Pak.j.stat.oper.res. Vol.21 No. 2 2025 pp 237-271 DOI: https://doi.org/10.18187/pjsor.v21i2.4653

Pakistan Journal of Statistics and Operation Research

Characterizations of Certain (2023-2024) Introduced Univariate Continuous Distributions II

G.G. Hamedani[1*], Amin Roshani[2], Partha Jyoti Hazarika[3]

*Corresponding author

1. Department of Mathematical and Statistical Sciences, Marquette University, Milwaukee, WI 53201-1881, USA, gholamhoss.hamedani@marquette.edu
2. Department of Statistics, Lorestan University, Khorramabad, Iran, roshani.amin@gmail.com
3. Department of Statistics, Dibrugarh University, Assam, India, parthajhazarika@gmail.com

Abstract

This paper is a continuation of our previous work with the same title, which deals with various characterizations of certain univariate continuous distributions proposed in (2023-2024) after the publication of our first paper in (2024). These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) reverse hazard function and (iv) conditional expectation of a single function of the random variable. It should be mentioned that for the characterization (i) the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

Key Words: Characterizations; Conditional expectation; Continuous distributions; Hazard function; Reverse hazard function.

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with certain characterizations of each of the following distributions: 1) Lindley Convolution with Gamma (LCG) distribution of Biswas et al. (2023); 2) Zubair-Exponentiated Weibull (ZEW) distribution of Alsultan (2023); 3) New Extended "A" (NEA) distribution of Al-Saiary (2023); 4) Type II Topp-Leone Generalized Inverted Exponential (TIITLGIE) distribution of Al-Saiary and Al-Jadaani (2022); 5) Balakrishnan-Alpha-Beta-Skew-Exponential (BABSE) distribution of Shah et al. (2023); 6) Double XLindley (DXL) distribution of Etaga et al. (2023); 7) Exponentiated Generalized Weibull Exponential (EGWE) distribution of Abonongo and Abonongo (2023); 8) Exponentiated Power-Weibull (EP-W) family of distributions of Zeenalabiden and Saracoglu (2023); 9) Ola Distribution (OD) of Al-Ta'ani and Gharaibeh (2023); 10) Odd Generalized Nadarajah-Haghighi Log-Logistic (OGNH-LL) distribution of Alabdulhadi (2024); 11) Power Unit Gumbel Type II (PUGT2) distribution of Nagy et al. (2023); 12) New XLindley (NXL) distribution of Khodja et al. (2023); 13) Compound Transmuted-Weibull (CT-W) family of distributions of Kus et al. (2023); 14) Doubly Bounded Exponential (DBE) distribution of Singh et al. (2023); 15) Odd Log Logistic Kumaraswamy (OLLK) distribution of Opone et al. (2023); 16) Odd Log-Logistic Power Inverse Lindley (OLL-PIL) distribution of Eltehiwy (2024); 17) Topp-Leone Odd Burr X-G (TLOBX-G) family of distributions of Oluyede et al. (2024); 18) Modified Generalized Linear Exponential (MGLE) distribution of Mahmoud et al. (2024); 19) Type II Exponentiated Half Logistic-Odd Burr X-G Power Series (TII-EHL-OBX-GPS) class of distributions of Dingalo et al. (2023); 20) Odd Lomax Gompertz (OLGo) distribution of Mazrouk et al. (2023); 21) Shifted Generalized Truncated Nadarajah-Haghighi (SGeTNH) distribution of Azimi and Esmailian (2023); 22) Inverted Exponentiated Chen (IEC) distribution of Azimi et al. (2023); 23) Three Parameter New Modified Weibull (NMW3) distribution of Ghazal (2023); 24) Type I Heavy-Tailed Odd Power Generalized Weibull-G (TI-HT-OPGW-G) family of distributions of Moakofi and Oluyede (2023a); 25) Type II Exponentiated Half-Logistic Gompertz-G (TIIEHL-Gom-G) distribution of Moakofi and Oluyede (2023b); 26) New Gamma Lindley (NGaL) distribution of Nawel et al. (2023); 27) Logistic Burr XII (LBXII) distribution of Guerra et al. (2023); 28) Inverted Nadarajah-Haghighi Power Series (INHPS) distribution of Ahsan-ul Haq et al. (2023); 29) Marshall-Olkin Bilal (MOB) distribution of Irshad et al. (2023); 30) Kumaraswamy Bell Exponential (KwBE) distribution of Imran et al. (2023); 31) New Kumaraswamy Exponential (NKwE) distribution of Naz et al. (2023); 32) Generalized Topp-Leone-G Power Series (GTL-GPS) class of distributions of Warahena-Liyanage et al. (2023); 33) New Two-Parameter Distribution-G (ND-G) family of distributions of Aidi et al. (2023); 34) Two-Component Mixture of generalized Rayleigh (GR) distribution of Dey et al. (2023); 35) Kumarswamy Tan Generalized (KwT-G) family of distributions of Alqawba et al. (2023); 36) Logarithmic Transformed Pareto I (LTPa-I) distribution of Aniyan and George (2023); 37) Poisson-Exponentiated Weibull (PEW) distribution of George and George (2023); 38) Exponentially Generated Modified Chen (EGMC) distribution of Abidemi and Abiodun (2023); 39) Maxwell-Lomax (M-L) distribution of Abiodun and Ishaq (2022); 40) Odd Beta Prime Inverted Kumaraswamy (OBPIK) distribution of Suleiman et al. (2023a); 41) Odd Beta Prime-Burr X (OBPBX) distribution of Suleiman et al. (2023c); 42) Log Burr III (LBIII) distribution of Ishaq et al. (2023b); 43) Odd Beta Prime-G (OBP-G) family of distributions of Suleiman et al. (2023e); 44) Log-Topp-Leone (LTL) distribution of Usman et al. (2023); 45) Odd Beta Prime Fréchet (OBPF) distribution of Suleiman et al. (2023b); 46) Odd Beta Prime-Logistic (OBP-Logistic) distribution of Suleiman et al. (2023d); 47) Log-Kumaraswamy (Log-Kw) distribution of Ishaq et al. (2023a); 48) Exponential-Lindley (Exp-Lindley) distribution of Sakthivel and G. (2024); 49) McDonald Generalized Power Weibull (MGPW) distribution of Sayibu et al. (2024); 50) Generalized Complementary Exponentiated Bell-Touchard (GCEBT) distribution of Imran and Mendy (2024); 51) Log-Lindley (Log-L) distribution of Alzawq et al. (2024); 52) MIT-Weibull (MITW) distribution of Lone et al. (2024); 53) New Two-Parameter Weighted Exponential (NTWE) distribution of Chesneau et al. (2024); 54) Generalized Marshall-Olkin Transmuted-G (GMOT-G) family of distributions of Handique et al. (2024); 55) Truncated Cauchy Power Kumarswamy Lomax (TCPKwL) distribution of Elbatal et al. (2024a); 56) Inverse Power Zeghdoudi (IPZ) distribution of Elbatal et al. (2024b).

We list below the cumulative distribution function (cdf) and probability density function (pdf) of each one of these distributions in the same order as listed above. We will be employing the same notation for the parameters as chosen by the original authors.

1) The cdf and pdf of (LCG) are given, respectively, by

$$F(x;\beta) = 1 - \frac{(1-x)^{\beta}(2+\beta+x\beta)}{2+\beta}, \quad 0 \le x \le 1, \tag{1}$$

and

$$f(x;\beta) = \frac{\beta(1+\beta)}{2+\beta}(1+x)(1-x)^{\beta-1}, \quad 0 < x < 1, \tag{2}$$

where $\beta > 0$ is a parameter.

2) The cdf and pdf of (ZEW) are given, respectively, by

$$F(x;\delta,b,\lambda,\beta) = \frac{e^{\delta\left[1-e^{-(bx)^{\lambda}}\right]^{2\beta}} - 1}{e^{\delta}-1}, \quad x \ge 0, \tag{3}$$

and

$$f\left(x;\delta,b,\lambda,\beta\right)=\frac{d}{dx}F\left(x;\delta,b,\lambda,\beta\right),\quad x>0, \tag{4}$$

where $\delta, b, \lambda, \beta$ are all positive parameters.

Remark 1.1. *Taking* $G\left(x\right)=\left[1-e^{-(bx)^{\lambda}}\right]^{2\beta}$, $x\geq 0$*, the cdf* (3) *can be written as*

$$F\left(x;\delta,b,\lambda,\beta\right)=\frac{e^{\delta G(x)}-1}{e^{\delta}-1},\quad x\geq 0,$$

which has been characterized in Hamedani (2021).

3) The cdf and pdf of (NEA) are given, respectively, by

$$F\left(x;\beta,\theta\right)=1-\left[\frac{1-\exp\left[\left(\frac{2}{\beta}\left(1-\exp\left(\frac{\beta}{x}\right)\right)\right)\right]}{1-\left(1-\theta\right)\exp\left[\left(\frac{2}{\beta}\left(1-\exp\left(\frac{\beta}{x}\right)\right)\right)\right]}\right]^{\theta},\quad x\geq 0, \tag{5}$$

and

$$f\left(x;\beta,\theta\right)=\frac{d}{dx}F\left(x;\beta,\theta\right),\quad x>0, \tag{6}$$

where $\beta>0,\theta>0$ are parameters.

Remark 1.2. *Taking* $G\left(x\right)=1-\frac{1-\exp\left[\left(\frac{2}{\beta}\left(1-\exp\left(\frac{\beta}{x}\right)\right)\right)\right]}{1-(1-\theta)\exp\left[\left(\frac{2}{\beta}\left(1-\exp\left(\frac{\beta}{x}\right)\right)\right)\right]}$, $x\geq 0$*, the cdf* (5) *can be expressed as*

$$F\left(x;\beta,\theta\right)=1-\left[1-G\left(x\right)\right]^{\theta},\quad x\geq 0,$$

which was mentioned in Hamedani et al. (2024).

4) The cdf and pdf of (TIITLGIE) are given, respectively, by

$$F\left(x;\lambda,\theta,\alpha\right)=1-\left[1-\left[1-\left[1-e^{-\lambda/x}\right]^{\theta}\right]^{2}\right]^{\alpha},\quad x\geq 0, \tag{7}$$

and

$$f\left(x;\lambda,\theta,\alpha\right)=\frac{d}{dx}F\left(x;\lambda,\theta,\alpha\right),\quad x>0, \tag{8}$$

where λ,θ,α are all positive parameters.

Remark 1.3. *Taking* $G\left(x\right)=e^{-\lambda/x}$, $x\geq 0$*, the cdf* (7) *can be written as*

$$F\left(x;\lambda,\theta,\alpha\right)=1-\left[1-\left[1-\left[1-G\left(x\right)\right]^{\theta}\right]^{2}\right]^{\alpha},\quad x\geq 0,$$

which was mentioned in Remark 1.39 in Hamedani (2023).

5) The cdf and pdf of (BABSE) are given, respectively, by

$$F\left(x;\alpha,\beta,\lambda\right)=\int_{0}^{x}f\left(u;\alpha,\beta,\lambda\right)du,\quad x\geq 0, \tag{9}$$

and

$$f\left(x;\alpha,\beta,\lambda\right)=Ce^{-\lambda x}P\left(x\right),\quad x>0, \tag{10}$$

where α,β,λ are all positive parameters, $P\left(x\right)=\left[\left(1-\alpha x-\beta x^{3}\right)^{2}+1\right]^{2}$ and C is the normalizing constant.

Remark 1.4. *Shah et al. (2023), proposed Balakrishnan-Alpha-Beta-Skew-Laplace (BABSL) distribution. Here, we replace "Laplace" with "Exponential" to have the support on* $(0, \infty)$.

6) The cdf and pdf of (DXL) are given, respectively, by

$$F(x;\theta) = \int_0^x f(u;\theta)\, du, \quad x \geq 0, \tag{11}$$

and

$$f(x;\theta) = Ce^{-\theta x} P(x), \quad x > 0, \tag{12}$$

where $\theta > 0$ is a parameter, $P(x) = 3 + 3\theta + \theta^2 + x$ and $C = \frac{\theta^2}{(\theta+1)^3}$ is the normalizing constant.

Remark 1.5. *The pdf* (12) *is similar to the pdf* (10).

7) The cdf and pdf of (EGWE) are given, respectively, by

$$F(x;a,b,c,d,\alpha,\beta) = \left[1 - \exp\left\{-\alpha\left(\frac{abx}{c}\right)^d\right\}\right]^\beta, \quad x \geq 0, \tag{13}$$

and

$$f(x;a,b,c,d,\alpha,\beta) = \frac{d}{dx} F(x;a,b,c,d,\alpha,\beta), \quad x > 0, \tag{14}$$

where $a, b, c, d, \alpha, \beta$ all positive parameters.

Remark 1.6. *Taking* $G(x) = \left[1 + \left(\frac{abx}{c}\right)^{-1}\right]^{-1}$, $x \geq 0$, *the cdf* (13) *can be written as*

$$F(x;a,b,c,d,\alpha,\beta) = \left[1 - \exp\left\{-\alpha\left(\frac{G(x)}{1-G(x)}\right)^d\right\}\right]^\beta, \quad x \geq 0,$$

which was mentioned in Remark 1.10 in Hamedani (2023).

8) The cdf and pdf of (EP-W) are given, respectively, by

$$\begin{aligned} F(x;\lambda,\delta,\psi) &= 1 - \exp\left\{1 - \exp\left(\frac{(\lambda x)^\delta}{\psi}\right)\right\} \\ &= 1 - \exp\left\{-\left[\exp\left(\frac{(\lambda x)^\delta}{\psi}\right) - 1\right]\right\}, \quad x \geq 0, \end{aligned} \tag{15}$$

and

$$f(x;\lambda,\delta,\psi) = \frac{d}{dx} F(x;\lambda,\delta,\psi), \quad x > 0, \tag{16}$$

where $\lambda > 0, \delta = \beta\theta > 0, \psi = \alpha^\beta > 0$ are parameters.

Remark 1.7. *Taking* $G(x) = 1 - \exp\left(-\frac{(\lambda x)^\delta}{\psi}\right)$, $x \geq 0$, *the cdf* (15) *can be expressed as*

$$F(x;\lambda,\delta,\psi) = 1 - \exp\left\{-\left(\frac{G(x)}{1-G(x)}\right)\right\}, \quad x \geq 0,$$

which is a special case of the cdf mentioned in Remark 1.6.

9) The cdf and pdf of (OD) are given, respectively, by

$$F(x;\beta)=\int_0^x f(u;\beta)\,du,\quad x\geq 0, \tag{17}$$

and

$$f(x;\beta)=Ce^{-\beta x}P(x),\quad x>0, \tag{18}$$

where $\beta>0$ is a parameter, $P(x)=x^7+x^3+1$ and $C=\frac{\beta^8}{\beta^7+6\beta^4+5040}$ is the normalizing constant.

Remark 1.8. *The pdf* (18) *is similar to the pdf* (12).

10) The cdf and pdf of (OGNH-LL) are given, respectively, by

$$\begin{aligned}F(x;\lambda,\sigma,\tau,\delta)&=1-ee^{-\left(1+\left(\frac{x}{\tau}\right)^{\delta\lambda}\right)^{\sigma}}\\&=1-e^{-\left[\left(1+\left(\frac{x}{\tau}\right)^{\delta\lambda}\right)^{\sigma}-1\right]},\quad x\geq 0,\end{aligned} \tag{19}$$

and

$$f(x;\lambda,\sigma,\tau,\delta)=\frac{d}{dx}F(x;\lambda,\sigma,\tau,\delta),\quad x>0, \tag{20}$$

where $\lambda,\sigma,\tau,\delta$ are all positive parameters.

Remark 1.9. *Taking* $G(x)=1-\left(1+\left(\frac{x}{\tau}\right)^{\delta\lambda}\right)^{-\sigma}$, $x\geq 0$, *the cdf* (19) *can be expressed as*

$$F(x;\lambda,\sigma,\tau,\delta)=1-\exp\left\{-\left(\frac{G(x)}{1-G(x)}\right)\right\},\quad x\geq 0,$$

which is a special case of the cdf mentioned in Remark 1.6.

11) The cdf and pdf of (PUGT2) are given, respectively, by

$$F(x;\delta,\sigma,\beta)=e^{-\delta\left(\frac{1}{x^{-\beta}-1}\right)^{-\sigma}},\quad 0\leq x\leq 1, \tag{21}$$

and

$$f(x;\delta,\sigma,\beta)=\beta\delta\sigma x^{-\beta-1}P(x),\quad 0<x<1, \tag{22}$$

where δ,σ,β are all positive parameters and $P(x)=\left(\frac{1}{x^{-\beta}-1}\right)^{1-\sigma}e^{-\delta\left(\frac{1}{x^{-\beta}-1}\right)^{-\sigma}}$.

12) The cdf and pdf of (NXL) are given, respectively, by

$$F(x;\theta)=1-\left(\frac{1}{2}\theta x+1\right)e^{-\theta x},\quad x\geq 0, \tag{23}$$

and

$$f(x;\theta)=\frac{\theta}{2}(1+\theta x)e^{-\theta x},\quad x>0, \tag{24}$$

where $\theta>0$ is a parameters.

13) The cdf and pdf of (CT-W) are given, respectively, by

$$F\left(x;\alpha,\beta,p\right)=\int_{0}^{x}f\left(u;\alpha,\beta,p\right)du,\quad x\geq 0, \tag{25}$$

and

$$f\left(x;\alpha,\beta,p\right)=\beta\alpha\left(1-p\right)x^{\beta-1}e^{-\alpha x^{\beta}}P\left(x\right),\quad x>0, \tag{26}$$

where $\alpha>0,\beta>0,p\in(0,1)$ are parameters and $P\left(x\right)=\frac{2p(p-1)^{2}e^{-\alpha x^{\beta}}-p^{2}e^{-2\alpha x^{\beta}}+2p^{2}-p^{3}-1}{\left[\left(1-p\left(1-e^{-\alpha x^{\beta}}\right)\right)\left(pe^{-\alpha x^{\beta}}-1\right)\right]^{2}}$.

Remarks 1.10. (a) *The pdf* (26) *is similar to the pdf* (52) *in Hamedani et al. (2024).* (b) *"$(p-1)$" in the equation (10) of Kus et al. should be "$(1-p)$".*

14) The cdf and pdf of (DBE) are given, respectively, by

$$F\left(x;\lambda,\theta_{1},\theta_{2}\right)=1-\exp\left[-\frac{\lambda\left(x-\theta_{1}\right)}{\left(\theta_{2}-x\right)}\right],\quad\theta_{1}\leq x\leq\theta_{2}, \tag{27}$$

and

$$f\left(x;\lambda,\theta_{1},\theta_{2}\right)=\frac{\lambda\left(\theta_{2}-\theta_{1}\right)}{\left(\theta_{2}-x\right)^{2}}\exp\left[-\frac{\lambda\left(x-\theta_{1}\right)}{\left(\theta_{2}-x\right)}\right],\quad\theta_{1}<x<\theta_{2}, \tag{28}$$

where $\lambda,\theta_{1},\theta_{2}$ are all positive parameters.

15) The cdf and pdf of (OLLK) are given, respectively, by

$$F\left(x;\alpha,a,b\right)=\frac{G\left(x\right)^{\alpha}}{G\left(x\right)^{\alpha}+\overline{G}\left(x\right)^{\alpha}},\quad 0\leq x\leq 1, \tag{29}$$

and

$$f\left(x;\alpha,a,b\right)=\frac{d}{dx}F\left(x;\alpha,a,b\right),\quad 0<x<1, \tag{30}$$

where α,a,b are all positive parameters and $G\left(x\right)=1-\left(1-x^{a}\right)^{b},0\leq x\leq 1$.

Remark 1.11. *The cdf* (29) *is similar to the cdf* (1.407) *in Hamedani (2023).*

16) The cdf and pdf of (OLL-PIL) are given, respectively, by

$$F\left(x;\alpha,\beta,\theta\right)=\frac{G\left(x\right)^{\theta}}{G\left(x\right)^{\theta}+\overline{G}\left(x\right)^{\theta}},\quad x\geq 0, \tag{31}$$

and

$$f\left(x;\alpha,\beta,\theta\right)=\frac{d}{dx}F\left(x;\alpha,\beta,\theta\right),\quad x>0, \tag{32}$$

where α,β,θ are all positive parameters and $G\left(x\right)=\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\beta/x^{\alpha}},x\geq 0$.

Remark 1.12. *The cdf* (31) *is similar to the cdf* (1.407) *in Hamedani (2023).*

17) The cdf and pdf of (TLOBX-G) are given, respectively, by

$$F\left(x;\alpha,\theta,\zeta\right)=\left[1-\left[1-\left\{1-\exp\left[-\left(\frac{G\left(x;\zeta\right)}{\overline{G}\left(x;\zeta\right)}\right)^{2}\right]\right\}^{\theta}\right]^{2}\right]^{\alpha},\quad x\geq 0, \tag{33}$$

and

$$f(x;\alpha,\theta,\zeta)=\frac{d}{dx}F(x;\alpha,\theta,\zeta), \quad x>0, \tag{34}$$

where $\alpha>0,\theta>0$ are parameters and $G(x;\zeta)$ is a baseline cdf depending on the parameter vector ζ.

Remark 1.13. *Taking* $K(x)=\left\{1-\exp\left[-\left(\frac{G(x;\zeta)}{\overline{G}(x;\zeta)}\right)^2\right]\right\}^{\theta}$, $x\geq 0$, *the cdf* (33) *can be expressed as*

$$F(x;\alpha,\theta,\zeta)=\left[1-[1-K(x)]^2\right]^{\alpha}, \quad x\geq 0,$$

which was mentioned in Remark 1.3.

18) The cdf and pdf of (MGLE) are given, respectively, by

$$F(x;c,b,\xi,\varphi)=1-\exp\left\{-\left(cx+\frac{b}{2}x^2\right)^{\xi}e^{\varphi\left(cx+\frac{b}{2}x^2\right)}\right\}, \quad x\geq 0, \tag{35}$$

and

$$f(x;c,b,\xi,\varphi)=(c+bx)\left(cx+\frac{b}{2}x^2\right)^{\xi-1}e^{-\left(cx+\frac{b}{2}x^2\right)^{\xi}}P(x), \quad x>0, \tag{36}$$

where $c\geq 0,b\geq 0,\xi>0,\varphi>0$ are parameters and $P(x)=\left(\xi+\varphi\left(cx+\frac{b}{2}x^2\right)\right)\exp\left\{-\left(cx+\frac{b}{2}x^2\right)\left[e^{\varphi\left(cx+\frac{b}{2}x^2\right)}-\varphi-1\right]\right\}$.

19) The cdf and pdf of (TII-EHL-OBX-GPS) are given, respectively, by

$$F(x;\beta,\upsilon,p)=1-\frac{C(\upsilon(1-G(x)))}{C(\upsilon)}, \quad x\in\mathbb{R}, \tag{37}$$

and

$$f(x;\beta,\upsilon,p)=\frac{d}{dx}F(x;\beta,\upsilon,p), \quad x\in\mathbb{R}, \tag{38}$$

where β,υ,p are all positive parameters, $G(x)=1-\left[\frac{1-H(x;\beta,\xi)}{1+H(x;\beta,\xi)}\right]^{p}$, $x\in\mathbb{R}$, $H(x;\beta,\xi)$ is a baseline cdf, $C(\upsilon)=\sum_{n=1}^{\infty}a_n\upsilon^n$ is finite and $\{a_n\}_{n\geq 1}$ is a sequence of non-negative integers.

Remark 1.14. *The version of the cdf* (37) *has been characterized in Hamedani (2017).*

20) The cdf and pdf of (OLGo) are given, respectively, by

$$F(x;\theta,\beta,b)=1-\left\{1+\frac{1}{\beta}\left[\frac{1-\exp\left(-b^{-1}\left(e^{bx}-1\right)\right)}{\exp\left(-b^{-1}\left(e^{bx}-1\right)\right)}\right]\right\}^{-\theta}, \quad x\geq 0, \tag{39}$$

and

$$f(x;\theta,\beta,b)=\frac{d}{dx}F(x;\theta,\beta,b), \quad x>0, \tag{40}$$

where θ,β,b are all positive parameters.

Remark 1.15. *Taking* $G(x)=1-\exp\left(-b^{-1}\left(e^{bx}-1\right)\right)$, $x\geq 0$, *the cdf* (39) *can be written as*

$$F(x;\theta,\beta,b)=1-\left\{1+\frac{1}{\beta}\left[\frac{G(x)}{1-G(x)}\right]\right\}^{-\theta}, \quad x\geq 0,$$

which is a special case of the cdf mentioned in Remark 1.80 in Hamedani (2023).

21) The cdf and pdf of (SGeTNH) are given, respectively, by

$$F(x;\alpha,\beta,\lambda,\theta)=\frac{e^{1-\left(1+\lambda e^{-(x-\theta)^{\beta}}\right)^{\alpha}}-e^{1-(1+\lambda)^{\alpha}}}{1-e^{1-(1+\lambda)^{\alpha}}},\quad x\geq\theta, \tag{41}$$

and

$$f(x;\alpha,\beta,\lambda,\theta)=\frac{d}{dx}F(x;\alpha,\beta,\lambda,\theta),\quad x>\theta, \tag{42}$$

where $\alpha,\beta,\lambda,\theta$ are all positive parameters.

Remark 1.16. *Taking* $G(x)=\frac{(1+\lambda)^{\alpha}-\left(1+\lambda e^{-(x-\theta)^{\beta}}\right)^{\alpha}}{(1+\lambda)^{\alpha}-1}$, $x\geq\theta$*, the cdf* (41) *can be expressed as*

$$F(x;\alpha,\beta,\lambda,\theta)=\frac{e^{\delta G(x)}-1}{e^{\delta}-1},\quad x\geq\theta,$$

which was mentioned in Remark 1.1 for $\delta=(1+\lambda)^{\alpha}-1$.

22) The cdf and pdf of (IEC) are given, respectively, by

$$F(x;\alpha,\beta,\lambda)=1-\left[1-e^{\lambda\left(1-e^{x^{-\beta}}\right)}\right]^{\alpha},\quad x\geq0, \tag{43}$$

and

$$f(x;\alpha,\beta,\lambda)=\frac{d}{dx}F(x;\alpha,\beta,\lambda),\quad x>0, \tag{44}$$

where α,β,λ are all positive parameters.

Remark 1.17. *Taking* $G(x)=e^{\lambda\left(1-e^{x^{-\beta}}\right)}$, $x\geq0$*, the cdf* (43) *can be written as*

$$F(x;\alpha,\beta,\lambda)=1-[1-G(x)]^{\alpha},\quad x\geq0,$$

which is a special case of the cdf was mentioned in Remark 1.3.

23) The cdf and pdf of (NMW3) are given, respectively, by

$$F(x;\alpha,\beta,\theta)=1-e^{1-e^{\alpha x^{\beta}e^{\theta x}}},\quad x\geq0, \tag{45}$$

and

$$f(x;\alpha,\beta,\theta)=\frac{d}{dx}F(x;\alpha,\beta,\theta),\quad x>0, \tag{46}$$

where α,β,θ are all positive parameters.

Remark 1.18. *Taking* $G(x)=1-e^{-\alpha x^{\beta}e^{\theta x}}$, $x\geq0$*, the cdf* (45) *can be expressed as*

$$F(x;\alpha,\beta,\theta)=1-\exp\left\{-\left(\frac{G(x)}{1-G(x)}\right)\right\},\quad x\geq0,$$

which is a special case of the cdf mentioned in Remark 1.6.

24) The cdf and pdf of (TI-HT-OPGW-G) are given, respectively, by

$$F(x;\theta,\alpha,\beta,\xi)=1-\left(\frac{1-G(x;\xi)}{1-(1-\theta)G(x;\xi)}\right)^{\theta},\quad x\in\mathbb{R}, \tag{47}$$

and

$$f(x;\theta,\alpha,\beta,\xi)=\frac{d}{dx}F(x;\theta,\alpha,\beta,\xi),\quad x\in\mathbb{R},\tag{48}$$

where θ,α,β are all positive parameters and $G(x;\xi)$ is a baseline cdf depending on the parameter ξ.

Remark 1.19. *The cdf* (47) *can be written as*

$$F(x;\theta,\alpha,\beta,\xi)=1-\left\{\frac{1-(1-\theta)G(x;\xi)}{1-G(x;\xi)}\right\}^{-\theta},\quad x\in\mathbb{R},$$

which was mentioned in Remark 1.17 of Hamedani (2023).

25) The cdf and pdf of (TIIEHL-Gom-G) are given, respectively, by

$$F(x;\alpha,\gamma,\psi)=1-\left(\frac{1-K(x)}{1+K(x)}\right)^{\alpha},\quad x\in\mathbb{R},\tag{49}$$

and

$$f(x;\alpha,\gamma,\psi)=\frac{d}{dx}F(x;\alpha,\gamma,\psi),\quad x\in\mathbb{R},\tag{50}$$

where $\alpha>0,\gamma>0$ are parameters, $K(x)=1-\exp\left(\frac{1}{\gamma}\left\{1-[1-G(x;\psi)]^{-\gamma}\right\}\right)$, $x\in\mathbb{R}$ and $G(x;\psi)$ is a baseline cdf depending on the parameter ψ.

Remark 1.20. *The new version of the cdf* (49) *is a special case of the cdf mentioned in Remark 1.19 for* $\theta=2$ *.*

26) The cdf and pdf of (NGaL) are given, respectively, by

$$F(x;\theta,\beta)=\int_0^x f(u;\theta,\beta)\,du,\quad x\geq 0,\tag{51}$$

and

$$f(x;\theta,\beta)=\left(\frac{\theta^2\left((\beta+1)x+1\right)}{1+\theta+\beta}\right)e^{-\theta x},\quad x>0,\tag{52}$$

where $\theta>0,\beta>0$ are parameters.

Remark 1.21. *The pdf* (52) *can be written as*

$$f(x;\theta,\beta)=\left(1-\frac{1+\beta+\theta(1-\theta)-\theta^2(\beta+1)x}{1+\theta+\beta}\right)e^{-\theta x},\quad x>0,$$

which is the old Lindley distribution.

27) The cdf and pdf of (LBXII) are given, respectively, by

$$F(x;\lambda,d,c,s)=\left\{1-\left[d\log\left\{1+\left(\frac{x}{s}\right)^c\right\}\right]^{-\lambda}\right\}^{-1},\quad x\geq 0,\tag{53}$$

and

$$f(x;\lambda,d,c,s)=C^*\left(\frac{x}{s}\right)^{c-1}\left(1+\left(\frac{x}{s}\right)^c\right)^{-2}P(x),\quad x>0,\tag{54}$$

where λ,d,c,s are all positive parameters, $C^*=\lambda c d^{-\lambda}s^{-1}$ and $P(x)=\frac{\left(1+\left(\frac{x}{s}\right)^c\right)\left[\log\left\{1+\left(\frac{x}{s}\right)^c\right\}\right]^{-(\lambda+1)}}{\left\{1-\left[d\log\left\{1+\left(\frac{x}{s}\right)^c\right\}\right]^{-\lambda}\right\}^2}$.

28) The cdf and pdf of (INHPS) are given, respectively, by

$$F(x;\alpha,\lambda,\theta)=1-\frac{A(\theta(1-G(x)))}{A(\theta)},\quad x\geq 0, \tag{55}$$

and

$$f(x;\alpha,\lambda,\theta)=\frac{d}{dx}F(x;\alpha,\lambda,\theta),\quad x>0, \tag{56}$$

where α,λ,θ are all positive parameters, $G(x)=e^{1-\left(1+\frac{\lambda}{x}\right)^{\alpha}}$, $x\geq 0$, $A(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ is finite and $\{a_n\}_{n\geq 1}$ is a sequence of non-negative integers.

Remark 1.22. *The version of the cdf* (55) *has been characterized in Hamedani (2017).*

29) The cdf and pdf of (MOB) are given, respectively, by

$$F(x;\lambda,\theta)=\frac{1-\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)}{\left(1-(1-\theta)\left[3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right]\right)},\quad x\geq 0, \tag{57}$$

and

$$f(x;\lambda,\theta)=\frac{d}{dx}F(x;\lambda,\theta),\quad x>0, \tag{58}$$

where $\lambda>0,\theta>0$ are parameters.

Remark 1.23. *Taking* $G(x)=1-\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)$, $x\geq 0$, *the cdf* (57) *can be written as*

$$F(x;\lambda,\theta)=\frac{G(x)}{\theta+(1-\theta)G(x)},\quad x\geq 0,$$

which is a special case of the cdf mentioned in Remark 1.80 in Hamedani (2023).

30) The cdf and pdf of (KwBE) are given, respectively, by

$$F(x;a,b,\omega,\theta)=1-\left[1-\left\{\frac{1-\exp\left(-e^{\omega}\left[1-e^{-\omega\left(1-e^{-\theta x}\right)}\right]\right)}{1-\exp\left(1-e^{\omega}\right)}\right\}^{a}\right]^{b},\quad x\geq 0, \tag{59}$$

and

$$f(x;a,b,\omega,\theta)=\frac{d}{dx}F(x;a,b,\omega,\theta),\quad x>0, \tag{60}$$

where a,b,ω,θ are all positive parameters.

Remark 1.24. *Taking* $G(x)=\left\{\frac{1-\exp\left(-e^{\omega}\left[1-e^{-\omega\left(1-e^{-\theta x}\right)}\right]\right)}{1-\exp(1-e^{\omega})}\right\}^{a}$, $x\geq 0$, *the cdf* (59) *can be expressed as*

$$F(x;a,b,\omega,\theta)=1-[1-G(x)]^{b},\quad x\geq 0,$$

which was mentioned in Hamedani et al. (2024).

31) The cdf and pdf of (NKwE) are given, respectively, by

$$F(x;a,s,\omega,\theta)=1-\left[1-\left\{1-e^{-\theta x}e^{1-e^{-\theta x}}\right\}^{a}\right]^{s},\quad x\geq 0, \tag{61}$$

and

$$f\left(x;a,s,\omega,\theta\right)=\frac{d}{dx}F\left(x;a,s,\omega,\theta\right),\quad x>0, \tag{62}$$

where a,s,ω,θ are all positive parameters.

Remark 1.25. *Taking* $G\left(x\right)=\left\{1-e^{-\theta x}e^{1-e^{-\theta x}}\right\}^{a}$, $x\geq 0$, *the cdf* (61) *can be written as*

$$F\left(x;a,s,\omega,\theta\right)=1-\left[1-G\left(x\right)\right]^{s},\quad x\geq 0,$$

which was mentioned in Hamedani et al. (2024).

32) The cdf and pdf of (GTL-GPS) are given, respectively, by

$$F\left(x;\beta,\theta,b,\xi\right)=1-\frac{C\left(\theta\left[1-\left(1-\overline{G}^{2}\left(x;\xi\right)\right)^{b}\right]^{\beta}\right)}{C\left(\theta\right)},\quad x\in\mathbb{R}, \tag{63}$$

and

$$f\left(x;\beta,\theta,b,\xi\right)=C^{*}g\left(x;\xi\right)\overline{G}\left(x;\xi\right)\left(1-\overline{G}^{2}\left(x;\xi\right)\right)^{b-1}P\left(x\right),\quad x\in\mathbb{R}, \tag{64}$$

where β,θ,b are all positive parameters, $C\left(\theta\right)=\sum_{n=1}^{\infty}a_{n}\theta^{n}$ is finite and $\left\{a_{n}\right\}_{n\geq 1}$ is a sequence of non-negative integers, $C^{*}=\frac{2b\beta\theta}{C(\theta)}$, $G\left(x;\xi\right)$ and $P\left(x\right)=\frac{C'\left(\theta\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]^{\beta}\right)}{\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]^{1-\beta}}$.

33) The cdf and pdf of (ND-G) are given, respectively, by

$$F\left(x;\alpha,\lambda,\xi\right)=1-\left(1+\alpha\left[\frac{G\left(x;\xi\right)}{\overline{G}\left(x;\xi\right)}\right]^{\lambda}\right)\exp\left\{-\alpha\left[\frac{G\left(x;\xi\right)}{\overline{G}\left(x;\xi\right)}\right]^{\lambda}\right\},\quad x\in\mathbb{R}, \tag{65}$$

and

$$f\left(x;\alpha,\lambda,\xi\right)=\frac{d}{dx}F\left(x;\alpha,\lambda,\xi\right),\quad x\in\mathbb{R}, \tag{66}$$

where $\alpha>0,\lambda>0$ are parameters and $G\left(x;\xi\right)$ is a baseline cdf depending on the parameter vector ξ.

Remark 1.26. *The cdf* (65) *is not new, please see cdf* (257) *in Hamedani (2021).*

34) The cdf and pdf of (GR) are given, respectively, by

$$F\left(x;\alpha_{1},\alpha_{2},\lambda_{1},\lambda_{2},p_{1},p_{2}\right)=\int_{0}^{x}f\left(u;\alpha_{1},\alpha_{2},\lambda_{1},\lambda_{2},p_{1},p_{2}\right)du,\quad x\geq 0, \tag{67}$$

and

$$f\left(x;\alpha_{1},\alpha_{2},\lambda_{1},\lambda_{2},p_{1},p_{2}\right)=Cxe^{-\left(\lambda_{1}x\right)^{2}}P\left(x\right),\quad x>0, \tag{68}$$

where $\alpha_{1},\alpha_{2},\lambda_{1},\lambda_{2},p_{1},p_{2}\left(p_{1}+p_{2}=1\right)$ are all positive parameters $C=2p_{1}\alpha_{1}\lambda_{1}^{2}$ and $P\left(x\right)=\left(1-e^{\left[-\left(\lambda_{1}x\right)^{2}\right]}\right)^{\alpha_{1}-1}+\frac{p_{2}\alpha_{2}\lambda_{2}^{2}}{p_{1}\alpha_{1}\lambda_{1}^{2}}e^{-\left(\lambda_{2}^{2}+\lambda_{1}^{2}\right)x^{2}}\left(1-e^{\left[-\left(\lambda_{2}x\right)^{2}\right]}\right)^{\alpha_{2}-1}$.

Remark 1.27. *The pdf* (68) *is similar to the pdf* (36).

35) The cdf and pdf of (KwT-G) are given, respectively, by

$$F(x;a,b,\xi) = 1 - \left\{1 - e^{-\alpha\left[\tan\left(\frac{\pi}{2}(1-G(x;\xi))\right)\right]}\right\}^{b}, \quad x \in \mathbb{R}, \tag{69}$$

and

$$f(x;a,b,\xi) = \frac{d}{dx}F(x;a,b,\xi), \quad x \in \mathbb{R}, \tag{70}$$

where $a>0, b>0$ are parameters and $G(x;\xi)$ is a baseline cdf depending on the parameter vector ξ.

Remark 1.28. *Taking* $K(x) = 1 - \left(1 + \left(1 + \tan\left[\frac{\pi}{2}(1-G(x;\xi))\right]\right)^{-1}\right)^{-1}$, $x \in \mathbb{R}$*, the cdf* (69) *can be expressed as*

$$F(x;a,b,\xi) = 1 - \exp\left\{-\alpha\left(\frac{K(x)}{1-K(x)}\right)\right\}^{b}, \quad x \geq 0,$$

which is a special case of the cdf mentioned in Remark 1.2 in Hamedani (2023).

36) The cdf and pdf of (LTPa-I) are given, respectively, by

$$F(x;\alpha,\lambda,\tau,\rho) = 1 - \frac{\log\left[\rho - (\rho-1)\left(1-\left(\frac{\lambda}{x}\right)^{\tau}\right)\right]}{\log(\rho)}, \quad x \geq \lambda, \tag{71}$$

and

$$f(x;\alpha,\lambda,\tau,\rho) = \frac{d}{dx}F(x;\alpha,\lambda,\tau,\rho), \quad x > \lambda, \tag{72}$$

where $\alpha,\lambda,\tau,\rho\,(\neq 1)$ are all positive parameters.

Remark 1.29. *Taking* $G(x) = 1 - \left(\frac{\lambda}{x}\right)^{\tau}$, $x \geq \lambda$*, the cdf* (51) *can be expressed as*

$$F(x;\alpha,\lambda,\tau,\rho) = 1 - \frac{\log\left[\rho - (\rho-1)G(x)\right]}{\log(\rho)}, \quad x \geq \lambda,$$

which was mentioned in Remark 1.5 in Hamedani (2023).

37) The cdf and pdf of (PEW) are given, respectively, by

$$F(x;\tau,\theta,m,\zeta) = \frac{1 - e^{-\left[1-e^{\frac{x}{\zeta}}\right]^{m\theta}}}{1-e^{-1}}, \quad x \geq 0, \tag{73}$$

and

$$f(x;\tau,\theta,m,\zeta) = \frac{d}{dx}F(x;\tau,\theta,m,\zeta), \quad x > 0, \tag{74}$$

where τ,θ,m,ζ are all positive parameters.

Remark 1.30. *Taking* $G(x) = \left[1-e^{\frac{x}{\zeta}}\right]^{m\theta}$, $x \geq 0$*, the cdf* (73) *can be written as*

$$F(x;\tau,\theta,m,\zeta) = \frac{1-e^{-G(x)}}{1-e^{-1}}, \quad x \geq 0,$$

which is a special case of the cdf mentioned in Remark 143 in Hamedani (2021).

38) The cdf and pdf of (EGMC) are given, respectively, by

$$F(x;s,k,\lambda,\beta)=\left[1-\exp\left\{k\lambda\left(1-e^{x^{\beta}}\right)\right\}\right]^{s}$$
$$=\left[1-\exp\left\{-k\lambda\left(e^{x^{\beta}}-1\right)\right\}\right]^{s},\quad x\geq 0, \tag{75}$$

and

$$f(x;s,k,\lambda,\beta)=\frac{d}{dx}F(x;s,k,\lambda,\beta),\quad x>0, \tag{76}$$

where s,k,λ,β all positive parameters.

Remark 1.31. *Taking* $G(x)=1-e^{-x^{\beta}}$, $x\geq 0$, *the cdf* (75) *can be written as*

$$F(x;s,k,\lambda,\beta)=\left[1-\exp\left\{-k\lambda\left(\frac{G(x)}{1-G(x)}\right)\right\}\right]^{s},\quad x\geq 0,$$

which is a special case of the cdf mentioned in Remark 1.6.

39) The cdf and pdf of (M-L) are given, respectively, by

$$F(x;\lambda,\beta,\theta)=\int_0^x f(u;\lambda,\beta,\theta)\,du,\quad x\geq 0, \tag{77}$$

and

$$f(x;\lambda,\beta,\theta)=C(1+\theta x)^{-(1+\beta)}\left((1+\theta x)^{-\beta}\right)^{-2}P(x),\quad x>0, \tag{78}$$

where λ,β,θ are all positive parameters, $C=\frac{2\beta\theta}{\lambda^3\sqrt{2\pi}}$ and $P(x)=\left(\frac{1-(1+\theta x)^{-\beta}}{(1+\theta x)^{-\beta}}\right)^2\exp\left(-\frac{1}{2\lambda^2}\left(\frac{1-(1+\theta x)^{-\beta}}{(1+\theta x)^{-\beta}}\right)^2\right)$.

Remark 1.32. *The pdf* (78) *is similar to the pdf* (54).

40) The cdf and pdf of (OBPIK) are given, respectively, by

$$F(x;a,b,c,d)=\int_0^x f(u;a,b,c,d)\,du,\quad x\geq 0, \tag{79}$$

and

$$f(x;a,b,c,d)=C(1+x)^{-(a+1)}P(x),\quad x>0, \tag{80}$$

where a,b,c,d are all positive parameters, $C=\frac{ab}{B(c,d)}$, $B(c,d)$ is the beta function and $P(x)=\frac{\left[1-(1+x)^{-a}\right]^{bc-1}}{\left\{1-\left[1-(1+x)^{-a}\right]^{b}\right\}^{1-d}}$.

Remark 1.33. *The pdf* (80) *is similar to the pdf* (78).

41) The cdf and pdf of (OBPBX) are given, respectively, by

$$F(x;\theta,\lambda,\alpha,\beta)=\int_0^x f(u;\theta,\lambda,\alpha,\beta)\,du,\quad x\geq 0, \tag{81}$$

and

$$f(x;\theta,\lambda,\alpha,\beta)=Cxe^{-(x\beta)^2}P(x),\quad x>0, \tag{82}$$

where $\theta, \lambda, \alpha, \beta$ are all positive parameters $C = \frac{2\alpha\beta^2}{B(\theta,\lambda)}$, $B(\theta, \lambda)$ is the beta function and $P(x) = \frac{\left(1-e^{-(x\beta)^2}\right)^{\theta\alpha-1}}{\left\{1-\left(1-e^{-(x\beta)^2}\right)^{\alpha}\right\}^{\theta+1}} \times \left[1 + \left(\frac{\left(1-e^{-(x\beta)^2}\right)^{\alpha}}{1-\left(1-e^{-(x\beta)^2}\right)^{\alpha}}\right)\right]^{-(\theta+\lambda)}$.

Remark 1.34. *The pdf* (82) *is similar to the pdf* (68).

42) The cdf and pdf of (LBIII) are given, respectively, by

$$F(x; \alpha, \beta, \lambda) = \left(1 + e^{-\beta x^{\alpha}}\right)^{-\lambda}, \quad x \geq 0, \tag{83}$$

and

$$f(x; \alpha, \beta, \lambda) = \alpha\beta\lambda x^{\alpha-1} e^{-\beta x^{\alpha}} \left(1 + e^{-\beta x^{\alpha}}\right)^{-\lambda-1}, \quad x > 0, \tag{84}$$

where α, β, λ are all positive parameters.

Remark 1.35. *The authors have $x \in \mathbb{R}$, which will not be correct unless they assume that α is an even number. Here we assume $x \geq 0$, then $\alpha > 0$ will be correct.*

43) The cdf and pdf of (OBP-G) are given, respectively, by

$$F(x; c, d, \varepsilon) = \int_{-\infty}^{x} f(u; c, d, \varepsilon)\, du, \quad x \in \mathbb{R}, \tag{85}$$

and

$$f(x; c, d, \varepsilon) = C^* \frac{g(x; \varepsilon)}{\left[1 - G(x; \varepsilon)\right]^2} \left(\frac{G(x; \varepsilon)}{1 - G(x; \varepsilon)}\right)^{-c-1} P(x), \quad x \in \mathbb{R}, \tag{86}$$

where $c > 0, d > 0$ are parameters, $C^* = \frac{1}{B(c,d)}$, $B(c, d)$ is the beta function, $P(x) = \left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)^{2c} \left[1 + \left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)\right]^{-(c+d)}$ and $G(x; \varepsilon)$ is a baseline cdf with the corresponding pdf $g(x; \varepsilon)$, which may depend on the parameter vector ε.

44) The cdf and pdf of (LTL) are given, respectively, by

$$F(x; \alpha) = \left(1 - e^{-2x}\right)^{\alpha}, \quad x \geq 0, \tag{87}$$

and

$$f(x; \alpha) = 2\alpha e^{-2x} P(x), \quad x > 0, \tag{88}$$

where $\alpha > 0$ is a parameter and $P(x) = \left(1 - e^{-2x}\right)^{\alpha-1}$.

Remarks 1.36. (a) *The pdf* (88) *is a special case of the pdf* (82). (b) *We will characterize the (LTL) distribution in Subsections 2.3 and 2.4.*

45) The cdf and pdf of (OBPF) are given, respectively, by

$$F(x; \alpha, \beta, \theta, \lambda) = \int_{0}^{x} f(u; \alpha, \beta, \theta, \lambda)\, du, \quad x \geq 0, \tag{89}$$

and

$$f(x; \alpha, \beta, \theta, \lambda) = C x^{-\theta-1} e^{-\left(\frac{\beta}{x}\right)^{\theta}} P(x), \quad x > 0, \tag{90}$$

where $\alpha, \beta, \theta, \lambda$ are all positive parameters, $C = \frac{\theta\beta^{\theta}}{B(\alpha,\lambda)}$, $B(\alpha, \lambda)$ is the beta function and $P(x) = e^{-(\alpha-1)\left(\frac{\beta}{x}\right)^{\theta}} \left\{1 - e^{-\left(\frac{\beta}{x}\right)^{\theta}}\right\}^{\lambda-1}$.

Remark 1.37. *The pdf* (90) *is similar to the pdf* (16) *in Hamedani et al. (2024).*

46) The cdf and pdf of (OBP-Logistic) are given, respectively, by

$$F(x;a,b,\mu,s)=\int_0^x f(u;a,b,\mu,s)\,du,\quad x\in\mathbb{R}, \tag{91}$$

and

$$f(x;a,b,\mu,s)=Ce^{-b\left(\frac{x-\mu}{s}\right)}P(x),\quad x\in\mathbb{R}, \tag{92}$$

where $a>0,b>0,\mu\in\mathbb{R},s>0$ are parameters, $C=\frac{1}{sB(a,b)}$, $B(a,b)$ is the beta function and $P(x)=\left[1+e^{-b\left(\frac{x-\mu}{s}\right)}\right]^{-(a+b)}$.

Remark 1.38. *The pdf* (92) *has a special form of the pdf* (84)*.*

47) The cdf and pdf of (Log-Kw) are given, respectively, by

$$F(x;\alpha,\beta)=1-\left[1-\left(1-e^{-x}\right)^{\alpha}\right]^{\beta},\quad x\geq 0, \tag{93}$$

and

$$f(x;\alpha,\beta)=\alpha\beta e^{-x}\left(1-e^{-x}\right)^{\alpha-1}\left[1-\left(1-e^{-x}\right)^{\alpha}\right]^{\beta-1},\quad x>0, \tag{94}$$

where $\alpha>0,\beta>0$ are parameter

48) The cdf and pdf of (Exp-Lindley) are given, respectively, by

$$F(x;\theta,\lambda,\alpha)=\int_0^x f(u;\theta,\lambda,\alpha)\,du,\quad x\geq 0, \tag{95}$$

and

$$f(x;\theta,\lambda,\alpha)=Ce^{-\theta x}P(x),\quad x>0, \tag{96}$$

where $\theta\geq 0,\lambda\geq 0,\alpha\geq 0$ are parameters $C=\frac{\theta^2(1-\alpha)}{\theta+1}$ and $P(x)=x+1+\frac{\alpha\lambda(\theta+1)}{\theta^2(1-\alpha)}e^{-(\lambda-\theta)x}$.

Remarks 1.39. (a)*The pdf* (96) *has the same form as the pdf* (82)*.* (b) *Similar statements can be said for Gompertz-Lindley and Log-Normal Lindley distributions mentioned in Sakthivel and G. (2024).*

49) The cdf and pdf of (MGPW) are given, respectively, by

$$F(x;\alpha,\beta,\lambda,a,b,c)=\frac{1}{B(a,b)}\int_0^{\left[1-\left(1+\lambda x^{\beta}\right)^{\alpha}\right]^{c}}u^{a-1}(1-u)^{b-1}\,du,\quad x\geq 0, \tag{97}$$

and

$$f(x;\alpha,\beta,\lambda,a,b,c)=\frac{d}{dx}F(x;\alpha,\beta,\lambda,a,b,c),\quad x>0, \tag{98}$$

where $\alpha,\beta,\lambda,a,b,c$ are all positive parameters.

Remark 1.40. *Taking* $Q(x)=\left[1-\left(1+\lambda x^{\beta}\right)^{\alpha}\right]^{c}$, $x\geq 0$*, the cdf* (97) *can be written as*

$$F(x;\alpha,\beta,\lambda,a,b,c)=\frac{1}{B(a,b)}\int_0^{Q(x)}u^{a-1}(1-u)^{b-1}\,du,\quad x\geq 0,$$

which was mentioned in Remark 1.37 in Hamedani (2023).

50) The cdf and pdf of (GCEBT) are given, respectively, by

$$F\left(x;\alpha,\theta,\zeta\right)=\frac{\exp\left[\theta\left(e^{\zeta H^{\alpha}(x)}-1\right)\right]-1}{\exp\left[\theta\left(e^{\zeta}-1\right)\right]-1},\quad x\in\mathbb{R}, \tag{99}$$

and

$$f\left(x;\alpha,\theta,\zeta\right)=\frac{d}{dx}F\left(x;\alpha,\theta,\zeta\right),\quad x\in\mathbb{R}, \tag{100}$$

where $\alpha>0,\theta\in\mathbb{R},\zeta>0$ are parameters and $H\left(x\right)$ is a baseline cdf.

Remark 1.41. *Taking* $G\left(x\right)=\frac{e^{\zeta H^{\alpha}(x)}-1}{e^{\zeta}-1}$, $x\in\mathbb{R}$, *the cdf* (99) *can be expressed as*

$$F\left(x;\alpha,\theta,\zeta\right)=\frac{\exp\left[\theta G\left(x\right)\right]-1}{\exp\left[\theta\right]-1},\quad x\in\mathbb{R},$$

which was mentioned in Remark 1.1.

51) The cdf and pdf of (Log-L) are given, respectively, by

$$F\left(x;\theta,p\right)=1-\frac{\log\left[1-\left(1-p\right)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]}{\log\left(\rho\right)},\quad x\geq0, \tag{101}$$

and

$$f\left(x;\theta,p\right)=\frac{d}{dx}F\left(x;\theta,p\right),\quad x>0, \tag{102}$$

where $\theta>0,p>0$ are parameters.

Remark 1.42. *Taking* $G\left(x\right)=1-\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}$, $x\geq0$, *the cdf* (101) *will reduce to the cdf mentioned in Remark 1.29.*

52) The cdf and pdf of (MITW) are given, respectively, by

$$F\left(x;\alpha,\beta,\lambda\right)=\frac{\alpha\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right)}{1-\left(1-\alpha\right)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right)},\quad x\geq0, \tag{103}$$

and

$$f\left(x;\alpha,\beta,\lambda\right)=\frac{d}{dx}F\left(x;\alpha,\beta,\lambda\right),\quad x>0, \tag{104}$$

where α,β,λ are all positive parameters.

Remark 1.43. *Taking* $G\left(x\right)=1-e^{-\left(\frac{x}{\lambda}\right)^{\beta}}$, $x\geq0$, *the cdf* (103) *can be written as*

$$F\left(x;\alpha,\beta,\lambda\right)=\frac{G\left(x\right)}{\frac{1}{\alpha}+\left(1-\frac{1}{\alpha}\right)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right)},\quad x\geq0,$$

which is a special case of the cdf mentioned in Remark 1.80 in Hamedani (2023).

53) The cdf and pdf of (NTWE) are given, respectively, by

$$F\left(x;\alpha,\beta\right)=\int_{0}^{x}f\left(u;\alpha,\beta\right)du,\quad x\geq0, \tag{105}$$

and

$$f\left(x;\alpha,\beta\right)=Ce^{-\alpha x}P\left(x\right),\quad x>0, \tag{106}$$

where $\alpha>0,\beta>0$ are parameters, $C=\frac{\alpha(\beta+1)^{2}}{\beta^{2}}$ and $P\left(x\right)=1-\left(1+\frac{\alpha}{\beta}\right)e^{-\alpha\beta x}$.

Remark 1.44. *The pdf* (106) *is similar to the pdf* (96).

54) The cdf and pdf of (GMOT-G) are given, respectively, by

$$F\left(x;\theta,\alpha,\beta,\lambda\right)=1-\left[\frac{\alpha\left[1-G\left(x\right)\left\{1+\lambda-\lambda G\left(x\right)\right\}\right]}{1-\left(1-\alpha\right)\left[1-G\left(x\right)\left\{1+\lambda-\lambda G\left(x\right)\right\}\right]}\right]^{\theta},\quad x\in\mathbb{R}, \tag{107}$$

and

$$f\left(x;\theta,\alpha,\beta,\lambda\right)=\frac{d}{dx}F\left(x;\theta,\alpha,\beta,\lambda\right),\quad x\in\mathbb{R}, \tag{108}$$

where $\theta>0,\alpha>0,\beta>0,\lambda\in[-1,1]$ are parameters and $G\left(x\right)$ is a baseline cdf.

Remark 1.45. *Taking* $K\left(x\right)=G\left(x\right)\left\{1+\lambda-\lambda G\left(x\right)\right\},\ x\in\mathbb{R}$*, the cdf* (107) *can be expressed as*

$$\begin{aligned}F\left(x;\theta,\alpha,\beta,\lambda\right)&=1-\left[\frac{\alpha\left[1-K\left(x\right)\right]}{1-\left(1-\alpha\right)\left[1-K\left(x\right)\right]}\right]^{\theta}\\&=1-\left[\frac{1-K\left(x\right)}{\frac{1}{\alpha}+\left(1-\frac{1}{\alpha}\right)\left[1-K\left(x\right)\right]}\right]^{\theta},\quad x\in\mathbb{R},\end{aligned}$$

which is the same as the cdf (47).

55) The cdf and pdf of (TCPKwL) are given, respectively, by

$$F\left(x;\lambda,\theta,\alpha,\beta,\mu\right)=\frac{4}{\pi}\arctan\left[1-\left[1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right]^{\theta}\right]^{\alpha},\quad x\geq 0, \tag{109}$$

and

$$f\left(x;\lambda,\theta,\alpha,\beta,\mu\right)=C\left(1+\frac{x}{\beta}\right)^{-\mu-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda-1}P\left(x\right),\quad x>0, \tag{110}$$

where $\lambda,\theta,\alpha,\beta,\mu$ are all positive parameters, $C=\frac{4\alpha^{2}\lambda\theta}{\pi\beta}$ and $P\left(x\right)=\frac{\left[1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right]^{\theta-1}\left[1-\left[1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right]^{\theta}\right]^{\alpha-1}}{1+\left[1-\left[1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right]^{\theta}\right]^{2\alpha}}$.

Remark 1.46. *Similar characterizations can be stated for Truncated Cauchy Power Kumaraswamy Exponential (TCPKwE) distribution and Truncated Cauchy Power Kumaraswamy Rayleigh (TCPKwE) distribution proposed by Elbatal et al. (2024a).*

56) The cdf and pdf of (IPZ) are given, respectively, by

$$F\left(x;\delta,\omega\right)=\int_{0}^{x}f\left(u;\delta,\omega\right)du,\quad x\geq 0, \tag{111}$$

and

$$f(x;\delta,\omega)=Cx^{-\omega-1}e^{-\delta x^{-\omega}}P(x), \quad x>0, \tag{112}$$

where $\delta>0,\omega>0$ are parameters, $C=\frac{\omega\delta^3}{\delta+2}$ and $P(x)=x^{-2\omega}(1+x^{\omega})$.

Remark 1.47. *The pdf* (112) *is similar to the pdf* (90).

2. Characterization of Distributions

As mentioned in the Introduction, characterizations of distributions is an important area of research which has recently attracted the attention of many researchers. This section deals with various characterizations of the distributions listed in the Introduction. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) the reverse hazard function and (iv) conditional expectation of a single function of the random variable. It should be mentioned that for the characterization (i) the cdf need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

2.1. Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of the distributions mentioned in the Introduction, in details, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem 2.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem 2.1. *Let $(\Omega,\mathcal{F},\mathbf{P})$ be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ $(d=-\infty,\ e=\infty$ might as well be allowed$)$. Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that*

$$\mathbf{E}[q_2(X)\mid X\geq x]=\mathbf{E}[q_1(X)\mid X\geq x]\,\eta(x), \quad x\in H,$$

is defined with some real function η. Assume that $q_1,q_2\in C^1(H)$, $\eta\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1=q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1,q_2 and η, particularly

$$F(x)=\int_a^x C\left|\frac{\eta'(u)}{\eta(u)\,q_1(u)-q_2(u)}\right|\exp(-s(u))\ du\ ,$$

where the function s is a solution of the differential equation $s'=\frac{\eta'\,q_1}{\eta\,q_1\,-\,q_2}$ and C is the normalization constant, such that $\int_H dF=1$.

Here is our first characterization.

Proposition 2.1. *Let $X:\Omega\to(0,1)$ be a continuous random variable and let $q_1(x)=(1-x)^{\beta}$ and $q_2(x)=q_1(x)(1-x)^{\beta}$ for $0<x<1$. The random variable X has pdf* (2) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}(1-x)^{\beta}, \quad 0<x<1.$$

Proof. Let X be a random variable with pdf (2), then

$$(1-F(x))\,E[q_1(X)\mid X\geq x]=\int_x^1\frac{\beta(1+\beta)}{2+\beta}(1-u)^{\beta-1}du=\frac{(1+\beta)}{2+\beta}(1-x)^{\beta}, \quad 0<x<1,$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^1\frac{\beta(1+\beta)}{2+\beta}(1-u)^{2\beta-1}\,du=\frac{(1+\beta)}{2(2+\beta)}(1-x)^{2\beta},\quad 0<x<1,$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}(1-x)^{\beta}<0\quad\text{for }\;0<x<1.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\beta(1-x)^{-1},\quad 0<x<1,$$

and hence

$$s(x)=-\beta\log(1-x),\quad 0<x<1.$$

Now, in view of Theorem 2.1, X has density (2). □

Corollary 2.1. *Let $X:\Omega\rightarrow(0,1)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1. The pdf of X is* (2) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\beta(1-x)^{-1},\quad 0<x<1.$$

Corollary 2.2. *The general solution of the differential equation in Corollary 2.1 is*

$$\eta(x)=(1-x)^{-\beta}\left[-\int\beta(1-x)^{\beta-1}(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. If X has pdf (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x)=\beta(1-x)^{-1}\eta(x)-\beta(1-x)^{-1}(q_1(x))^{-1}q_2(x),$$

or

$$(1-x)^{\beta}\eta'(x)-\beta(1-x)^{\beta-1}\eta(x)=-\beta(1-x)^{\beta-1}(q_1(x))^{-1}q_2(x),$$

or

$$\frac{d}{dx}\left\{(1-x)^{\beta}\eta(x)\right\}=-\beta(1-x)^{\beta-1}(q_1(x))^{-1}q_2(x),$$

from which we arrive at

$$\eta(x)=(1-x)^{-\beta}\left[-\int\beta(1-x)^{\beta-1}(q_1(x))^{-1}q_2(x)\,dx+D\right].$$

□

Note that a set of functions satisfying the differential equation in Corollary 2.1, is given in Proposition 2.1 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.2. *Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x>0$. The random variable X has pdf* (10) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}e^{-\lambda x},\quad x>0.$$

Proof. Let X be a random variable with pdf (10), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\int_x^{\infty}Ce^{-\lambda u}du=\frac{C}{\lambda}e^{-\lambda x},\quad x>0,$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^{\infty}Ce^{-2\lambda u}du=\frac{C}{2\lambda}e^{-2\lambda x},\quad x>0,$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}e^{-\lambda x}<0\ \ \text{for}\ \ x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\lambda,\quad x>0,$$

and hence

$$s(x)=\lambda x,\quad x>0.$$

Now, in view of Theorem 2.1, X has density (10). □

Corollary 2.3. *Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.2. The pdf of X is* (10) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\lambda,\quad x>0.$$

Corollary 2.4. *The general solution of the differential equation in Corollary 2.3 is*

$$\eta(x)=e^{\lambda x}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.3, is given in Proposition 2.2 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.3. *Let $X:\Omega\rightarrow(0,1)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,x^{-\beta}$ for $0<x<1$. The random variable X has pdf* (22) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}\left(x^{-\beta}+1\right),\quad 0<x<1.$$

Proof. Let X be a random variable with pdf (22), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\int_x^{1}\beta\delta\sigma u^{-\beta-1}du=\delta\sigma\left(x^{-\beta}-1\right),\quad 0<x<1,$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^{1}\beta\delta\sigma u^{-2\beta-1}du=\frac{\delta\sigma}{2}\left(x^{-2\beta}-1\right),\quad 0<x<1,$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left(x^{-\beta}-1\right) < 0 \quad \text{for } 0<x<1.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\beta x^{-\beta-1}}{x^{-\beta}-1}, \quad 0<x<1,$$

and hence

$$s(x) = -\log\left(x^{-\beta}-1\right), \quad 0<x<1.$$

Now, in view of Theorem 2.1, X has density (22). □

Corollary 2.5. *Let $X : \Omega \rightarrow (0,1)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.3. The pdf of X is* (22) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\beta x^{-\beta-1}}{x^{-\beta}-1}, \quad 0<x<1.$$

Corollary 2.6. *The general solution of the differential equation in Corollary 2.5 is*

$$\eta(x) = \left(x^{-\beta}-1\right)^{-1}\left[-\int \beta x^{-\beta-1}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.5, is given in Proposition 2.3 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.4. *Let $X : \Omega \rightarrow (0,\infty)$ be a continuous random variable and let $q_1(x) = (1+\theta x)^{-1}$ and $q_2(x) = q_1(x) e^{-\theta x}$ for $x>0$. The random variable X has pdf* (24) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2} e^{-\theta x}, \quad x>0.$$

Proof. Let X be a random variable with pdf (24), then

$$(1-F(x)) E\left[q_1(X) \mid X \geq x\right] = \int_x^\infty \frac{\theta}{2} e^{-\theta u} du = \frac{1}{2} e^{-\theta x}, \quad x>0,$$

and

$$(1-F(x)) E\left[q_2(X) \mid X \geq x\right] = \int_x^\infty \frac{\theta}{2} e^{-2\theta u} du = \frac{1}{4} e^{-2\theta x}, \quad x>0,$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\theta x} < 0 \quad \text{for } x>0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \theta, \quad x>0,$$

and hence

$$s(x) = \theta x, \quad x>0.$$

Now, in view of Theorem 2.1, X has density (24). □

Corollary 2.7. *Let $X : \Omega \rightarrow (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.4. The pdf of X is* (24) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \theta, \quad x > 0.$$

Corollary 2.8. *The general solution of the differential equation in Corollary 2.7 is*

$$\eta(x) = e^{\theta x}\left[-\int \theta e^{-\theta x}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.7, is given in Proposition 2.4 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.5. *Let $X : \Omega \rightarrow (\theta_1, \theta_2)$ be a continuous random variable and let $q_1(x) \equiv 1$ and $q_2(x) = \exp\left[-\frac{\lambda(x-\theta_1)}{(\theta_2-x)}\right]$, for $\theta_1 < x < \theta_2$. The random variable X has pdf* (28) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\exp\left[-\frac{\lambda(x-\theta_1)}{(\theta_2-x)}\right], \quad \theta_1 < x < \theta_2.$$

Proof. Let X be a random variable with pdf (28), then

$$\begin{aligned}(1-F(x))\, E\left[q_1(X) \mid X \geq x\right] &= \int_x^{\theta_2} \frac{\lambda(\theta_2-\theta_1)}{(\theta_2-u)^2}\exp\left[-\frac{\lambda(u-\theta_1)}{(\theta_2-u)}\right] du \\ &= \exp\left[-\frac{\lambda(x-\theta_1)}{(\theta_2-x)}\right], \quad \theta_1 < x < \theta_2,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\, E\left[q_2(X) \mid X \geq x\right] &= \int_x^{\theta_2} \frac{\lambda(\theta_2-\theta_1)}{(\theta_2-u)^2}\exp\left[-\frac{2\lambda(u-\theta_1)}{(\theta_2-u)}\right] du, \\ &= \frac{1}{2}\exp\left[-\frac{2\lambda(x-\theta_1)}{(\theta_2-x)}\right], \quad \theta_1 < x < \theta_2,\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\exp\left[-\frac{\lambda(x-\theta_1)}{(\theta_2-x)}\right] < 0, \text{ for } \theta_1 < x < \theta_2.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\lambda(\theta_2-\theta_1)}{(\theta_2-x)^2}, \quad \theta_1 < x < \theta_2,$$

and hence

$$s(x) = \frac{\lambda(\theta_2-\theta_1)}{(\theta_2-x)}, \quad \theta_1 < x < \theta_2.$$

Now, in view of Theorem 2.1, X has density (28). □

Corollary 2.9. *Let $X : \Omega \to (\theta_1, \theta_2)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.5. The pdf of X is* (28) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\lambda(\theta_2 - \theta_1)}{(\theta_2 - x)^2}, \quad \theta_1 < x < \theta_2.$$

Corollary 2.10. *The general solution of the differential equation in Corollary 2.9 is*

$$\eta(x) = \exp\left[\frac{\lambda(x - \theta_1)}{(\theta_2 - x)}\right]\left[-\int \frac{\lambda(\theta_2 - \theta_1)}{(\theta_2 - x)^2} \exp\left[-\frac{\lambda(x - \theta_1)}{(\theta_2 - x)}\right](q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.9, is given in Proposition 2.5 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.6. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\left(cx + \frac{b}{2}x^2\right)^{\xi}}$, for $x > 0$. The random variable X has pdf* (36) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2} e^{-\left(cx + \frac{b}{2}x^2\right)^{\xi}}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (36), then

$$\begin{aligned}(1 - F(x))\, E[q_1(X) \mid X \geq x] &= \int_x^{\infty} (c + bu)\left(cu + \frac{b}{2}u^2\right)^{\xi - 1} e^{-\left(cu + \frac{b}{2}u^2\right)^{\xi}} du, \\ &= \frac{1}{\xi} e^{-\left(cx + \frac{b}{2}x^2\right)^{\xi}}, \quad x > 0,\end{aligned}$$

and

$$\begin{aligned}(1 - F(x))\, E[q_2(X) \mid X \geq x] &= \int_x^{\infty} (c + bu)\left(cu + \frac{b}{2}u^2\right)^{\xi - 1} e^{-2\left(cu + \frac{b}{2}u^2\right)^{\xi}} du, \\ &= \frac{1}{2\xi} e^{-2\left(cx + \frac{b}{2}x^2\right)^{\xi}}, \quad x > 0,\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\left(cx + \frac{b}{2}x^2\right)^{\xi}} < 0, \quad \text{for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \xi(c + bx)\left(cx + \frac{b}{2}x^2\right)^{\xi - 1}, \quad x > 0,$$

and hence

$$s(x) = \left(cx + \frac{b}{2}x^2\right)^{\xi}, \quad x > 0.$$

Now, in view of Theorem 2.1, X has density (36). □

Corollary 2.11. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.6. The pdf of X is* (36) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\xi\,(c+bx)\left(cx+\frac{b}{2}x^2\right)^{\xi-1},\quad x>0.$$

Corollary 2.12. *The general solution of the differential equation in Corollary 2.11 is*

$$\eta(x)=e^{\left(cx+\frac{b}{2}x^2\right)^{\xi}}\left[-\int\xi\,(c+bx)\left(cx+\frac{b}{2}x^2\right)^{\xi-1}e^{-\left(cx+\frac{b}{2}x^2\right)^{\xi}}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.11, is given in Proposition 2.6 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.7. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\left(1+\left(\frac{x}{s}\right)^c\right)^{-1}$, for $x>0$. The random variable X has pdf* (54) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}\left(1+\left(\frac{x}{s}\right)^c\right)^{-1},\quad x>0.$$

Proof. Let X be a random variable with pdf (54), then

$$\begin{aligned}(1-F(x))\,E\left[q_1(X)\mid X\ge x\right]&=\int_x^{\infty}C^*\left(\frac{u}{s}\right)^{c-1}\left(1-\left(\frac{u}{s}\right)^c\right)^{-2}du\\&=\frac{C^*s}{c}\left(1+\left(\frac{x}{s}\right)^c\right)^{-1},\quad x>0,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\ge x\right]&=\int_x^{\infty}C^*\left(\frac{u}{s}\right)^{c-1}\left(1+\left(\frac{u}{s}\right)^c\right)^{-3}du,\\&=\frac{C^*s}{2c}\left(1+\left(\frac{x}{s}\right)^c\right)^{-2},\quad x>0,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\left(1+\left(\frac{x}{s}\right)^c\right)^{-1}<0,\quad\text{for }x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\left(\frac{c}{s}\right)\left(\frac{x}{s}\right)^{c-1}}{\left(1+\left(\frac{x}{s}\right)^c\right)},\quad x>0,$$

and hence

$$s(x)=\log\left\{\left(1+\left(\frac{x}{s}\right)^c\right)\right\},\quad x>0.$$

Now, in view of Theorem 2.1, X has density (54). □

Corollary 2.13. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.7. The pdf of X is* (54) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\left(\frac{c}{s}\right)\left(\frac{x}{s}\right)^{c-1}}{\left(1+\left(\frac{x}{s}\right)^{c}\right)},\quad x>0.$$

Corollary 2.14. *The general solution of the differential equation in Corollary 2.13 is*

$$\eta(x)=\left(1+\left(\frac{x}{s}\right)^{c}\right)\left[-\int\left(\frac{c}{s}\right)\left(\frac{x}{s}\right)^{c-1}\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-2}(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.13, is given in Proposition 2.7 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.8. *Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\left(1-\overline{G}^2(x;\xi)\right)^{b}$, for $x\in\mathbb{R}$. The random variable X has pdf (64) if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}\left\{1+\left(1-\overline{G}^2(x;\xi)\right)^{b}\right\},\quad x\in\mathbb{R}.$$

Proof. Let X be a random variable with pdf (64), then

$$\begin{aligned}(1-F(x))\,E[q_1(X)\mid X\geq x]&=\int_x^{\infty}C^{*}g(u;\xi)\,\overline{G}(u;\xi)\left(1-\overline{G}^2(u;\xi)\right)^{b-1}du\\&=\frac{C^{*}}{2b}\left\{1-\left(1-\overline{G}^2(x;\xi)\right)^{b}\right\},\quad x\in\mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E[q_2(X)\mid X\geq x]&=\int_x^{\infty}C^{*}g(u;\xi)\,\overline{G}(u;\xi)\left(1-\overline{G}^2(u;\xi)\right)^{2b-1}du\\&=\frac{C^{*}}{4b}\left\{1-\left(1-\overline{G}^2(x;\xi)\right)^{2b}\right\},\quad x\in\mathbb{R},\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-\left(1-\overline{G}^2(x;\xi)\right)^{b}\right\}>0,\quad \text{for } x\in\mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{b g(x;\xi)\,\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{1-\left(1-\overline{G}^2(x;\xi)\right)^{b}},\quad x\in\mathbb{R},$$

and hence

$$s(x)=-\log\left\{1-\left(1-\overline{G}^2(x;\xi)\right)^{b}\right\},\quad x>0.$$

Now, in view of Theorem 2.1, X has density (64). □

Corollary 2.15. *Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.8. The pdf of X is* (64) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{bg(x;\xi)\,\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{1-\left(1-\overline{G}^2(x;\xi)\right)^b},\quad x\in\mathbb{R}.$$

Corollary 2.16. *The general solution of the differential equation in Corollary 2.15 is*

$$\eta(x)=\left\{1-\left(1-\overline{G}^2(x;\xi)\right)^b\right\}^{-1}\left[-\int bg(x;\xi)\,\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.15, is given in Proposition 2.8 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.9. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=\left(1+e^{-\beta x^\alpha}\right)^{\lambda+1}$ and $q_2(x)=q_1(x)\,e^{-\beta x^\alpha}$, for $x>0$. The random variable X has pdf* (84) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}e^{-\beta x^\alpha},\quad x>0.$$

Proof. Let X be a random variable with pdf (84), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\int_x^\infty\alpha\beta\lambda u^{\alpha-1}e^{-\beta u^\alpha}du=\lambda e^{-\beta x^\alpha},\quad x>0,$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^\infty\alpha\beta\lambda u^{\alpha-1}e^{-2\beta u^\alpha}du=\frac{\lambda}{2}e^{-2\beta x^\alpha},\quad x>0,$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}e^{-\beta x^\alpha}<0,\quad \text{for }x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\alpha\beta x^{\alpha-1},\quad x>0,$$

and hence

$$s(x)=\beta x^\alpha,\quad x>0.$$

Now, in view of Theorem 2.1, X has density (84). □

Corollary 2.17. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.9. The pdf of X is* (84) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\alpha\beta x^{\alpha-1},\quad x>0.$$

Corollary 2.18. *The general solution of the differential equation in Corollary 2.17 is*

$$\eta(x) = e^{\beta x^{\alpha}} \left[-\int \alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D \right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.17, is given in Proposition 2.9 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.10. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)^{-1}$, for $x \in \mathbb{R}$. The random variable X has pdf* (86) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{c}{c+1}\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)^{-1}, \quad x \in \mathbb{R}.$$

Proof. Let X be a random variable with pdf (86), then

$$\begin{aligned}(1-F(x))\, E[q_1(X) \mid X \geq x] &= \int_x^{\infty} C^* \frac{g(u;\varepsilon)}{[1-G(u;\varepsilon)]^2}\left(\frac{G(u;\varepsilon)}{1-G(u;\varepsilon)}\right)^{-c-1} du \\ &= \frac{C^*}{c}\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)^{-c}, \quad x \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\, E[q_2(X) \mid X \geq x] &= \int_x^{\infty} C^* \frac{g(u;\varepsilon)}{[1-G(u;\varepsilon)]^2}\left(\frac{G(u;\varepsilon)}{1-G(u;\varepsilon)}\right)^{-c-2} du \\ &= \frac{C^*}{c+1}\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)^{-c-1}, \quad x \in \mathbb{R},\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{c+1}\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)^{-1} < 0, \quad \text{for } x \in \mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{c\left(\frac{g(x;\varepsilon)}{[1-G(x;\varepsilon)]^2}\right)}{\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)} = \frac{c g(x;\varepsilon)}{G(x;\varepsilon)\left[1-G(x;\varepsilon)\right]}, \quad x \in \mathbb{R},$$

and hence

$$s(x) = c\log\left\{\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)\right\}, \quad x \in \mathbb{R}.$$

Now, in view of Theorem 2.1, X has density (86). □

Corollary 2.19. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.10. The pdf of X is* (86) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{c g(x;\varepsilon)}{G(x;\varepsilon)\left[1-G(x;\varepsilon)\right]}, \quad x \in \mathbb{R}.$$

Corollary 2.20. *The general solution of the differential equation in Corollary 2.19 is*

$$\eta(x)=\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)^{c}\left[-\int c\left(\frac{g(x;\varepsilon)}{[1-G(x;\varepsilon)]^{2}}\right)\left(\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right)^{-c-1}(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.19, is given in Proposition 2.10 with $D=0$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.11. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=\left[1-(1-e^{-x})^{\alpha}\right]^{1-\beta}$ and $q_2(x)=q_1(x)(1-e^{-x})^{\alpha}$, for $x>0$. The random variable X has pdf* (94) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}\left\{1+\left(1-e^{-x}\right)^{\alpha}\right\},\quad x>0.$$

Proof. Let X be a random variable with pdf (94), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\int_x^{\infty}\alpha\beta e^{-u}\left(1-e^{-u}\right)^{\alpha-1}du=\beta\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\},\quad x>0,$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^{\infty}\alpha\beta e^{-u}\left(1-e^{-u}\right)^{2\alpha-1}du=\frac{\beta}{2}\left\{1-\left(1-e^{-x}\right)^{2\alpha}\right\},\quad x>0,$$

and finally

$$\eta(x)q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\}>0,\quad \text{for } x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\alpha e^{-x}(1-e^{-x})^{\alpha-1}}{1-(1-e^{-x})^{\alpha}},\quad x>0,$$

and hence

$$s(x)=-\log\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\},\quad x\in\mathbb{R}.$$

Now, in view of Theorem 2.1, X has density (94). □

Corollary 2.21. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.11. The pdf of X is* (94) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\alpha e^{-x}(1-e^{-x})^{\alpha-1}}{1-(1-e^{-x})^{\alpha}},\quad x>0.$$

Corollary 2.22. *The general solution of the differential equation in Corollary 2.21 is*

$$\eta(x)=\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\}^{-1}\left[-\int\alpha e^{-x}\left(1-e^{-x}\right)^{\alpha-1}(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.21, is given in Proposition 2.11 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.12. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}$, for $x > 0$. The random variable X has pdf* (110) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{1+\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right\}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (110), then

$$\begin{aligned}(1-F(x))\,E[q_1(X) \mid X \geq x] &= \int_x^\infty C\left(1+\frac{u}{\beta}\right)^{-\mu-1}\left(1-\left(1+\frac{u}{\beta}\right)^{-\mu}\right)^{\lambda-1} du \\ &= \frac{C\beta}{\mu\lambda}\left\{1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right\}, \quad x > 0,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E[q_2(X) \mid X \geq x] &= \int_x^\infty C\left(1+\frac{u}{\beta}\right)^{-\mu-1}\left(1-\left(1+\frac{u}{\beta}\right)^{-\mu}\right)^{2\lambda-1} du \\ &= \frac{C\beta}{2\mu\lambda}\left\{1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{2\lambda}\right\}, \quad x > 0,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left\{1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right\} > 0, \quad \text{for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \frac{\left(\frac{\lambda\mu}{\beta}\right)\left(1+\frac{x}{\beta}\right)^{-\mu-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda-1}}{1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}}, \quad x > 0,$$

and hence

$$s(x) = -\log\left\{1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right\}, \quad x \in \mathbb{R}.$$

Now, in view of Theorem 2.1, X has density (110). □

Corollary 2.23. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.12. The pdf of X is* (110) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\left(\frac{\lambda\mu}{\beta}\right)\left(1+\frac{x}{\beta}\right)^{-\mu-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda-1}}{1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}},\quad x>0.$$

Corollary 2.24. *The general solution of the differential equation in Corollary 2.23 is*

$$\eta(x)=\left\{1-\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda}\right\}^{-1}\times$$
$$\left[-\int\left(\frac{\lambda\mu}{\beta}\right)\left(1+\frac{x}{\beta}\right)^{-\mu-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\mu}\right)^{\lambda-1}(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. The proof is similar to that Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.23, is given in Proposition 2.12 with $D=\frac{1}{2}$. However, it should also be noted that there are
other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

2.2. Characterization in Terms of Hazard Function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h'_F(x)}{h_F(x)}-h_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of 5 of the new distributions in terms of the hazard function, which are not of the above trivial form.

Proposition 2.13. *Let $X:\Omega\to(0,1)$ be a continuous random variable. The random variable X has pdf (2) if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$h'_F(x)-(1-x)^{-1}h_F(x)=2\beta(1-x)^{-1}(2+\beta+x\beta)^{-2},\ \ 0<x<1,$$

with the boundary condition $\lim_{x\to 0}h_F(x)=\frac{\beta(1+\beta)}{2+\beta}$.

Proof. Multiplying both sides of the above equation by $(1-x)^{\beta}$, we have

$$\frac{d}{dx}\left\{(1-x)^{\beta}h_F(x)\right\}=2\beta(1-x)^{\beta-1}(2+\beta+x\beta)^{-2},$$

or

$$(1-x)^{\beta}h_F(x)=\int 2\beta(1-x)^{\beta-1}(2+\beta+x\beta)^{-2}\,dx,$$

or

$$h_F(x)=(1-x)^{-\beta}\left[\beta(1+\beta)(1+x)(1-x)^{\beta-1}(2+\beta+x\beta)^{-1}\right]=\frac{\beta(1+\beta)(1+x)(1-x)^{\beta-1}}{(1-x)^{\beta}(2+\beta+x\beta)},$$

which is the hazard function corresponding to the pdf (2). □

Proposition 2.14. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf* (24) *if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$h'_F(x) + \theta(\theta x + 2)^{-1} h_F(x) = \theta^2 (\theta x + 2)^{-1}, \quad x > 0,$$

with the boundary condition $\lim_{x\to 0} h_F(x) = \frac{\theta}{2}$.

Proposition 2.15. *Let $X : \Omega \to (\theta_1, \theta_2)$ be a continuous random variable. The random variable X has pdf* (28) *if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$h'_F(x) - 2(\theta_2 - x)^{-1} h_F(x) = 0, \;\; \theta_1 < x < \theta_2,$$

with the boundary condition $\lim_{x\to \theta_1} h_F(x) = \frac{\lambda}{\theta_2 - \theta_1}$.

Proposition 2.16. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf* (36) *if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$\begin{aligned} &h'_F(x) - (\xi - 1)(c + bx)\left(cx + \frac{b}{2}x^2\right)^{-1} h_F(x) \\ &= \left(cx + \frac{b}{2}x^2\right)^{\xi - 1} \frac{d}{dx}\left\{(c + bx)\left(\xi + \varphi\left(cx + \frac{b}{2}x^2\right) e^{\varphi\left(cx + \frac{b}{2}x^2\right)}\right)\right\}, \end{aligned}$$

with the boundary condition $\lim_{x\to 0} h_F(x) = 0$ *for* $\xi > 1$.

Proposition 2.17. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf* (94) *if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$h'_F(x) + h_F(x) = \alpha\beta e^{-x} \frac{d}{dx}\left\{\frac{(1 - e^{-x})^{\alpha - 1}}{1 - (1 - e^{-x})^{\alpha}}\right\},$$

with the boundary condition $\lim_{x\to 0} h_F(x) = 0$ *for* $\alpha > 1$.

2.3. Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \;\; x \in \text{support of } F.$$

In this subsection we present characterizations of 3 of the new distributions in terms of the reverse hazard function.

Proposition 2.18. *Let $X : \Omega \to (0, 1)$ be a continuous random variable. The random variable X has pdf* (22) *if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation*

$$r'_F(x) + (\beta + 1) x^{-1} r_F(x) = -\beta^2 \delta\sigma (\sigma - 1) x^{-2(\beta+1)} \left(x^{-\beta} + 1\right)^{\sigma - 2}, \quad 0 < x < 1,$$

with boundary condition $\lim_{x\to 1} r_F(x) = 0$ *for* $\sigma > 1$.

Proposition 2.19. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf* (84) *if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation*

$$r'_F(x) - (\alpha - 1) x^{-1} r_F(x) = \alpha\beta\lambda x^{\alpha - 1} \frac{d}{dx}\left\{\frac{e^{-\beta x^{\alpha}}}{1 + e^{-\beta x^{\alpha}}}\right\}, \quad x > 0,$$

with boundary condition $r_F(0) = 0$ *for* $\alpha > 1$.

Proposition 2.20. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf* (88) *if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation*

$$r_F'(x) + 2r_F(x) = -2e^{-2x}\left(1 - e^{-2x}\right)^{-2}, \quad x > 0,$$

with boundary condition $\lim_{x\to\infty} r_F(x) = 0$.

2.4. Characterization Based on the Conditional Expectation of Certain Function of the Random Variable

In this subsection we employ a single function ψ (or ψ_1) of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$ (or $\psi_1(X)$). The following propositions have already appeared in Hamedani's previous work (2013), so we will just state them here which can be used to characterize 6 of the new distributions listed in Section 1.

Proposition 2.21. *Let $X : \Omega \to (e, f)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (e, f) with $\lim_{x\to e^+} \psi(x) = 1$. Then for $\delta \neq 1$,*

$$E[\psi(X) \mid X \geq x] = \delta\psi(x), \qquad x \in (e, f),$$

if and only if

$$\psi(x) = (1 - F(x))^{\frac{1}{\delta}-1}, \qquad x \in (e, f)$$

Proposition 2.22. *Let $X : \Omega \to (e, f)$ be a continuous random variable with cdf F. Let $\psi_1(x)$ be a differentiable function on (e, f) with $\lim_{x\to f^-} \psi_1(x) = 1$. Then for $\delta_1 \neq 1$,*

$$E[\psi_1(X) \mid X \leq x] = \delta_1\psi_1(x), \qquad x \in (e, f)$$

implies

$$\psi_1(x) = (F(x))^{\frac{1}{\delta_1}-1}. \qquad x \in (e, f)$$

Remarks 2.1.

(A) *For $(e, f) = (0, 1)$, $\psi_1(x) = e^{-\left(\frac{1}{x^{-\beta}-1}\right)^{-\sigma}}$ and $\delta_1 = \frac{\delta}{\delta+1}$, Proposition 2.22 provides a characterization of the PUGT2 distribution.*

(B) *For $(e, f) = (0, \infty)$, $\psi(x) = \left(\frac{1}{2}\theta x + 1\right)^{1/\theta} e^{-x}$ and $\delta = \frac{\theta}{\theta+1}$, Proposition 2.21 provides a characterization of the NXL distribution.*

(C) *For $(e, f) = (\theta_1, \theta_2)$, $\psi(x) = \exp\left[-\frac{(x-\theta_1)}{(\theta_2-x)}\right]$ and $\delta = \frac{\lambda}{\lambda+1}$, Proposition 2.21 provides a characterization of the DBE distribution.*

(D) *For $(e, f) = (0, \infty)$, $\psi_1(x) = \left(1 + e^{-\beta x^{\alpha}}\right)^{-1}$ and $\delta_1 = \frac{\lambda}{\lambda+1}$, Proposition 2.22 provides a characterization of the LBIII distribution.*

(E) *For $(e, f) = (0, \infty)$, $\psi_1(x) = \left(1 + e^{-2x}\right)$ and $\delta_1 = \frac{\alpha}{\alpha+1}$, Proposition 2.22 provides a characterization of the LTL distribution.*

(F) *For $(e, f) = (0, \infty)$, $\psi(x) = 1 - (1 + e^{-x})^{\alpha}$ and $\delta = \frac{\beta}{\beta+1}$, Proposition 2.21 provides a characterization of the Log-Kw distribution.*

References

Abidemi, A. K. and Abiodun, A. A. (2023). Exponentially generated modified Chen distribution with applications to lifetime dataset. *Journal of Probability and Statistics*, Article ID 4458562:25 pages.

Abiodun, A. A. and Ishaq, A. I. (2022). On Maxwell-Lomax distribution: properties and applications. *Arab Journal of Basic and Applied Sciences*, 29(1):221–232.

Abonongo, A. I. L. and Abonongo, J. (2023). Exponentiated generalized Weibull exponential distribution: properties, estimation and applications. *Computational Journal of Mathematical and Statistical Sciences*, 3(1):57–84.

Ahsan-ul Haq, M., Shahzad, M. K., and Tariq, S. (2023). The inverted Nadarajah-Haghighi power series distributions. *International Journal of Applied and Computational Mathematics*, 10(11):16 pages.

Aidi, K., Abd El-Baset, A. A., and Seddik, N. (2023). A new two-parameter distribution-G family: properties and applications. *Electronic Journal of Applied Statistical Analysis*, 16(3):614–638.

Al-Saiary, Z. A. (2023). A new extended "A" distribution with real data applications. *Advances and Applications in Statistics*, 90(2):257–287.

Al-Saiary, Z. A. and Al-Jadaani, S. H. (2022). Type II Topp-Leone generalized inverted exponential distribution with real data applications. *Asian Journal of Probability and Statistics*, 16(2):37–51.

Al-Ta'ani, O. and Gharaibeh, M. M. (2023). Ola distribution: a new one parameter model with applications to engineering and covid-19 data. *Applied Mathematics & Information Sciences*, 17(2):243–252.

Alabdulhadi, M. H. (2024). Odd generalized Nadarajah-Haghighi log-logistic model with modeling. *Applied Mathematics & Information Sciences*, 18(1):49–57.

Alqawba, M., Altayab, Y., Zaidi, S. M., and Afify, A. Z. (2023). The extended Kumaraswamy generated family: properties, inference and applications in applied fields. *Electronic Journal of Applied Statistical Analysis*, 16(3):740–763.

Alsultan, R. (2023). Zubair-exponentiated Weibull distribution with group acceptance sampling schemes and applications to blader cancer. *Advances and Applications in Statistics*, 90(2):235–256.

Alzawq, F. S. A., ElKholy, A. K., and Ahmed, A. H. N. (2024). A new log Lindley distribution with applications. *Journal of Statistics Applications & Probability*, 13(1):379–396.

Aniyan, A. S. and George, D. (2023). A new class of Pareto distribution: estimation and its applications. *Applications & Applied Mathematics*, 18(2):23 pages.

Azimi, R. and Esmailian, M. (2023). A new generalization of Nadarajah-Haghighi distribution with application to cancer and COVID-19 death data. *Mathematica Slovaca*, 73(1):221–244.

Azimi, R., Esmailian, M., and Gallardo, D. I. (2023). The inverted exponentiated Chen distribution with application to cancer data. *Japanese Journal of Statistics and Data Science*, 6(1):213–241.

Biswas, A., Chakraborty, S., and Gosh, I. (2023). A novel method of generating distributions on the unit interval with applications. *Communications in Statistics - Theory and Methods*. (forthcoming).

Chesneau, C., Tomy, L., and Jose, M. (2024). On a new two-parameter weighted exponential distribution and its application to the COVID-19 and censored data. *Thailand Statistician*, 22(1):17–30.

Dey, S., Shakhatreh, M. K., Singh, K., Mahto, A. K., and Tripathi, Y. M. (2023). Classical and Bayesian estimations for a 2-component generalized Rayleigh mixture model under type I censoring. *Electronic Journal of Applied Statistical Analysis*, 16(3):654–693.

Dingalo, N., Oluyede, B., and Chipepa, F. (2023). The type II exponentiated half logistic-odd Burr X-G power series class of distributions with applications. *Heliyon*, 9(e22402):1–25.

Elbatal, I., Handique, L., and Chakraborty, S. (2024a). Truncated Cauchy power Kumaraswamy generalized family of distributions: theory and applications. *Statistics, Optimization & Information Computing*, 12(2):364–380.

Elbatal, I., Hassan, A. S., Gemeay, A. M., Diab, L. S., Ben-Ghorbal, A., and Elgarhy, M. (2024b). The inverse power Zeghdoudi distribution with twelve different methods of estimation and applications to engineering and environmental data. (Preprint).

Eltehiwy, M. A. (2024). A new odd log-logistic inverse Lindley distribution with properties and applications. *Statistics, Optimization & Information Computing*, 12(1):91–108.

Etaga, H. O., Nwankwo, M. P., Oramulu, D. O., Anabike, I. C., and Obulezi, O. J. (2023). The double XLindley distribution: properties and applications. *Scholars Journal of Physics, Mathematics and Statistics*, 10(10):192–202.

George, A. and George, D. (2023). Poisson-Exponentiated Weibull distribution: properties, applications and extension. *Applications & Applied Mathematics*, 18(2):21 pages.

Ghazal, M. G. M. (2023). A new extension of the modified Weibull distribution with applications for engineering data. *Probabilistic Engineering Mechanics*, 74(103523):1–14.

Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. In *Mathematical statistics and probability theory: volume B statistical inference and methods proceedings of the 6th Pannonian symposium on mathematical statistics, bad Tatzmannsdorf, Austria, September 14–20, 1986*, pages 75–84. Springer.

Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments. *Statistics*, 21(4):613–618.

Guerra, R. R., Pena-Ramirez, F. A., and Cordeiro, G. M. (2023). the logistic Burr XII distribution: properties and applications to income data. *STATS*, 6:1260–1279.

Hamedani, G. G. (2017). Characterizations and infinite divisibility of certain 2016-2017 univariate continuous distributions ii. *International Mathematical Forum*, 12(12):565–609.

Hamedani, G. G. (2021). *Characterizations of recently introduced univariate continuous distributions III*. (Research Monograph, NOVA, May 2021).

Hamedani, G. G. (2023). *Characterizations of recently introduced univariate continuous distributions IV*. (Research Monograph, NOVA, May 2023).

Hamedani, G. G., Roshani, A., and Butt, N. S. (2024). Characterizations of certain (2023-2024) introduced univariate continuous distributions.

Handique, L., Chakraborty, E.-M. M., Afify, A. Z., and Eliwa, M. S. (2024). Modeling veterinary medical data utilizing a new generalized Marshall-Olkin transmuted generator of distributions with statistical properties. *Thailand Statistician*, 22(1):219–236.

Imran, M., Bakouch, H. S., Tahir, M. H., Ameeq, M., Jamal, F., and Mendy, J. T. (2023). A new Bell-exponential model: properties and applications. *Cognet Engineering*, 10(2281062):1–25.

Imran, M. and Mendy, J. T. (2024). Bell-Touchard-G family of distributions: applications to quality control and actuarial data. *Journal of Statistics Applications & Probability*, 13(1):339–355.

Irshad, M. R., Ahammed, M., Maya, R., and Al-Omari, A. I. (2023). Marshall-Olkin Bilal distribution with associated minification process and acceptance sampling plans. *Hacettepe Journal of Mathematics and Statistics*. 29 pages. (To appear).

Ishaq, A. I., Suleiman, A. A., Daud, H., Singh, N. S. S., Othman, M., Sokkalingam, R., Wiratchotistian, P., Usman, A. G., and Abba, S. I. (2023a). Log-Kumaraswamy distribution: its features and applications. *Frontiers in Applied Mathematics and Statistics*, page 13 pages. DOI: 10.3389/fams.2023.1258961.

Ishaq, A. I., Suleiman, A. A., Usman, A., Daud, H., and Sokkalingam, R. (2023b). Transformed log Burr III distribution: structural features and application to milk production. *Engineering Proceedings*, 56(1):7 pages.

Khodja, N., Gemeay, A. M., Zeghdoudi, H., Karakaya, K., Alshangiti, A. M., Bakr, M. E., Balogun, O. S., Muse, A. H., and Hussam, E. (2023). Modeling voltage real data set by a new version of Lindley distribution. *IEEE Access*, 11:67220–67233.

Kus, C., Karakaya, K., Tanis, C., Akdogan, Y., Sert, S., and Kalkan, F. (2023). Compound transmuted family of distributions: properties and applications. *Ricerche di Matematica*. DOI: 10.1007/s11587-023-00808-7.

Lone, M. A., Dar, I. H., and Jan, T. R. (2024). A new family of generalized distributions with an application to Weibull distribution. *Thailand Statistician*, 22(1):1–16.

Mahmoud, M. A. W., Ghazal, M. G. M., and Radwan, H. M. M. (2024). Modified generalized linear exponential distribution: properties and applications. *Statistics, Optimization & Information Computing*, 12(1):231–255.

Mazrouk, W., Shafiq, S., Naz, S., Jamal, F., Sapkota, L. P., Nagy, M., Mansi, A. H., Hussam, E., and Gemeay, A. M. (2023). A new univariate continuous distribution with applications in reliability. *AIP Advances*, 13(115126):1–24.

Moakofi, T. and Oluyede, B. (2023a). The type I heavy-tailed odd power generalized Weibull-G family of distributions with applications. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 72(4):921–958.

Moakofi, T. and Oluyede, B. (2023b). Type II exponentiated half-logistic Gompertz-G family of distributions: properties and applications. *Mathematica Slovaca*, 73(3):785–810.

Nagy, M., Gemeay, A. M., Rajitha, C. S., Karakaya, K., Saglam, S., Mansi, A. H., and Kilai, M. (2023). Power unit Gumbel type II distribution: statistical properties, regression analysis, and applications. *AIP Advances*, 13(115124):1–14.

Nawel, A. K., Zeghdoudi, H., Natalia, D., and Lazri, N. (2023). New gamma Lindley distribution: properties and applications. (Preprint).

Naz, S., Tahir, M. H., Jamal, F., Ameeq, M., Shafiq, S., and Mendy, J. T. (2023). A group acceptance sampling plan based on flexible new Kumaraswamy exponential distribution: an application to quality control reliability. *Cogent Engineering*, 10(22579445):1–18.

Oluyede, B., Tlhaloganyang, B., and Sengweni, W. (2024). The Topp-Leone odd Burr X-G family of distributions: properties and applications. *Statistics, Optimization & Information Computing*, 12(1):109–132.

Opone, F. C., Ubaka, N. O., and Karakaya, K. (2023). Statistical analysis of Covid-19 data using the odd log logistic Kumaraswamy distribution. *Statistics, Optimization & Information Computing*, 12(1):210–230.

Sakthivel, K. M. and G., V. (2024). An empirical study on modeling skewed data using a mixture of probability distributions. *Pakistan Journal of Statistics*, 40(1):81–104.

Sayibu, S. B., Luguterah, A., and Nasiru, S. (2024). McDonald generalized power Weibull distribution: properties, and applications. *Journal of Statistics Applications & Probability*, 13(1):297–322.

Shah, S., Hazarika, P. J., Chakrobarty, S., and Alizadeh, M. (2023). The Balakrishnan-alpha-beta-skew-Laplace distribution: properties and applications. *Statistics, Optimization & Information Computing*, 11(3):755–772.

Singh, B. P., Das, U. D., Karakaya, K., Bakouch, H. S., and Abba, B. (2023). Doubly bounded exponential model: some information measures and estimation. *Communications in Statistics-Theory and Methods*, pages 1–18.

Suleiman, A. A., Daud, H., Ishaq, A. I., Othman, M., Sokkalingam, R., Usman, A., and Osi, A. O. (2023a). The odd beta prime inverted Kumaraswamy distribution with application to COVID-19 mortality rate in Italy. *Engineering Proceedings*, 56(1):218.

Suleiman, A. A., Daud, H., Othman, M., Singh, N. S. S., Ishaq, A. I., Sokkalingam, R., and Husin, A. (2023b). A novel extension of the Fréchet distribution: Statistical properties and application to groundwater pollutant concentrations. *Data Science Insights*, 1(1):1–24.

Suleiman, A. A., Daud, H., Singh, N. S. S., Ishaq, A. I., and Othman, M. (2023c). A new odd beta prime-Burr X distribution with applications to petroleum rock sample data and COVID-19 mortality rate. *Data (MDPI)*, 8(143):24 pages.

Suleiman, A. A., Daud, H., Singh, N. S. S., Othman, M., Ishaq, A. I., and Sokkalingam, R. (2023d). A novel odd beta prime-Logistic distribution: desirable mathematical properties and applications to engineering and environmental data. *Sustainability (MDPI)*, 15(10239):25 pages.

Suleiman, A. A., Othman, M., Ishaq, A. I., Daud, H., Indawati, R., Abdullah, M. L., and Husin, A. (2023e). The odd beta prime-G family of probability distributions: properties and applications to engineering and environmental data. *Computer Sciences & Mathematics Forum*, 2:7 pages.

Usman, A., Ishaq, A. I., Suleiman, A. A., Othman, M., Daud, H., and Aliyu, Y. (2023). Univariate and bivariate log-Topp-Leone distribution using censored and uncensored datasets. *Computer Sciences & Mathematics Forum*, 7(32):6 pages.

Warahena-Liyanage, G., Gabanakgosi, M., and Oluyede, B. (2023). Generalized Topp-Leone-G power series class of distributions: properties and applications. *Electronic Journal of Applied Statistical Analysis*, 16(3):564–583.

Zeenalabiden, N. A. and Saracoglu, B. (2023). A new family of distributions: exponential power-X family of distributions and its some properties. *Iraqi Journal of Statistical Sciences*, 20(3):235–248.