Pak.j.stat.oper.res. Vol.21 No.4 2025 pp 601-617 DOI: https://doi.org/10.18187/pjsor.v21i4.4662

Pakistan Journal of Statistics and Operation Research

Characterizations of the Recently Introduced Discrete Distributions II

G.G. Hamedani[1*], Amin Roshani[2], Nadeem S. Butt[3]

*Corresponding author

1. Department of Mathematical and Statistical Sciences, Marquette University, Milwaukee, WI 53201-1881, USA, gholamhoss.hamedani@marquette.edu
2. Department of Statistics, Lorestan University, Khorramabad, Iran, roshani.amin@gmail.com
3. Department of Family and Community Medicine, King Abdulaziz University, Jeddah, Saudi Arabia, nshafique@kau.edu.sa

Abstract

Certain characterizations of 19 recently introduced discrete distributions are presented in three directions: (i) based on an appropriate function of the random variable; (ii) in terms of the reverse hazard function and (iii) in terms of the hazard function. This is a continuation of our previous work with the same title.

Key Words: Characterizations; Conditional expectation; Discrete distributions; Hazard function; Reverse hazard function.

1. Introduction

As we mentioned in our previous works, sometimes in real life cases, it is very difficult to obtain samples from a continuous distribution. The observed values are generally discrete due to the fact that they are not measured in continuum. In some cases, it may be possible to measure the observations via a continuous scale, however, they may be recorded in a manner in which a discrete model seems more suitable. Consequently, the discrete models are appearing quite frequently in applied fields and have attracted the attention of many researchers.
Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution.
We list below the cumulative distribution functions and probability mass functions of 19 new discrete distributions which have been introduced after our previous paper (Hamedani et al., 2024) was published. Section 2 provides the characterizations of these distributions based on the conditional expectations of certain functions of the random variables. Section 3 takes up the characterizations of 1 of the 19 distributions in terms of its reverse hazard function and Section 4 deals with the characterizations of 2 of the 19 distributions in terms of the hazard function.

The cumulative distribution (cdf), the corresponding probability mass function (pmf), the reverse hazard and hazard functions of each of the 19 distributions are listed below in (1)-(19).

(1) The cdf, pmf, reverse hazard and hazard functions of Discretized Generalized Gompertz (DGG) distribution of Das et al. (2023), are given, respectively, by

$$F\left(x;\lambda,\alpha,\theta\right)=\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta},\quad x\in\mathbb{N}^{*}, \tag{1}$$

$$f\left(x;\lambda,\alpha,\theta\right)=\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta}-\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\right]^{\theta},\quad x\in\mathbb{N}^{*}, \tag{2}$$

$$r_{F}\left(x\right)=1-\frac{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\right]^{\theta}}{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta}},\quad x\in\mathbb{N}^{*}, \tag{3}$$

$$h_{F}\left(x\right)=\frac{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta}-\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\right]^{\theta}}{1-\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta}},\quad x\in\mathbb{N}^{*}, \tag{4}$$

where $\lambda>0,\alpha\geq 0,\theta>0$ are parameters, $\mathbb{N}^{*}=\mathbb{N}\cup\{0\}$, and $\mathbb{N}$ is the set of all positive integers.

(2) The cdf, pmf, reverse hazard and hazard functions of Weighted Discretized Fréchet-Weibull (WDFW) distribution of Das and Das (2023), are given, respectively, by

$$F\left(x;\alpha,\beta,m,j\right)=C\sum_{u=0}^{x}p\left(u;\alpha,\beta,m,j\right),\quad x\in\mathbb{N}^{*}, \tag{5}$$

$$f\left(x;\alpha,\beta,m,j\right)=C\left[1-e^{-\left(\frac{x+1}{m}\right)}\right]\left\{e^{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha j}}-e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha j}}\right\},\quad x\in\mathbb{N}^{*}, \tag{6}$$

$$r_{F}\left(x\right)=\frac{\left[1-e^{-\left(\frac{x+1}{m}\right)}\right]\left\{e^{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha j}}+e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha j}}\right\}}{\sum_{u=0}^{x}p\left(u;\alpha,\beta,m,j\right)},\quad x\in\mathbb{N}^{*}, \tag{7}$$

$$h_{F}\left(x\right)=\frac{C\left[1-e^{-\left(\frac{x+1}{m}\right)}\right]\left\{e^{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha j}}+e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha j}}\right\}}{1-C\sum_{u=0}^{x}p\left(u;\alpha,\beta,m,j\right)},\quad x\in\mathbb{N}^{*}, \tag{8}$$

where α,β,m,j are all positive parameters, $p\left(u;\alpha,\beta,m,j\right)=\left[1-e^{-\left(\frac{x+1}{m}\right)}\right]\left\{e^{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha j}}+e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha j}}\right\}$, $x\in\mathbb{N}^{*}$ and $C=\left(\sum_{c=0}^{\infty}p\left(c;\alpha,\beta,m,j\right)\right)^{-1}$ is the normalizing constant.

(3) The cdf, pmf, reverse hazard and hazard functions of Poisson Epanechnikov-Exponential (PEE) distribution of Karakaya (2023), are given, respectively, by

$$F\left(x;\beta\right)=C\sum_{u=0}^{x}f\left(u;\beta\right),\quad x\in\mathbb{N}^{*}, \tag{9}$$

$$f\left(x;\beta\right)=C\left(2\beta+1\right)^{-x}P\left(x\right),\quad x\in\mathbb{N}^{*}, \tag{10}$$

$$r_{F}\left(x\right)=\frac{\left(2\beta+1\right)^{-x}P\left(x\right)}{\sum_{u=0}^{x}f\left(u;\beta\right)},\quad x\in\mathbb{N}^{*}, \tag{11}$$

$$h_{F}\left(x\right)=\frac{C\left(2\beta+1\right)^{-x}P\left(x\right)}{1-C\sum_{u=0}^{x}f\left(u;\beta\right)},\quad x\in\mathbb{N}^{*}, \tag{12}$$

where $\beta>0$ is a parameter, $P\left(x\right)=1-\frac{1}{2}\left(\frac{2\beta+1}{3\beta+1}\right)^{x+1}$ and $C=\frac{3\beta}{2\beta+1}$.

(4) The cdf, pmf, reverse hazard and hazard functions of Mixed Poisson Transmuted Exponential (MPTE) distribution of Adetunji and Sabri (2023b), are given, respectively, by

$$F(x;\theta,p)=C\sum_{u=0}^{x}f(u;\theta,p)\,,\quad x\in\mathbb{N}^{*}, \tag{13}$$

$$f(x;\theta,p)=C(1+\theta)^{-x}P(x)\,,\quad x\in\mathbb{N}^{*}, \tag{14}$$

$$r_F(x)=\frac{(1+\theta)^{-x}P(x)}{\sum_{u=0}^{x}f(u;\theta,p)},\quad x\in\mathbb{N}^{*}, \tag{15}$$

$$h_F(x)=\frac{C(1+\theta)^{-x}P(x)}{1-C\sum_{u=0}^{x}f(u;\theta,p)},\quad x\in\mathbb{N}^{*}, \tag{16}$$

where $\theta>0, p\in[-1,1]$ are parameters, $P(x)=(1-p)+6p\left[\left(\frac{1+\theta}{1+2\theta}\right)^{x+1}-\left(\frac{1+\theta}{1+3\theta}\right)^{x+1}\right]$ and $C=\frac{\theta}{(1+\theta)}$.

Remark 1.1. The pmf (14) is similar to the pmf (10).

(5) The cdf, pmf, reverse hazard and hazard functions of Mixed Poisson Cubic Ranked Transmuted Exponential (MPCRTE) distribution of Adetunji and Sabri (2023a), are given, respectively, by

$$F(x;\alpha,\beta)=C\sum_{u=0}^{x}f(u;\alpha,\beta)\,,\quad x\in\mathbb{N}^{*}, \tag{17}$$

$$f(x;\alpha,\beta)=C(1+\alpha)^{-x}P(x)\,,\quad x\in\mathbb{N}^{*}, \tag{18}$$

$$r_F(x)=\frac{(1+\alpha)^{-x}P(x)}{\sum_{u=0}^{x}f(u;\alpha,\beta)},\quad x\in\mathbb{N}^{*}, \tag{19}$$

$$h_F(x)=\frac{C(1+\alpha)^{-x}P(x)}{1-C\sum_{u=0}^{x}f(u;\alpha,\beta)},\quad x\in\mathbb{N}^{*}, \tag{20}$$

where $\alpha>0,\beta>0$ are parameters, $P(x)=1-2\beta\left(\frac{1+\alpha}{1+2\alpha}\right)^{x+1}+3\beta\left(\frac{1+\alpha}{1+3\alpha}\right)^{x+1}$ and $C=\frac{\alpha}{(1+\alpha)}$.

Remark 1.2. The pmf (18) is almost the same as the pmf (14).

(6) The cdf, pmf, reverse hazard and hazard functions of Poisson XLindley (PXL) distribution of Seghier et al. (2023b), are given, respectively, by

$$F(x;\theta)=C\sum_{u=0}^{x}f(u;\theta)\,,\quad x\in\mathbb{N}^{*}, \tag{21}$$

$$f(x;\theta)=C(1+\theta)^{-x}P(x)\,,\quad x\in\mathbb{N}^{*}, \tag{22}$$

$$r_F(x)=\frac{(1+\theta)^{-x}P(x)}{\sum_{u=0}^{x}f(u;\theta)},\quad x\in\mathbb{N}^{*}, \tag{23}$$

$$h_F(x)=\frac{C(1+\theta)^{-x}P(x)}{1-C\sum_{u=0}^{x}f(u;\theta)},\quad x\in\mathbb{N}^{*}, \tag{24}$$

where $\theta > 0$ is a parameter, $P(x) = \theta^2 + 3\theta + 3 + x$ and $C = \frac{\theta^2}{(1+\theta)^4}$.

Remark 1.3. The pmf (22) is similar to the pmf (18).

(7) The cdf, pmf, reverse hazard and hazard functions of New Discrete Ramos-Louzada (NDRL) distribution of Ahsan-ul Haq and Zafar (2023), are given, respectively, by

$$F(x; \upsilon) = C \sum_{u=0}^{x} f(u; \upsilon), \quad x \in \mathbb{N}^*, \tag{25}$$

$$f(x; \upsilon) = C\upsilon^x P(x), \quad x \in \mathbb{N}^*, \tag{26}$$

$$r_F(x) = \frac{\upsilon^x P(x)}{\sum_{u=0}^{x} f(u; \upsilon)}, \quad x \in \mathbb{N}^*, \tag{27}$$

$$h_F(x) = \frac{C\upsilon^x P(x)}{1 - C\sum_{u=0}^{x} f(u; \upsilon)}, \quad x \in \mathbb{N}^*, \tag{28}$$

where $0.61 \leq \upsilon \leq 1$ is a parameter, $P(x) = 1 + 2\log(\upsilon)(\log(\upsilon))^2$ and $C = \frac{(1-\upsilon)^2}{(1+2\log(\upsilon))(1-\upsilon)+\upsilon(\log(\upsilon))^2}$.

Remark 1.4. The pmf (26) is similar to the pmf (14) with $\upsilon = (1+\theta)^{-1}$.

(8) The cdf, pmf, reverse hazard and hazard functions of Poisson Entropy-Based Weighted Exponential (PEBWE) distribution of Alomair and Ahsan-ul Haq (2023), are given, respectively, by

$$F(x; \beta) = 1 - \frac{(1 + \beta(2+x) - (1+\beta)\ln(\beta))}{(1+\beta)^{2+x}(1-\ln(\beta))}, \quad x \in \mathbb{N}^*, \tag{29}$$

$$f(x; \beta) = \frac{(1 + \beta(1+x) - (1+\beta)\ln(\beta))}{(1+\beta)^{1+x}(1-\ln(\beta))} - \frac{(1 + \beta(2+x) - (1+\beta)\ln(\beta))}{(1+\beta)^{2+x}(1-\ln(\beta))}, \quad x \in \mathbb{N}^*, \tag{30}$$

$$r_F(x) = \frac{\frac{(1+\beta(1+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{1+x}(1-\ln(\beta))} - \frac{(1+\beta(2+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+x}(1-\ln(\beta))}}{1 - \frac{(1+\beta(2+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+x}(1-\ln(\beta))}}, \quad x \in \mathbb{N}^*, \tag{31}$$

$$h_F(x) = \frac{\frac{(1+\beta(1+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{1+x}(1-\ln(\beta))}}{\frac{(1+\beta(2+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+x}(1-\ln(\beta))}} - 1 = \frac{(1+\beta)(1 + \beta(1+x) - (1+\beta)\ln(\beta))}{(1 + \beta(2+x) - (1+\beta)\ln(\beta))} - 1, \quad x \in \mathbb{N}^*, \tag{32}$$

where $\beta > 0$ is a parameter.

(9) The cdf, pmf, reverse hazard and hazard functions of Poisson New XLindley (Poisson-NXL) distribution of Seghier et al. (2023a), are given, respectively, by

$$F(x; \theta) = 1 - \frac{(2 + \theta(3+x))}{2(\theta+1)^{x+2}}, \quad x \in \mathbb{N}^*, \tag{33}$$

$$f(x; \theta) = C(\theta+1)^{-x} P(x), \quad x \in \mathbb{N}^*, \tag{34}$$

$$r_F(x) = \frac{f(x; \theta)}{F(x; \theta)}, \quad x \in \mathbb{N}^*, \tag{35}$$

$$h_F(x) = \frac{2C(\theta+1)^2 P(x)}{(2+\theta(3+x))}, \quad x \in \mathbb{N}^*, \tag{36}$$

where $\theta > 0$ is a parameter, $C = \frac{\theta}{2(\theta+1)^2}$ and $P(x) = \theta x + 2\theta + 1$.

Remark 1.5. The pmf (34) is similar to the pmf (22).

(10) The cdf, pmf, reverse hazard and hazard functions of New Discrete Bilal (NDB) distribution of Ahsan-ul Haq et al. (2023), are given, respectively, by

$$F(x;\lambda) = C\sum_{u=0}^{x} \lambda^{2u} P(u), \quad x \in \mathbb{N}^*, \tag{37}$$

$$f(x;\lambda) = C\lambda^{2x} P(x), \quad x \in \mathbb{N}^*, \tag{38}$$

$$r_F(x) = \frac{\lambda^{2x} P(x)}{\sum_{u=0}^{x} \lambda^{2u} P(u)}, \quad x \in \mathbb{N}^*, \tag{39}$$

$$h_F(x) = \frac{C\lambda^{2x} P(x)}{1 - C\sum_{u=0}^{x} \lambda^{2u} P(u)}, \quad x \in \mathbb{N}^*, \tag{40}$$

where $0 < \lambda < 1$ is a parameter, $C = \frac{(1+\lambda)(1-\lambda^3)}{\lambda^2}$ and $P(x) = 1 - \lambda^x$.

Remark 1.6. The pmf (38) is similar to the pmf (34) for $\lambda = (\theta+1)^{-1/2}$.

(11) The cdf, pmf, reverse hazard and hazard functions of New Weighted Poisson (NWP) distribution of Diafouka et al. (2023), are given, respectively, by

$$F(x;\lambda) = C\sum_{u=0}^{x} \lambda^{u} P(u), \quad x \in \mathbb{N}^*, \tag{41}$$

$$f(x;\lambda) = C\lambda^{x} P(x), \quad x \in \mathbb{N}^*, \tag{42}$$

$$r_F(x) = \frac{\lambda^{x} P(x)}{\sum_{u=0}^{x} \lambda^{u} P(u)}, \quad x \in \mathbb{N}^*, \tag{43}$$

$$h_F(x) = \frac{C\lambda^{x} P(x)}{1 - C\sum_{u=0}^{x} \lambda^{u} P(u)}, \quad x \in \mathbb{N}^*, \tag{44}$$

where C is the normalizing constatnt, $P(x) = (x+a)^{\alpha}(x!)^{-\nu}$ and $\lambda \in (0,1), \alpha \in \mathbb{R}, \nu > 0, a > 0$ are parameters..

Remark 1.7. The pmf (42) is similar to the pmf (38).

(12) The cdf, pmf, reverse hazard and hazard functions of Compound Conway-Maxwell-Poisson (CCMP) distribution of Merupula et al. (2023), are given, respectively, by

$$F(x;\lambda,\upsilon) = C\sum_{u=0}^{x} \lambda^{u} P(u), \quad x \in \mathbb{N}^*, \tag{45}$$

$$f(x;\lambda,\upsilon) = C\lambda^{x} P(x), \quad x \in \mathbb{N}^*, \tag{46}$$

$$r_F(x) = \frac{\lambda^x P(x)}{\sum_{u=0}^{x} \lambda^u P(u)}, \quad x \in \mathbb{N}^*, \tag{47}$$

$$h_F(x) = \frac{C\lambda^x P(x)}{1 - C\sum_{u=0}^{x} \lambda^u P(u)}, \quad x \in \mathbb{N}^*, \tag{48}$$

where $C = \left[\sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^{\upsilon}}\right]$, $P(x) = (x!)^{-\nu}$ and $\lambda > 0, \upsilon \geq 0$ are parameters.

Remark 1.8. The pmf (46) is similar to the pmf (10) for $\lambda = (2\beta + 1)^{-1}$.

(13) The cdf, pmf, reverse hazard and hazard functions of Discrete Half-Logistic (DHL) distribution of Abd-Elmonem et al. (2024), are given, respectively, by

$$F(x; a) = 1 - 2^a e^{-a} e^{-a(x+1)} \left[1 - \left(\frac{a-2}{a}\right) e^{-(x+1)}\right]^{-a}, \ \ x \in \mathbb{N}^*, \tag{49}$$

$$f(x; a) = 2^a e^{-a} \left\{ e^{-ax} \left[1 - \left(\frac{a-2}{a}\right) e^{-x}\right]^{-a} - e^{-a(x+1)} \left[1 - \left(\frac{a-2}{a}\right) e^{-(x+1)}\right]^{-a} \right\}, \quad x \in \mathbb{N}^*, \tag{50}$$

$$r_F(x) = \frac{2^a e^{-a} \left\{ e^{-ax} \left[1 - \left(\frac{a-2}{a}\right) e^{-x}\right]^{-a} - e^{-a(x+1)} \left[1 - \left(\frac{a-2}{a}\right) e^{-(x+1)}\right]^{-a} \right\}}{1 - 2^a e^{-a} e^{-a(x+1)} \left[1 - \left(\frac{a-2}{a}\right) e^{-(x+1)}\right]^{-a}}, \quad x \in \mathbb{N}^*, \tag{51}$$

$$h_F(x) = \frac{\left\{ e^{-ax} \left[1 - \left(\frac{a-2}{a}\right) e^{-x}\right]^{-a} \right\}}{e^{-a(x+1)} \left[1 - \left(\frac{a-2}{a}\right) e^{-(x+1)}\right]^{-a}} - 1, \quad x \in \mathbb{N}^*, \tag{52}$$

where $a > 0$ is a parameters.

(14) The cdf, pmf, reverse hazard and hazard functions of Discrete Gompertz-Weibull-Fréchet (DGo-WFr) distribution of Akkanphudit (2023), are given, respectively, by

$$F(x; \lambda, \theta, \alpha, \beta, \gamma, \omega) = 1 - \exp\left\{\frac{\lambda}{\theta}\left[1 - \left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x+1}\right)^{\alpha}} - 1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\}, \quad x \in \mathbb{N}^*, \tag{53}$$

$$\begin{aligned} f(x; \lambda, \theta, \alpha, \beta, \gamma, \omega) = & \exp\left\{\frac{\lambda}{\theta}\left[1 - \left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x}\right)^{\alpha}} - 1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\} - \\ & \exp\left\{\frac{\lambda}{\theta}\left[1 - \left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x+1}\right)^{\alpha}} - 1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\}, \quad x \in \mathbb{N}^*, \end{aligned} \tag{54}$$

$$r_F\left(x\right)=\frac{\exp\left\{\frac{\lambda}{\theta}\left[1-\left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x}\right)^{\alpha}}-1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\}}{1-\exp\left\{\frac{\lambda}{\theta}\left[1-\left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x+1}\right)^{\alpha}}-1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\}}-\frac{\exp\left\{\frac{\lambda}{\theta}\left[1-\left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x+1}\right)^{\alpha}}-1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\}}{1-\exp\left\{\frac{\lambda}{\theta}\left[1-\left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x+1}\right)^{\alpha}}-1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\}},\quad x\in\mathbb{N}^{*},\tag{55}$$

$$h_F\left(x\right)=\frac{\exp\left\{\frac{\lambda}{\theta}\left[1-\left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x}\right)^{\alpha}}-1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\}}{\exp\left\{\frac{\lambda}{\theta}\left[1-\left\{\exp\left[-\omega\left(e^{\left(\frac{\beta}{x+1}\right)^{\alpha}}-1\right)^{-\gamma}\right]\right\}^{-\theta}\right]\right\}}-1,\quad x\in\mathbb{N}^{*},\tag{56}$$

where $\lambda,\theta,\alpha,\beta,\gamma,\omega$ are all positive parameters.

Remark 1.9. The pmf (54) is similar to the pmf (50).

(15) The cdf, pmf, reverse hazard and hazard functions of Discrete Hypo Exponential (DHE) distribution of Krishnakumari and George (2023), are given, respectively, by

$$F\left(x;\phi_1,\phi_2,\alpha\right)=\frac{V\left(x+1;\phi_1,\phi_2,\alpha\right)}{\left(\phi_2-\phi_1\right)^2},\quad x\in\mathbb{N}^{*}\ ,\tag{57}$$

$$f\left(x;\phi_1,\phi_2,\alpha\right)=\frac{V\left(x+1;\phi_1,\phi_2,\alpha\right)-V\left(x;\phi_1,\phi_2,\alpha\right)}{\left(\phi_2-\phi_1\right)^2},\quad x\in\mathbb{N}^{*}\ ,\tag{58}$$

$$r_F\left(x\right)=1-\frac{V\left(x;\phi_1,\phi_2,\alpha\right)}{V\left(x+1;\phi_1,\phi_2,\alpha\right)},\quad x\in\mathbb{N}^{*},\tag{59}$$

$$h_F\left(x\right)=\frac{V\left(x+1;\phi_1,\phi_2,\alpha\right)-V\left(x;\phi_1,\phi_2,\alpha\right)}{1-V\left(x+1;\phi_1,\phi_2,\alpha\right)},\quad x\in\mathbb{N}^{*},\tag{60}$$

where ϕ_1,ϕ_2,α are all positive parameters.

Remark 1.10. The pmf (58) is similar to the pmf (54).

(16) The cdf, pmf, reverse hazard and hazard functions of Poisson New X-Lindley (PNXL) distribution of Irshad et al. (2024), are given, respectively, by

$$F\left(x;\theta\right)=C\sum_{u=0}^{x}f\left(u;\theta\right),\quad x\in\mathbb{N}^{*},\tag{61}$$

$$f\left(x;\theta\right)=C\left(\theta+1\right)^{-x}P\left(x\right),\quad x\in\mathbb{N}^{*},\tag{62}$$

$$r_F\left(x\right)=\frac{\left(\theta+1\right)^{-x}P\left(x\right)}{\sum_{u=0}^{x}f\left(u;\theta\right)},\quad x\in\mathbb{N}^{*},\tag{63}$$

$$h_F(x) = \frac{C(\theta+1)^{-x} P(x)}{1 - C\sum_{u=0}^{x} f(u;\theta)}, \quad x \in \mathbb{N}^*, \tag{64}$$

where $\theta > 0$ is a parameter, $P(x) = 2\theta + \theta x + 1$ and $C = \frac{\theta}{2(\theta+1)^2}$.

Remark 1.11. The pmf (62) is similar to the pmf (10).

(17) The cdf, pmf, reverse hazard and hazard functions of Poisson Generalized Lindley (PGL) distribution of Atikankul (2023), are given, respectively, by

$$F(x;\theta,\alpha) = C\sum_{u=0}^{x} f(u;\theta,\alpha), \quad x \in \mathbb{N}^*, \tag{65}$$

$$f(x;\theta,\alpha) = C(\theta+1)^{-x} P(x), \quad x \in \mathbb{N}^*, \tag{66}$$

$$r_F(x) = \frac{(\theta+1)^{-x} P(x)}{\sum_{u=0}^{x} f(u;\theta,\alpha)}, \quad x \in \mathbb{N}^*, \tag{67}$$

$$h_F(x) = \frac{C(\theta+1)^{-x} P(x)}{1 - C\sum_{u=0}^{x} f(u;\theta,\alpha)}, \quad x \in \mathbb{N}^*, \tag{68}$$

where $\theta > 0, \alpha > 1$ are parameters, $P(x) = \frac{\Gamma(\alpha+x-1)(\alpha(\theta+2)-\theta+x-2)}{\Gamma(x+1)}, \Gamma(a)$ is the gamma function and $C = \frac{\theta^\alpha}{(\theta+1)^{\alpha+1}}$.

Remark 1.12. The pmf (66) is similar to the pmf (62).

(18) The cdf, pmf, reverse hazard and hazard functions of Discrete New Generalized Rayleigh (DNGR) distribution of Haj Ahmad et al. (2024), are given, respectively, by

$$F(x;\theta,\beta,\alpha) = 1 - \alpha\left(\frac{1-\left(1-e^{-\theta(x+1)^2}\right)^\beta}{\alpha-\left(1-e^{-\theta(x+1)^2}\right)^\beta}\right), \quad x \in \mathbb{N}^*, \tag{69}$$

$$f(x;\theta,\beta,\alpha) = \alpha\left\{\left(\frac{1-\left(1-e^{-\theta x^2}\right)^\beta}{\alpha-\left(1-e^{-\theta x^2}\right)^\beta}\right) - \left(\frac{1-\left(1-e^{-\theta(x+1)^2}\right)^\beta}{\alpha-\left(1-e^{-\theta(x+1)^2}\right)^\beta}\right)\right\}, \quad x \in \mathbb{N}^*, \tag{70}$$

$$r_F(x) = \frac{\alpha\left\{\left(\frac{1-\left(1-e^{-\theta x^2}\right)^\beta}{\alpha-\left(1-e^{-\theta x^2}\right)^\beta}\right) - \left(\frac{1-\left(1-e^{-\theta(x+1)^2}\right)^\beta}{\alpha-\left(1-e^{-\theta(x+1)^2}\right)^\beta}\right)\right\}}{1-\alpha\left(\frac{1-\left(1-e^{-\theta(x+1)^2}\right)^\beta}{\alpha-\left(1-e^{-\theta(x+1)^2}\right)^\beta}\right)}, x \in \mathbb{N}^*, \tag{71}$$

$$h_F(x) = \frac{\left(\frac{1-\left(1-e^{-\theta x^2}\right)^\beta}{\alpha-\left(1-e^{-\theta x^2}\right)^\beta}\right)}{\left(\frac{1-\left(1-e^{-\theta(x+1)^2}\right)^\beta}{\alpha-\left(1-e^{-\theta(x+1)^2}\right)^\beta}\right)} - 1, \quad x \in \mathbb{N}^*, \tag{72}$$

where θ, β, α are all positive parameters.

Remark 1.13. The pmf (70) is similar to the pmfs (2) and (30).

(19) The cdf, pmf, reverse hazard and hazard functions of Zero-and-One Inflated Cosine Geometric (ZOICG) distribution of Junnumtuam et al. (2022), are given, respectively, by

$$F\left(x;\omega_1,\omega_2,p,\theta\right)=\begin{cases}C_1, & x=0,\\ C_1+C_2, & x=1,\\ C_3\sum_{u=2}^{x}p^{u}P\left(u\right), & x\left(\geq 2\right)\in\mathbb{N}^*\end{cases}, \tag{73}$$

$$F\left(x;\omega_1,\omega_2,p,\theta\right)=\begin{cases}C_1, & x=0,\\ C_2, & x=1,\\ C_3p^{x}P\left(x\right), & x\left(\geq 2\right)\in\mathbb{N}^*\end{cases}, \tag{74}$$

$$r_F\left(x\right)=\frac{f\left(x;\omega_1,\omega_2,p,\theta\right)}{F\left(x;\omega_1,\omega_2,p,\theta\right)},\quad x\in\mathbb{N}^*, \tag{75}$$

$$h_F\left(x\right)=\frac{f\left(x;\omega_1,\omega_2,p,\theta\right)}{1-F\left(x;\omega_1,\omega_2,p,\theta\right)},\quad x\in\mathbb{N}^*, \tag{76}$$

where $\omega_1\in\left(0,1\right),\omega_2\in\left(0,1\right),p\in\left(0,1\right),\theta\in\left[0,\frac{\pi}{2}\right]$ are parameters, $C_1=\omega_1+\left(1-\omega_1-\omega_2\right)C_{p,\theta},C_2=\omega_2+\left(1-\omega_1-\omega_2\right)C_{p,\theta}p\left(\cos\theta\right)^2,C_3=\left(1-\omega_1-\omega_2\right)C_{p,\theta}p^{x}\left(\cos\left(\theta x\right)\right)^2$ and $P\left(x\right)=\left(\cos\left(\theta x\right)\right)^2$.

Remark 1.14. The pmf (74) is similar to the pmf (46).

2. Characterizations Based on Conditional Expectation

In this Section, we present our characterizations of all the distributions listed in Section 1 in terms of the conditional expectations of certain functions of the random variables. The choice of each function depends on the form of the pmf. Most of the proofs follow the same scheme, we will give all of them for the sake of completeness.

2.1. Discretized Generalized Gompertz (DGG) Distribution

Proposition 2.1. *Let $X:\Omega\rightarrow\mathbb{N}^*$ be a random variable. The pmf of X is* (2) *if and only if*

$$E\left\{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(X+1)}-1\right)}\right]^{\theta}+\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha X}-1\right)}\right]^{\theta}\mid X\leq k\right\}=\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}. \tag{77}$$

Proof. If X has pmf (2), then for $k\in\mathbb{N}^*$, the left-hand side of (77), using telescoping sum formula, will be

$$\begin{aligned}&\left(F\left(k\right)\right)^{-1}\sum_{x=0}^{k}\left\{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{2\theta}-\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\right]^{2\theta}\right\}\\&=\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{-\theta}\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{2\theta}\\&=\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}.\end{aligned}$$

Conversely, if (77) holds, then

$$\sum_{x=0}^{k}\left\{\left[\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta}+\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\right]^{\theta}\right]f\left(x\right)\right\}=F\left(k\right)\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}. \tag{78}$$

From (78), we also have

$$\sum_{x=0}^{k-1}\left\{\left[\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta}+\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\right]^{\theta}\right]f\left(x\right)\right\}=\left(F\left(k\right)-f\left(k\right)\right)\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha k}-1\right)}\right]^{\theta}. \tag{79}$$

Now, subtracting (79) from (78), yields

$$f(k)\left[\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}+\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha k}-1\right)}\right]^{\theta}\right]$$
$$=F(k)\left\{\left[\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}-\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha k}-1\right)}\right]^{\theta}\right]\right\}+f(k)\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha k}-1\right)}\right]^{\theta}.$$

From the above equality, we have

$$\frac{f(k)}{F(k)}=\frac{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}-\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha k}-1\right)}\right]^{\theta}}{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}}=1-\frac{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha k}-1\right)}\right]^{\theta}}{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}},$$

which is the reverse hazard function corresponding to the pmf (2), so X has pmf (2). □

2.2. Weighted Discretized Fréchet-Weibull (WDFW) Distribution

Proposition 2.2. *Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is* (6) *if and only if*

$$E\left\{\left[\left[1-e^{-\left(\frac{X+1}{m}\right)}\right]^{-1}\left\{e^{-\beta^{\alpha}\left(\frac{m}{X+1}\right)^{\alpha j}}+e^{-\beta^{\alpha}\left(\frac{m}{X}\right)^{\alpha j}}\right\}\right]\mid X\leq k\right\}=\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha j}}}{\sum_{x=0}^{k}p(x;\alpha,\beta,m,j)}. \tag{80}$$

Proof. If X has pmf (6), then for $k\in\mathbb{N}$, the left-hand side of (80), using telescoping sum formula, will be

$$(F(k))^{-1}\sum_{x=0}^{k}\left\{e^{-2\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha j}}-e^{-2\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha j}}\right\}=\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha j}}}{\sum_{x=0}^{k}p(x;\alpha,\beta,m,j)}.$$

Conversely, if (80) holds, then

$$\sum_{x=0}^{k}\left\{\left[e^{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha j}}+e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha j}}\right]f(x)\right\}=F(k)\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha j}}}{\sum_{x=0}^{k}p(x;\alpha,\beta,m,j)}\right). \tag{81}$$

From (81), we also have

$$\sum_{x=0}^{k-1}\left\{\left[e^{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha j}}+e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha j}}\right]f(x)\right\}=F(k-1)\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}}{\sum_{x=0}^{k-1}p(x;\alpha,\beta,m,j)}\right)$$
$$=(F(k)-f(k))\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}}{\sum_{x=0}^{k-1}p(x;\alpha,\beta,m,j)}\right). \tag{82}$$

Now, subtracting (82) from (81), yields

$$f(k)\left[e^{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha j}}+e^{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}\right]$$
$$=F(k)\left\{\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha j}}}{\sum_{x=0}^{k}p(x;\alpha,\beta,m,j)}\right)-\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}}{\sum_{x=0}^{k-1}p(x;\alpha,\beta,m,j)}\right)\right\}+f(k)\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}}{\sum_{x=0}^{k-1}p(x;\alpha,\beta,m,j)}\right).$$

From the above equality, after some manipulations, we have

$$\frac{f(k)}{F(k)}=\frac{\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha j}}}{\sum_{x=0}^{k}p(x;\alpha,\beta,m,j)}\right)-\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}}{\sum_{x=0}^{k-1}p(x;\alpha,\beta,m,j)}\right)}{\left(e^{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha j}}+e^{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}\right)-\left(\frac{e^{-2\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}}{\sum_{x=0}^{k-1}p(x;\alpha,\beta,m,j)}\right)}$$
$$=\frac{\left[1-e^{-\left(\frac{k+1}{m}\right)}\right]\left(e^{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha j}}-e^{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha j}}\right)}{\sum_{x=0}^{k}p(x;\alpha,\beta,m,j)},$$

which is the reverse hazard function corresponding to the pmf (6), so X has pmf (6). □

2.3. Poisson Epanechnikov-Exponential (PEE) Distribution

Proposition 2.3. *Let $X:\Omega\to\mathbb{N}^{*}$ be a random variable. The pmf of X is* (10) *if and only if*

$$E\left\{[P(X)]^{-1}\mid X\le k\right\}=\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-(k+1)}\right]}{\sum_{x=0}^{k}(2\beta+1)^{-x}P(x)}. \tag{83}$$

Proof. If X has pmf (10), then for $k\in\mathbb{N}$, the left-hand side of (83), using finite geometric sum formula, will be

$$(F(k))^{-1}\sum_{x=0}^{k}\left\{C(2\beta+1)^{-x}\right\}=\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-(k+1)}\right]}{\sum_{x=0}^{k}(2\beta+1)^{-x}P(x)}.$$

Conversely, if (83) holds, then

$$\sum_{x=0}^{k}\left\{[P(x)]^{-1}f(x)\right\}=F(k)\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-(k+1)}\right]}{\sum_{x=0}^{k}(2\beta+1)^{-x}P(x)}\right). \tag{84}$$

From (84), we also have

$$\sum_{x=0}^{k-1}\left\{[P(x)]^{-1}f(x)\right\}=F(k-1)\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-k}\right]}{\sum_{x=0}^{k-1}(2\beta+1)^{-x}P(x)}\right)$$
$$=(F(k)-f(k))\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-k}\right]}{\sum_{x=0}^{k-1}(2\beta+1)^{-x}P(x)}\right). \tag{85}$$

Now, subtracting (85) from (84), yields

$$f(k)\left\{[P(x)]^{-1}-\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-k}\right]}{\sum_{x=0}^{k-1}(2\beta+1)^{-x}P(x)}\right)\right\}$$
$$=F(k)\left\{\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-(k+1)}\right]}{\sum_{x=0}^{k}(2\beta+1)^{-x}P(x)}\right)-\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-k}\right]}{\sum_{x=0}^{k-1}(2\beta+1)^{-x}P(x)}\right)\right\}.$$

From the above equality, we have

$$\frac{f\left(k\right)}{F\left(k\right)}=\frac{\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-(k+1)}\right]}{\sum_{x=0}^{k}(2\beta+1)^{-x}P(x)}\right)-\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-k}\right]}{\sum_{x=0}^{k-1}(2\beta+1)^{-x}P(x)}\right)}{\left[P\left(x\right)\right]^{-1}-\left(\frac{(1+1/2\beta)\left[1-(2\beta+1)^{-k}\right]}{\sum_{x=0}^{k-1}(2\beta+1)^{-x}P(x)}\right)}=\frac{(2\beta+1)^{-k}P(k)}{\sum_{x=0}^{k}f(x;\beta)},$$

which is the reverse hazard function corresponding to the pmf (10), so X has pmf (10). □

2.4. Poisson Entropy-Based Weighted Exponential (PEBWE)

Proposition 2.4. *Let* $X:\Omega\to\mathbb{N}^{*}$ *be a random variable. The pmf of* X *is* (30) *if and only if*

$$E\left\{\left[\frac{\left(1+\beta\left(1+X\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{1+X}}+\frac{\left(1+\beta\left(2+X\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+X}}\right]\mid X>k\right\}$$
$$=\left(\frac{\left(1+\beta\left(2+k\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+k}}\right). \tag{86}$$

Proof. If X has pmf (30), then for $k\in\mathbb{N}$, the left-hand side of (86), using telescoping sum formula, will be

$$\left(1-F\left(k\right)\right)^{-1}\sum_{x=k+1}^{\infty}\left(1-\ln\left(\beta\right)\right)^{-1}\left\{\left(\frac{\left(1+\beta\left(1+x\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{1+x}}\right)^{2}-\left(\frac{\left(1+\beta\left(2+x\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+x}}\right)^{2}\right\}$$
$$=\left(\frac{\left(1+\beta\left(2+k\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+k}}\right)^{-1}\left(\frac{\left(1+\beta\left(2+k\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+k}}\right)^{2}$$
$$=\left(\frac{\left(1+\beta\left(2+k\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+k}}\right).$$

Conversely, if (86) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\left(\frac{\left(1+\beta\left(1+x\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{1+x}}\right)+\left(\frac{\left(1+\beta\left(2+x\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+x}}\right)\right]f\left(x\right)\right\}$$
$$=\left(1-F\left(k\right)\right)\left(\frac{\left(1+\beta\left(2+k\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+k}}\right)$$
$$=\left(1-F\left(k+1\right)+f\left(k+1\right)\right)\left(\frac{\left(1+\beta\left(2+k\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+k}}\right). \tag{87}$$

From (87), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\left(\frac{\left(1+\beta\left(1+x\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{1+x}}\right)+\left(\frac{\left(1+\beta\left(2+x\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{2+x}}\right)\right]f\left(x\right)\right\}$$
$$=\left(1-F\left(k+1\right)\right)\left(\frac{\left(1+\beta\left(3+k\right)-\left(1+\beta\right)\ln\left(\beta\right)\right)}{\left(1+\beta\right)^{3+k}}\right). \tag{88}$$

Now, subtracting (88) from (87), yields

$$f(k+1)\left[\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right)+\left(\frac{(1+\beta(3+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{3+k}}\right)\right]$$
$$=(1-F(k+1))\left\{\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right)-\left(\frac{(1+\beta(3+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{3+k}}\right)\right\}$$
$$+f(k+1)\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right).$$

From the above equality, we have

$$\frac{f(k+1)}{1-F(k+1)}=\frac{\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right)-\left(\frac{(1+\beta(3+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{3+k}}\right)}{\left(\frac{(1+\beta(3+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{3+k}}\right)}=\frac{\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right)}{\left(\frac{(1+\beta(3+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{3+k}}\right)}-1,$$

which is the hazard function corresponding to the pmf (30), so X has pmf (30). □

2.5. Discrete Half-Logistic (DHL)

Proposition 2.5. *Let* $X:\Omega\to\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (50) *if and only if*

$$E\left\{\left[e^{-aX}\left[1-\left(\frac{a-2}{a}\right)e^{-X}\right]^{-a}+e^{-a(X+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(X+1)}\right]^{-a}\right]\mid X>k\right\}$$
$$=\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right). \tag{89}$$

Proof. If X has pmf (50), then for $k\in\mathbb{N}$, the left-hand side of (89), using telescoping sum formula, will be

$$(1-F(k))^{-1}\sum_{x=k+1}^{\infty}2^ae^{-a}\left\{e^{-2ax}\left[1-\left(\frac{a-2}{a}\right)e^{-x}\right]^{-2a}-e^{-2a(x+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(x+1)}\right]^{-2a}\right\}$$
$$=\left(2^ae^{-a}\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right)^{-1}2^ae^{-a}\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right)^2$$
$$=\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right).$$

Conversely, if (89) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[e^{-ax}\left[1-\left(\frac{a-2}{a}\right)e^{-x}\right]^{-a}+e^{-a(x+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(x+1)}\right]^{-a}\right]f(x)\right\}$$
$$=(1-F(k))\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right)$$
$$=(1-F(k+1)+f(k+1))\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right). \tag{90}$$

From (90), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[e^{-ax}\left[1-\left(\frac{a-2}{a}\right)e^{-x}\right]^{-a}+e^{-a(x+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(x+1)}\right]^{-a}\right]f\left(x\right)\right\}$$
$$=\left(1-F\left(k+1\right)\right)\left(\left\{e^{-2a(k+2)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+2)}\right]^{-2a}\right\}\right). \tag{91}$$

Now, subtracting (91) from (90), yields

$$f\left(k+1\right)\left[e^{-a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-a}+e^{-a(k+2)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+2)}\right]^{-a}\right]$$
$$=\left(1-F\left(k+1\right)\right)\left\{\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right)-\left(\left\{e^{-2a(k+2)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+2)}\right]^{-2a}\right\}\right)\right\}$$
$$+f\left(k+1\right)\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right).$$

From the above equality, we have

$$\frac{f\left(k+1\right)}{1-F\left(k+1\right)}=\frac{\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right)-\left(\left\{e^{-2a(k+2)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+2)}\right]^{-2a}\right\}\right)}{\left(\left\{e^{-2a(k+2)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+2)}\right]^{-2a}\right\}\right)}$$
$$=\frac{\left(\left\{e^{-2a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-2a}\right\}\right)}{\left(\left\{e^{-2a(k+2)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+2)}\right]^{-2a}\right\}\right)}-1,$$

which is the hazard function corresponding to the pmf (50), so X has pmf (50). □

3. Characterizations Based on Conditional Expectation

This chapter deals with 1 of the distributions listed in Section 1. The characterizations presented here are in terms of the reverse hazard function.

3.1. Discretized Generalized Gompertz (DGG) Distribution

Proposition 3.1. *Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is (2) if and only if its reverse hazard function, r_F, satisfies the difference equation*

$$r_F\left(k+1\right)-r_F\left(k\right)=\frac{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha k}-1\right)}\right]^{\theta}}{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}}-\frac{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}}{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+2)}-1\right)}\right]^{\theta}},\quad k\in\mathbb{N}^*, \tag{92}$$

with the initial condition $r_F\left(0\right)=1$.

Proof. If X has pmf (2), then clearly (92) holds. Now, if (92) holds, then for every $x\in\mathbb{N}$, we have

$$\sum_{k=1}^{x-1}\left\{r_F\left(k+1\right)-r_F\left(k\right)\right\}=\sum_{k=0}^{x-1}\left\{\frac{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha k}-1\right)}\right]^{\theta}}{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}}-\frac{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+1)}-1\right)}\right]^{\theta}}{\left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(k+2)}-1\right)}\right]^{\theta}}\right\},$$

or, using telescoping sum

$$r_F(x) - r_F(0) = -\frac{\left[1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right]^{\theta}}{\left[1 - e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta}},$$

or in view of the initial condition

$$r_F(x) = 1 - \frac{\left[1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right]^{\theta}}{\left[1 - e^{-\frac{\lambda}{\alpha}\left(e^{\alpha(x+1)}-1\right)}\right]^{\theta}}, \quad x \in \mathbb{N}^*,$$

which is the reverse hazard function corresponding to the pmf (2). □

4. Characterizations of distributions based on hazard function

This subsection is devoted to 2 distributions listed in the Introduction. The characterizations presented here are in terms of the hazard function.

4.1. Poisson Entropy-Based Weighted Exponential (PEBWE)

Proposition 4.1. *Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is* (30) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \frac{\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right)}{\left(\frac{(1+\beta(3+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{3+k}}\right)} - \frac{\left(\frac{(1+\beta(1+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{1+k}}\right)}{\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right)}, \tag{93}$$

with the initial condition $h_F(0) = \frac{(1+\beta)^2(1-\ln(\beta))}{(1+2\beta-(1+\beta)\ln(\beta))} - 1.$

Proof. If X has pmf (30), then clearly (93) holds. Now, if (93) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\} = \sum_{k=0}^{x-1} \left\{ \frac{\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right)}{\left(\frac{(1+\beta(3+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{3+k}}\right)} - \frac{\left(\frac{(1+\beta(1+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{1+k}}\right)}{\left(\frac{(1+\beta(2+k)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+k}}\right)} \right\}$$
$$= \frac{\left(\frac{(1+\beta(1+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{1+x}}\right)}{\left(\frac{(1+\beta(2+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+x}}\right)} - \frac{(1+\beta)^2(1-\ln(\beta))}{(1+2\beta-(1+\beta)\ln(\beta))},$$

or

$$h_F(x) - h_F(0) = \frac{\left(\frac{(1+\beta(1+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{1+x}}\right)}{\left(\frac{(1+\beta(2+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+x}}\right)} - \frac{(1+\beta)^2(1-\ln(\beta))}{(1+2\beta-(1+\beta)\ln(\beta))},$$

or, in view of the initial condition

$$h_F(x) = \frac{\left(\frac{(1+\beta(1+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{1+x}}\right)}{\left(\frac{(1+\beta(2+x)-(1+\beta)\ln(\beta))}{(1+\beta)^{2+x}}\right)} - 1, \quad x \in \mathbb{N}^*,$$

which is the hazard function corresponding to the pmf (30). □

4.2. Discrete Half-Logistic (DHL)

Proposition 4.2. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (50) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \frac{\left\{e^{-a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-a}\right\}}{\left\{e^{-a(k+2)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+2)}\right]^{-a}\right\}} - \frac{\left\{e^{-ak}\left[1-\left(\frac{a-2}{a}\right)e^{-k}\right]^{-a}\right\}}{\left\{e^{-a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-a}\right\}}, \tag{94}$$

with the initial condition $h_F(0) = \frac{\left[1-\left(\frac{a-2}{a}\right)\right]^{-a}}{e^{-a}\left[1-\left(\frac{a-2}{a}\right)e^{-1}\right]^{-a}} - 1.$

Proof. If X has pmf (50), then clearly (94) holds. Now, if (94) holds, then for every $x \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{k=0}^{x-1}\{h_F(k+1) - h_F(k)\} \\
&= \sum_{k=0}^{x-1}\left\{\frac{\left\{e^{-a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-a}\right\}}{\left\{e^{-a(k+2)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+2)}\right]^{-a}\right\}} - \frac{\left\{e^{-ak}\left[1-\left(\frac{a-2}{a}\right)e^{-k}\right]^{-a}\right\}}{\left\{e^{-a(k+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(k+1)}\right]^{-a}\right\}}\right\} \\
&= \frac{\left\{e^{-ax}\left[1-\left(\frac{a-2}{a}\right)e^{-x}\right]^{-a}\right\}}{\left\{e^{-a(x+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(x+1)}\right]^{-a}\right\}} - \frac{\left[1-\left(\frac{a-2}{a}\right)\right]^{-a}}{e^{-a}\left[1-\left(\frac{a-2}{a}\right)e^{-1}\right]^{-a}},
\end{aligned}$$

or

$$h_F(x) - h_F(0) = \frac{\left\{e^{-ax}\left[1-\left(\frac{a-2}{a}\right)e^{-x}\right]^{-a}\right\}}{\left\{e^{-a(x+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(x+1)}\right]^{-a}\right\}} - \frac{\left[1-\left(\frac{a-2}{a}\right)\right]^{-a}}{e^{-a}\left[1-\left(\frac{a-2}{a}\right)e^{-1}\right]^{-a}},$$

or, in view of the initial condition

$$h_F(x) = \frac{\left\{e^{-ax}\left[1-\left(\frac{a-2}{a}\right)e^{-x}\right]^{-a}\right\}}{\left\{e^{-a(x+1)}\left[1-\left(\frac{a-2}{a}\right)e^{-(x+1)}\right]^{-a}\right\}} - 1, \quad x \in \mathbb{N}^*,$$

which is the hazard function corresponding to the pmf (50). □

References

Abd-Elmonem, A. M., Teamah, A. A. E., and Gemeay, A. M. (2024). Discrete half-logistic distribution: statistical properties, estimation, and application. *Journal of Statistics Applications & Probability*, 13(1):273–284.

Adetunji, A. A. and Sabri, S. R. M. (2023a). An alternative count distribution for modeling dispersed observations. *Pertanika Journal of Science & Technology*, 31(3):1587–1603.

Adetunji, A. A. and Sabri, S. R. M. (2023b). On zero-inflated mixed Poisson exponential distribution: properties and applications to observation with excess zeros. *Maejo International Journal Of Science & Technology*, 17(01):68–80.

Ahsan-ul Haq, M., Irshad, M. R., Ahammed, E. S. M., and Maya, R. (2023). New discrete Bilal distribution and associated INAR(1) process. *Lobachevskii Journal of Mathematics*, 44(9):3647–3662.

Ahsan-ul Haq, M. and Zafar, J. (2023). A new one-parameter discrete probability distribution with its neutrosophic extension: mathematical properties and applications. *International Journal of Data Science and Analysis*. 11 pages, DOI: 10.1007/s41060-023-00382-z.

Akkanphudit, T. (2023). The discrete Gompertz-Weibull-Fréchet distribution: properties and applications. *Lobachevskii Journal of Mathematics*, 44(9):3663–3672.

Alomair, A. and Ahsan-ul Haq, M. (2023). A new extension of Poisson distribution for asymmetric count data: theory, classical and Bayesian estimation with application to lifetime data. *Peer Journal of Computer Science*, 9:e1748:1–19. DOI: 10.7717/peerj-cs.1748.

Atikankul, Y. (2023). A new Poisson generalized Lindley regression model. *Austrian Journal of Statistics*, 52(1):39–50.

Das, D. and Das, B. (2023). Weighted discretized Fréchet-Weibull distribution with application to real life data. *Applied Mathematics & Information Sciences*, 7(5):791–806.

Das, D., Das, B., and Hazarika, P. J. (2023). Discretized version of generalized Gompertz distribution with application to real life data. *International Journal of Agriculture & Statistical Sciences*, 19(1):441–456.

Diafouka, M. K., Louzayadio, C. G., and Malouata, R. O. (2023). A new weighted Poisson distribution for over-and under-dispersion situations. *Applications and Applied Mathematics (AAM)*, 18(1):1–15. Article 6.

Haj Ahmad, H., Ramadan, D. A., and Almetwally, E. M. (2024). Evaluating the discrete generalized Rayleigh distribution: statistical inferences and applications to real data analysis. *Mathematics (MDPI)*, 12(183):23 pages.

Hamedani, G. G., Roshani, A., and Butt, N. S. (2024). Characterizations of recently introduced discrete distributions. *Pakistan Journal of Statistics and Operation Research (PJSOR)*.

Irshad, M. R., Aswathy, S., Maya, R., and Nadarajah, S. (2024). New one-parameter over-dispersed discrete distribution and its application to the nonnegative integer-valued autoregressive model of order one. *Mathematics (MDPI)*, 12(81):14 pages.

Junnumtuam, S., Niwitpong, S.-A., and Niwitpong, S. (2022). A zero-and-one inflated cosine geometric distribution and its application. *Mathematics (MDPI)*, 10(4012):22 pages.

Karakaya, K. (2023). A new discrete distribution with applications to radiation, smoking, and health data. *Journal of Radiation Research and Applied Sciences*, 16(100735):1–11.

Krishnakumari, K. and George, D. (2023). Exponentiated discrete hypo exponential distribution and its generalizations. *Reliability: Theory & Applications*, 4(76):998–1010.

Merupula, J., Vaidyanathan, V. S., and Chesneau, C. (2023). Prediction interval for compound Conway-Maxwell-Poisson regression model with application to vehicle insurance claim data. *Mathematical and Computational Applications (MDPI)*, 28(2):39. 14 pages.

Seghier, F. Z., Ahsan-ul Haq, M., Zeghdoudi, H., and Hashmi, S. (2023a). A new generalization of Poisson distribution for over-dispersed, count data: mathematical properties, regression model and applications. *Lobachevskii Journal of Mathematics*, 44(9):3850–3859.

Seghier, F. Z., Zeghdoudi, H., and Raman, V. (2023b). A novel discrete distribution: properties and application using Nipah virus infection data set. *European Journal of Statistics*, 3:1–17.