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On The Efficiency of Receiving a Stepwise Gaussian Random Disturbance With an Unknown Moment of Appearance and Central Frequency

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Abstract

The synthesis of the computationally simple maximum likelihood algorithms for detecting and measuring the moment of appearance and the central frequency of a fast-fluctuating Gaussian random disturbance is carried out. Using the method of multiplicative and additive local Markov approximation of the decision-determining statistics or its increment, the closed analytical expressions are found for the false alarm and missing probabilities (the detection task), as well as for the conditional biases and variances of the desired estimates (the measurement task). By statistical simulation methods, it is established that the proposed detector and measurer are operable, and the analytical formulas describing their performance are in good agreement with the corresponding experimental data in a wide range of parameter values of the random process being analyzed.

Key Words: Gaussian random disturbance, unknown moment of appearance and band center, maximum likelihood method, adaptive approach, detection, estimated parameter, local Markov approximation method, false alarm and missing probabilities, bias and variance of estimate, statistical simulation.

1. Introduction

In a number of applications of control theory, technical and medical diagnostics, pattern recognition, measurement data processing, etc., it is necessary to solve the problem of statistical analysis of abrupt random disturbances of diagnosed processes and systems (by Basseville and Benveniste. (1986)), (by Golpaiegany et al., (2019)). In practice, the objects being diagnosed can be of a stochastic nature or be subject to external random influences. An abrupt disturbance of such objects is often a random process that uniformly occupies the entire operating frequency band and occurs at some a priori unknown point in time (by Trifonov et al., (1991)), (by Zakharov et al., (2001)), (by Chernoyarov. et al., (2015)).

In (by Chernoyarov. et al., (2025)), a technique for detecting a fast-fluctuating bandpass Gaussian random disturbance and measuring its unknown moment of appearance is considered. It is shown that the use of the proposed approach makes it possible to obtain effective algorithms for processing random disturbances under conditions of parametric a priori uncertainty. Below, based on the results obtained in (by Chernoyarov. et al., (2025)), the structure and characteristics of the detector and measurer of a stepwise Gaussian random disturbance with the unknown moment of appearance and central frequency are found, assuming the rapidity of its fluctuations and the relative uniformity of its spectral density in a specified frequency band. It is shown that the synthesized detector and measurer are computationally significantly simpler compared to the commonly used analogues. Using statistical simulation methods, the operability and sufficiently high efficiency of the considered processing algorithms are established.

2. The Problem Statement

Under a stepwise random disturbance, we understand the multiplicative combination of the form

$$s(t)=\xi(t)\theta(t-\lambda_0), \qquad \theta(x)=\begin{cases}1, & x\geq 0,\\ 0, & x<0.\end{cases} \tag{1}$$

Here λ_0 is the moment of appearance (beginning) of disturbance, and $\xi(t)$ is the stationary centered Gaussian random process with the spectral density

$$G_\xi(\omega)=\frac{d}{2}\left[I\left(\frac{\vartheta_0-\omega}{\Omega}\right)+I\left(\frac{\vartheta_0+\omega}{\Omega}\right)\right], \qquad I(x)=\begin{cases}1, & |x|\leq 1/2,\\ 0, & |x|>1/2.\end{cases} \tag{2}$$

In (2), the notations are: ϑ_0 is the central frequency, Ω is the bandwidth, and d is the intensity of the process $\xi(t)$ which also determines its dispersion (i.e., the average power) $D=\Omega d/2\pi$. This type of spectral density shape approximation can be used if the real spectral density decreases rapidly outside the bandwidth Ω. Indeed, it is well known (by Engelson. (1971)) that the resolution capability of any spectrum analyzer is of the order of the value of $2\pi/\tau_0$, where $\tau_0=T-\lambda_0$ is the analyzed signal duration. We denote the bandwidth within which the real spectral density decreases from its maximum value to almost zero as $\Delta\Omega$. Then the conditions of applicability of the approximation (2) can be written as $\Delta\Omega << (2\pi/\tau_0) << \Omega$.

The practically important case arises when the duration of the disturbance (1) significantly exceeds the correlation time $2\pi/\Omega$ of the process $\xi(t)$ (i.e., the fluctuations of the process $\xi(t)$ are "fast"), thus meaning that the following condition is satisfied:

$$\tau_0 >> 2\pi/\Omega \quad (\mu_0=\tau_0\Omega/2\pi >> 1). \tag{3}$$

It is presupposed that interferences and registration errors can be described by Gaussian white noise $n(t)$ with the one-sided spectral density N_0. As a result, over the observation interval $t\in[0,T]$, and in the presence of the random disturbance (1), an additive mixture of the form

$$x(t)=s(t)+n(t), \tag{4}$$

is received, while in its absence one gets

$$x(t)=n(t). \tag{5}$$

Based on the observations (4) and (5), it is necessary to synthesize algorithms that are optimal in one sense or another for detecting and estimating the moment of appearance and the central frequency of a random disturbance (1), and to determine their characteristics. It is assumed that the parameters λ_0 and ϑ_0 can take values from a priori intervals $[\Lambda_1,\Lambda_2]$, $[\Theta_1,\Theta_2]$, and $0\leq\Lambda_1<\Lambda_2<T$.

3. Detecting a Stepwise Random Disturbance with the Unknown Moment of Appearance and Center Frequency

In order to detect a random disturbance (1), the adaptive (maximum likelihood) approach is used (by Helstrom. (2013)), (by Van Trees. et al., (2013)), (by Trifonov. et al., (1991)), (by Trifonov. et al., (1986)). For this purpose, after applying the results of (by Chernoyarov. et al., (2025)), the expression for the decision-determining statistics (the logarithm of the functional of the likelihood ratio (FLR)), when receiving the realization (4) in view of the alternative (5), can be written in the form:

$$L(\lambda,\vartheta)=\frac{q}{N_0(1+q)}\int_{\lambda}^{T} y^2(t,\vartheta)\,\mathrm{d}t-\frac{\Omega(T-\lambda)}{2\pi}\ln(1+q) \quad . \quad (6)$$

Here

$$q=d/N_0 \quad , \quad (7)$$

λ and ϑ are the current values of the unknown parameters λ_0 and ϑ_0, $y(t,\vartheta)=\int_{-\infty}^{\infty}x(t')h(t-t',\vartheta)\,\mathrm{d}t'$, and $h(t)$ is the function belonging to the spectrum $H(\omega,\vartheta)=\int_{-\infty}^{\infty}h(t,\vartheta)\exp(-i\omega t)\,\mathrm{d}t$ which satisfies the condition $|H(\omega,\vartheta)|^2=I[(\omega-\vartheta)/\Omega]+I[(\omega+\vartheta)/\Omega]$.

Then a decision in favor of the hypothesis (4) or (5) is made based on a comparison of the value of

$$L_m=\max_{\lambda\in[\Lambda_1,\Lambda_2],\,\vartheta\in[\Theta_1,\Theta_2]} L(\lambda,\vartheta) \quad (8)$$

with the threshold c selected in accordance with the accepted optimality criterion. When the threshold c is exceeded by the value of (8), it is assumed that the random disturbance (1) is present in the observed data, otherwise it is decided that such disturbance is absent.

To quantitatively describe the detection efficiency, we use the false alarm (type I error) probability notated as α and the missing (type II error) probability notated as β (by Helstrom. (2013)), (by Van Trees. et al., (2013)), (by Trifonov. et al., (1991)), (by Trifonov. et al., (1986)).

To find the false alarm probability, the decision-determining statistics (6) is presented in the form

$$L(\lambda,\vartheta)=L(l,\eta)=\frac{qT}{N_0(1+q)}\int_{l}^{1} y^2(T\tilde{t},\Omega\eta)\,\mathrm{d}\tilde{t}-\mu(1-l)\ln(1+q) \quad , \quad (9)$$

where

$$l=\lambda/T\,,\quad \eta=\vartheta/\Omega\,,\quad \tilde{t}=t/T\,,\quad \mu=\Omega T/2\pi \quad (10)$$

are dimensionless variables. Further, the false alarm probability is written as follows:

$$\alpha=P\Big[\max_{l\in[\tilde{\Lambda}_1,\tilde{\Lambda}_2],\,\vartheta\in[\tilde{\Theta}_1,\tilde{\Theta}_2]} L(l,\vartheta)>c\,\big|\,x(t)=n(t)\Big]\,,\quad \tilde{\Lambda}_{1,2}=\Lambda_{1,2}/T\,,\quad \tilde{\Theta}_{1,2}=\Theta_{1,2}/\Omega\,. \quad (11)$$

As in (by Trifonov. et al., (1991)), it can be shown that the functional (9) is asymptotically (at $\mu\to\infty$) Gaussian random field. Then, when the condition (3) is satisfied, its complete statistical description is given by the first two moments, namely, the mathematical expectation or signal function $S(l,\eta)=\langle L(l,\eta)\rangle$ and the correlation function $B(l_1,l_2,\eta_1,\eta_2)=\langle N(l_1,\eta_1)N(l_2,\eta_2)\rangle$ of the noise function $N(l,\eta)=L(l,\eta)-\langle L(l,\eta)\rangle$ (by Chernoyarov. et al., (2015)), (by Trifonov. et al., (1986)), (by Chernoyarov. et al., (2014)). Here and below $\langle\ \rangle$ mean the operation of averaging over all possible observations $x(t)$ (and, if it is required, at the fixed values of λ_0 and ϑ_0 also).

From (9), directly performing averaging operations, taking into account the inequality (3) and neglecting the values of the order μ^{-1} or less, for the signal function $S(l,\eta)$ and the correlation function $B(l_1,l_2,\eta_1,\eta_2)$ of the logarithm of FLR, in the absence of random disturbance (1) in the realization of the observed data, one gets

$$S(l)=\mu(1-l)\left[q/(1+q)-\ln(1+q)\right], \quad (12)$$

$$B(l_1,l_2,\eta_1,\eta_2)=\mu q^2\left[1-\max(l_1,l_2)\right]\max\left(0,1-|\eta_2-\eta_1|\right)/(1+q)^2,$$

$$l,l_1,l_2\in\left[\widetilde{\Lambda}_1,\widetilde{\Lambda}_2\right],\ \eta,\eta_1,\eta_2\in\left[\widetilde{\Theta}_1,\widetilde{\Theta}_2\right].$$

Now it is time to pass to the new variable in (9), (12):

$$\widetilde{l}=1-l,\quad \widetilde{l}\in\left[1-\widetilde{\Lambda}_2,1-\widetilde{\Lambda}_1\right]. \quad (13)$$

In that case,

$$S(\widetilde{l})=\mu\widetilde{l}\left[q/(1+q)-\ln(1+q)\right],\quad B(\widetilde{l}_1,\widetilde{l}_2,\eta_1,\eta_2)=\mu q^2\min(\widetilde{l}_1,\widetilde{l}_2)\max\left(0,1-|\eta_2-\eta_1|\right)/(1+q)^2. \quad (14)$$

According to (12) and (14), the Gaussian random field $L(\widetilde{l},\eta)$ (9) in the statistical sense is equivalent to the product

$$U(\widetilde{l})V(\eta),\quad l\in\left[\widetilde{\Lambda}_1,\widetilde{\Lambda}_2\right],\quad \eta\in\left[\widetilde{\Theta}_1,\widetilde{\Theta}_2\right] \quad (15)$$

of the two statistically independent Gaussian random processes $U(\widetilde{l})$ and $V(\eta)$ with the mathematical expectations and correlation functions of the form

$$S_U(\widetilde{l})=\langle U(\widetilde{l})\rangle=\mu\widetilde{l}\left[q/(1+q)-\ln(1+q)\right], \quad (16)$$

$$B_U(\widetilde{l}_1,\widetilde{l}_2)=\left\langle\left[U(\widetilde{l}_1)-\langle U(\widetilde{l}_1)\rangle\right]\left[U(\widetilde{l}_2)-\langle U(\widetilde{l}_2)\rangle\right]\right\rangle=\mu q^2\min(\widetilde{l}_1,\widetilde{l}_2)/(1+q)^2,$$

$$S_V(\eta)=\langle V(\eta)\rangle=1,\quad B_V(\eta_1,\eta_2)=\left\langle\left[V(\eta_1)-\langle V(\eta_1)\rangle\right]\left[V(\eta_2)-\langle V(\eta_2)\rangle\right]\right\rangle=\max\left(0,1-|\eta_2-\eta_1|\right). \quad (17)$$

Therefore, the probability (11) can be represented as follows:

$$\alpha=P\left[\max_{\widetilde{l}\in\left[1-\widetilde{\Lambda}_2,1-\widetilde{\Lambda}_1\right]}U(\widetilde{l})\max_{\eta\in\left[\widetilde{\Theta}_1,\widetilde{\Theta}_2\right]}V(\eta)>c\right].$$

The formula for the distribution function of a product of independent random variables is given, for example, in (by Rohatgi. (1976)) and (by Grami. (2019)). Let

$$F_U(x)=P\left[\max_{\widetilde{l}\in\left[1-\widetilde{\Lambda}_2,1-\widetilde{\Lambda}_1\right]}U(\widetilde{l})<x\right],\quad F_V(y)=P\left[\max_{\eta\in\left[\widetilde{\Theta}_1,\widetilde{\Theta}_2\right]}V(\eta)<y\right] \quad (18)$$

are the distribution functions of the largest maxima of random processes $U(\widetilde{l})$ and $V(\eta)$ on the intervals $\left[1-\widetilde{\Lambda}_2,1-\widetilde{\Lambda}_1\right]$ and $\left[\widetilde{\Theta}_1,\widetilde{\Theta}_2\right]$, respectively. Then, according to (by Rohatgi. (1976)) and (Grami. (2019)), for the probability α (11), it can be written

$$\alpha=1-\int_0^{\infty}F_V(c/x)\,\mathrm{d}F_U(x)-\int_{-\infty}^{0}\left[1-F_V(c/x)\right]\mathrm{d}F_U(x). \quad (19)$$

From (16), it follows that the process $U(\widetilde{l})$ (15) can be approximately represented by a Markov random process (by Dynkin. (2006)) with the drift coefficient

$$K_1 = -\left[\left.\frac{\mathrm{d}S_U(\widetilde{l}_1)}{\mathrm{d}\widetilde{l}_1}\right|_{\widetilde{l}_1=\widetilde{l}+0} + \frac{y-S_U(\widetilde{l})}{B_U(\widetilde{l},\widetilde{l})}\left.\frac{\partial B_U(\widetilde{l},\widetilde{l}_1)}{\partial \widetilde{l}_1}\right|_{\widetilde{l}_1=\widetilde{l}+0'}\right] = \mu\left[\ln(1+q) - \frac{q}{1+q}\right] \tag{20}$$

and the diffusion coefficient

$$K_2 = \left.\frac{\mathrm{d}B_U(\widetilde{l}_1,\widetilde{l}_1)}{\mathrm{d}\widetilde{l}_1}\right|_{\widetilde{l}_1=\widetilde{l}+0} - 2\left.\frac{\partial B_U(\widetilde{l},\widetilde{l}_1)}{\partial \widetilde{l}_1}\right|_{\widetilde{l}_1=\widetilde{l}+0} = \frac{\mu q^2}{(1+q)^2}.$$

The "minus" sign in front of the general expression in (20) means taking into account the fact that, for the convenience of mathematical calculations, the interval of change of the variable $\widetilde{l}$ in (13) is chosen to be the mirrored one instead of the real interval $\left[1-\widetilde{\Lambda}_1, 1-\widetilde{\Lambda}_2\right]$.

The distribution function of the largest maximum of a Gaussian Markov random process with the constant drift and diffusion coefficients is found in (by Chernoyarov. et al., (2017)). Using the results of (by Chernoyarov. et al., (2017)), for the probability $F_U(x)$ (18), one obtains

$$F_U(x) = \frac{1}{\sigma_1\sqrt{2\pi}}\int_0^\infty \exp\left[-\frac{(u-x+M_1)^2}{2\sigma_1^2}\right]\times$$
$$\times\left\{\Phi\left(\frac{u-K_1m_\lambda}{\sqrt{K_2m_\lambda}}\right) - \exp\left(\frac{2K_1u}{K_2}\right)\left[1-\Phi\left(\frac{u+K_1m_\lambda}{\sqrt{K_2m_\lambda}}\right)\right]\right\}\mathrm{d}u. \tag{21}$$

Here

$M_1 = S_U\left(1-\widetilde{\Lambda}_2\right) = \mu\left(1-\widetilde{\Lambda}_2\right)\left[q/(1+q) - \ln(1+q)\right]$, $\sigma_1^2 = B_U\left(1-\widetilde{\Lambda}_2, 1-\widetilde{\Lambda}_2\right) = \mu q^2\left(1-\widetilde{\Lambda}_2\right)/(1+q)^2$,

$m_\lambda = 1-\widetilde{\Lambda}_1 - \left(1-\widetilde{\Lambda}_2\right) = \widetilde{\Lambda}_2 - \widetilde{\Lambda}_1$ is the reduced length of the a priori interval of possible values of the parameter λ_0, and

$$\Phi(x) = \int_{-\infty}^{x}\exp\left(-t^2/2\right)\mathrm{d}t/\sqrt{2\pi} \tag{22}$$

is the probability integral (by Abramowitz. et al., (1964)). Thus, one gets

$$\frac{\mathrm{d}F_U(x)}{\mathrm{d}x} = \frac{1}{\sigma_1^3\sqrt{2\pi}}\int_0^\infty (u-x+M_1)\exp\left[-\frac{(u-x+M_1)^2}{2\sigma_1^2}\right]\times$$
$$\times\left\{\Phi\left(\frac{u-K_1m_\lambda}{\sqrt{K_2m_\lambda}}\right) - \exp\left(\frac{2K_1u}{K_2}\right)\left[1-\Phi\left(\frac{u+K_1m_\lambda}{\sqrt{K_2m_\lambda}}\right)\right]\right\}\mathrm{d}u. \tag{23}$$

The accuracy of the formulas (21) and (23) increases with x and μ increasing and m_λ decreasing.

The distribution function of the largest maximum of a Gaussian random process with a nonzero constant mathematical expectation and triangular correlation function (17) is obtained, for example, in (18) and (19). Using the results of (by Shepp. (1966)) and (by Siegmund. (1986)), for the probability $F_V(y)$ (18), one gets

$$F_V(x)=\begin{cases}\exp\left[-\dfrac{m_\text{v}(x-1)}{\sqrt{2\pi}}\exp\left(-\dfrac{(x-1)^2}{2}\right)\right], & x\geq 2,\\ 0, & x<2,\end{cases} \tag{24}$$

where $m_\text{v}=\widetilde{\Theta}_2-\widetilde{\Theta}_1$. The accuracy of the formula (24) increases together with m_v and x.

Substituting (23) and (24) into (19), the false alarm probability can be calculated for a specified threshold c. And, taking into account the relation $F_V(x)=0$ at $x\leq 0$, the formula (19) can be somewhat simplified by transforming it into the following:

$$\alpha=1-\int_0^\infty F_V(c/x)\mathrm{d}F_U(x)-F_U(0) \quad . \tag{25}$$

Now it is time to find the probability β of missing the moment of appearance of a random disturbance. By definition, according to (by Helstrom. (2013)), (by Van Trees. et al., (2013)), (by Trifonov. et al., (1991)), (by Trifonov. et al., (1986)),

$$\beta=P[\max_{l\in[\widetilde{\Lambda}_1,\widetilde{\Lambda}_2],\,\vartheta\in[\widetilde{\Theta}_1,\widetilde{\Theta}_2]} L(l,\vartheta)<c\,|\,x(t)=s(t)+n(t)] \quad , \tag{26}$$

where $L(l,\vartheta)$ is defined in the same way as in (9), at the observations (4), while $\widetilde{\Lambda}_i$, $\widetilde{\Theta}_i$ coincide with those specified in (11).

Following (by Trifonov. et al., (1991)), it can be shown that in the presence of a random disturbance (1) in the realization of the observed data, the functional $L(l,\eta)$ (9) is asymptotically Gaussian one at $\mu_0\to\infty$. Then, when the condition (3) is satisfied, for its complete statistical description, it is enough to specify the first two moments that are the signal function $S(l,\eta)=\langle L(l,\eta)\rangle$ and the correlation function $B(l_1,l_2,\eta_1,\eta_2)=\langle N(l_1,\eta_1)N(l_2,\eta_2)\rangle$ of the noise function $N(l,\eta)=L(l,\eta)-\langle L(l,\eta)\rangle$ (by Chernoyarov. et al., (2015)), (by Trifonov. et al., (1986)), (by Chernoyarov. et al., (2014)), so that

$$L(l,\eta)=S(l,\eta)+N(l,\eta) \quad . \tag{27}$$

Performing averaging in (9) over all possible realizations (4) at the fixed values of the parameters λ_0 and ϑ_0, and neglecting the terms of the infinitesimal order of μ^{-1} and less, one obtains

$$S(l,\eta)=\mu\left[q^2\max(0,1-|\eta-\eta_0|)(1-\max(l_0,l))/(1+q)+(1-l)(q/(1+q)-\ln(1+q))\right], \tag{28}$$

$$B(l_1,l_2,\eta_1,\eta_2)=\mu q^2\left[(1-\max(l_1,l_2))\max(0,1-|\eta_2-\eta_1|)+q(2+q)(1-\max(l_0,l_1,l_2))\times\right.$$
$$\left.\times\max(0,1+\min(0,\eta_1-\eta_0,\eta_2-\eta_0)-\max(0,\eta_1-\eta_0,\eta_2-\eta_0))\right]/(1+q)^2,$$

where

$$l_0=\lambda_0/T\,, \qquad \eta_0=\vartheta_0/\Omega\,. \tag{29}$$

It is presupposed that, when processing the realization (4), the condition of high a posteriori accuracy is satisfied (by Trifonov. et al., (1986)), i.e., the output power signal-to-noise ratio (SNR) z^2 for the algorithm (8) is quite large:

$$z^2 = S^2(l_0,\eta_0) \Big/ \left\langle N^2(l_0,\eta_0) \right\rangle = \mu(1-l_0)\left[1-\ln(1+q)/q\right]^2 >> 1, \qquad (30)$$

so that the position of the maximum of the functional $L(l,\eta)$ (9) is in a small δ-neighborhood of the point (l_0,η_0) with a probability tending to 1. With an unlimited increase in the SNR z^2 ($z^2 \to \infty$), the value of the indicated neighborhood is

$$\delta = \max\left(|l-l_0|,|\eta-\eta_0|\right) \to 0. \qquad (31)$$

Thus, if the condition (30) is satisfied, then $\delta << 1$, and the following asymptotic representations are valid for the signal function and the correlation function of the noise function (28):

$$S(l,\eta) = S(\breve{l},\breve{\eta}) = \mu\left[(1-l_0)(q-\ln(1+q)) - \frac{q^2}{1+q}\max(0,\breve{l}) - \right.$$
$$\left. -\breve{l}\left(\frac{q}{1+q} - \ln(1+q)\right) - \frac{q^2(1-l_0)}{1+q}|\breve{\eta}|\right] + o(\delta), \qquad (32)$$

$$B(l_1,l_2,\eta_1,\eta_2) = B(\breve{l}_1,\breve{l}_2,\breve{\eta}_1,\breve{\eta}_2) = \frac{\mu q^2}{(1+q)^2}\left[(1-l_0)(1+q)^2 - q(2+q)\max(0,\breve{l}_1,\breve{l}_2) - \right.$$
$$\left. - \max(\breve{l}_1,\breve{l}_2) - q(2+q)(\max(0,\breve{\eta}_1,\breve{\eta}_2) - \min(0,\breve{\eta}_1,\breve{\eta}_2)) - (1-l_0)|\breve{\eta}_2 - \breve{\eta}_1|\right] + o(\delta),$$

where $\breve{l} = l - l_0$, $\breve{\eta} = \eta - \eta_0$, and $o(\delta)$ includes the higher-order infinitesimal terms compared with δ.

At this stage, one can pass from the decision-determining statistics (9) to the decision-determining statistics of the form

$$\upsilon(\breve{l},\breve{\eta}) = \left[L(\breve{l},\breve{\eta}) - L_0\right]/\sigma, \qquad (33)$$

where $L_0 = L(l_0,\eta_0)$ is the asymptotically Gaussian random variable (in the case of an increasing μ_0 (3)) with the characteristics $\langle L_0\rangle = \mu(q-\ln(1+q))(1-l_0)$, $\left\langle (L_0 - \langle L_0\rangle)^2\right\rangle = \sigma^2$ and $\sigma = q\sqrt{\mu(1-l_0)}$.

Using the representation (33), we can rewrite the missing probability (26) when receiving a random disturbance (1) with the unknown moment of appearance and central frequency as

$$\beta = P\left[\max_{\breve{l}\in[\tilde{\Lambda}_1-l_0,\tilde{\Lambda}_2-l_0],\,\breve{\eta}\in[\tilde{\Upsilon}_1-\eta_0,\tilde{\Upsilon}_2-\eta_0]} \upsilon(\breve{l},\breve{\eta}) < (c-L_0)/\sigma \,|\, x(t) = s(t)+n(t)\right]. \qquad (34)$$

From (32), when the conditions (30) and (31) are satisfied, for the mathematical expectation and the correlation function of the functional $\upsilon(\breve{l},\breve{\eta})$ (33), one gets

$$\left\langle \upsilon(\breve{l},\breve{\eta})\right\rangle = S_{\upsilon 1}(\breve{l}) + S_{\upsilon 2}(\breve{\eta}) + o(\delta), \qquad (35)$$

$$\left\langle\left[\upsilon(\breve{l}_1,\breve{\eta}_1) - \left\langle \upsilon(\breve{l}_1,\breve{\eta}_1)\right\rangle\right]\left[\upsilon(\breve{l}_2,\breve{\eta}_2) - \left\langle \upsilon(\breve{l}_2,\breve{\eta}_2)\right\rangle\right]\right\rangle = B_{\upsilon 1}(\breve{l}_1,\breve{l}_2) + B_{\upsilon 2}(\breve{\eta}_1,\breve{\eta}_2) + o(\delta),$$

where

$$S_{\upsilon 1}(\breve{l}) = \sqrt{\frac{\mu}{1-l_0}} \begin{cases} A_1\breve{l}, & \breve{l} < 0, \\ -B_1\breve{l}, & \breve{l} \ge 0, \end{cases} \qquad S_{\upsilon 2}(\breve{\eta}) = \sqrt{\mu(1-l_0)}\, C_1|\breve{\eta}|, \quad (36)$$

$$B_{\upsilon 1}(\breve{l}_1, \breve{l}_2) = \frac{1}{1-l_0} \begin{cases} A_2 \min\left(|\breve{l}_1|, |\breve{l}_2|\right), & \breve{l}_1\breve{l}_2 \ge 0, \\ 0, & \breve{l}_1\breve{l}_2 < 0, \end{cases} \qquad B_{\upsilon 2}(\breve{\eta}_1, \breve{\eta}_2) = \begin{cases} B_2 \min\left(|\breve{\eta}_1|, |\breve{\eta}_2|\right), & \breve{\eta}_1\breve{\eta}_2 \ge 0, \\ 0, & \breve{\eta}_1\breve{\eta}_2 < 0, \end{cases} \quad (37)$$

$$A_1 = \ln(1+q)/q - 1/(1+q), \quad B_1 = 1 - \ln(1+q)/q, \quad C_1 = q/(1+q),$$

$$A_2 = \begin{cases} A_{21}, & \breve{l}_1 < 0, \breve{l}_2 < 0, \\ A_{22}, & \breve{l}_1 \ge 0, \breve{l}_2 \ge 0, \end{cases} \quad A_{21} = \frac{1}{(1+q)^2}, \quad A_{22} = 1, \quad B_2 = \frac{1+(1+q)^2}{(1+q)^2}.$$

Let $\iota_1(\breve{l})$, $\iota_2(\breve{\eta})$ are statistically independent Gaussian random processes with the mathematical expectations (36)

$$\langle \iota_1(\breve{l}) \rangle = S_{\upsilon 1}(\breve{l}), \qquad \langle \iota_2(\breve{\eta}) \rangle = S_{\upsilon 2}(\breve{\eta}) \quad (38)$$

and the correlation functions (37)

$$\langle \iota_1(\breve{l}_1)\iota_1(\breve{l}_2) \rangle - \langle \iota_1(\breve{l}_1) \rangle \langle \iota_1(\breve{l}_2) \rangle = B_{\upsilon 1}(\breve{l}_1, \breve{l}_2), \qquad \langle \iota_2(\breve{\eta}_1)\iota_2(\breve{\eta}_2) \rangle - \langle \iota_2(\breve{\eta}_1) \rangle \langle \iota_2(\breve{\eta}_2) \rangle = B_{\upsilon 2}(\breve{\eta}_1, \breve{\eta}_2). \quad (39)$$

If $\delta \to 0$ (31), then, while neglecting the terms of a higher infinitesimal order than δ, one can assume that the characteristics (35) of the random field $\upsilon(\breve{l}, \breve{\eta})$ coincide with the corresponding characteristics of the sum of the random processes $\iota_1(\breve{l}) + \iota_2(\breve{\eta})$. Then, at sufficiently large SNRs, for the probability (34), one obtains

$$\beta \approx P\left[\max_{\breve{l}\in[-\delta,\delta]} \iota_1(\breve{l}) + \max_{\breve{\eta}\in[-\delta,\delta]} \iota_2(\breve{\eta}) < \frac{c-L_0}{\sigma}\right] = \int_{-\infty}^{c/\sigma} w_0(x)\left[\int_0^{c/\sigma - x} F_{\iota 1}\left(\frac{c}{\sigma} - x - u\right) w_{\iota 2}(u)\,\mathrm{d}u\right]\mathrm{d}x. \quad (40)$$

Here $w_0(x) = \exp\left[-(x-z)^2/2\right]/\sqrt{2\pi}$ is the probability density of the Gaussian random variable; L_0/σ, $F_{\iota 1}(x) = P[\max_{\breve{l}\in[-\delta,\delta]} \iota_1(\breve{l}) < x]$ is the distribution function of the largest maximum of the process $\iota_1(\breve{l})$, $w_{\iota 2}(x) = \mathrm{d}F_{\iota 2}(x)/\mathrm{d}x$; $F_{\iota 2}(x) = P[\max_{\breve{\eta}\in[-\delta,\delta]} \iota_2(\breve{\eta}) < x]$ is the distribution function of the largest maximum of the process $\iota_2(\breve{\eta})$, and z is determined according to (30). The lower limit of the internal integral in (40) is taken equal to 0, since $\iota_1(0) = 0$, $\iota_2(0) = 0$ and, therefore, $F_{\iota 1}(x) = 0$, $w_{\iota 2}(x) = 0$ at $x < 0$.

From (37), (39), it is easy to see that the values of the process $\iota_i(\zeta_i)$, $i = 1,2$ (where $\zeta_1 = \breve{l}$, $\zeta_2 = \breve{\eta}$) on the intervals $[-\delta, 0)$ and $[0, \delta]$ are not correlated. Therefore, due to the asymptotic Gaussianity of this process, these values appear to be approximately statistically independent at the finite values of μ. Then, for the probabilities $F_{\iota 1}(x)$ and $F_{\iota 2}(x)$, the following representations are valid:

$$F_{\iota 1}(x) = F_{11}(x)F_{12}(x), \qquad F_{\iota 2}(x) = F_{21}(x)F_{22}(x), \quad (41)$$

where

$$F_{11}(x)=P[\max_{\breve{l}\in[-\delta,0)} \iota_1(\breve{l})<x], \qquad F_{12}(x)=P[\max_{\breve{l}\in[0,\delta]} \iota_1(\breve{l})<x], \quad (42)$$

$$F_{21}(x)=P[\max_{\breve{\eta}\in[-\delta,0)} \iota_2(\breve{\eta})<x], \qquad F_{22}(x)=P[\max_{\breve{\eta}\in[0,\delta]} \iota_2(\breve{\eta})<x].$$

Let

$$\upsilon_{11}(\breve{l})=x-\iota_1(-\breve{l}),\ \upsilon_{21}(\breve{\eta})=x-\iota_2(-\breve{\eta}),\ \upsilon_{12}(\breve{l})=x-\iota_1(\breve{l}),\ \upsilon_{22}(\breve{\eta})=x-\iota_2(\breve{\eta}),\ \breve{l}\in[0,\delta],\ \breve{\eta}\in[0,\delta]. \quad (43)$$

Following (by Kailath. (1986)), it is easy to show that in the subareas $\breve{l}\ge 0$, $\breve{\eta}\ge 0$, taking into account (35)-(39), the conditions of Doob's theorem (by Doob. (1986)) are satisfied for the processes $\upsilon_{1i}(\breve{l})$, $\upsilon_{2i}(\breve{\eta})$, $i=1,2$ (43). According to this theorem, the processes $\upsilon_{1i}(\breve{l})$, $\upsilon_{2i}(\breve{\eta})$, $i=1,2$ are the Markov random processes characterized, as it follows from (38), (39), by the drift coefficients

$$a_{l1}=A_1\sqrt{\mu/(1-l_0)}, \quad a_{l2}=B_1\sqrt{\mu/(1-l_0)}, \quad a_{\eta 1}=a_{\eta 2}=C_1\sqrt{\mu/(1-l_0)}. \quad (44)$$

and the diffusion coefficients

$$b_{l1}=A_{21}/(1-l_0), \quad b_{l2}=A_{22}/(1-l_0), \quad b_{\eta 1}=b_{\eta 2}=B_2, \quad (45)$$

respectively. Here A_1, B_1, A_{21}, A_{22}, C_1 and B_2 are determined based on (36) and (37).

Using the relations (43), the desired distribution functions (42), as in (by Chernoyarov. et al., (2014)), (by Chernoyarov. et al., (2014)), (by Chernoyarov. et al., (2014)), can be written in the form

$$F_{1i}(x)=P[\max_{\breve{l}\in[0,\delta]} \upsilon_{1i}(\breve{l})>0]=\int_0^\infty w_{1i}(y,\delta)\,\mathrm{d}y, \quad F_{2i}(x)=P[\max_{\breve{\eta}\in[0,\delta]} \upsilon_{2i}(\breve{\eta})>0]=\int_0^\infty w_{2i}(y,\delta)\,\mathrm{d}y, \quad i=1,2, \quad (46)$$

where $w_{1i}(y,\breve{l})$, $w_{2i}(y,\breve{\eta})$ are the probability densities that the processes $\upsilon_{1i}(\breve{l})$ and $\upsilon_{2i}(\breve{\eta})$, whose values are $\upsilon_{1i}(0)=x$, $\upsilon_{2i}(0)=x$ at the initial moments $\breve{l}=0$ and $\breve{\eta}=0$, will reach the values $\upsilon_{1i}(\delta)=y$, $\upsilon_{2i}(\delta)=y$ by the end of the analysis intervals ($\breve{l}=\delta$, $\breve{\eta}=\delta$), while $\underset{\breve{l}\in[0,\delta]}{0<\upsilon_{1i}(\breve{l})<\infty}$, $\underset{\breve{\eta}\in[0,\delta]}{0<\upsilon_{2i}(\breve{\eta})<\infty}$.

The probability densities $w_{1i}(y,\breve{l})$, $w_{2i}(y,\breve{\eta})$, introduced in (46), can be found as the solutions of the Fokker-Planck-Kolmogorov equations (by Dynkin. (2006)) with the drift coefficients a_{li}, $a_{\eta i}$ and the diffusion coefficients b_{li}, $b_{\eta i}$ ($i=1,2$) under the initial conditions $w_{1i}(y,0)=\delta(x-y)$, $w_{2i}(y,0)=\delta(x-y)$ and the boundary conditions $w_{1i}(0,\breve{l})=w_{1i}(\infty,\breve{l})=0$, $w_{2i}(0,\breve{\eta})=w_{2i}(\infty,\breve{\eta})=0$. While solving these equations using the characteristic function method (by Dynkin. (2006)), we also take into account the specified boundary conditions by means of the reflection method involving a change of sign, as it is described in (by Chernoyarov. et al., (2014)) and (by Chernoyarov. et al., (2014)). Then, after integrating the solutions found and substituting the results into (42), for the functions (40) and (41), one gets

$$F_{\iota 1}(x)=f(x,a_{l1},b_{l1},\delta)\,f(x,a_{l2},b_{l2},\delta), \quad F_{\iota 2}(x)=f(x,a_{\eta 1},b_{\eta 1},\delta)\,f(x,a_{\eta 2},b_{\eta 2},\delta), \quad (47)$$

$$w_{\iota 2}(x)=g(x,a_{\eta 1},b_{\eta 1},\delta)\,f(x,a_{\eta 2},b_{\eta 2},\delta)+g(x,a_{\eta 2},b_{\eta 2},\delta)\,f(x,a_{\eta 1},b_{\eta 1},\delta),$$

if $x\ge 0$, and $F_{\iota 1}(x)=F_{\iota 2}(x)=0$, $w_{\iota 2}(x)=0$, if $x<0$. Here

$$f(x,a,b,\delta)=\Phi\left(\frac{a\delta+x}{\sqrt{b\delta}}\right)-\exp\left(-\frac{2ax}{b}\right)\Phi\left(\frac{a\delta-x}{\sqrt{b\delta}}\right). \quad (48)$$

$$g(x,a,b,\delta)=\frac{2a}{b}\exp\left(-\frac{2ax}{b}\right)\Phi\left(\frac{a\delta-x}{\sqrt{b\delta}}\right)+\sqrt{\frac{2}{\pi b\delta}}\exp\left[-\frac{(a\delta+x)^2}{2b\delta}\right],$$

and $\Phi(x)$ is the probability integral (22).

In practice, to calculate the probability (26) by applying the formulas (40), (47), (48) is a quite difficult task, since there is no well-founded method for determining the value of δ. However, under the conditions of a high a posteriori accuracy, when the relations (30) and (31) are satisfied, instead of (48) in (47), and at $\delta\to 0$, one can use the asymptotic expressions of the form (by Chernoyarov. et al., (2014))

$$f(x,a,b,0)=1-\exp(-2ax/b), \qquad g(x,a,b,0)=(2a/b)\exp(-2ax/b). \quad (49)$$

Substituting (49) into (47), then (47) into (40), and then integrating the resulting expression by the variables *x*, *u*, one obtains

$$\beta=\Phi(c/\sigma-z)+r_1(z_{l1}+z_{l2})\mathrm{B}(c/\sigma,z_{l1}+z_{l2})-r_1(z_{l1})\mathrm{B}(c/\sigma,z_{l1})-r_1(z_{l2})\mathrm{B}(c/\sigma,z_{l2})+$$
$$+r_2(2z_\eta)\mathrm{B}(c/\sigma,2z_\eta)-2r_2(z_\eta)\mathrm{B}(c/\sigma,z_\eta), \quad (50)$$

where

$$\mathrm{B}(x,u)=\exp[-u(x-z-u/2)]\Phi(x-z-u), \qquad r_1(Z)=2z_\eta^2/(Z-z_\eta)(Z-2z_\eta),$$
$$r_2(Z)=z_{l1}z_{l2}(z_{l1}+z_{l2}-2Z)/(z_{l1}+z_{l2}-Z)(z_{l1}-Z)(z_{l2}-Z),$$

$$z_{li}=2a_{li}/b_{li}, \quad i=1,2, \quad z_\eta=2a_{\eta1}/b_{\eta1}=2a_{\eta2}/b_{\eta2},$$

The accuracy of the formula (50) increases when μ_0 (3), *z* (30) and *c* are increasing.

4. Estimating the Moment of Appearance and the Central Frequency of a Stepwise Random Disturbance

Now it is presupposed that the random disturbance (1) is present on the interval $[0,T]$ with the probability 1. Based on the observed realization (4), it is necessary to estimate the parameters λ_0 and ϑ_0. The synthesis of the estimation algorithm is carried out based on the maximum likelihood method (by Chernoyarov et al., (2015)), (by Van Trees. et al., (2013)), (by Trifonov. et al., (1986)). Using the representation of the logarithm of FLR (6), for the maximum likelihood estimates (MLEs) λ_m and ϑ_m of the moment of appearance and the central frequency of a random disturbance, one gets

$$(\lambda_m,\vartheta_m)=\underset{\lambda\in[\Lambda_1,\Lambda_2],\vartheta\in[\Theta_1,\Theta_2]}{\arg\max} L(\lambda,\vartheta). \quad (51)$$

In order to evaluate the performance of the algorithm (51), it is necessary to determine the characteristics of the estimates λ_m, ϑ_m. As it is noted above, when fulfilling the condition (3), the functional (9) is an approximately Gaussian random field (by Trifonov. et al., (1991)), i.e., in a statistical sense, it can be completely described by the moment or correlation functions of the first two orders (i.e., the mathematical expectation and the correlation function). Taking into account the last remark, the functional (9) is presented in the form of the sum (27), and the dimensionless estimates

$$\breve{l}_m = (\lambda_m - \lambda_0)/T \,, \qquad \breve{\eta}_m = (\vartheta_m - \vartheta_0)/\Omega \,.$$

are introduced instead of (51).

If the inequality (3) holds and the value of q (7) is not too small, so that the output SNR $z^2 >> 1$ (30), then, with the probability tending to 1, the coordinates $(\breve{l}_m, \breve{\eta}_m)$ of the position of the largest maximum of the decision-determining statistics $L(\breve{l}, \breve{\eta})$ belong to a small δ-neighborhood of the point $(0,0)$. With an unlimited increase in SNR ($z^2 \to \infty$), the value of this neighborhood satisfies the relation (31). Then, the asymptotic representations (32) are valid for the moments of the functional $L(\breve{l}, \breve{\eta})$.

Following (by Trifonov. et al., (1986)), (by Chernoyarov. et al., (2014)), (by Chernoyarov. et al., (2014)), the difference functional $\upsilon(\breve{l}, \breve{\eta})$ (33) with the characteristics (35)-(37) and the statistically independent Gaussian random processes $\iota_1(\breve{l})$, $\iota_2(\breve{\eta})$ with the characteristics (38), (39) are introduced into consideration. In this case

$$(\breve{l}_m, \breve{\eta}_m) = \arg\max L(\breve{l}, \breve{\eta}) = \arg\max \upsilon(\breve{l}, \breve{\eta}) \,. \quad (52)$$

As it is noted above, the characteristics (35) of the random field $\upsilon(\breve{l}, \breve{\eta})$ asymptotically (at $z \to \infty$ (30) and $\delta \to 0$ (31)) coincide with the corresponding characteristics of the sum of the independent random processes $\iota_1(\breve{l}) + \iota_2(\breve{\eta})$. Therefore, there is the convergence in a distribution of the form $\upsilon(\breve{l}, \breve{\eta}) \xrightarrow[z \to \infty]{} \iota_1(\breve{l}) + \iota_2(\breve{\eta})$. Then, the estimates (52) with an increasing SNR z converge in a distribution to the estimates

$$\breve{l}'_m = \underset{\breve{l} \in [-\delta, \delta]}{\arg\max}\, \iota_1(\breve{l}) \,, \qquad \breve{\eta}'_m = \underset{\breve{\eta} \in [-\delta, \delta]}{\arg\max}\, \iota_2(\breve{\eta}) \,,$$

while the MLEs λ_m and ϑ_m (51) converge in a distribution to the estimates $T(\breve{l}'_m + l_0)$, $\Omega(\breve{\eta}'_m + \eta_0)$.

Using the results of (by Kailath. (1986)), it is easy to demonstrate that the processes $\iota_1(\breve{l})$ and $\iota_2(\breve{\eta})$, possessing the characteristics (36)-(39) at $\breve{l} \in [-\delta, \delta]$ and $\breve{\eta} \in [-\delta, \delta]$, satisfy the conditions of Doob's theorem (by Doob. (1986)). Therefore, in the δ-neighborhood of the point $(0,0)$, they are Markov random processes, whose drift coefficients a_l, a_η and diffusion coefficients b_l, b_η take the form

$$a_l = \begin{cases} a_{l1}\,, & \breve{l} < 0\,, \\ -a_{l2}\,, & \breve{l} \geq 0\,, \end{cases} \qquad b_l = \begin{cases} b_{l1}, & \breve{l} < 0\,, \\ b_{l2}, & \breve{l} \geq 0\,, \end{cases} \qquad a_\eta = a_{\eta 1} \begin{cases} 1, & \breve{\eta} < 0\,, \\ -1, & \breve{\eta} \geq 0\,, \end{cases} \qquad b_\eta = b_{\eta 1}\,,$$

where a_{l1}, a_{l2}, b_{l1}, b_{l2}, $a_{\eta 1}$, $b_{\eta 1}$ are determined in the same way as in (44), (45).

Now, one can see the probability densities and the distribution functions of the magnitude and position of the largest maximum of a Markov random process $\rho(x)$, $x \in [X_1, X_2]$ with the drift coefficient $K_{\rho 1}$ and the diffusion coefficient $K_{\rho 2}$ changing stepwise at some internal point x_0 of the interval $[X_1, X_2]$:

$$K_{\rho 1} = \begin{cases} a_{\rho 1}\,, & X_1 \leq x < x_0\,, \\ -a_{\rho 2}\,, & x_0 \leq x \leq X_2\,, \end{cases} \qquad K_{\rho 2} = \begin{cases} b_{\rho 1}\,, & X_1 \leq x < x_0\,, \\ b_{\rho 2}\,, & x_0 \leq x \leq X_2 \end{cases}$$

Explicit expressions for all that are found in (by Chernoyarov. et al., (2014)). In particular, for the probability density of the random variable $x_m = \underset{x\in[X_1,X_2]}{\arg\max}\,\rho(x)$, one obtains

$$w(X)=\begin{cases} z_{\rho 1}^2\Psi\left(z_{\rho 1}^2(x_0-X), z_{\rho 1}^2(x_0-X_1), z_{\rho 2}^2(X_2-x_0), \dfrac{1}{R_\rho}\right), & X<x_0, \\ z_{\rho 2}^2\Psi\left(z_{\rho 2}^2(X-x_0), z_{\rho 2}^2(X_2-x_0), z_{\rho 1}^2(x_0-X_1), R_\rho\right), & X\ge x_0, \end{cases} \tag{53}$$

where $z_{\rho i}^2 = 2a_{\rho i}^2/b_{\rho i}$, $i=1,2$, $R_\rho = a_{\rho 1}b_{\rho 2}/a_{\rho 2}b_{\rho 1}$,

$$\Psi(y,y_1,y_2,y_3)=\frac{1}{2\sqrt{\pi}|y|^{3/2}}\left\{\frac{\exp[-(y_1-y)/4]}{\sqrt{\pi(y_1-y)}}+\Phi\left(\sqrt{\frac{y_1-y}{2}}\right)\right\}\times$$
$$\times\int_0^\infty \xi\exp\left[-\frac{(\xi+y)^2}{4y}\right]\left[\Phi\left(\frac{y_3\xi+y_2}{\sqrt{2y_2}}\right)-\exp(-y_3\xi)\Phi\left(\frac{-y_3\xi+y_2}{\sqrt{2y_2}}\right)\right]d\xi. \tag{54}$$

Assuming in (53) that $a_{\rho 1}=a_{\rho 2}=a_\rho$, $b_{\rho 1}=b_{\rho 2}=b_\rho$ ($R_\rho=1$), one gets the expression for the probability density of the position of the largest maximum of a Markov random process of a diffusion type with a stepwise changing drift coefficient $K_{\rho 1}=a_\rho\begin{cases}1, & X_1\le x<x_0, \\ -1, & x_0\le x\le X_2,\end{cases}$ and a constant diffusion coefficient $K_{\rho 2}=b_\rho$.

Using (53), for the first two moments of the random deviation $X_m = x_m - x_0$, it can be written

$$\langle X_m\rangle=\int_{X_1}^{X_2}(x-x_0)\,w(x)\,\mathrm{d}x=\frac{1}{z_{\rho 2}^2}F_b\left(z_{\rho 2}^2(X_2-x_0), z_{\rho 1}^2(x_0-X_1), R_\rho\right)-$$
$$-\frac{1}{z_{\rho 1}^2}F_b\left(z_{\rho 1}^2(x_0-X_1), z_{\rho 2}^2(X_2-x_0), \frac{1}{R_\rho}\right), \tag{55}$$

$$\left\langle X_m^2\right\rangle=\int_{X_1}^{X_2}(x-x_0)^2 w(x)\,\mathrm{d}x=\frac{1}{z_{\rho 2}^4}F_V\left(z_{\rho 2}^2(X_2-x_0), z_{\rho 1}^2(x_0-X_1), R_\rho\right)+$$
$$+\frac{1}{z_{\rho 1}^2}F_V\left(z_{\rho 1}^4(x_0-X_1), z_{\rho 2}^2(X_2-x_0), \frac{1}{R_\rho}\right),$$

where

$$F_b(y_1,y_2,y_3)=\int_0^{y_1} y\,\Psi(y,y_1,y_2,y_3)\,\mathrm{d}y=\frac{1}{2}\int_0^{y_1}\frac{1}{\sqrt{\pi y}}\left\{\frac{\exp[-(y_1-y)/4]}{\sqrt{\pi(y_1-y)}}+\Phi\left(\sqrt{\frac{y_1-y}{2}}\right)\right\}\times$$
$$\times\int_0^\infty \xi\exp\left[-\frac{(\xi+y)^2}{4y}\right]\left[\Phi\left(\frac{y_3\xi+y_2}{\sqrt{2y_2}}\right)-\exp(-y_3\xi)\Phi\left(\frac{-y_3\xi+y_2}{\sqrt{2y_2}}\right)\right]\mathrm{d}\xi\mathrm{d}y, \tag{56}$$

$$F_V(y_1,y_2,y_3)=\int_0^{y_1} y^2\,\Psi(y,y_1,y_2,y_3)\,\mathrm{d}y=\frac{1}{2}\int_0^{y_1}\sqrt{\frac{y}{\pi}}\left\{\frac{\exp[-(y_1-y)/4]}{\sqrt{\pi(y_1-y)}}+\Phi\left(\sqrt{\frac{y_1-y}{2}}\right)\right\}\times$$
$$\times\int_0^{\infty}\xi\exp\left[-\frac{(\xi+y)^2}{4y}\right]\left[\Phi\left(\frac{y_3\xi+y_2}{\sqrt{2y_2}}\right)-\exp(-y_3\xi)\Phi\left(\frac{-y_3\xi+y_2}{\sqrt{2y_2}}\right)\right]\mathrm{d}\xi\mathrm{d}y\,. \qquad (57)$$

In a general case, the calculation of the integrals (55)-(57) at the fixed values of $a_{\rho i}$, $b_{\rho i}$, $i=1,2$ is possible only using numerical methods. However, if the condition $a_{\rho i}^2 >> b_{\rho i}$ is satisfied, then, without a noticeable loss in accuracy, instead of (54), a simpler approximation of the form

$$\Psi(x,\infty,\infty,y)=w_a(x,y)=\Phi\left(\sqrt{|x|/2}\right)-1+$$
$$+(2y+1)\exp\left[|x|y(y+1)\right]\left[1-\Phi\left((2y+1)\sqrt{|x|/2}\right)\right] \qquad (58)$$

can be used for the function $\Psi(y,y_1,y_2,y_3)$. In the special case, when $a_{\rho 1}=a_{\rho 2}=a_\rho$, $b_{\rho 1}=b_{\rho 2}=b_\rho$ that corresponds to $R_\rho=1$, from (58), one obtains $w_a(x,1)=\Phi\left(\sqrt{|x|/2}\right)-1+3\exp(2|x|)\left[1-\Phi\left(3\sqrt{|x|/2}\right)\right]$.

Substituting (58) into (55)-(57) and turning the limits of integration into infinite ones, for the first two moments of the random deviation X_m, one finds

$$\langle X_m\rangle=\left[a_{\rho 1}^2 b_{\rho 2}R_\rho\left(R_\rho+2\right)-a_{\rho 2}^2 b_{\rho 1}\left(2R_\rho+1\right)\right]\Big/2a_{\rho 1}^2 a_{\rho 2}^2\left(R_\rho+1\right)^2, \quad (59)$$
$$\left\langle X_m^2\right\rangle=\left[a_{\rho 1}^4 b_{\rho 2}^2 R_\rho\left(2R_\rho^2+6R_\rho+5\right)+a_{\rho 2}^4 b_{\rho 1}^2\left(5R_\rho^2+6R_\rho+2\right)\right]\Big/2a_{\rho 1}^4 a_{\rho 2}^4\left(R_\rho+1\right)^3.$$

If $R=1$ ($a_{\rho 1}=a_{\rho 2}=a_\rho$, $b_{\rho 1}=b_{\rho 2}=b_\rho$), then the formulas (59) are significantly simplified and take the form

$$\langle X_m\rangle=0, \qquad \left\langle X_m^2\right\rangle=13b_\rho^2\big/8a_\rho^4. \quad (60)$$

Based on it, for the fixed values of λ_0 and ϑ_0, it is easy to write down the asymptotic expressions for the conditional biases (the systematic errors) $b(\lambda_m|\lambda_0,\vartheta_0)=\langle\lambda_m-\lambda_0\rangle$, $b(\vartheta_m|\lambda_0,\vartheta_0)=\langle\vartheta_m-\vartheta_0\rangle$ and variances (the mean square errors) $V(\lambda_m|\lambda_0,\vartheta_0)=\left\langle(\lambda_m-\lambda_0)^2\right\rangle$, $V(\vartheta_m|\lambda_0,\vartheta_0)=\left\langle(\vartheta_m-\vartheta_0)^2\right\rangle$ of the estimates (51):

$$b(\lambda_m|\lambda_0,\vartheta_0)=T\langle\breve{l}'_m\rangle, \qquad b(\vartheta_m|\lambda_0,\vartheta_0)=\Omega\langle\breve{\eta}'_m\rangle \quad (61)$$
$$V(\lambda_m|\lambda_0,\vartheta_0)=T^2\left\langle\breve{l}'^2_m\right\rangle, \qquad V(\vartheta_m|\lambda_0,\vartheta_0)=\Omega^2\left\langle\breve{\eta}'^2_m\right\rangle.$$

Here $\left\langle\breve{l}'_{m\nu}\right\rangle$, $\left\langle\breve{l}'^2_m\right\rangle$ and $\langle\breve{\eta}'_m\rangle$, $\left\langle\breve{\eta}'^2_m\right\rangle$ are determined from (55), (59) or (55), (60) by substituting the values $\widetilde{\Lambda}_1$, $\widetilde{\Lambda}_2$, l_0 or $\widetilde{\Theta}_1$, $\widetilde{\Theta}_2$, η_0 (11), (29) instead of X_1, X_2, x_0. Besides, the drift coefficients a_{li}, $a_{\eta i}$ and diffusion coefficients b_{li}, $b_{\eta i}$ from (44), (45) are substituted instead of $a_{\rho i}$ and $b_{\rho i}$, and the relations $R_l=a_{l1}b_{l2}/a_{l2}b_{l1}$ or $R_\eta=1$ replace R_ρ. In this case, the accuracy of the formulas (55), (59), (60), (61) increases with μ (10) and z (30).

From (61), in particular, it follows that, in a general case, for the finite values of z (30), the MLE λ_m is conditionally biased while the MLE ϑ_m is approximately conditionally unbiased. At the same time, with an unlimited increase in SNR (30), the estimates (51) are asymptotically conditionally unbiased and, therefore, consistent. Moreover, under the conditions of high a posteriori accuracy (when $z^2 >> 1$), the quality of the estimate of the moment of appearance does not depend on an ignorance of the central frequency of the random disturbance (1), and vice versa. It should be also noted that, in the case of small SNR z, the characteristic values of the estimates λ_m and ϑ_m calculated using (59), (60), (61) may appear largely erroneous. This is due to the fact that the formulas (59), (60), in the contrast to (55), are obtained without taking into account the restrictions on the finite lengths of a priori intervals of possible values of the estimated parameters λ_0 and ϑ_0. Thus, in the case when $z \to 0$, they describe the characteristics of a random variable uniformly distributed over an interval whose length tends to infinity.

5. The Statistical Simulation Results

In order to determine the errors in the found approximate formulas for the characteristics of the synthesized algorithms of processing Gaussian random disturbances at finite SNRs, computer statistical simulation of the detector (8) and the measurer (51) operation is performed. During the simulation, on the interval $\tilde{t} \in [0,1]$ (10) at the discrete points in time $\tilde{t}_j = j\Delta\tilde{t}$, $j = \overline{0, \mathrm{int}(1/\Delta\tilde{t})}$, and for each value of $\eta_k = \tilde{\Theta}_1 + (k+1/2)\Delta\eta$, $k = \overline{0, \mathrm{int}((\tilde{\Theta}_2 - \tilde{\Theta}_1)/\Delta\eta - 1)}$ of the normalized central frequency η (10) at the specified values of μ, q, l_0, η_0, the dimensionless samples $\tilde{y}_{jk} = y(T\tilde{t}_j, \Omega\eta_k)\sqrt{T/N_0}$ (6) are formed, as described in (by Chernoyarov. et al., (2014)), (by Chernoyarov. et al., (2015)). Here $\mathrm{int}(\cdot)$ is the integer part of the number, and $\Delta\eta$ is the discrete step of the current normalized central frequency value change. Based on the generated samples $\tilde{y}_{jk}$, following (by Chernoyarov. et al., (2014)), (by Chernoyarov. et al., (2015)), the samples $L_{nk} = L(n\Delta l, k\Delta\eta)$ of the random field $L(l,\eta)$ (9) are calculated in the area $[\tilde{\Lambda}_1, \tilde{\Lambda}_2] \cup [\tilde{\Theta}_1, \tilde{\Theta}_2]$, where Δl is the discrete step of the current normalized moment of appearance value change (10). The discretization step for the variable $\tilde{t}$ (10) is chosen equal to $\Delta\tilde{t} = 0.01/\mu$, and for the variables l and η – equal to $\Delta l = \Delta\eta = 10^{-3}$. As a result, the relative root-mean-square error of the stepwise approximation of the functional $L(l,\eta)$ (9) based on the generated samples, calculated according to the technique described in (by Zakharov et al., (2001)), does not exceed 10%.

In each test performed for each generated realization $L(l,\eta)$, a decision on the presence or absence of a random disturbance (1) in the observations is made by comparing the maximum sample $L_{\max} = \max_{n,k} L_{nk}$ with the selected threshold c: $L_{\max} \gtrless c$. Also, at $q \neq 0$, the normalized estimates $l_m = \lambda_m / T$ и $\eta_m = \vartheta_m / \Omega$ (51) are found as follows:

$$l_m = \tilde{\Lambda}_1 + N_{\max}\Delta l, \quad \eta_m = \tilde{\Theta}_1 + K_{\max}\Delta\eta,$$
$$N_{\max}, K_{\max} : L_{N_{\max}K_{\max}} \geq L_{nk}, 0 \leq n, N_{\max} \leq \mathrm{int}((\tilde{\Lambda}_2 - \tilde{\Lambda}_1)/\Delta l), 0 \leq k, K_{\max} \leq \mathrm{int}((\tilde{\Theta}_2 - \tilde{\Theta}_1)/\Delta\eta),$$

In Figures. 1-4, some results obtained during the statistical simulation, as well as the corresponding theoretical dependencies, are presented. To obtain each experimental value, at least $5 \cdot 10^4$ realizations (4) or (5) are processed at the fixed $\tilde{\Lambda}_1 = 0.3$, $\tilde{\Lambda}_2 = 0.7$, $l_0 = (\tilde{\Lambda}_1 + \tilde{\Lambda}_2)/2$, $\tilde{\Theta}_1 = 7.5$, $\tilde{\Theta}_2 = 12.5$, $\eta_0 = (\tilde{\Theta}_1 + \tilde{\Theta}_2)/2$. Thus, the deviation of the boundaries of the confidence intervals from the experimental data is not greater than 10% with the probability of 0.9.

In Figure. 1, by solid lines the theoretical dependences of the false alarm probability α (11) on the threshold value c are drawn. The curve 1 is calculated using formula (25) for $\mu = 100$, $q = 0.25$; 2 – for $\mu = 100$, $q = 0.5$; 3 – for $\mu = 500$, $q = 0.25$. The corresponding experimental values of the probability α are plotted with circles, triangles and squares. In Figure. 2, the solid lines show the theoretical dependences of the missing probability (26) on the value q (7) of the normalized average power of the process (1) calculated using formula (50). The value of the threshold c is determined from (25) using the Neyman-Pearson criterion at a selected false alarm probability of 0.01. The curve 1 corresponds to $\mu = 200$; 2 – to $\mu = 500$; 3 – to $\mu = 1000$. In Figure. 3, 4, by solid and dashed lines the theoretical dependences (61) of normalized conditional variances $V_l(q) = V(\lambda_m|\lambda_0, \vartheta_0)/T^2$, $V_\eta(q) = V(\vartheta_m|\lambda_0, \vartheta_0)/\Omega^2$ of the estimates (51) of the moment of appearance and the central frequency of a stepwise random disturbance (1) calculated using (55) and (59) or (60). The curves 1 are plotted for $\mu = 1000$; 2 – for $\mu = 2000$; 3 – for $\mu = 4000$. The dotted line in Figure. 3 demonstrates the maximum normalized conditional variance $(\tilde{\Lambda}_2 - \tilde{\Lambda}_1)^2/12 \approx 0.013$ of the MLE λ_m. The experimental values of the probability β for $\mu = 200$, 500, 1000 and the variances V_l, V_η for $\mu = 1000$, 2000, 4000 are indicated in Figure. 2-4 by circles, triangles and squares, respectively.

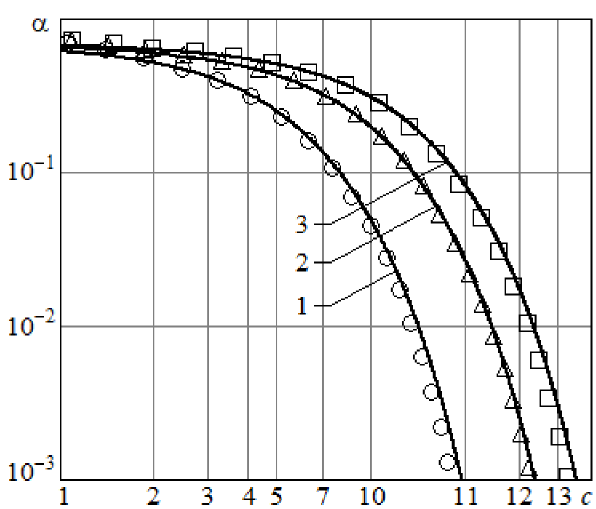

Figure. 1. The false alarm probability.

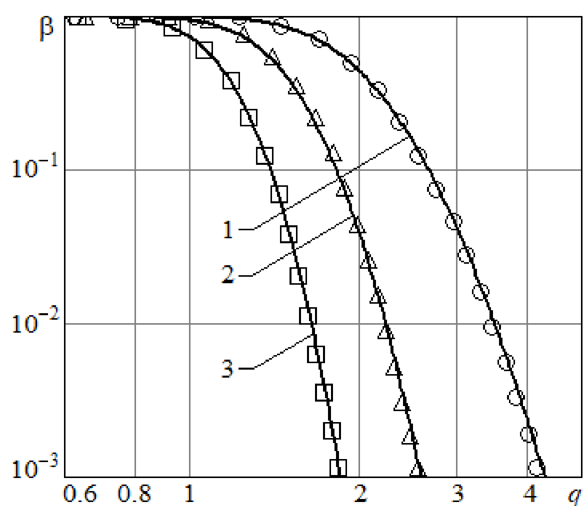

Figure. 2. The missing probability.

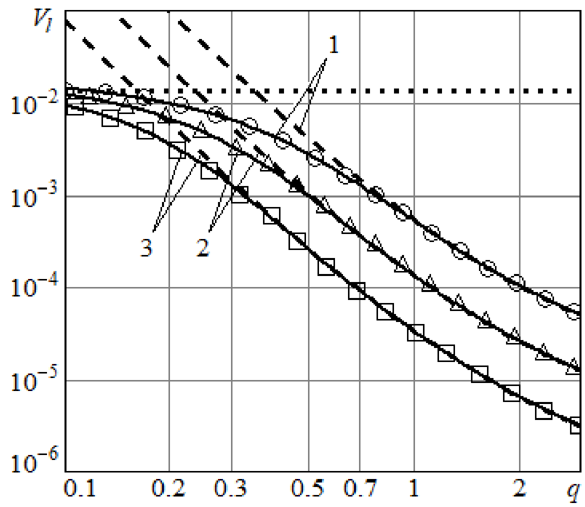

Figure. 3. The variance of the estimate of the moment of appearance of the random disturbance.

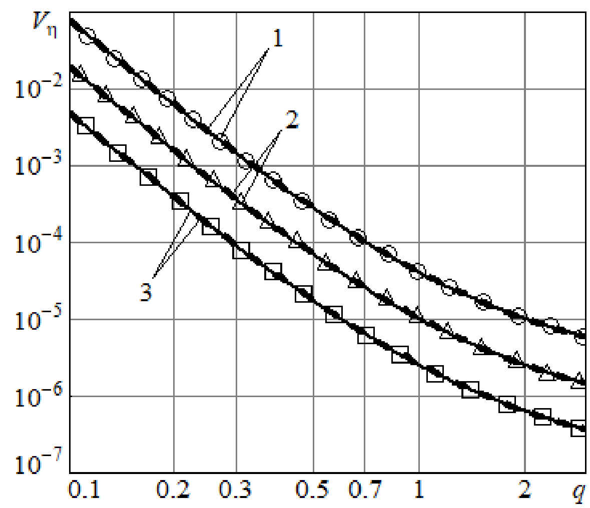

Figure. 4. The variance of the estimate of the central frequency of the random disturbance.

As it follows from Figure. 1-4, the obtained theoretical dependences (25), (50), (61) (using (55)) for the probabilities α (11), β (2.14) and the variances $V(\lambda_m|\lambda_0,\vartheta_0)$, $V(\vartheta_m|\lambda_0,\vartheta_0)$ are in good agreement with experimental data, at least, if $\mu_0 \geq 100$ (3), $\mu \geq 200$ (in the detection task) or $\mu_0 \geq 500$, $\mu \geq 1000$ (in the estimation task) and $q \geq 0{,}1$, $\tilde{\Lambda}_1 \geq 0.3$, $\tilde{\Lambda}_2 \leq 0.7$, $\tilde{\Theta}_2 - \tilde{\Theta}_1 \geq 5$. Under conditions of not too small output SNR (30), when $z > 1$ (when estimating the central frequency of a stepwise disturbance) and $z > 3.5$ (when estimating the moment of appearance of a stepwise disturbance), simpler approximations based on (59), (60) can be used to calculate the variances (61) of the corresponding estimates (51).

It should be noted that the theoretical dependencies (61) may deviate from the corresponding experimental values at $q > 2\ldots3$. This is due to the fact that formulas (61) have been derived by neglecting terms of the order of the correlation time of the random disturbance (1). Consequently, when the normalized variances $V_l(q) = V(\lambda_m|\lambda_0,\vartheta_0)/T^2$ and $V_\eta(q) = V(\vartheta_m|\lambda_0,\vartheta_0)/\Omega^2$ decrease to the magnitude of the order of μ^{-2}, the error of formulas (61) becomes significant. However, the variance values presented in Figures. 3 and 4 significantly exceed this threshold. For this reason, no deviation between experimental and theoretical variance values is observed here at large q.

6. Conclusion

Using the condition of fast fluctuations in the realization of the observed data, which allows neglecting the values of the order and less than the correlation time of the analyzed stepwise Gaussian random disturbance, a relatively simple approximation of the decision-determining statistics (i.e, FLR logarithm) can be found. Based on this approximation, it is possible to obtain new algorithms for detecting and estimating the moment of appearance and the central frequency of a random disturbance. Such algorithms are computationally significantly simpler compared to the optimal analogues synthesized based on other approaches.

In order to analytically determine the characteristics of the synthesized processing algorithms, the multiplicative or additive local Markov approximation method can be effectively applied. This method includes the approximation of the decision-determining statistics or its increment by the product or sum of Gaussian Markov processes of the diffusion type. Then, by determining the drift and diffusion coefficients of the indicated Gaussian Markov processes and solving the corresponding Fokker-Planck-Kolmogorov equations with the specified initial and boundary conditions, one can find the closed expressions for the false alarm and missing probabilities when detecting a

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random disturbance with the unknown moment of appearance and central frequency. Such expressions are also produced for the conditional central moments of the maximum likelihood estimates of the moment of appearance and the central frequency of a random disturbance. Particularly important are the ones that are for the first-order moments being the biases and for the second-order moments being the variances.
Theoretical dependencies for the characteristics of detection and estimates of the moment of appearance and the central frequency of an abrupt random disturbance successfully describe the corresponding experimental data in the case when the ratio of the duration of the random disturbance to its correlation time is at least 100 (when detecting) or 500 (when estimating), and the ratio of the disturbance intensity to background noise intensity is not less than 0.1. The calculation of the variances of the estimates of the time and frequency parameters of the disturbance can be carried out using rather simple arithmetic operations. But it is possible only if the output voltage signal-to-noise ratio exceeds 0.5-3.5; i.e., it is 0.5 when estimating the central frequency and 3.5 when estimating the moment of appearance.

Ignorance of the central frequency leads to some deterioration in the quality of detection but does not reduce the accuracy of measuring the moment of appearance of a random disturbance. In addition, the root-mean-square error of the maximum likelihood estimate of the central frequency in the case of signal-to-noise ratios greater than 1 can be significantly less than the root-mean-square error of the maximum likelihood estimate of the moment of appearance.

The found expressions for the characteristics of the considered algorithms for detecting an abrupt random disturbance and estimating its unknown moment of appearance and central frequency allow making an informed choice between these and other algorithms depending on a relative simplicity of their practical implementation, the information available a priori, and the requirements to the algorithm performance.

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