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Characterizations of Certain (2023-2024) Introduced Univariate Continuous Distributions III

G.G. Hamedani[1*], Amin Roshani[2], Nadeem Shafique Butt[3]

*Corresponding author

1. Department of Mathematical and Statistical Sciences, Marquette University, Milwaukee, WI 53201-1881, USA, gholamhoss.hamedani@marquette.edu
2. Department of Statistics, Lorestan University, Khorramabad, Iran, roshani.a@lu.ac.com
3. Department of Family and Community Medicine, King Abdulaziz University, Jeddah, Saudi Arabia, nshafique@kau.edu.sa

Abstract

This paper is a continuation of our previous works with the same title, which deals with various characterizations of certain univariate continuous distributions proposed in (2023-2024) after the publication of our first two papers in (2024). These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) reverse hazard function and (iv) conditional expectation of a single function of the random variable. It should be mentioned that for the characterization (i) the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

Key Words: Characterizations; Conditional expectation; Continuous distributions; Hazard function; Reverse hazard function.

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with certain characterizations of each of the following distributions: 1) Exponentiated Odd Log-Logistics Weibull (EOLLW) distribution of Rodrigues et al. (2023a); 2) Truncated Weibull-Exponential (TWE) distribution of Abbas et al. (2023); 3) New Extended Normal Quantile Regression (NENQR) distribution of Rodrigues et al. (2023b); 4) New Half-Logistic Fréchet (NHLFr) distribution of Khan et al. (2023); 5) Alpha Power Exponentiated Pareto (APEP) distribution of Pimsap et al. (2023); 6) Power Exponentiated Weibull (PEW) distribution of Rajitha and Vaishnavi (2023); 7) Left Truncated Topp-Leone (LTTL) distribution of Zaninetti (2023); 8) Partly-Exponential (PE) distribution of Atikankul et al. (2021); 9) Generalized Lindley (GL) distribution of Cha and Badia (2024); 10) Zubair-Kumaraswamy (Z-Kum) distribution of Manu et al. (2023a); 11) Zubair-Ikum (Z-Ikum) distribution of Manu et al. (2023b); 12) Novel Updated Weibull (NU-Weibull) distribution of Alnssyan et al. (2023); 13) New Flexible Weibull Extension (NFWE) distribution of Alshanbari et al. (2023b); 14) Weighted Cosine-Weibull (WC-Weibull) distribution of Odhah et al. (2023); 15) Unit Upper Truncated Weibull (UUTW) distribution of Okorie et al. (2023); 16) New Modified Cosine-G (NMC-G) distribution of Alshanbari et al. (2023d); 17) Neutrosophic Generalized Exponential (NGE) distribution of Rao et al. (2023); 18) Neutrosophic Generalized Rayleigh (NGR) distribution of Norouzirad et al. (2023); 19) New Modified Sine-G (NMS-G) family of distributions of Alshanbari et al. (2023c); 20) New Tangent-G (NT-G) family of distributions of Alshanbari et al. (2023a); 21) Scale Mixture of Exponentials (SME) distribution of Barahona et al. (2024); 22) Modified Unit-Half-Normal (MUHN) distribution of Alvarez et al. (2024); 23) Truncated Positive Symmetric (TPS) family of distributions of Gómez et al. (2023); 24) Slash-Weighted Lindley (SWL) distribution of Castillo et al. (2023); 25) Modified Skew-t-Normal ($\mathrm{MS_tN}$) distribution of Arrué et al. (2023); 26) Modified Gamma (MG) distribution of Alshehri and Kayid (2023); 27) Generalized Alpha Power Exponentiated Inverse Exponential (GAPEIEx) distribution of Kargbo et al. (2023); 28) Exponentiated Power Chris-Jerry (EPCJ) distribution of Obulezi et al. (2023); 29) New Generalized Odd Fréchet-Odd Exponential-G (NGOF-OE-G) family of distributions of Sadiq et al. (2023); 30) Sabur (Sabur) distribution of Ahmad et al. (2023); 31) Quartic Transmuted Fréchet (QTF) distribution of Moloy et al. (2024); 32) New Cubic Transmuted Power Function (NCTPF) distribution of Hafeez et al. (2023); 33) Topp-Leone Kumarswamy Marshall-Olkin Generated (TLKMO-G) family of distributions of Atchadé et al. (2024); 34) Modified Cubic Transformed-Rayleigh (MCT-Rayleigh) distribution of Rahman et al. (2023); 35) New Logarithmic Transformation (NLT) distribution of Alshawarbeh et al. (2024); 36) New Extension-X-Length-Baised Exponential (NEXLBE) distribution of Alamri et al. (2023); 37) Arctan Inverse Weibull (AIW) distribution of Alrashidi (2023); 38) Right Truncated Fréchet-Inverted Weibull (RTFIW) distribution of Nader et al. (2023); 39) Unit Gompertz-Power Series (UGPS) distribution of Yousef et al. (2024); 40) Truncated Inverse Power Ailamujia (TIPA) distribution of El Gazar et al. (2023); 41) Generalized Doubly Bounded Exponential (GDBE) distribution of Okereke et al. (2023); 42) Burr XII-G (BXII-G) family of distributions of Al-Shomrani and Shawky (2024); 43) Cosine Fréchet Loss (CFrL) distribution of Abonongo et al. (2024); 44) Sine Burr III Loss (SBIIIL) distribution of Abonongo et al. (2023); 45) Exponentiated Generalized Weibull Exponential (EGWE) distribution of Abonongo and Abonongo (2024); 46) Odd Weibull Lindley (OWL) distribution of Rajitha and Anisha (2024); 47) A Weighted Weibull (AWW) distribution of Kanwal et al. (2024); 48) Odd Exponential Logarithmic-G (OEL-G) distribution of Hussain et al. (2024); 49) Exponentiated Alpha Power Inverted Exponential (EAPIE) distribution of Eghwerido and Eghwerido (2024); 50) Half Logistic-Truncated Exponential (HL-TEXP) distribution of Gul et al. (2023); 51) Exponentiated XLindley (EXL) distribution of Alomari et al. (2024); 52) Modified Extended Kumarswamy Exponential (MEKwE) distribution of Chaudhary et al. (2023a); 53) Exponentiated Inverse Chen (EIC) distribution of Telee and Kumar (2023a); 54) Extended Kumarswamy Exponential (EKwE) distribution of Chaudhary et al. (2023b); 55) New Exponential-G (NE-G) class of distributions of Al Mutairi et al. (2022); 56) Inverse Exponentiated Exponential Poisson (IEEP) distribution of Telee and Kumar (2023b);

We list below the cumulative distribution function (cdf) and probability density function (pdf) of each one of these distributions in the same order as listed above. We will be employing the same notation for the parameters as chosen by the original authors.

1) The cdf and pdf of (EOLLW) are given, respectively, by

$$F(x;\nu,\lambda,\gamma,\sigma)=\frac{\left\{1-\exp\left[-\left(\frac{x}{\gamma}\right)^{\sigma}\right]\right\}^{\nu\lambda}}{\left[\left\{1-\exp\left[-\left(\frac{x}{\gamma}\right)^{\sigma}\right]\right\}^{\nu}+\left\{\exp\left[-\left(\frac{x}{\gamma}\right)^{\sigma}\right]\right\}^{\nu}\right]^{\lambda}},\quad x\geq 0, \tag{1}$$

and

$$f(x;\nu,\lambda,\gamma,\sigma)=\frac{d}{dx}F(x;\nu,\lambda,\gamma,\sigma),\quad x>0, \tag{2}$$

where $\nu,\lambda,\gamma,\sigma$ are all positive parameters.

Remark 1.1. *Taking $G(x)=1-\exp\left[-\left(\frac{x}{\gamma}\right)^{\sigma}\right]$, $x\geq 0$, the cdf (1) can be written as*

$$F(x;\nu,\lambda,\gamma,\sigma)=\left[\frac{G^{\nu}(x)}{G^{\nu}(x)+(1-G(x))^{\nu}}\right]^{\lambda},\quad x\geq 0,$$

which was mentioned in Remark 1.20 in Hamedani (2023) for power λ.

2) The cdf and pdf of (TWE) are given, respectively, by

$$F(x;\gamma,\lambda,k)=\frac{1-\exp\left\{-\gamma^{-k}\left(1-e^{-x/\lambda}\right)^{k}\right\}}{1-e^{-\gamma^{-k}}},\quad x\geq 0, \tag{3}$$

and

$$f(x;\gamma,\lambda,k)=Ce^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}P(x),\quad x>0, \tag{4}$$

where γ,λ,k are all positive parameters, $C=\frac{k\left(1-e^{-\gamma^{-k}}\right)^{-1}}{\lambda\gamma^{k}}$ and $P(x)=\exp\left[-\left(\frac{1-e^{-x/\lambda}}{\gamma}\right)^{k}\right]$.

Remarks 1.2. (a) *We will characterize the cdf* (3) *in subsections 2.1 and 2.3 in details.* (b) *Also, we like to mention that taking* $G(x)=\left(1-e^{-x/\lambda}\right)^{k}$, $x\geq 0$, *the cdf* (3) *can be expressed as*

$$F(x;\gamma,\lambda,k)=\frac{1-\exp\left\{-\gamma^{-k}G(x)\right\}}{1-e^{-\gamma^{-k}}},\quad x\geq 0,$$

which was mentioned in Remark 1.31 in Hamedani et al. (2024b).

3) The cdf and pdf of (NENQR) are given, respectively, by

$$\begin{aligned}F(x;a,b,\lambda,\sigma,\tau)&=\frac{\Phi\left(\frac{x-w}{\sigma}\right)^{ab}}{\left[\Phi\left(\frac{x-w}{\sigma}\right)^{a}+\left\{1-\Phi\left(\frac{x-w}{\sigma}\right)\right\}^{a}\right]^{b}}\\&=\left(\frac{\Phi\left(\frac{x-w}{\sigma}\right)^{a}}{\left[\Phi\left(\frac{x-w}{\sigma}\right)^{a}+\left\{1-\Phi\left(\frac{x-w}{\sigma}\right)\right\}^{a}\right]}\right)^{b},\quad x\in\mathbb{R},\end{aligned} \tag{5}$$

and

$$f(x;a,b,\lambda,\sigma,\tau)=\frac{d}{dx}F(x;a,b,\lambda,\sigma,\tau),\quad x\in\mathbb{R}, \tag{6}$$

where $a>0,b>0,\lambda\in\mathbb{R},\sigma>0,\tau\in(0,1)$ are parameters, $\Phi(x)$ is the cdf of the standard normal random variable and $w=\lambda-\sqrt{2}\sigma\times\operatorname{erf}^{-1}\left[2\left(\frac{\tau^{1/(ab)}}{\tau^{1/(ab)}+\left(1-\tau^{1/(ab)}\right)^{1/a}}\right)-1\right]$.

Remark 1.3. *Taking* $G(x)=\Phi\left(\frac{x-w}{\sigma}\right)$, $x\in\mathbb{R}$, *the cdf* (5) *can be written as*

$$F(x;a,b,\lambda,\sigma,\tau)=\left[\frac{G(x)^{a}}{G(x)^{a}+(1-G(x))^{a}}\right]^{b},\quad x\in\mathbb{R},$$

which was mentioned in Remark 1.20 in Hamedani (2023) for power b.

4) The cdf and pdf of (NHLFr) are given, respectively, by

$$F(x;a,b,\lambda)=\frac{G(x)}{2-G(x)},\quad x\geq 0, \tag{7}$$

and

$$f(x;a,b,\lambda)=\frac{d}{dx}F(x;a,b,\lambda),\quad x>0, \tag{8}$$

where a,b,λ are all positive parameters and $G(x)=1-\left(1-e^{-ax^{-b}}\right)^{\lambda\left(e^{-ax^{-b}}\right)}$.

Remarks 1.4. (a)*The cdf* (7) *as expressed above was mentioned in Remark 1.5 in Hamedani et al. (2024a).* (b) *Similar statement can be stated for the following 4 distributions mentioned in Khan et al. (2023): New Half Logistic Beta (NHLB); New Half Logistic Kumaraswamy (NHLKw); New Half Logistic Weibull (NHLW); New Half Logistic Gumbel (NHLGu).*

5) The cdf and pdf of (APEP) are given, respectively, by

$$F\left(x;\alpha,a,k\right)=\frac{\alpha^{G(x)}-1}{\alpha-1},\quad x\geq\log k, \tag{9}$$

and

$$f\left(x;\alpha,a,k\right)=\frac{d}{dx}F\left(x;\alpha,a,k\right),\quad x>\log k, \tag{10}$$

where $\alpha\left(\neq 1\right),a,k$ are all positive parameters and $G\left(x\right)=1-k^{a}\left(e^{-ax}\right)$.

Remark 1.5. *The cdf* (9) *as expressed above was mentioned in Remark 1.1 in Hamedani et al. (2024b) for* $\alpha=e^{\delta}$.

6) The cdf and pdf of (PEW) are given, respectively, by

$$F\left(x;k,\lambda,\beta,\upsilon\right)=\frac{\upsilon^{G(x)}-1}{\upsilon-1},\quad x\geq 0, \tag{11}$$

and

$$f\left(x;k,\lambda,\beta,\upsilon\right)=\frac{d}{dx}F\left(x;k,\lambda,\beta,\upsilon\right),\quad x>0, \tag{12}$$

where $k,\lambda,\beta,\upsilon\left(\neq 1\right)$ are all positive parameters and $G\left(x\right)=\left(1-e^{-x^{k}/\lambda}\right)^{\beta}$.

Remark 1.6. *The cdf* (11) *as expressed above was mentioned in Remark 1.1 in Hamedani et al. (2024b) for* $\upsilon=e^{\delta}$.

7) The cdf and pdf of (LTTL) are given, respectively, by

$$F\left(x;\beta,x_{l},b\right)=\int_{x_{l}}^{x}f\left(u;\beta,x_{l},b\right)du,\quad x_{l}\leq x\leq b, \tag{13}$$

and

$$f\left(x;\beta,x_{l},b\right)=C\left(b-x\right)P\left(x\right),\quad x_{l}<x<b, \tag{14}$$

where β,x_{l},b are all positive parameters, $C=\frac{2\beta}{b^{2}\left(1-x_{l}^{\beta}b^{-2\beta}\left(2b-x_{l}\right)^{\beta}\right)}$ and $P\left(x\right)=\left(-\frac{x^{2}}{b^{2}}+\frac{2x}{bb}\right)^{\beta-1}$.

8) The cdf and pdf of (PE) are given, respectively, by

$$F\left(x;\gamma,\delta,\tau\right)=1-e^{-x/\tau}\left(\frac{\delta\left(1-\gamma e^{-x/\delta}\right)+\tau}{C_{1}}\right),\quad x\geq 0, \tag{15}$$

and

$$f\left(x;\gamma,\delta,\tau\right)=Ce^{-x/\tau}P\left(x\right),\quad x>0, \tag{16}$$

where $0\leq\gamma\leq 1,\delta>0,0<\tau<\delta\left(1+\gamma\right)$ are parameters, $C_{1}=\delta\left(1-\gamma\right)+\tau,C=\frac{\delta+\tau}{\tau\left[\delta\left(1-\gamma\right)+\tau\right]}$ and $P\left(x\right)=\left(1-\gamma e^{-x/\delta}\right)$.

Remarks 1.7. (a) *The pdf* (16) *is similar to the pdf* (4). (b) *We will, however, characterize cdf* (16) *in subsections 2.2 and 2.4.*

9) The cdf and pdf of (GL) are given, respectively, by

$$F(x;\alpha,k,\theta)=\int_0^x f(u;\alpha,k,\theta)\,du,\quad x\geq 0, \tag{17}$$

and

$$f(x;\alpha,k,\theta)=Ce^{-\theta x}P(x),\quad x>0, \tag{18}$$

where $\alpha>-1, k\in\mathbb{N}, \theta>0$ are parameters, $C=\left(\sum_{j=0}^{k}\frac{\Gamma(\alpha+1+j)}{\theta^{\alpha+1+j}}\right)^{-1}$ and $P(x)=x^{\alpha}\sum_{i=0}^{k}x^{i}$.

Remark 1.8. *The pdf* (18) *is similar to the pdf* (16).

10) The cdf and pdf of (Z-Kum) are given, respectively, by

$$F(x;a,b)=\frac{e^{aG(x)}-1}{e^{a}-1},\quad 0\leq x\leq 1, \tag{19}$$

and

$$f(x;a,b)=\frac{d}{dx}F(x;a,b),\quad 0<x<1, \tag{20}$$

where $a>0, b>0$ are parameters and $G(x)=\left(1-(1-x^{a})^{b}\right)^{2}$.

Remark 1.9. *The cdf* (19) *was mentioned in Remark 1.6.*

11) The cdf and pdf of (Z-Ikum) are given, respectively, by

$$F(x;\alpha,\lambda,\beta)=\frac{e^{\alpha G(x)}-1}{e^{\alpha}-1},\quad x\geq 0, \tag{21}$$

and

$$f(x;\alpha,\lambda,\beta)=\frac{d}{dx}F(x;\alpha,\lambda,\beta),\quad x>0, \tag{22}$$

where α,λ,β are all positive parameters and $G(x)=\left(\left(1-(1+x)^{-\lambda}\right)^{\beta}\right)^{2}$.

Remark 1.10. *The cdf* (21) *is the similar to the cdf* (19).

12) The cdf and pdf of (NU-Weibull) are given, respectively, by

$$F(x;\alpha,\phi,\sigma,\xi)=\int_0^x f(u;\alpha,\phi,\sigma,\xi)\,du,\quad x\geq 0, \tag{23}$$

and

$$f(x;\alpha,\phi,\sigma,\xi)=g(x;\sigma,\xi)\,P(x),\quad x>0, \tag{24}$$

where $\alpha>0, \phi\,(|\phi|\geq 1), \sigma>0, \xi>0$ are parameters, $G(x;\sigma,\xi)=1-e^{-\sigma x^{\xi}}$, $x\geq 0$ is a baseline cdf with the corresponding pdf $g(x;\sigma,\xi)$ and $P(x)=\frac{1}{\phi}\left\{\phi+\alpha\left[G(x;\sigma,\xi)\right]^{\alpha-1}-(\alpha+1)\left[G(x;\sigma,\xi)\right]^{\alpha}\right\}$.

Remark 1.11. *In view of the definition of $G(x;\sigma,\xi)=1-e^{-\sigma x^{\xi}}$, $x\geq 0$, the assumption "$x\in\mathbb{R}$" in equation 2 on page 2 of the authors paper (Alnssyan et al., 2023) should be "$x\geq 0$".*

13) The cdf and pdf of (NFWE) are given, respectively, by

$$F\left(x;\delta_1,\delta_2\right)=1-\frac{e^{-e^{\delta_1 x-\delta_2/x}}}{\exp\left\{1-e^{-e^{\delta_1 x-\delta_2/x}}\right\}},\quad x\geq 0, \tag{25}$$

and

$$f\left(x;\delta_1,\delta_2\right)=\frac{d}{dx}F\left(x;\delta_1,\delta_2\right),\quad x>0, \tag{26}$$

where $\delta_1>0,\delta_2>0$ are parameters.

Remark 1.12. *Taking* $G\left(x\right)=1-e^{-e^{\delta_1 x-\delta_2/x}}$, $x\geq 0$, *the cdf* (25) *can be written as*

$$F\left(x;\delta_1,\delta_2\right)=1-\overline{G}\left(x\right)e^{-G(x)},\quad x\geq 0,$$

which was mentioned in Remark 1.125 in Hamedani (2023).

14) The cdf and pdf of (WC-Weibull) are given, respectively, by

$$F\left(x;\gamma,\theta\right)=\frac{e^{\left(1-\cos\left[\frac{\pi\left(1-e^{-\gamma x^\theta}\right)}{2-e^{-\gamma x^\theta}}\right]\right)}-1}{e-1},\quad x\geq 0, \tag{27}$$

and

$$f\left(x;\gamma,\theta\right)=\frac{d}{dx}F\left(x;\gamma,\theta\right),\quad x>0, \tag{28}$$

where $\gamma>0,\theta>0$ are parameters.

Remark 1.13. *Taking* $G\left(x\right)=1-\cos\left[\frac{\pi\left(1-e^{-\gamma x^\theta}\right)}{2-e^{-\gamma x^\theta}}\right]$, $x\geq 0$, *the cdf* (27) *can be expressed as*

$$F\left(x;\gamma,\theta\right)=\frac{e^{G(x)}-1}{e-1},\quad x\geq 0,$$

which is the same as the cdf mentioned in Remark 1.1 in Hamedani et al. (2024b) for $\delta=1$.

15) The cdf and pdf of (UUTW) are given, respectively, by

$$F\left(x;\beta,\lambda\right)=1-\frac{1-e^{-\lambda(1-x)^\beta}}{1-e^{-\lambda}},\quad 0\leq x\leq 1, \tag{29}$$

and

$$f\left(x;\beta,\lambda\right)=\frac{d}{dx}F\left(x;\beta,\lambda\right),\quad 0<x<1, \tag{30}$$

where $\beta>0,\lambda>0$ are parameters.

Remark 1.14. *Taking* $G\left(x\right)=1-\left(1-x\right)^\beta$, $0\leq x\leq 1$, *the cdf* (29) *can be written as*

$$F\left(x;\beta,\lambda\right)=\frac{e^{\lambda G(x)}-1}{e^\lambda-1},\quad 0\leq x\leq 1,$$

which is the same as the cdf mentioned in Remark 1.1 in Hamedani et al. (2024b).

16) The cdf and pdf of (NMC-G) are given, respectively, by

$$F(x;\xi) = 1 - \left(\frac{1 - \cos\left[\frac{\pi}{2}\overline{G}(x;\xi)\right]}{1 + \cos\left[\frac{\pi}{2}\overline{G}(x;\xi)\right]} \right)^2, \quad x \in \mathbb{R}, \tag{31}$$

and

$$f(x;\xi) = \frac{d}{dx} F(x;\xi), \quad x \in \mathbb{R}, \tag{32}$$

where $\overline{G}(x;\xi) = 1 - G(x;\xi)$ and $G(x;\xi)$ is a baseline cdf which depends on the parameter vector ξ.

Remark 1.15. *Taking* $K(x) = 1 - \left(\frac{1-\cos\left[\frac{\pi}{2}\overline{G}(x;\xi)\right]}{1+\cos\left[\frac{\pi}{2}\overline{G}(x;\xi)\right]} \right)$, $x \in \mathbb{R}$*, the cdf* (31) *can be expressed as*

$$F(x;\xi) = 1 - (1 - K(x))^2, \quad x \in \mathbb{R},$$

which is a special case of the cdf mentioned in Remark 1.3 in Hamedani et al. (2024b).

17) The cdf and pdf of (NGE) are given, respectively, by

$$F(x_N;\delta_N,\upsilon_N) = \left(1 - \exp\left\{-\frac{x_N}{\upsilon_N}\right\}\right)^{\delta_N}, \quad x_N \geq 0, \tag{33}$$

and

$$f(x_N;\delta_N,\upsilon_N) = \frac{d}{dx_N} F(x_N;\delta_N,\upsilon_N), \quad x_N > 0, \tag{34}$$

where $\delta_N > 0, \upsilon_N > 0$ are parameters.

Remark 1.16. *As far as the characterizations of the cdf* (33) *are concerned, the cdf* (33) *is a generalized exponential distribution.*

18) The cdf and pdf of (NGR) are given, respectively, by

$$F(x_N;\upsilon_N,\sigma_N) = \left(1 - \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\}\right)^{\upsilon_N}, \quad x_N \geq 0, \tag{35}$$

and

$$f(x_N;\upsilon_N,\upsilon_N) = \frac{d}{dx_N} F(x_N;\delta_N,\upsilon_N), \quad x_N > 0, \tag{36}$$

where $\upsilon_N > 0, \sigma_N > 0$ are parameters.

Remark 1.17. *As far as the characterizations of the cdf* (35) *are concerned, the cdf* (35) *is a generalized Rayleigh distribution.*

19) The cdf and pdf of (NMS-G) are given, respectively, by

$$F(x;\lambda) = 1 - \left(\frac{1 - \sin\left[\frac{\pi}{2}G(x;\lambda)\right]}{1 + \sin\left[\frac{\pi}{2}G(x;\lambda)\right]} \right)^2, \quad x \in \mathbb{R}, \tag{37}$$

and

$$f(x;\lambda) = \frac{d}{dx} F(x;\lambda), \quad x \in \mathbb{R}, \tag{38}$$

where $G(x;\lambda)$ is a baseline cdf which depends on the parameter vector λ.

Remark 1.18. *Taking* $K(x) = 1 - \left(\frac{1-\sin\left[\frac{\pi}{2}G(x;\lambda)\right]}{1+\sin\left[\frac{\pi}{2}G(x;\lambda)\right]}\right)$, $x \in \mathbb{R}$, *the cdf* (37) *can be written as*

$$F(x;\lambda) = 1 - (1 - K(x))^2, \quad x \in \mathbb{R},$$

which is a special case of the cdf mentioned in Remark 1.3 in Hamedani et al. (2024b).

20) The cdf and pdf of (NT-G) are given, respectively, by

$$F(x) = 1 - \left(\frac{1 - \tan\left[\frac{\pi}{4}G(x)\right]}{1 + \tan\left[\frac{\pi}{4}G(x)\right]}\right)^2, \quad x \in \mathbb{R}, \tag{39}$$

and

$$f(x) = \frac{d}{dx}F(x), \quad x \in \mathbb{R}, \tag{40}$$

where $G(x)$ is a baseline cdf.

Remark 1.19. *Taking* $K(x) = 1 - \left(\frac{1-\tan\left[\frac{\pi}{4}G(x)\right]}{1+\tan\left[\frac{\pi}{4}G(x)\right]}\right)$, $x \in \mathbb{R}$, *the cdf* (39) *can be expressed as*

$$F(x) = 1 - (1 - K(x))^2, \quad x \in \mathbb{R},$$

which is a special case of the cdf mentioned in Remark 1.3 in Hamedani et al. (2024b).

21) The cdf and pdf of (SME) are given, respectively, by

$$F(x;\lambda,p) = \int_0^x f(u;\lambda,p)\,du, \quad x \geq 0, \tag{41}$$

and

$$f(x;\lambda,p) = \lambda e^{-2\lambda x}P(x), \quad x > 0, \tag{42}$$

where $\lambda > 0, p > 0$ are parameters, $P(x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 \upsilon^{a-1}(1-\upsilon)^{b-a-1}e^{x\upsilon}d\upsilon$, $b > a > 0$ and $\Gamma(a)$ is the gamma function.

Remark 1.20. *The pdf* (42) *is similar to the pdf* (16).

22) The cdf and pdf of (MUHN) are given, respectively, by

$$F(x;\sigma) = \int_0^x f(u;\sigma)\,du, \quad 0 \leq x \leq 1, \tag{43}$$

and

$$f(x;\sigma) = \frac{2}{\sigma}x^{-2}P(x), \quad 0 < x < 1, \tag{44}$$

where $\sigma > 0$ is a parameter, $P(x) = \phi\left(\frac{1-x}{\sigma x}\right)$ and $\phi(x)$ is the pdf of the standard normal random variable.

23) The cdf and pdf of (TPS) are given, respectively, by

$$F(x;\sigma,\mu) = \int_0^x f(u;\sigma,\mu)\,du, \quad x \geq 0, \tag{45}$$

and

$$f(x;\sigma,\mu) = Cg_0\left(\frac{x-\mu}{\sigma}\right), \quad x > 0, \tag{46}$$

where $\sigma > 0, \mu \in \mathbb{R}$ are parameters, $g_0(x)$ is a symmetric around zero pdf with the corresponding cdf $G_0(x)$ and $C = \frac{1}{\sigma G_0\left(\frac{\mu}{\sigma}\right)}$.

24) The cdf and pdf of (SWL) are given, respectively, by

$$F(x;\alpha,\beta,p) = \int_0^x f(u;\alpha,\beta,p)\,du, \quad x \geq 0, \tag{47}$$

and

$$f(x;\alpha,\beta,p) = Cx^{-(p+1)}P(x), \quad x > 0, \tag{48}$$

where α, β, p are all positive parameters, $C = \frac{p}{(\alpha+\beta)\Gamma(\beta)\alpha^p}$, $\Gamma(\beta)$ is the gamma function, $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ and $P(x) = \alpha\gamma(\beta+p,\alpha x) + \gamma(\beta+p+1,\alpha x)$.

Remark 1.21. *The pdf* (48) *is similar to the pdf* (44).

25) The cdf and pdf of (MS_tN) are given, respectively, by

$$F(x;\lambda,\nu) = \int_{-\infty}^x f(u;\lambda,\nu)\,du, \quad x \in \mathbb{R}, \tag{49}$$

and

$$f(x;\lambda,\nu) = 2t_\nu(x)P(x), \quad x \in \mathbb{R}, \tag{50}$$

where $\lambda \in \mathbb{R}, \nu \in \mathbb{N}$ are parameters, $t_\nu(x)$ is the pdf of the t-distribution with ν degrees of freedom, $P(x) = \Phi\left(\frac{\lambda x}{\sqrt{1+x^2}}\right)$ and $\Phi(x)$ is the cdf of the standard normal random variable.

Remark 1.22. *The pdf* (50) *is similar to the pdf* (24).

26) The cdf and pdf of (MG) are given, respectively, by

$$F(x;\alpha,\beta,\lambda) = \int_0^x f(u;\alpha,\beta,\lambda)\,du, \quad x \geq 0, \tag{51}$$

and

$$f(x;\alpha,\beta,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-x^\alpha}P(x), \quad x > 0, \tag{52}$$

where α, β, λ are all positive parameters, $P(x) = \exp\left\{x^\alpha + \alpha\beta x - \lambda x e^{\beta x}\right\}(1+\beta x)$ and $\Gamma(\alpha)$ is the gamma function.

27) The cdf and pdf of (GAPEIEx) are given, respectively, by

$$F(x;\alpha,a,b,k) = \left(\frac{\alpha^{e^{-ak/x}} - 1}{\alpha - 1}\right)^b, \quad x \geq 0, \tag{53}$$

and

$$f(x;\alpha,a,b,k) = \frac{d}{dx}F(x;\alpha,a,b,k), \quad x > 0, \tag{54}$$

where $\alpha\,(\neq 1), a, b, k$ are all positive parameters.

Remark 1.23. *The cdf* (53) *is simply the b power of the cdf* (9).

28) The cdf and pdf of (EPCJ) are given, respectively, by

$$F(x;c,\theta,\alpha)=\int_0^x f(u;c,\theta,\alpha)\,du,\quad x\geq 0, \tag{55}$$

and

$$f(x;c,\theta,\alpha)=Cx^{\alpha-1}e^{-\theta x^{\alpha}}P(x),\quad x>0, \tag{56}$$

where c,θ,α are all positive parameters, $C=\frac{c\alpha\theta^2}{\theta+2}$ and $P(x)=\left(1+\theta x^{2\alpha}\right)\left\{1-\left[1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta+2}\right]e^{-\theta x^{\alpha}}\right\}^{c-1}$.

Remarks 1.24. *(a)The pdf* (56) *is similar to the pdf* (52). (b) *We have*

$$F(x;c,\theta,\alpha)=\left\{1-\left[1+\frac{\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta+2}\right]e^{-\theta x^{\alpha}}\right\}^{c}=\{G(x)\}^{c},\quad x\geq 0,$$

which is simply c power of a distribution.

29) The cdf and pdf of (NGOF-OE-G) are given, respectively, by

$$F(x;\alpha,\beta,\delta,\gamma,\phi)=\exp\left\{-\left(\alpha\left(\left(1-\exp\left\{-\frac{\delta\left(1-e^{-\frac{\phi x^2}{2}}\right)}{e^{-\frac{\phi x^2}{2}}}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\},\quad x\geq 0, \tag{57}$$

and

$$f(x;\alpha,\beta,\delta,\gamma,\phi)=\frac{d}{dx}F(x;\alpha,\beta,\delta,\gamma,\phi),\quad x>0, \tag{58}$$

where $\alpha,\beta,\delta,\gamma,\phi$ are all positive parameters.

Remark 1.25. *Taking* $G(x)=\left(1-\exp\left\{-\frac{\delta\left(1-e^{-\frac{\phi x^2}{2}}\right)}{e^{-\frac{\phi x^2}{2}}}\right\}\right)^{\gamma}$, $x\geq 0$, *the cdf* (57) *can be written as*

$$F(x;\alpha,\beta,\delta,\gamma,\phi)=\exp\left\{-\left(\alpha\left(\left[\frac{1-G(x)}{G(x)}\right]\right)\right)^{\beta}\right\},\quad x\geq 0,$$

which was mentioned in Remark 1.32 in Hamedani (2023).

30) The cdf and pdf of (SABUR) are given, respectively, by

$$F(x;\alpha,\beta)=\int_0^x f(u;\alpha,\beta)\,du,\quad x\geq 0, \tag{59}$$

and

$$f(x;\alpha,\beta)=Ce^{-\beta x}P(x),\quad x>0, \tag{60}$$

where $\alpha>0,\beta>0$ are parameters, $C=\frac{\beta^2}{\alpha\beta+\beta^2+1}$ and $P(x)=\alpha+\beta+\frac{\beta}{2}x^2$.

Remark 1.26. *As far as characterizations are concerned, the pdf* (60) *is a special case of the pdf* (4) *for* $k=1$.

31) The cdf and pdf of (QTF) are given, respectively, by

$$F(x;\alpha,\beta)=\int_0^x f(u;\alpha,\beta)\,du,\quad x\geq 0, \tag{61}$$

and

$$f(x;\alpha,\beta)=\alpha\beta^{\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}P(x), \quad x>0, \tag{62}$$

where $\alpha>0,\beta>0$ are parameters, $P(x)=a_1+2a_2e^{-\left(\frac{x}{\beta}\right)^{\alpha}}+3a_3e^{-2\left(\frac{x}{\beta}\right)^{\alpha}}+4a_4e^{-3\left(\frac{x}{\beta}\right)^{\alpha}}$, $a_1=2\lambda_1, a_2=3(\lambda_2-\lambda_1), a_3=2(\lambda_1-2\lambda_2+\lambda_3), a_4=1-\lambda_1+\lambda_2-2\lambda_3$ and $\lambda_1\in(0,2), \lambda_2\in(0,2), \lambda_3\in(0,2), 0\leq\lambda_1+\lambda_2+\lambda_3\leq 2$.

Remark 1.27. *As far as characterizations are concerned, the pdf* (62) *is similar to the pdf* (52).

32) The cdf and pdf of (NCTPF) are given, respectively, by

$$F(x;\alpha,\theta,\lambda_1,\lambda_2,\lambda_3)=(1+\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}}+(\lambda_2-\lambda_1)\frac{x^{2\alpha}}{\theta^{2\alpha}}-\lambda_2\frac{x^{3\alpha}}{\theta^{3\alpha}}, \quad 0\leq x\leq\theta, \tag{63}$$

and

$$f(x;\alpha,\theta,\lambda_1,\lambda_2,\lambda_3)=\frac{d}{dx}F(x;\alpha,\theta,\lambda_1,\lambda_2,\lambda_3), \quad 0<x<\theta, \tag{64}$$

where $\alpha>0,\beta>0,\lambda_1\in[-1,1], \lambda_2\in[-1,1], -2\leq\lambda_1+\lambda_2\leq 1$ are parameters.

Remark 1.28. *The cdf* (63) *is a special case of the cdf* (101) *in Hamedani (2021) for* $b=1$.

33) The cdf and pdf of (TLKMO-G) are given, respectively, by

$$F(x;a,b,\alpha,p,\sigma)=\left(1-\left\{1-\left(\frac{R(x;\sigma)}{1-p\overline{R}(x;\sigma)}\right)^{a}\right\}^{2b}\right)^{\alpha}, \quad x\in\mathbb{R}, \tag{65}$$

and

$$f(x;a,b,\alpha,p,\sigma)=\frac{d}{dx}F(x;a,b,\alpha,p,\sigma), \quad x\in\mathbb{R}, \tag{66}$$

where $a>0,b>>0,\alpha>0,p\in(0,1)$ are parameters, $\overline{R}(x;\sigma)=1-R(x;\sigma)$ and $R(x;\sigma)$ is a baseline cdf which depends on the parameter σ.

Remark 1.29. *Taking* $G(x)=\left(\frac{R(x;\sigma)}{1-p\overline{R}(x;\sigma)}\right)^{a}$, $x\in\mathbb{R}$*, the cdf* (65) *can be expressed as*

$$F(x;a,b,\alpha,p,\sigma)=\left\{1-(1-G(x))^{2b}\right\}^{\alpha}, \quad x\in\mathbb{R},$$

which is a special case of the cdf mentioned in Remark 1.3 in Hamedani et al. (2024b).

34) The cdf and pdf of (MCT-Rayleigh) are given, respectively, by

$$F(x;\sigma,\lambda,\theta)=\int_0^x f(u;\sigma,\lambda,\theta)\,du, \quad x\geq 0, \tag{67}$$

and

$$f(x;\sigma,\lambda,\theta)=\frac{1}{\sigma}xe^{-\frac{3x^2}{2\sigma^2}}P(x), \quad x>0, \tag{68}$$

where $\sigma>0,\lambda\in[-1,1],\theta\in[0,2]$ are parameters and $P(x)=2(\theta+1)\lambda e^{\frac{x^2}{\sigma^2}}-3\theta\lambda$.

Remark 1.30. *As far as characterizations are concerned, the pdf* (68) *is similar to the pdfs* (52) *and* (62).

35) The cdf and pdf of (NLT) are given, respectively, by

$$F\left(x;\rho\right)=\frac{\log\left[e+G\left(x;\rho\right)\right]-1}{C},\quad x\in\mathbb{R},\tag{69}$$

and

$$f\left(x;\rho\right)=\frac{d}{dx}F\left(x;\rho\right),\quad x\in\mathbb{R},\tag{70}$$

where $C=\log\left(e+1\right)-1$ and ρ is a parameter and $G\left(x;\rho\right)$ is a baseline cdf which depends on the parameter vector ρ.

Remark 1.31. *As far as characterizations are concerned, the cdf* (69) *is similar to the cdf mentioned in Remark 1.18 in Hamedani et al. (2024b).*

36) The cdf and pdf of (NEXLBE) are given, respectively, by

$$F\left(x;\alpha,\beta\right)=1-\left(\frac{1-\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{2}}{1-\left(1-\alpha\right)\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{2}}\right)^{\alpha},\quad x\geq 0,\tag{71}$$

and

$$f\left(x;\alpha,\beta\right)=\frac{d}{dx}F\left(x;\alpha,\beta\right),\quad x>0,\tag{72}$$

where $\alpha>0,\beta>0$ are parameters.

Remark 1.32. *Taking* $K\left(x\right)=1-\left(\frac{1-\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{2}}{1-\left(1-\alpha\right)\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{2}}\right)$, $x\geq 0$, *the cdf* (71) *can be written as*

$$F\left(x;\alpha,\beta\right)=1-\left(1-G\left(x\right)\right)^{\alpha},\quad x\geq 0,$$

which is a special case of the cdf mentioned in Remark 1.3 in Hamedani et al. (2024b).

37) The cdf and pdf of (AIW) are given, respectively, by

$$F\left(x;\alpha,\beta\right)=\frac{4}{\pi}\arctan\left[e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right],\quad x\geq 0,\tag{73}$$

and

$$f\left(x;\alpha,\beta\right)=\frac{d}{dx}F\left(x;\alpha,\beta\right),\quad x>0,\tag{74}$$

where $\alpha>0,\beta>0$ are parameters.

Remark 1.33. *Taking* $K\left(x\right)=e^{-\left(\frac{\alpha}{x}\right)^{\beta}}$, $x\geq 0$, *the cdf* (73) *can be written as*

$$F\left(x;\alpha,\beta\right)=\frac{4}{\pi}\arctan\left[K\left(x\right)\right],\quad x\geq 0,$$

which was mentioned in Remark 1.384 in Hamedani (2023).

38) The cdf and pdf of (RTFIW) are given, respectively, by

$$F\left(x;\alpha,\beta,b\right)=e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)},\quad 0\leq x\leq b,\tag{75}$$

and

$$f(x;\alpha,\beta,b)=\alpha\beta x^{-(\beta+1)}e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)},\quad 0<x<b, \tag{76}$$

where α,β,b are all positive parameters.

39) The cdf and pdf of (UGPS) are given, respectively, by

$$F(x;\xi,\zeta,\gamma)=1-\frac{C(\gamma(1-G(x)))}{C(\gamma)},\quad 0\leq x\leq 1, \tag{77}$$

and

$$f(x;\xi,\zeta,\gamma)=\frac{d}{dx}F(x;\xi,\zeta,\gamma),\quad 0<x<1, \tag{78}$$

where ξ,ζ,γ are all positive parameters, $G(x)=e^{-\xi\left(x^{-\zeta}-1\right)}$, $0\leq x\leq 1$, $C(\gamma)=\sum_{n=1}^{\infty}a_n\gamma^n$ is finite and $\{a_n\}_{n\geq 1}$ is a sequence of non-negative integers.

Remark 1.34. *The version of the cdf* (77) *has been characterized in Hamedani (2017) and also mentioned in Remark 1.14 in Hamedani et al. (2024b).*

40) The cdf and pdf of (TIPA) are given, respectively, by

$$F(x;\alpha,\beta)=\int_0^x f(u;\alpha,\beta)\,du,\quad x\geq 0, \tag{79}$$

and

$$f(x;\alpha,\beta)=Cx^{-\beta-1}e^{-\alpha x^{-\beta}}P(x),\quad x>0, \tag{80}$$

where $\alpha>0,\beta>0$ are parameters, $C=\frac{\alpha^2}{(1+\alpha)e^{-\alpha}}$ and $P(x)=x^{-\beta}$.

Remark 1.35. *The pdf* (80) *was mentioned in Remark 1.47 in Hamedani et al. (2024b).*

41) The cdf and pdf of (GDBE) are given, respectively, by

$$F(x;\alpha,\beta,\gamma,\lambda)=1-e^{-\lambda\left(\frac{x-\beta}{\gamma-x}\right)^{\alpha}},\quad \beta\leq x\leq\gamma, \tag{81}$$

and

$$f(x;\alpha,\beta,\gamma,\lambda)=C(x-\beta)^{\alpha-1}P(x),\quad \beta<x<\gamma, \tag{82}$$

where $\alpha>0,\beta\geq 0,\beta<\gamma,\gamma>0,\lambda>0$ are parameters, $C=\alpha\lambda(\gamma-\beta)$ and $P(x)=\frac{e^{-\lambda\left(\frac{x-\beta}{\gamma-x}\right)}}{(\gamma-x)^{\alpha+1}}$.

42) The cdf and pdf of (BXII-G) are given, respectively, by

$$F(x;a,b,c,k,\xi)=1-\left[1+\left(\frac{-\log[1-G(x)]}{a}\right)^{c}\right]^{-k},\quad x\in\mathbb{R}, \tag{83}$$

and

$$f(x;a,b,c,k,\xi)=\left(\frac{ck}{a}\right)g(x)P(x),\quad x\in\mathbb{R}, \tag{84}$$

where $a>0,b>0,c>0,k>0,\xi=(\theta_1,\theta_2,...,\theta_p)^T$ are parameters, $G(x)$ is a baseline cdf with the corresponding pdf $g(x)$ and $P(x)=\frac{1}{1-G(x)}\left(\frac{-\log[1-G(x)]}{a}\right)^{c-1}\left[1+\left(\frac{-\log[1-G(x)]}{a}\right)^{c}\right]^{-(k+1)}$.

Remark 1.36. *The pdf* (84)*, as far as our characterizations are concerned, is similar to the cdf* (24).

43) The cdf and pdf of (CFrL) are given, respectively, by

$$F\left(x;\alpha,\beta,\sigma\right)=1-\cos\left[\frac{\pi}{2}G\left(x\right)\right],\quad x\geq 0, \tag{85}$$

and

$$f\left(x;\alpha,\beta,\sigma\right)=\frac{d}{dx}F\left(x;\alpha,\beta,\sigma\right),\quad x>0, \tag{86}$$

where α,β,σ are all positive parameters and $G\left(x\right)=1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}$, $x\geq 0$.

Remark 1.37. *The cdf* (85)*, as far as our characterizations are concerned, is similar to the cdf mentioned in Remark 1.16 in Hamedani (2023).*

44) The cdf and pdf of (SBIIIL) are given, respectively, by

$$F\left(x;c,k,\sigma\right)=\sin\left[\frac{\pi}{2}G\left(x\right)\right],x\geq 0, \tag{87}$$

and

$$f\left(x;c,k,\sigma\right)=\frac{d}{dx}F\left(x;c,k,\sigma\right),\quad x>0, \tag{88}$$

where c,k,σ are all positive parameters and $G\left(x\right)=1-\frac{\sigma\left(1-\left(1+x^{-c}\right)^{-k}\right)}{\sigma-\log\left(1-\left(1+x^{-c}\right)^{-k}\right)}$, $x\geq 0$.

Remark 1.38. *The cdf* (87)*, as far as our characterizations are concerned, was mentioned in Remark 1.7 in Hamedani et al. (2024a).*

45) The cdf and pdf of (EGWE) are given, respectively, by

$$F\left(x;a,b,c,d,\alpha,\beta\right)=\left[G\left(x\right)\right]^{\beta},\quad x\geq 0, \tag{89}$$

and

$$f\left(x;a,b,c,d,\alpha,\beta\right)=\frac{d}{dx}F\left(x;a,b,c,d,\alpha,\beta\right),\quad x>0, \tag{90}$$

where a,b,c,d,α,β are all positive parameters and $G\left(x\right)=1-\exp\left\{-\alpha\left(\frac{abx}{c}\right)^{d}\right\}$, $x\geq 0$.

Remark 1.39. *The cdf* (89)*, as far as our characterizations are concerned, is the β power of a cdf which was characterized in Hamedani (2016).*

46) The cdf and pdf of (OWL) are given, respectively, by

$$F\left(x;\alpha,\lambda,\theta\right)=1-\exp\left\{-\lambda\left(\frac{\theta+1}{\theta+1+\theta x}e^{-\theta x}-1\right)^{\alpha}\right\},\quad x\geq 0, \tag{91}$$

and

$$f\left(x;\alpha,\lambda,\theta\right)=C\left(\theta+1+\theta x\right)^{-2}P\left(x\right),\quad x>0, \tag{92}$$

where α,λ,θ are all positive parameters, $C=\alpha\lambda\theta^{2}\left(\theta+1\right)$ and
$P\left(x\right)=\left\{\left(1+x\right)e^{\theta x}\left(\frac{\theta+1}{\theta+1+\theta x}e^{-\theta x}-1\right)^{\alpha-1}\right\}\exp\left\{-\lambda\left(\frac{\theta+1}{\theta+1+\theta x}e^{-\theta x}-1\right)^{\alpha}\right\}$.

47) The cdf and pdf of (AWW) are given, respectively, by

$$F\left(x;\lambda,\gamma\right)=\int_{0}^{x}f\left(u;\lambda,\gamma\right)du,\quad x\geq 0, \tag{93}$$

and

$$f\left(x;\lambda,\gamma\right)=Cx^{\gamma-1}e^{-\lambda x^{\gamma}}P\left(x\right),\quad x>0, \tag{94}$$

where $\lambda>0,\gamma>0$ are parameters, $C=\frac{2^{1/\gamma}\lambda^{1/\gamma}}{2^{(1+1/\gamma)}-\Gamma(1+1/\gamma)}$, $\Gamma\left(a\right)$ is a gamma function and $P\left(x\right)=x^{1-\gamma}\left(2-e^{-\lambda x^{\gamma}}\right)$.

48) The cdf and pdf of (OEL-G) are given, respectively, by

$$F\left(x;\theta,\lambda,\tau\right)=1-\frac{\log\left\{1-\left(1-\theta\right)e^{-\lambda\left[\frac{G(x;\tau)}{1-G(x;\tau)}\right]}\right\}}{\log\left(\theta\right)},\quad x\in\mathbb{R}, \tag{95}$$

and

$$f\left(x;\theta,\lambda,\tau\right)=\frac{d}{dx}F\left(x;\theta,\lambda,\tau\right),\quad x\in\mathbb{R}, \tag{96}$$

where θ,λ,τ are all positive parameters and $G\left(x;\tau\right)$ is a baseline cdf.

Remark 1.40. *Taking* $K\left(x\right)=1-e^{-\lambda\left[\frac{G(x;\tau)}{1-G(x;\tau)}\right]}$, $x\in\mathbb{R}$, *the cdf* (95) *can be written as*

$$F\left(x;\theta,\lambda,\tau\right)=1-\frac{\log\left\{\theta+\left(1-\theta\right)K\left(x\right)\right\}}{\log\left(\theta\right)},\quad x\in\mathbb{R},$$

which was mentioned in Remark 1.30 in Hamedani et al. (2024b) as well as in Remark 1.5 in Hamedani (2023).

49) The cdf and pdf of (EAPIE) are given, respectively, by

$$F\left(x;\alpha,\beta,a\right)=\left(\frac{\alpha^{\exp\left(-\frac{\beta}{x}\right)}-1}{\alpha-1}\right)^{a},\quad x\geq 0, \tag{97}$$

and

$$f\left(x;\alpha,\beta,a\right)=\frac{d}{dx}F\left(x;\alpha,\beta,a\right),\quad x>0, \tag{98}$$

where α,β,a are all positive parameters.

Remark 1.41. *The cdf* (97) *is not new, please see cdf* (53).

50) The cdf and pdf of (HL-TEXP) are given, respectively, by

$$F\left(x;\theta,a\right)=\frac{1-e^{-\theta(x-a)}}{1+e^{-\theta(x-a)}},\quad x\geq a, \tag{99}$$

and

$$f\left(x;\theta,a\right)=\frac{d}{dx}F\left(x;\theta,a\right),\quad x>a, \tag{100}$$

where $\theta>0,a>0$ are parameters.

Remark 1.42. *Taking* $G\left(x\right)=1-e^{-\theta(x-a)}$, $x\geq a$, *the cdf* (99) *can be written as*

$$F\left(x;\theta,a\right)=\frac{G\left(x\right)}{2-G\left(x\right)},\quad x\geq a,$$

which is a special case of the cdf mentioned in Remark 1.1 for $\lambda=\nu=1$.

51) The cdf and pdf of (EXL) are given, respectively, by

$$F\left(x;\alpha,\delta\right)=\left[1-\left(1+\frac{\delta x}{(1+\delta)}\right)e^{-\delta x}\right]^{\alpha},\quad x\geq 0, \tag{101}$$

and

$$f\left(x;\alpha,\delta\right)=\frac{d}{dx}F\left(x;\alpha,\delta\right),\quad x>0, \tag{102}$$

where $\alpha>0,\delta>0$ are parameters.

Remark 1.43. *Taking* $G\left(x\right)=1-\left(1+\frac{\delta x}{(1+\delta)}\right)e^{-\delta x}$, $x\geq 0$, *the cdf* (101) *can be expressed as*

$$F\left(x;\alpha,\delta\right)=\left[G\left(x\right)\right]^{\alpha},\quad x\geq 0,$$

which is simply α *power of the cdf* $G\left(x\right)$.

52) The cdf and pdf of (MEKwE) are given, respectively, by

$$F\left(x;\alpha,\lambda,\beta\right)=\int_{0}^{x}f\left(u;\alpha,\lambda,\beta\right)du,\quad x\geq 0, \tag{103}$$

and

$$f\left(x;\alpha,\lambda,\beta\right)=\beta\lambda e^{\alpha x}e^{-\lambda xe^{\alpha x}\left(1-e^{-\lambda x}\right)}\left[\lambda xe^{-\lambda x}+\left(1+\alpha x\right)\left(1-e^{-\lambda x}\right)\right]P\left(x\right),\quad x>0, \tag{104}$$

where α,λ,β are all positive parameters and $P\left(x\right)=\left\{1-e^{-\lambda xe^{\alpha x}\left(1-e^{-\lambda x}\right)}\right\}^{\beta-1}$.

53) The cdf and pdf of (EIC) are given, respectively, by

$$F\left(x;\alpha,\lambda,\theta\right)=1-\left[1-e^{\lambda\left(1-e^{x^{-\alpha}}\right)}\right]^{\theta},\quad x\geq 0, \tag{105}$$

and

$$f\left(x;\alpha,\lambda,\theta\right)=\frac{d}{dx}F\left(x;\alpha,\lambda,\theta\right),\quad x>0, \tag{106}$$

where α,λ,θ are all positive parameters.

Remark 1.44. *Taking* $G\left(x\right)=e^{\lambda\left(1-e^{x^{-\alpha}}\right)}$, $x\geq 0$, *the cdf* (105) *can be written as*

$$F\left(x;\alpha,\lambda,\theta\right)=1-\left[1-G\left(x\right)\right]^{\theta},\quad x\geq 0,$$

which was mentioned in Remark 1.3 in Hamedani et al. (2024b).

54) The cdf and pdf of (EKwE) are given, respectively, by

$$F\left(x;\lambda,\beta\right)=\left[1-\left(e^{-\lambda x}\right)^{\left(1-e^{-\lambda x}\right)}\right]^{\beta}=\left[1-e^{-\lambda x\left(1-e^{-\lambda x}\right)}\right]^{\beta},\quad x\geq 0, \tag{107}$$

and

$$f\left(x;\lambda,\beta\right)=\frac{d}{dx}F\left(x;\lambda,\beta\right),x>0, \tag{108}$$

where $\lambda>0,\beta>0$ are parameters.

Remark 1.45. *Taking $G(x) = \frac{x\left(1-e^{-\lambda x}\right)}{1+x\left(1-e^{-\lambda x}\right)}$, $x \geq 0$, the cdf* (107) *can be expressed as*

$$F(x;\lambda,\beta) = \left[1 - e^{-\lambda\left(\frac{G(x)}{1-G(x)}\right)}\right]^{\beta}, \quad x \geq 0,$$

which was mentioned in Remark 1.6.

55) The cdf and pdf of (NE-G) are given, respectively, by

$$F(x;\alpha,\beta) = \frac{[1+G(x;\beta)]^{\alpha} - 1}{2^{\alpha} - 1}, \quad x \in \mathbb{R}, \tag{109}$$

and

$$f(x;\alpha,\beta) = \frac{\alpha g(x;\beta)[1+G(x;\beta)]^{\alpha-1}}{2^{\alpha} - 1}, \quad x \in \mathbb{R}, \tag{110}$$

where $\alpha > 0$ is a parameter and $G(x;\beta)$ is a baseline cdf with the corresponding pdf $g(x;\beta)$ which depends on the parameter vector β.

56) The cdf and pdf of (IEEP) are given, respectively, by

$$\begin{aligned} F(x;\alpha,\beta,\lambda) &= 1 - \frac{1 - \exp\left\{-\lambda\left(1 - e^{-\beta/x}\right)^{\alpha}\right\}}{1 - e^{-\lambda}} \\ &= \frac{\exp\left\{\lambda[1 - \left(1 - e^{-\beta/x}\right)^{\alpha}]\right\} - 1}{e^{\lambda} - 1}, \quad x \geq 0, \end{aligned} \tag{111}$$

and

$$f(x;\alpha,\beta,\lambda) = \frac{d}{dx}F(x;\alpha,\beta,\lambda), \quad x > 0, \tag{112}$$

where α, β, λ are all positive parameters.

Remark 1.46. *Taking $G(x) = 1 - \left(1 - e^{-\beta/x}\right)^{\alpha}$, $x \geq 0$, the cdf* (111) *can be written as*

$$F(x;\alpha,\beta,\lambda) = \frac{\exp\{\lambda G(x)\} - 1}{e^{\lambda} - 1}, \quad x \geq 0,$$

which was mentioned in Remark 1.1 in Hamedani et al. (2024b).

2. Characterization of Distributions

As mentioned in the Introduction, characterizations of distributions is an important area of research which has recently attracted the attention of many researchers. This section deals with various characterizations of the distributions listed in the Introduction. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) the reverse hazard function and (iv) conditional expectation of a single function of the random variable. It should be mentioned that for the characterization (i) the cdf need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

2.1. Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of the distributions mentioned in the Introduction, in details, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem 2.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem 2.1. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that*

$$\mathbf{E}\left[q_2\left(X\right)\ |\ X \geq x\right] = \mathbf{E}\left[q_1\left(X\right)\ |\ X \geq x\right]\eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1\left(H\right)$, $\eta \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F\left(x\right) = \int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right)q_1\left(u\right) - q_2\left(u\right)}\right|\exp\left(-s\left(u\right)\right)\ du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\, q_1}{\eta\, q_1\, -\, q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Here is our first characterization.

Proposition 2.1. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1\left(x\right) = \left[P\left(x\right)\right]^{-1}$ and $q_2\left(x\right) = q_1\left(x\right)\left(1 - e^{-x/\lambda}\right)^k$ for $x > 0$. The random variable X has pdf* (4) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta\left(x\right) = \frac{1}{2}\left\{1 + \left(1 - e^{-x/\lambda}\right)^k\right\}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (4), then

$$\begin{aligned}\left(1 - F\left(x\right)\right)E\left[q_1\left(X\right)\ |\ X \geq x\right] &= \int_x^\infty Ce^{-u/\lambda}\left(1 - e^{-u/\lambda}\right)^{k-1}du \\ &= \frac{C\lambda}{k}\left\{1 - \left(1 - e^{-x/\lambda}\right)^k\right\}, \quad x > 0,\end{aligned}$$

and

$$\begin{aligned}\left(1 - F\left(x\right)\right)E\left[q_2\left(X\right)\ |\ X \geq x\right] &= \int_x^\infty Ce^{-u/\lambda}\left(1 - e^{-u/\lambda}\right)^{2k-1}du \\ &= \frac{C\lambda}{2k}\left\{1 - \left(1 - e^{-x/\lambda}\right)^{2k}\right\}, \quad x > 0,\end{aligned}$$

and finally

$$\eta\left(x\right)q_1\left(x\right) - q_2\left(x\right) = \frac{q_1\left(x\right)}{2}\left\{1 - \left(1 - e^{-x/\lambda}\right)^k\right\} > 0 \quad \text{for} \ \ x > 0.$$

Conversely, if η is given as above, then

$$s'\left(x\right) = \frac{\eta'\left(x\right)q_1\left(x\right)}{\eta\left(x\right)q_1\left(x\right) - q_2\left(x\right)} = \frac{\frac{k}{\lambda}e^{-x/\lambda}\left(1 - e^{-x/\lambda}\right)^{k-1}}{1 - \left(1 - e^{-x/\lambda}\right)^k}, \quad x > 0,$$

and hence

$$s\left(x\right) = -\log\left\{1 - \left(1 - e^{-x/\lambda}\right)^k\right\}, \quad x > 0.$$

Now, in view of Theorem 2.1, X has density (4). □

Corollary 2.1. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1. The pdf of X is* (4) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\frac{k}{\lambda} e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}}{1-\left(1-e^{-x/\lambda}\right)^{k}}, \quad x>0.$$

Corollary 2.2. *The general solution of the differential equation in Corollary 2.1 is*

$$\eta(x) = \left\{1-\left(1-e^{-x/\lambda}\right)^{k}\right\}^{-1}\left[-\int \frac{k}{\lambda} e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}(q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. If X has pdf (4), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \left(\frac{\frac{k}{\lambda} e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}}{1-\left(1-e^{-x/\lambda}\right)^{k}}\right)\eta(x) - \left(\frac{\frac{k}{\lambda} e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}}{1-\left(1-e^{-x/\lambda}\right)^{k}}\right)(q_1(x))^{-1} q_2(x),$$

or

$$\begin{aligned}&\left\{1-\left(1-e^{-x/\lambda}\right)^{k}\right\}\eta'(x) - \left(\frac{k}{\lambda} e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}\right)\eta(x)\\ &= -\left(\frac{k}{\lambda} e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}\right)(q_1(x))^{-1} q_2(x),\end{aligned}$$

or

$$\frac{d}{dx}\left\{\left\{1-\left(1-e^{-x/\lambda}\right)^{k}\right\}\eta(x)\right\} = -\left(\frac{k}{\lambda} e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}\right)(q_1(x))^{-1} q_2(x),$$

from which we arrive at

$$\eta(x) = \left\{1-\left(1-e^{-x/\lambda}\right)^{k}\right\}^{-1}\left[-\int \frac{k}{\lambda} e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)^{k-1}(q_1(x))^{-1} q_2(x)\, dx + D\right].$$

□

Note that a set of functions satisfying the differential equation in Corollary 2.1, is given in Proposition 2.1 with $D=\frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.2. *Let $X : \Omega \to (x_l, b)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)(b-x)$ for $x \in (x_l, b)$. The random variable X has pdf* (14) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{2}{3}(b-x), \quad x \in (x_l, b).$$

Proof. Let X be a random variable with pdf (14), then

$$\begin{aligned}(1-F(x))\, E[q_1(X) \mid X \geq x] &= \int_x^b C(b-u)\, du\\ &= \frac{C}{2}(b-x)^2, \quad x \in (x_l, b),\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]&=\int_x^b C\,(b-u)^2\,du\\&=\frac{C}{3}\,(b-x)^3\,,\quad x\in(x_l,b)\,,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{3}\,(b-x)<0\quad\text{for}\;\;x\in(x_l,b)\,.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{2}{b-x},\quad x\in(x_l,b)\,,$$

and hence

$$s(x)=-2\log(b-x)\,,\quad x\in(x_l,b)\,.$$

Now, in view of Theorem 2.1, X has density (14). □

Corollary 2.3. *Let $X:\Omega\rightarrow\in(x_l,b)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.2. The pdf of X is* (14) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{2}{b-x},\quad x\in(x_l,b)\,.$$

Corollary 2.4. *The general solution of the differential equation in Corollary 2.3 is*

$$\eta(x)=(b-x)^{-2}\left[-\int 2\,(b-x)\,(q_1(x))^{-1}\,q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. If X has pdf (14), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x)=2\,(b-x)^{-1}\,\eta(x)-2\,(b-x)^{-1}\,(q_1(x))^{-1}\,q_2(x)\,,$$

or

$$(b-x)^2\,\eta'(x)-2\,(b-x)\,\eta(x)=-2\,(b-x)\,(q_1(x))^{-1}\,q_2(x)\,,$$

or

$$\frac{d}{dx}\left\{(b-x)^2\,\eta(x)\right\}=-2\,(b-x)\,(q_1(x))^{-1}\,q_2(x)\,,$$

from which we arrive at

$$\eta(x)=(b-x)^{-2}\left[-\int 2\,(b-x)\,(q_1(x))^{-1}\,q_2(x)\,dx+D\right].$$

□

Note that a set of functions satisfying the differential equation in Corollary 2.3, is given in Proposition 2.2 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.3. *Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,G(x;\sigma,\xi)$ for $x>0$. The random variable X has pdf* (24) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{1 + G(x;\sigma,\xi)\right\}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (24), then

$$\begin{aligned}(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] &= \int_x^\infty g(u;\sigma,\xi)\, du \\ &= 1 - G(x;\sigma,\xi), \quad x > 0,\end{aligned}$$

and

$$\begin{aligned}(1 - F(x))\, E\left[q_2(X) \mid X \geq x\right] &= \int_x^\infty g(u;\sigma,\xi)\, G(u;\sigma,\xi)\, du \\ &= \frac{1}{2}\left\{1 - G(x;\sigma,\xi)^2\right\}, \quad x > 0,\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left\{1 - G(x;\sigma,\xi)\right\} > 0 \quad \text{for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{g(x;\sigma,\xi)}{1 - G(x;\sigma,\xi)}, \quad x > 0,$$

and hence

$$s(x) = -\log\left\{1 - G(x;\sigma,\xi)\right\}, \quad x > 0.$$

Now, in view of Theorem 2.1, X has density (24). □

Corollary 2.5. *Let $X : \Omega \to\in (o,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.3. The pdf of X is* (24) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{g(x;\sigma,\xi)}{1 - G(x;\sigma,\xi)}, \quad x > 0.$$

Corollary 2.6. *The general solution of the differential equation in Corollary 2.5 is*

$$\eta(x) = \left\{1 - G(x;\sigma,\xi)\right\}^{-1}\left[-\int g(x;\sigma,\xi)\,(q_1(x))^{-1}\, q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.5, is given in Proposition 2.3 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.4. *Let $X : \Omega \to (0,1)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, x^{-1}$ for $0 < x < 1$. The random variable X has pdf* (44) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{1 + x^{-1}\right\}, \quad 0 < x < 1.$$

Proof. Let X be a random variable with pdf (44), then

$$\begin{aligned}(1-F(x))\,E[q_1(X)\mid X\geq x] &= \int_x^1 \frac{2}{\sigma}u^{-2}du\\ &= \frac{2}{\sigma}\left\{x^{-1}-1\right\},\quad 0<x<1,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E[q_2(X)\mid X\geq x] &= \int_x^1 \frac{2}{\sigma}u^{-3}du\\ &= \frac{1}{\sigma}\left\{x^{-2}-1\right\},\quad 0<x<1,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x) = -\frac{q_1(x)}{2}\left\{x^{-1}-1\right\}<0 \quad \text{for } 0<x<1.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)} = \frac{x^{-2}}{x^{-1}-1},\quad 0<x<1,$$

and hence

$$s(x) = -\log\left(x^{-1}-1\right),\quad 0<x<1.$$

Now, in view of Theorem 2.1, X has density (44). □

Corollary 2.7. *Let $X:\Omega\to(0,1)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.4. The pdf of X is* (44) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)} = \frac{x^{-2}}{x^{-1}-1},\quad 0<x<1.$$

Corollary 2.8. *The general solution of the differential equation in Corollary 2.7 is*

$$\eta(x) = \left(x^{-1}-1\right)^{-1}\left[-\int x^{-2}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.7, is given in Proposition 2.4 with $D=\frac{1}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.5. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)\equiv 1$ and $q_2(x)=G_0\left(\frac{x-\mu}{\sigma}\right)$ for $x>0$. The random variable X has pdf* (46) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{1+G_0\left(\frac{x-\mu}{\sigma}\right)\right\},\quad x>0.$$

Proof. Let X be a random variable with pdf (46), then

$$\begin{aligned}(1-F(x))\,E\left[q_1(X)\mid X\geq x\right] &= \int_x^\infty Cg_0\left(\frac{u-\mu}{\sigma}\right)du\\ &= C\sigma\left\{1-G_0\left(\frac{x-\mu}{\sigma}\right)\right\},\quad x>0,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^\infty Cg_0\left(\frac{u-\mu}{\sigma}\right)G_0\left(\frac{u-\mu}{\sigma}\right)du\\ &= \frac{C\sigma}{2}\left\{1-G_0\left(\frac{x-\mu}{\sigma}\right)^2\right\},\quad x>0,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-G_0\left(\frac{x-\mu}{\sigma}\right)\right\}>0\quad\text{for}\ \ x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{g_0\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left\{1-G_0\left(\frac{x-\mu}{\sigma}\right)\right\}},\quad x>0,$$

and hence

$$s(x)=-\log\left(1-G_0\left(\frac{x-\mu}{\sigma}\right)\right),\quad x>0.$$

Now, in view of Theorem 2.1, X has density (46). □

Corollary 2.9. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.5. The pdf of X is* (46) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{g_0\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left\{1-G_0\left(\frac{x-\mu}{\sigma}\right)\right\}},\quad x>0.$$

Corollary 2.10. *The general solution of the differential equation in Corollary 2.9 is*

$$\eta(x)=\left\{1-G_0\left(\frac{x-\mu}{\sigma}\right)\right\}^{-1}\left[-\int\frac{1}{\sigma}g_0\left(\frac{x-\mu}{\sigma}\right)(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.9, is given in Proposition 2.5 with $D=\frac{1}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.6. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-x^\alpha}$ for $x>0$. The random variable X has pdf* (52) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}e^{-x^\alpha},\quad x>0.$$

Proof. Let X be a random variable with pdf (52), then

$$\begin{aligned}(1-F(x))\,E\left[q_1(X)\mid X\geq x\right] &= \int_x^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-u^\alpha} du \\ &= \frac{\lambda^\alpha}{\alpha\Gamma(\alpha)} e^{-x^\alpha}, \quad x>0,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-2u^\alpha} du \\ &= \frac{\lambda^\alpha}{2\alpha\Gamma(\alpha)} e^{-2x^\alpha}, \quad x>0,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-x^\alpha} < 0 \quad \text{for} \quad x>0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \alpha x^{\alpha-1}, \quad x>0,$$

and hence

$$s(x) = x^\alpha, \quad x>0.$$

Now, in view of Theorem 2.1, X has density (52). □

Corollary 2.11. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.6. The pdf of X is* (52) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \alpha x^{\alpha-1}, \quad x>0.$$

Corollary 2.12. *The general solution of the differential equation in Corollary 2.11 is*

$$\eta(x) = e^{x^\alpha}\left[-\int \alpha x^{\alpha-1} e^{-x^\alpha} (q_1(x))^{-1} q_2(x)\,dx + D\right],$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.11, is given in Proposition 2.6 with $D=0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.7. *Let $X:\Omega\to(0,b)$ be a continuous random variable and let $q_1(x) = e^{\alpha\left(x^{-\beta}-b^{-\beta}\right)}$ and $q_2(x) = q_1(x)\,x^{-\beta}$ for $0<x<b$. The random variable X has pdf* (76) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left(x^{-\beta}+b^{-\beta}\right), \quad 0<x<b.$$

Proof. Let X be a random variable with pdf (76), then

$$\begin{aligned}(1-F(x))\,E[q_1(X)\mid X\geq x]&=\int_x^b \alpha\beta u^{-(\beta+1)}du\\&=\alpha\left(x^{-\beta}-b^{-\beta}\right),\quad 0<x<b,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E[q_2(X)\mid X\geq x]&=\int_x^b \alpha\beta u^{-(2\beta+1)}du\\&=\frac{\alpha}{2}\left(x^{-2\beta}-b^{-2\beta}\right),\quad 0<x<b,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\left(x^{-\beta}-b^{-\beta}\right)<0\quad\text{for}\quad 0<x<b.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\beta x^{-(\beta+1)}}{x^{-\beta}-b^{-\beta}},\quad 0<x<b,$$

and hence

$$s(x)=-\log\left(x^{-\beta}-b^{-\beta}\right),\quad 0<x<b.$$

Now, in view of Theorem 2.1, X has density (76). □

Corollary 2.13. *Let $X:\Omega\rightarrow(0,b)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.7. The pdf of X is* (76) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\beta x^{-(\beta+1)}}{x^{-\beta}-b^{-\beta}},\quad 0<x<b.$$

Corollary 2.14. *The general solution of the differential equation in Corollary 2.13 is*

$$\eta(x)=\left(x^{-\beta}-b^{-\beta}\right)^{-1}\left[-\int\beta x^{-(\beta+1)}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.13, is given in Proposition 2.7 with $D=\frac{1}{2}b^{-\beta}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.8. *Let $X:\Omega\rightarrow(\beta,\gamma)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)(x-\beta)^{\alpha}$ for $\beta<x<\gamma$. The random variable X has pdf* (82) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}\left\{(\gamma-\beta)^{\alpha}+(x-\beta)^{\alpha}\right\},\quad \beta<x<\gamma.$$

Proof. Let X be a random variable with pdf (82), then

$$
\begin{aligned}
(1-F(x))\,E\left[q_1(X)\mid X\geq x\right] &= \int_x^\gamma C\,(u-\beta)^{\alpha-1}\,du \\
&= \frac{C}{\alpha}\left\{(\gamma-\beta)^\alpha-(x-\beta)^\alpha\right\},\quad \beta<x<\gamma,
\end{aligned}
$$

and

$$
\begin{aligned}
(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^\gamma C\,(u-\beta)^{2\alpha-1}\,du \\
&= \frac{C}{2\alpha}\left\{(\gamma-\beta)^{2\alpha}-(x-\beta)^{2\alpha}\right\},\quad \beta<x<\gamma,
\end{aligned}
$$

and finally

$$
\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{(\gamma-\beta)^\alpha-(x-\beta)^\alpha\right\}>0\quad\text{for }\ \beta<x<\gamma.
$$

Conversely, if η is given as above, then

$$
s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha\,(x-\beta)^{\alpha-1}}{\left\{(\gamma-\beta)^\alpha-(x-\beta)^\alpha\right\}},\quad \beta<x<\gamma,
$$

and hence

$$
s(x)=-\log\left\{(\gamma-\beta)^\alpha-(x-\beta)^\alpha\right\},\quad \beta<x<\gamma.
$$

Now, in view of Theorem 2.1, X has density (82). □

Corollary 2.15. *Let $X:\Omega\to(\beta,\gamma)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.8. The pdf of X is* (82) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$
\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha\,(x-\beta)^{\alpha-1}}{\left\{(\gamma-\beta)^\alpha-(x-\beta)^\alpha\right\}},\quad \beta<x<\gamma.
$$

Corollary 2.16. *The general solution of the differential equation in Corollary 2.15 is*

$$
\eta(x)=\left\{(\gamma-\beta)^\alpha-(x-\beta)^\alpha\right\}^{-1}\left[-\int\alpha\,(x-\beta)^{\alpha-1}\,(q_1(x))^{-1}\,q_2(x)\,dx+D\right],
$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.15, is given in Proposition 2.8 with $D=\frac{1}{2}(\gamma-\beta)^{2\alpha}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.9. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,(\theta+1+\theta x)^{-1}$ for $x>0$. The random variable X has pdf* (92) *if and only if the function η defined in Theorem 2.1 has the form*

$$
\eta(x)=\frac{1}{2}(\theta+1+\theta x)^{-1},\quad x>0.
$$

Proof. Let X be a random variable with pdf (92), then

$$\begin{aligned}(1-F(x))\,E\left[q_1(X)\mid X\geq x\right] &= \int_x^\infty C\left(\theta+1+\theta u\right)^{-2}du \\ &= \frac{C}{\theta}\left(\theta+1+\theta x\right)^{-1}, \quad x>0,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^\infty C\left(\theta+1+\theta u\right)^{-3}du \\ &= \frac{C}{2\theta}\left(\theta+1+\theta x\right)^{-2}, \quad x>0,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\left(\theta+1+\theta x\right)^{-1}<0 \quad \text{for } x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\theta\left(\theta+1+\theta x\right)^{-1}, \quad x>0,$$

and hence

$$s(x)=\log\left(\theta+1+\theta x\right), \quad x>0.$$

Now, in view of Theorem 2.1, X has density (92). □

Corollary 2.17. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.9. The pdf of X is* (92) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\theta\left(\theta+1+\theta x\right)^{-1}, \quad x>0.$$

Corollary 2.18. *The general solution of the differential equation in Corollary 2.17 is*

$$\eta(x)=\left(\theta+1+\theta x\right)\left[-\int\alpha\left(\theta+1+\theta x\right)^{-2}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.17, is given in Proposition 2.9 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.10. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=\left[P(x)\right]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x^\gamma}$ for $x>0$. The random variable X has pdf* (94) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}e^{-\lambda x^\gamma}, \quad x>0.$$

Proof. Let X be a random variable with pdf (94), then

$$
\begin{aligned}
(1-F(x))\,E\left[q_1(X)\mid X\geq x\right] &= \int_x^{\infty} C u^{\gamma-1} e^{-\lambda u^{\gamma}}\,du \\
&= \frac{C}{\lambda\gamma} e^{-\lambda x^{\gamma}}, \quad x>0,
\end{aligned}
$$

and

$$
\begin{aligned}
(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^{\infty} C u^{\gamma-1} e^{-2\lambda u^{\gamma}}\,du \\
&= \frac{C}{2\lambda\gamma} e^{-2\lambda x^{\gamma}}, \quad x>0,
\end{aligned}
$$

and finally

$$
\eta(x)\,q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\lambda x^{\gamma}} < 0 \quad \text{for } x>0.
$$

Conversely, if η is given as above, then

$$
s'(x) = \frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)} = \lambda\gamma x^{\gamma-1}, \quad x>0,
$$

and hence

$$
s(x) = \lambda x^{\gamma}, \quad x>0.
$$

Now, in view of Theorem 2.1, X has density (94). □

Corollary 2.19. *Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.10. The pdf of X is* (94) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$
\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)} = \lambda\gamma x^{\gamma-1}, \quad x>0.
$$

Corollary 2.20. *The general solution of the differential equation in Corollary 2.19 is*

$$
\eta(x) = e^{\lambda x^{\gamma}}\left[-\int \lambda\gamma x^{\gamma-1} e^{-\lambda x^{\gamma}} \left(q_1(x)\right)^{-1} q_2(x)\,dx + D\right],
$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.19, is given in Proposition 2.10 with $D=0$. However, it should also be noted that
there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.11. *Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)=\left[P(x)\right]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right)}$ for $x>0$. The random variable X has pdf* (104) *if and only if the function η defined in Theorem 2.1 has the form*

$$
\eta(x) = \frac{1}{2} e^{-\lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right)}, \quad x>0.
$$

Proof. Let X be a random variable with pdf (104), then

$$
\begin{aligned}
(1-F(x))\,E\left[q_1(X)\mid X\geq x\right] &= \int_x^\infty \beta\lambda e^{\alpha u} e^{-\lambda u e^{\alpha u}\left(1-e^{-\lambda u}\right)}\left[\lambda u e^{-\lambda u}+(1+\alpha u)\left(1-e^{-\lambda u}\right)\right]du \\
&= \beta e^{-\lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right)}, \quad x>0,
\end{aligned}
$$

and

$$
\begin{aligned}
(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^\infty \beta\lambda e^{\alpha u} e^{-2\lambda u e^{\alpha u}\left(1-e^{-\lambda u}\right)}\left[\lambda u e^{-\lambda u}+(1+\alpha u)\left(1-e^{-\lambda u}\right)\right]du \\
&= \frac{\beta}{2} e^{-2\lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right)}, \quad x>0,
\end{aligned}
$$

and finally

$$
\eta(x)\,q_1(x)-q_2(x) = -\frac{q_1(x)}{2} e^{-\lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right)} < 0 \quad \text{for } x>0.
$$

Conversely, if η is given as above, then

$$
s'(x) = \frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)} = \lambda e^{\alpha x}\left[\lambda x e^{-\lambda x}+(1+\alpha x)\left(1-e^{-\lambda x}\right)\right], \quad x>0,
$$

and hence

$$
s(x) = \lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right), \quad x>0.
$$

Now, in view of Theorem 2.1, X has density (104). □

Corollary 2.21. *Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.11. The pdf of X is* (104) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$
\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)} = \lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right), \quad x>0.
$$

Corollary 2.22. *The general solution of the differential equation in Corollary 2.21 is*

$$
\eta(x) = e^{\lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right)}\left[\begin{array}{c} -\int \lambda e^{\alpha x}\left[\lambda x e^{-\lambda x}+(1+\alpha x)\left(1-e^{-\lambda x}\right)\right]\times \\ e^{-\lambda x e^{\alpha x}\left(1-e^{-\lambda x}\right)}\left(q_1(x)\right)^{-1} q_2(x)\,dx + D \end{array}\right],
$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.21, is given in Proposition 2.11 with $D=0$. However, it should also be noted that
there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.12. *Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)=\left[1+G(x;\beta)\right]^{1-\alpha}$ and $q_2(x)=q_1(x)\,G(x;\beta)$ for $x\in\mathbb{R}$. The random variable X has pdf* (110) *if and only if the function η defined in Theorem 2.1 has the form*

$$
\eta(x) = \frac{1}{2}\left\{1+G(x;\beta)\right\}, \quad x\in\mathbb{R}.
$$

Proof. Let X be a random variable with pdf (110), then

$$
(1-F(x))\,E\left[q_1(X)\mid X\geq x\right] = \int_x^\infty\left(\frac{\alpha}{2^\alpha-1}\right) g(u;\beta)\,du = \left(\frac{\alpha}{2^\alpha-1}\right)\left\{1-G(x;\beta)\right\}, \quad x\in\mathbb{R},
$$

and

$$(1-F(x))\,E\left[q_2(X)\ \mid X\geq x\right]=\int_x^\infty\left(\frac{\alpha}{2^\alpha-1}\right)g(u;\beta)\,G(u;\beta)\,du=\frac{1}{2}\left(\frac{\alpha}{2^\alpha-1}\right)\left\{1-G(x;\beta)^2\right\},\quad x\in\mathbb{R},$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-G(x;\beta)\right\}>0\quad\text{for }\ x\in\mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{g(x;\beta)}{1-G(x;\beta)},\quad x\in\mathbb{R},$$

and hence

$$s(x)=-\ln\left\{1-G(x;\beta)\right\},\quad x\in\mathbb{R}.$$

Now, in view of Theorem 2.1, X has density (110). □

Corollary 2.23. *Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.12. The pdf of X is* (110) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{g(x;\beta)}{1-G(x;\beta)},\quad x\in\mathbb{R}.$$

Corollary 2.24. *The general solution of the differential equation in Corollary 2.23 is*

$$\eta(x)=\left\{1-G(x;\beta)\right\}^{-1}\left[-\int g(x;\beta)\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. Is straightforward and hence omitted. □

Note that a set of functions satisfying the differential equation in Corollary 2.23, is given in Proposition 2.12 with $D=\frac{1}{2}$. However, it should also be noted that
there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

2.2. Characterization in Terms of Hazard Function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of 4 of the new distributions in terms of the hazard function, which are not of the above trivial form.

Proposition 2.13. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable. The random variable X has pdf* (16) *if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$h_F'(x)-\frac{\gamma}{\delta}e^{-x/\delta}\left(1-\gamma e^{-x/\delta}\right)^{-1}h_F(x)=CC_1\left(1-\gamma e^{-x/\delta}\right)\frac{d}{dx}\left\{\left[\delta\left(1-\gamma e^{-x/\delta}\right)+\tau\right]^{-1}\right\},\quad x>0,$$

with the boundary condition $\lim_{x\to 0}h_F(x)=\frac{CC_1(1-\gamma)}{\delta(1-\gamma)+\tau}$.

Proof. Multiplying both sides of the above equation by $\left(1-\gamma^{-x/\delta}\right)^{-1}$, we have

$$\frac{d}{dx}\left\{\left(1-\gamma^{-x/\delta}\right)^{-1}h_F\left(x\right)\right\}=CC_1\frac{d}{dx}\left\{\left[\delta\left(1-\gamma e^{-x/\delta}\right)+\tau\right]^{-1}\right\},$$

or

$$\left(1-\gamma^{-x/\delta}\right)^{-1}h_F\left(x\right)=CC_1\left[\delta\left(1-\gamma e^{-x/\delta}\right)+\tau\right]^{-1},$$

or

$$h_F\left(x\right)=CC_1\left[\delta\left(1-\gamma e^{-x/\delta}\right)+\tau\right]^{-1}\left(1-\gamma^{-x/\delta}\right),$$

which is the hazard function corresponding to the pdf (16). □

Proposition 2.14. *Let $X:\Omega\rightarrow\left(\beta,\gamma\right)$ be a continuous random variable. The random variable X has pdf* (82) *if and only if its hazard function $h_F\left(x\right)$ satisfies the following differential equation*

$$h_F'\left(x\right)+\left(\alpha+1\right)\left(\delta-x\right)^{-1}h_F\left(x\right)=\alpha\left(\alpha-1\right)\lambda\left(\gamma-\beta\right)\left(\gamma-x\right)^{-\left(\alpha+1\right)}\left(x-\beta\right)^{\alpha-2},\quad\beta<x<\gamma,$$

with the boundary condition $\lim_{x\rightarrow\beta}h_F\left(x\right)=0$ *for* $\alpha>1$.

Proof. It is similar to that of Proposition 2.13. □

Proposition 2.15. *Let $X:\Omega\rightarrow\left(0,\infty\right)$ be a continuous random variable. The random variable X has pdf* (92) *if and only if its hazard function $h_F\left(x\right)$ satisfies the following differential equation*

$$h_F'\left(x\right)-\theta h_F\left(x\right)=\alpha\lambda\theta^2\left(\theta+1\right)e^{\theta x}\frac{d}{dx}\left\{\frac{\left(1+x\right)\left(\frac{\left(\theta+1\right)e^{-\theta x}}{\left(\theta+1+\theta x\right)}-1\right)^{\alpha-1}}{\left(\theta+1+\theta x\right)^2}\right\},\quad x>0,$$

with the boundary condition $\lim_{x\rightarrow0}h_F\left(x\right)=0$ *for* $\alpha>1$.

Proof. It is similar to that of Proposition 2.13. □

Proposition 2.16. *Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable. The random variable X has pdf* (110) *if and only if its hazard function $h_F\left(x\right)$ satisfies the following differential equation*

$$h_F'\left(x\right)-\frac{\left(\alpha-1\right)g\left(x;\beta\right)}{1+G\left(x;\beta\right)}h_F\left(x\right)=\alpha\left[1+G\left(x;\beta\right)\right]^{\alpha-1}\frac{d}{dx}\left\{\frac{g\left(x;\beta\right)}{2^\alpha-\left[1+G\left(x;\beta\right)\right]^\alpha}\right\},\quad x\in\mathbb{R},$$

with the boundary condition $\lim_{x\rightarrow0}h_F\left(x\right)=\frac{\alpha g\left(0;\beta\right)}{2^\alpha-1}$.

Proof. It is similar to that of Proposition 2.13. □

2.3. Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F\left(x\right)=\frac{f\left(x\right)}{F\left(x\right)},\ \ x\in\text{support of }F$$

In this subsection we present characterizations of 3 of the new distributions in terms of the reverse hazard function.

Proposition 2.17. *Let $X:\Omega\rightarrow\left(0,\infty\right)$ be a continuous random variable. The random variable X has pdf* (4) *if and only if its reverse hazard function $r_F\left(x\right)$ satisfies the following differential equation*

$$r_F'(x)-\frac{(k-1)}{\lambda}e^{-x/\lambda}\left(1-e^{-x/\lambda}\right)r_F(x)=\frac{k}{\lambda\gamma^k}\left(1-e^{-x/\lambda}\right)\frac{d}{dx}\left\{\frac{e^{-x/\lambda}\exp\left[-\left(\frac{\left(1-e^{-x/\lambda}\right)}{\gamma}\right)^k\right]}{1-\exp\left[-\left(\frac{\left(1-e^{-x/\lambda}\right)}{\gamma}\right)^k\right]}\right\},\quad x>0,$$

with boundary condition $\lim_{x\to\infty} r_F(x)=0$.

Proposition 2.18. *Let* $X:\Omega\to(0,b)$ *be a continuous random variable. The random variable* X *has pdf* (76) *if and only if its reverse hazard function* $r_F(x)$ *satisfies the following differential equation*

$$r_F'(x)+(\beta+1)\,x^{-1}r_F(x)=0,\quad 0<x<b,$$

with boundary condition $\lim_{x\to b} r_F(x)=\alpha\beta b^{-(\beta+1)}$.

Proposition 2.19. *Let* $X:\Omega\to\mathbb{R}$ *be a continuous random variable. The random variable* X *has pdf* (110) *if and only if its reverse hazard function* $r_F(x)$ *satisfies the following differential equation*

$$r_F'(x)-\frac{(\alpha-1)\,g(x;\beta)}{1+G(x;\beta)}r_F(x)=\alpha\left[1+G(x;\beta)\right]^{\alpha-1}\frac{d}{dx}\left\{\frac{g(x;\beta)}{\left[1+G(x;\beta)\right]^{\alpha}-1}\right\},\quad x\in\mathbb{R},$$

with boundary condition $\lim_{x\to\infty} r_F(x)=\left(\frac{\alpha2^{\alpha-1}}{2^{\alpha}-1}\right)\lim_{x\to\infty} g(x;\beta)$.

2.4. Characterization Based on the Conditional Expectation of Certain Function of the Random Variable

In this subsection we employ a single function ψ (or ψ_1) of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$ (or $\psi_1(X)$). The following propositions have already appeared in Hamedani's previous work (2013), so we will just state them here which can be used to characterize 4 of the new distributions listed in Section 1.

Proposition 2.20. *Let* $X:\Omega\to(e,f)$ *be a continuous random variable with cdf* F. *Let* $\psi(x)$ *be a differentiable function on* (e,f) *with* $\lim_{x\to e^+}\psi(x)=1$. *Then for* $\delta\neq1$,

$$E\left[\psi(X)\mid X\geq x\right]=\delta\psi(x),\qquad x\in(e,f),$$

if and only if

$$\psi(x)=(1-F(x))^{\frac{1}{\delta}-1},\qquad x\in(e,f)$$

Proposition 2.21. *Let* $X:\Omega\to(e,f)$ *be a continuous random variable with cdf* F. *Let* $\psi_1(x)$ *be a differentiable function on* (e,f) *with* $\lim_{x\to f^-}\psi_1(x)=1$. *Then for* $\delta_1\neq1$,

$$E\left[\psi_1(X)\mid X\leq x\right]=\delta_1\psi_1(x),\qquad x\in(e,f)$$

implies

$$\psi_1(x)=(F(x))^{\frac{1}{\delta_1}-1}.\qquad x\in(e,f)$$

Remarks 2.1.

(A) *For* $(e,f)=(0,\infty)$, $\psi(x)=e^{-x}\left(\frac{\delta\left(1-\gamma e^{-x/\delta}\right)}{C_1}\right)^{1/\tau}$ *and* $\delta=\frac{1}{\tau+1}$, *Proposition 2.20 provides a characterization of the PE distribution.*

(B) *For* $(e,f)=(0,b)$, $\psi(x)=e^{-\left(x^{-\beta}-b^{-\beta}\right)}$ *and* $\delta=\frac{\alpha}{\alpha+1}$, *Proposition 2.20 provides a characterization of the RTFIW distribution.*

(C) *For* $(e,f)=(\beta,\gamma)$, $\psi(x)=e^{-\left(\frac{x-\beta}{\gamma-x}\right)^{\alpha}}$ *and* $\delta=\frac{\lambda}{\lambda+1}$, *Proposition 2.20 provides a characterization of the GDBE distribution.*

(D) *For* $(e,f)=(0,\infty)$, $\psi(x)=\exp\left\{-\left(\frac{\theta+1}{\theta+1+\theta x}e^{-\theta x}-1\right)^{\alpha}\right\}$ *and* $\delta=\frac{\lambda}{\lambda+1}$, *Proposition 2.20 provides a characterization of the OWL distribution.*

References

Abbas, S., Farooq, M., Darwish, J. A., Shahbaz, S. A., and Shabaz, M. O. (2023). Truncated Weibull-exponential distribution: methods and applications. *Scientific Reports*, 13(20849):1–11.

Abonongo, A. I. L. and Abonongo, J. (2024). Exponentiated generalized Weibull exponential distribution: properties, estimation and applications. *Computational Journal of Mathematical and Statistical Sciences*, 3(1):57–84.

Abonongo, J., Mwanik, I. J., and Aduda, J. A. (2023). Actuarial measures, estimation and applications of sine Burr III loss distribution. *Mathematics and Statistics*, 11(2):405–420.

Abonongo, J., Mwanik, I. J., and Aduda, J. A. (2024). Cosine Fréchet loss distribution: properties, actuarial measures and insurance applications. *Computational Journal of Mathematical and Statistical Sciences*, 3(1):1–32.

Ahmad, A., Ahmad, A., and Rather, A. A. (2023). The Sabur distribution: properties and application related to engineering data. *Reliability: Theory & Applications*, 8(4 (76)):882–892.

Al Mutairi, A., Iqbal, M. Z., Arshad, M. Z., and Afify, A. Z. (2022). A new class of the power function distribution: theory and inference with an application to engineering data. *Journal of Mathematics (Hindawi)*, 2022. Article ID 1206254, 15 pages.

Al-Shomrani, A. A. and Shawky, A. I. (2024). A new family of distributions: properties and applications. *Advances and Applications in Statistics*, 91(3):257–300.

Alamri, O. A., Alessa, A. H., Hussam, E., Alhelali, M. H., and Kilai, M. (2023). Statistical modelling for the Covid-19 mortality rate in the Kingdom of Saudi Arabia. *Alexandria Engineering Journal*, 68:517–526.

Alnssyan, B., Ahmad, Z., Malela-Majika, J.-C., Seong, J.-T., and Shafik, W. (2023). On the identifiability statistical features of a new distributional approach with reliability applications. *AIP Advances*, 13(12):1–15.

Alomari, A. M., Ahmed, M., Tariq, S., Ahsan-ul Haq, M., and Talib, J. (2024). An exponentiated XLindley distribution with properties, inference and applications. *Heliyon*, 10(e25472):21 pages.

Alrashidi, A. (2023). Arctan inverse Weibull distribution: estimation, simulation and application. In *3rd International Conference on Computing and Information Technology (IICCIT)*, pages 668–673. IEEE.

Alshanbari, H. M., Al-Mofleh, H., Seong, J.-T., and Khosa, S. K. (2023a). A new tangent-generated probabilistic approach with symmetrical and asymmetrical natures: Monte Carlo simulation with reliability applications. *Symmetry (MDPI)*, 15(11):2066. 39 pages.

Alshanbari, H. M., Odhah, O. H., Al-Mofleh, H., Ahmad, Z., Khosa, S. K., and El-Bagoury, A. A. H. (2023b). A new flexible Weibull extension model: different estimation methods and modeling an extreme value data. *Heliyon*, 9(e21704):1–21.

Alshanbari, H. M., Rao, G. S., Seong, J.-T., and Khosa, S. K. (2023c). A new sine-based distributional method with symmetrical and asymmetrical natures: control chart with industrial implication. *Symmetry (MDPI)*, 15(1892):22.

Alshanbari, H. M., Rao, G. S., Seong, J.-T., Salem, S., and Khosa, S. K. (2023d). A new cosine-originated probability distribution with symmetrical and asymmetrical behaviors: repetitive acceptance sampling with reliability application. *Symmetry (MDPI)*, 15(2187):15 pages.

Alshawarbeh, E., Arshad, M. Z., Iqbal, M. Z., Ghamkhar, M., Al Mutairi, A., Meraou, M. A., Hussam, E., and Alrashidi, A. (2024). Modeling medical and engineering data using a new power function distribution: theory and inference. *Journal of Radiation Research and Applied Sciences*, 17(1):1–15.

Alshehri, M. A. and Kayid, M. (2023). A modified gamma model: properties, estimation, and applications. *Axioms (MDPI)*, 12(262):17 pages.

Alvarez, P. I., Varela, H., Cortés, I. E., Venegas, O., and Gómez, H. W. (2024). Modified unit-half-normal distribution with applications. *Mathematics (MDPI)*, 12(136):16 pages.

Arrué, J., Arellano, R. B., Calderin-Ojeda, E., Venegas, O., and Gómez, H. W. (2023). Likelihood based inference and bias reduction in the modified skew-t-normal distribution. *Mathematics (MDPI)*, 11(3287):21 pages.

Atchadé, M. N., N'bouké, M. A. G., Djibril, A. M., Al Mutairi, A., Mustafa, M. S., Hussam, E., Alsuhabi, H., and Nassr, S. G. (2024). A new Topp-Leone Kumarswamy Marshall-Olkin generated family of distributions with applications. *Heliyon*, 10(e24001):1–20.

Atikankul, Y., Thongteeraparp, A., Bodhisuwan, W., Qin, J., and Boonto, S. (2021). The zero-truncated Poisson-weighted exponential Distribution with Applications. *Lobachevskii Journal of Mathematics*, 42(13):3088–3097.

Barahona, J. A., Gómez, Y. M., Gómez-Déniz, E., Venegas, O., and Gómez, H. W. (2024). Scale mixture of exponential distribution with an application. *Mathematics (MDPI)*, 12(156):17 pages.

Castillo, J. S., Barranco-Chamorro, I., Venegas, O., and Gómez, H. W. (2023). Slash-weighted Lindley distribution: properties, inference, and applications. *Mathematics (MDPI)*, 11(3980):14 pages.

Cha, J. H. and Badia, F. G. (2024). Poisson generalized Lindley process and its properties. *Metrika*, 87:61–74.

Chaudhary, A. K., Telee, L. B. S., Karki, M., and Kumar, V. (2023a). Modified extended Kumaraswamy exponential distribution: model and properties. *Journal of Statistics and Computer Science*, 2(2):133–146.

Chaudhary, A. K., Telee, L. B. S., and Kumar, V. (2023b). Extended Kumaraswamy exponential distribution with application to COVID-19 data set. *Journal of Nepal Mathematical Society (JNMS)*, 6(1):1–10.

Eghwerido, J. T. and Eghwerido, O. S. (2024). Theoretical Anallysis of the exponentiated alpha power inverted exponential distribution: properties and application. *FUPRE Journal of Scientific and Industrial Research*, 8(1):1–10.

El Gazar, A. M., Elgarhy, M., and El-Desouky, B. S. (2023). Classical and Bayesian estimation for the truncated inverse power Ailamujia distribution with applications. *AIP Advances*, 13(125122):1–19.

Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. In *Mathematical statistics and probability theory: volume B statistical inference and methods proceedings of the 6th Pannonian symposium on mathematical statistics, bad Tatzmannsdorf, Austria, September 14–20, 1986*, pages 75–84. Springer.

Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments. *Statistics*, 21(4):613–618.

Gómez, H. J., Santoro, K. I., Barranco-Chamorro, I., Venegas, O., Gallardo, D. J., and Gómez, H. W. (2023). A family of truncated positive distributions. *Mathematics (MDPI)*, 11(4431):15 pages.

Gul, A., Sandhu, A. J., Farooq, M., Adil, M., Hassan, Y., and Khan, F. (2023). half logistic-truncated exponential distribution: characteristics and applications. *PLOS ONE*, 18(11):e0285992. DOI: 10.1371/journal.pone.0285992, 22 pages.

Hafeez, M., Khalil, R., and Amjad, N. (2023). A new cubic transmuted power function distribution: properties, inference and application. *Jordan Journal of Mathematics and Statistics (JJMS)*, 16(4):719–739.

Hamedani, G. (2017). Characterizations and infinite divisibility of certain 2016-2017 univariate continuous distributions II. In *International Mathematical Forum*, volume 12, pages 565–609.

Hamedani, G. G. (2016). On characterizations and infinite divisibility of recently introduced distributions. *Studia Scientiarum Mathematicarum Hungarica*, 53(4):467–511.

Hamedani, G. G. (2021). *Characterizations of recently introduced univariate continuous distributions III*. (Research Monograph, NOVA, May 2021).

Hamedani, G. G. (2023). Characterizations of recently introduced univariate continuous distributions IV. *Research Monograph, NOVA, May*, 2023.

Hamedani, G. G., Roshani, A., and Butt, N. S. (2024a). Characterizations of certain (2023-2024) introduced univariate continuous distributions.

Hamedani, G. G., Roshani, A., and Butt, N. S. (2024b). Characterizations of certain (2023-2024) introduced univariate continuous distributions II.

Hussain, N., Tahir, M. H., Jamal, F., Ameeq, M., Shafiq, S., and Mendy, J. T. (2024). An acceptance sampling plan for the odd exponential-logarithmic Fréchet distribution: applications to quality control data. *Cogent Engineering*, 11(1):15 pages.

Kanwal, S., Tahir, M. H., Jamal, F., Ameeq, M., and Mendy, J. T. (2024). A weighted Weibull detection model for line transect sampling: application on wooden stake perpendicular distance data. *Cognet Engineering*, 11(1):17 pages.

Kargbo, M., Gichuhi, A. W., and Wanjoya, A. K. (2023). A novel extended inverse-exponential distribution and its application to COVID-19 data. *Engineering Reports*, e12828:1–21.

Khan, S., Tahir, M. H., and Jamal, F. (2023). Analysis of COVID-19 and cancer data using new half-logistic generated family of distributions. *Operations Research and Decisions*, pages 71–95.

Manu, F., Usman, U., Audu, A., and Isah, B. Y. (2023a). A new life-time model with bathtub and inverted bathtub failure rate function. *Asian Journal of Probability and Statistics*, 25(4):107–119.

Manu, F., Usman, U., Audu, A., and Isah, B. Y. (2023b). New life-time continuous playability distribution with flexible failure rate function. *FUDMA Journal of Sciences (FJS)*, 7(1):323–329.

Moloy, D. J., Ali, M. A., and Alam, F. M. A. (2024). Quartic transmuted Fréchet distribution: properties, estimation and applications. *Advances and Applications in Statistics*, 91(2):175–204.

Nader, N., Ramadan, D. A., Haj Ahmad, H., El-Damcese, M. A., and El-Desouky, B. S. (2023). Optimizing analgesic pain relief time analysis through Bayesian and non-Bayesian approaches to new right truncated Fréchet-inverted Weibull distribution. *Mathematics (AIMS)*, 8(12):31217–31245.

Norouzirad, M., Rao, G. S., and Mazarei, D. (2023). Neutrosophic generalized Rayleigh distribution with application. *Neutrosophic Sets and Systems (NSS)*, 58:248–262.

Obulezi, O. J., Ujunwa, O. E., Anabike, I. C., and Igbokwe, C. P. (2023). The exponentiated power Chris-Jerry distribution: properties, regression, simulation and applications to infant mortality rate and lifetime of COVID-19 patients. *TWIST*, 18(4):328–337.

Odhah, O. H., Alshanbari, H. M., Ahmad, Z., and Rao, G. S. (2023). A weighted cosine-G family of distributions: properties and illustration using time-to-event data. *Axioms (MDPI)*, 12(849):20 pages.

Okereke, E. W., Bakouch, H. S., and Chesneau, C. (2023). Development of an efficient bounded distribution with application to practical data. *Säo Paulo Journal of Mathematical Sciences*. DOI: 10.1007/s40863-023-00390-8, pages 1–24.

Okorie, I. E., Afuecheta, E., and Bakouch, H. S. (2023). Unit upper truncated Weibull distribution with extension to 0 and 1 inflated model-theory and applications. *Heliyon*, 9(11):1–22.

Pimsap, P., Bodhisuwan, W., and Volodin, A. (2023). Alpha power exponentiated Pareto distribution. *Lobachevskii Journal of Mathematics*, 44(9):3779–3788.

Rahman, M. M., Gemeay, A. M., Khan, M. A. I., Meraou, M. A., Bakr, M. E., Muse, A. H., Hussam, E., and Balogun, O. S. (2023). A new modified cubic transmuted-G family of distributions: properties and different methods of estimation with applications to real-life data. *AIP Advances*, 13(095025):1–18.

Rajitha, C. S. and Anisha, K. (2024). The odd Weibull Lindley distribution for modeling wind energy data. *International Journal of Data Science and Analytics*. DOI: 10.1007/s41060-023-00500-x.

Rajitha, C. S. and Vaishnavi, M. (2023). Power exponentiated Weibull distribution: application in survival rate of cancer patients. *Lobachevskii Journal of Mathematics*, 44(9):3806–3824.

Rao, G. S., Norouzirad, M., and Mazarei, D. (2023). Neutrosophic Generalized Exponential Distribution with Application. *Neutrosophic Sets and Systems (NSS)*, 55:471–485.

Rodrigues, G. M., Ortega, E. M. M., Cordeiro, G. M., and Vila, R. (2023a). Quantile regression with a new exponentiated odd log-logistic Weibull distribution. *Mathematics (MDPI)*, 11(1518):20 pages.

Rodrigues, G. M., Ortega, E. M. M., Vila, R., and Cordeiro, G. M. (2023b). A new extended normal quantile regression model: properties and applications. *Communications in Statistics-Simulation and Computation*, pages 1–22.

Sadiq, I. A., Doguwa, S. I. S., Yahaya, A., and Garba, J. (2023). New generalized odd Fréchet-odd exponential-G family of distribution with statistical properties and applications. *FUDMA Journal of Sciences (FJS)*, 7(6):41–51.

Telee, L. B. S. and Kumar, V. (2023a). Exponentiated inverse Chen distribution: properties and applications. *Journal of Nepalese Management Academia*, 1(1):53–62.

Telee, L. B. S. and Kumar, V. (2023b). Inverse exponentiated exponential Poisson distribution with theory and applications. *International Journal of Engineering Science Technologies*, 7(5):17–36.

Yousef, M. M., Hassan, A. S., and Almetwally, E. M. (2024). Statistical inference for the unit Gompertz power series distribution using ranked set sampling with applications. *Assiut University Journal of Multidisciplinary Scientific Research (AUNJMSR)*, 53(1):154–189.

Zaninetti, L. (2023). New probability distributions in astrophysics: XI. left truncation for the Topp-Leone distribution. *International Journal of Astronomy and Astrophysics*, 13(3):154–165.