 and t=2 when q=0.55 for minimum cost and maximum profit respectively.
Figure 11 & Figure 12 show the variation of cost and profit with time t by varying ξ(=0.0001, 0.0002, 0.0003). The value of both cost and profit increases with increase in t upto t(=2.00) when ξ=0.0001, 0.0002 and 0.0003 then decreases. Hence we get the optimal value at t=5 when ξ=0.0001 and t=2 when ξ=0.0001 for minimum cost and maximum profit respectively. Finally, the variation in rate of catastrophes shows the minor effect on cost and profit.

Conclusions

The time-dependent solution, for the two-dimensional state M/M/2 queuing model with multiple vacation, correlated servers, feedback and catastrophes has been obtained. The numerical analysis clearly demonstrates the meaningful impact of arrival rate, service rate, vacations rate, correlation parameters, feedback probability and catastrophes rate on the system performances. Finally, the model estimates the total expected cost and total expected profit and obtained the best optimal value at t=5 when service rate=2.75 and t=2 when feedback probability=0.55 for minimum cost and maximum profit respectively. These key measures give a greater understanding of model behaviour. Finally, the numerical analysis clearly demonstrates the meaningful impact of the correlated servers and multiple vacation on the system performances. This model finds its applications in communication networks, computer networks, supermarkets, hospital administrations, financial sector and many others.

Acknowledgement:

One of the authors (Sharvan Kumar) acknowledges the Kurukshetra University for providing the University research scholarship which made this research possible.

References

1. Ammar, S. I. (2015). Transient analysis of an M/M/1 queue with impatient behavior and multiple vacations. Applied Mathematics and Computation, 260, 97-105.
2. Ammar, S. I., Samanta, S. K., Kilany, N. M., and Jiang, T. (2022). Analysis of Stationary Fluid Queue Driven by State-Dependent Birth-Death Process Subject to Catastrophes. Scientia Iranica.
3. Cooper, R. B. (1970). Queues served in cyclic order waiting times. Bell System Technical Journal, 49(3), 399-413.
4. D'Avignon, G., and Disney, R. (1976). Single-server queues with state-dependent feedback. INFOR: Information Systems and Operational Research, 14(1), 71-85.

5. Finch, P. D. (1959). Cyclic queues with feedback. Journal of the Royal Statistical Society Series B: Statistical Methodology, 21, 153-157
6. Gahlawat, V. K., Indra, Rahul, K., Mor, R. S., and Malik, M. (2021). A two state time-dependent bulk queue model with intermittently available server. Reliability: Theory and Applications, 16,114-122.
7. Giorno, V., Nobile, A. G., and Spina, S. (2014). On some time non-homogeneous queueing systems with catastrophes. Applied Mathematics and Computation, 245, 220-234.
8. Indra and Sharda (2004). A batch arrival two-state M/M/1 queueing system with latest arrival run (LAR) having maximum effective length one. International journal of information and management sciences, 15, 71-80.
9. Indra and Sweety, B. (2010). The transient solution of an unreliable M/G/1 queue with vacations. International journal of information and management sciences, 391-406.
10. Ke, J. C., and Pearn, W. L. (2004). Optimal management policy for heterogeneous arrival queueing systems with server breakdowns and vacations. Quality Technology& Quantitative Management, 1, 149-162.
11. Ke, J. C., Lee, S. L., and Liou, C. H. (2009). Machine repair problem in production systems with spares and server vacations. RAIRO-Operations Research, 43, 35-54.
12. Kumar, B. K., andArivudainambi, D. (2000). Transient solution of an M/M/1 queue with catastrophes. Computers & Mathematics with applications, 40, 1233-1240.
13. Kumar, B. K., and Madheswari, S. P. (2005). Transient analysis of an M/M/1 queue subject to catastrophes and server failures. Stochastic Analysis and Applications, 23, 329-340.
14. Kumar, R. (2017). A transient solution to the M/M/c queuing model equation with balking and catastrophes. Croatian Operational Research Review, 8, 577-591.
15. Kumar, S., and Indra (2023). Cost & profit analysis of two-dimensional state M/M/2 queuing model with correlated servers, multiple vacation, balking and catastrophes. Reliability: Theory and Applications, 18, 893-908.
16. Marshall, A. W., and Olkin, I. (1967). A multivariate exponential distribution. Journal of the American Statistical Association, 62, 30-44.
17. Niranjan, S. P., Chandrasekaran, V. M., and Indhira, K. (2019). Optimum cost analysis of batch service retrial queuing system with server failure, threshold and multiple vacations. Songklanakarin Journal of Science & Technology, 41, 363-374.
18. Nishida, T., Watanabe, R. andTahara, A. (1974). Poisson queue with correlated two servers. Technology Reports of Osaka University, 24, 1193.
19. Panta, A. P., Ghimire, R. P., Panthi, D., and Pant, S. R. (2021). A Review of Vacation Queueing Models in Different Framework. Annals of Pure and Applied Mathematics, 24, 99-121.
20. Pegden, C. D. andRosenshine, M. (1982). Some new results for the M/M/1 queue. Management Science, 28, 821-828.
21. Sampath, M. S. (2020). Transient analysis of an M/M/1/N queue with balking, catastrophes, server failures and repairs. 23rd International Conference, Distributed Computer and Communication Networks. 441-448.
22. Sharda and Indra (1996). Utilization of an intermittently available time and vacation time of a two-state queueing model. International Journal of Information and Management. Sciences, 7, 13-23.
23. Sharda and Indra (1999). Utilizing the intermittently available time and the vacation time. International journal of information and management sciences, 10, 1-11.
24. Sharda, Garg, P. C., and Garg, I. (1986). An M/M/1∞ queuing system with feedback.Microelectronics Reliability, 26, 261-264.
25. Sharma, O. P., and Maheswar, M. V. R. (1994). Analysis of M| M| 2| N queue with correlated servers. Optimization, 30, 379-385.
26. Sharma, O.P. (1990). Markovian queues. Elis Horwood, ltd, England.
27. Sharma, R., and Indra (2022). Transient analysis for a multiple vacations queueing model with impatient customers. Journal of Scientific Research, 66, 393-403.
28. Takacs, L. (1963). A single-server queue with feedback. Bell System Technical Journal, 42, 505-519.
29. Tarabia, A. M. (2011). Transient and steady-state analysis of an M/M/1 queue with balking, catastrophes, server failures and repairs. Journal of Industrial and Management optimization, 7, 811-823.
30. Tian, N., Li, Q. L., and Gao, J. (1999). Conditional stochastic decompositions in the M/M/c queue with server vacations. Stochastic Models, 15, 367-377.