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Properties and Application of Trimodal Skew Normal Distribution

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Abstract

A new type of continuous distribution that extends the skew distribution developed by Azzalini (1985) is presented in this paper. This new distribution is designed to effectively model real-life data that may have up to three modes. The primary objective of this study is to provide a comprehensive understanding of the structural properties of this distribution, including moments, moments generating function, Fisher's information matrix, characterization, and parameter estimation through the method of maximum likelihood. Additionally, the distribution's flexibility and usefulness are evaluated by analyzing two real-life datasets. The analysis findings suggest that, as measured by AIC and BIC values, the new distribution demonstrates superior performance in fitting the datasets compared to other distributions. The lower values of AIC and BIC suggest that the new distribution better fits the datasets compared to other alternatives.

Key Words: AIC, BIC, Characterization, Skew Distribution, Tri-modal Distribution

Mathematical Subject Classification: 60E05, 62H10, 62H12

1. Introduction

Many researchers have proposed different distributions to accurately model real-life data that exhibit skewness. These distributions are built upon the pioneering work of Azzalini (1985). One well-known distribution in this context is Azzalini's skew-normal distribution, which offers a probability density function (pdf) defined as follows:

$$f_X(x;\lambda) = 2\phi(x)\Phi(\lambda x)\ ; \quad x \in R\ , \lambda \in R \tag{1}$$

where the parameter λ represents the degree of asymmetry, while $\phi(.)$ and $\Phi(.)$ correspond to the pdf and cumulative density function (cdf) of the standard normal distribution, respectively, while this distribution exhibits a broad spectrum of skewness and kurtosis, it does not possess the capability to model datasets with multiple modes.

In many complex modeling situations, datasets often exhibit two or more distinct patterns or clusters, referred to as modes. To address this, researchers have recently introduced extensions to the skew-normal distribution, which deviate from the traditional unimodal model. Some of these extensions include the flexible skew-symmetric distribution, two-piece skew-normal distributions, bimodal exponential power distribution, bimodal skew-elliptical distribution, skew-flexible normal distribution, alpha skew Laplace distribution, and more (Kim, (2005); Hassan and Hijazi (2005,2010); Elal-Olivero *et al.*(2009); Gomez *et al.* (2011); Harandi and Alamatsaz, (2013); Shah *et al.*, (2020a, 2021b, 2022); Shah and Hazarika (2021); Das *et al.*, (2023)).

Recently, Elal-Olivero (2010) introduced a new type of bimodal distribution called the alpha skew-normal distribution and its density is given by

$$f_X(x;\alpha) = \left(\frac{(1-\alpha x)^2+1}{2+\alpha^2}\right)\phi(x); x \in R\ , \alpha \in R \tag{2}$$

The parameter α in this distribution controls the shape and number of modes present. Hazarika *et al.* (2020) developed Balakrishnan alpha skew normal distribution which is more flexible than Equation (2). After that, Hazarika and Chakraborty (2014) and Shah *et al.* (2020b) extended this concept by introducing the alpha skew logistic distribution and Balakrishnan alpha skew logistic distribution respectively and investigating their properties.Furthermore, Sharafei *et al.* (2017), Shah *et al.* (2021a)and Shah *et al.* (2024) proposed a generalization of the alpha skew-normal distribution, alpha beta skew-normal distribution and Balakrishnan alpha skew-normal distribution respectively by adding an additional parameter(s) to capture further asymmetry. In a related study, Elal-Olivero *et al.* (2020) introduced the bimodal skew-normal distribution, focusing on its properties and conducting inferential analysis. The density function of this distribution is represented by a specific mathematical expression as

$$f_X(x;\alpha,\lambda)=2\left(\frac{1+\alpha x^2}{1+\alpha}\right)\phi(x)\Phi(\lambda x); x\in R\ ,\lambda\in R,\alpha\geq 0 \tag{3}$$

Das *et al.* (2024) introduced a class of skew distributions known as the Bimodal Tanh Skew Normal (BTSN) distributions. These distributions incorporate a novel skewing mechanism utilizing the hyperbolic tangent function. Subsequently, Das *et al.* (2025) introduced the Generalized Alpha Skew Laplace distribution, specifically designed to effectively model datasets exhibiting both unimodal and bimodal characteristics.

The distributions mentioned earlier provide an alternative approach to address the computational challenges associated with the mixer distribution. When dealing with tri-modal data containing three distinct groups or patterns, it is necessary to estimate the mixing proportions (w_1, w_2 and $w_3=1-w_1-w_2$) for each group. For example, in a tri-modal normal distribution, it is necessary to estimate parameters such as means (μ_1,μ_2,μ_3) standard deviations ($\sigma_1,\sigma_2,\sigma_3$) and the mixing proportions (w_1,w_2). This involves estimating a total of eight parameters. However, estimating these parameters becomes complicated due to the large number involved, making optimization of the likelihood function challenging. Moreover, the presence of numerous parameters can lead to numerical errors during optimization. Consequently, there is a need to develop effective methods for fitting tri-modal behaviour in real-life data. Unfortunately, there is limited research in the statistical field focusing on distributions capable of accurately capturing tri-modal patterns. Recently, Martinez-Florez *et al.* (2022) addressed this issue by introducing a distribution that can handle symmetric data with three modes. They initially introduced the tri-modal normal distribution, denoted as $X\sim TN$, and provided a mathematical expression to describe its density function as

$$f(x)=\left(\frac{\left(x^2-1\right)^2+2}{4}\right)\phi(x) \tag{4}$$

Furthermore, Shah *et al.*(2023) introduced a new multimodal extension of alpha skew normal distribution using Balakrishnan mechanism. Subsequently, Pathak *et al.* (2023) and Das *et al.* (2023) introduced a tri-modal extension of the Skew Logistic Distribution and the Flexible Alpha Skew Normal Distribution. They further developed and proposed the characterization theory for these distributions.

The main objective of this research paper is to introduce a model that exhibits the necessary flexibility to accurately capture both symmetric and asymmetric behaviours observed in tri-modal datasets. To achieve this, the paper is structured as follows: Section 2 introduces the tri-modal skew-normal distribution and thoroughly explores its shape and the particular cases of the same. Section 3 presents several structural properties related to the tri-modal skew-normal distribution, offering valuable insights into its characteristics. Section 4 presents the characterization of the study distribution. Section 5 focuses on parameter estimation for the newly introduced distribution, utilizing the maximum likelihood estimation approach along with the Fisher information matrix and Monte Carlo Simulation techniques. Section 6involves the application of the studied model to two real-life datasets, followed by a comparative analysis with specific rival models. Finally, the concluding remarks are presented in the final section, summarizing the findings and highlighting noteworthy aspects of the studied distribution.

2. Trimodal Skew Normal Distribution

This section introduced a novel skew-normal distribution and examined its fundamental properties. The proposed distribution provides a unique framework for modeling skewed data and demonstrates unique characteristics that set it apart from conventional distributions. Various properties and features of this new distribution were discussed, providing an understanding of its behavior and potential applications across different fields of study.

Definition: A random variable X follows tri-modal skew normal distribution, denoted by $TSN(\lambda)$, if it has the pdf

$$f_X(x;\lambda)=\left(\frac{\left(x^2-1\right)^2+2}{2}\right)\phi(x)\Phi(\lambda x)\ \ ;\ x\in R,\lambda\in R \tag{5}$$

The pdf of the proposed distribution takes the form $w(x)\phi(x)\Phi(\lambda x)$, where $w(x)$ represents a fourth-order polynomial that allows for the presence of at least three modes (refer to Martinez-Florez *et al.* (2022)). The term $\phi(x)$ corresponds to the pdf of the normal distribution, while $\Phi(\lambda x)$ represents the cdf of the standard normal distribution with real-valued parameter λ.

2.1. Special cases of $TSN(\lambda)$ distribution:

- If $\lambda=0$, then one get the trimodal normal distribution of Martinez-Florez *et al.* (2022) given by
$$f_X(x)=\left(\frac{\left(x^2-1\right)^2+2}{4}\right)\phi(x)$$
- If $\lambda\to\pm\infty$ then $f_X(x)=\left(\frac{\left(x^2-1\right)^2+2}{2}\right)\phi(x)$
- If $X\sim TSN(\lambda)$, then $-X\sim TSN(-\lambda)$

2.2. Plot of the Probability density function of $TSN(\lambda)$ distribution:

Figure 1 displays the density plot of the tri-modal skew-normal distribution for various values of λ. The figure reveals that the distribution can have up to three modes, and the skewness of the distribution depends on the λ value. Specifically, when λ is positive, the distribution exhibits positive skewness; when λ is negative, the skewness becomes negative. Analyzing Figure 1, it is evident that the tri-modal pattern emerges when λ falls within the range of -0.5 and 0.5. Notably, when λ equals zero, the distribution transforms into a tri-modal normal distribution.

3. Distribution Properties

This section presents an analysis of the properties of the studied distribution, focusing on the examination of its defining characteristics and features.

3.1. Cumulative distribution function (cdf)

Theorem 1: *The cdf of $TSN(\lambda)$ distribution is given by*

$$F_X(x)=F_Y(x)-\frac{1}{4}x\left(1+x^2\right)f_Y(x)-\frac{\lambda b}{4\left(1+\lambda^2\right)}\left\{2+\left(1+\lambda^2\right)\left(1+x^2\right)\right\}\phi\left(x\sqrt{1+\lambda^2}\right) \tag{6}$$

where $F_Y(x)$ and $f_Y(x)$ are the cdf and pdf of skew normal distribution respectively and $b=\sqrt{\frac{2}{\pi}}$.

Proof:

$$\begin{aligned}F_X(x)&=P(X\le x)=\frac{1}{2}\int_{-\infty}^{x}\left(t^4-2t^2+3\right)\phi(t)\,\Phi(\lambda t)\,dt\\&=\frac{1}{2}\int_{-\infty}^{x}t^4\ \phi(t)\,\Phi(\lambda t)\,dt-\int_{-\infty}^{x}t^2\ \phi(t)\,\Phi(\lambda t)\,dt+\frac{3}{2}\int_{-\infty}^{x}\phi(t)\,\Phi(\lambda t)\,dt\\&=\frac{1}{2}I_1-I_2+\frac{3}{2}I_3\end{aligned} \tag{7}$$

The integral I_2 represents the cdf of the bimodal skew normal distribution (Elal-Olivero *et al.*, 2020) while I_3 corresponds to the cdf of the skew-normal distribution (Azzalini, 1985). The remaining term was calculated using the method of integration by parts. As a result, the cumulative distribution function of the tri-modal skew normal distribution can be expressed as follows:

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$$F_X(x) = F_Y(x) - \frac{1}{4}x(1+x^2)f_Y(x) - \frac{\lambda b}{4(1+\lambda^2)}\left\{2+(1+\lambda^2)(1+x^2)\right\}\phi\left(x\sqrt{1+\lambda^2}\right)$$

Figure 2 illustrates the cdf of the $TSN(\lambda)$ distribution. This graph facilitates the analysis of how the shape of the distribution varies with changes in the parameter λ. Examining the plot provides a deeper understanding of the behavior of the TSN distribution and the influence of the λ on its shape.

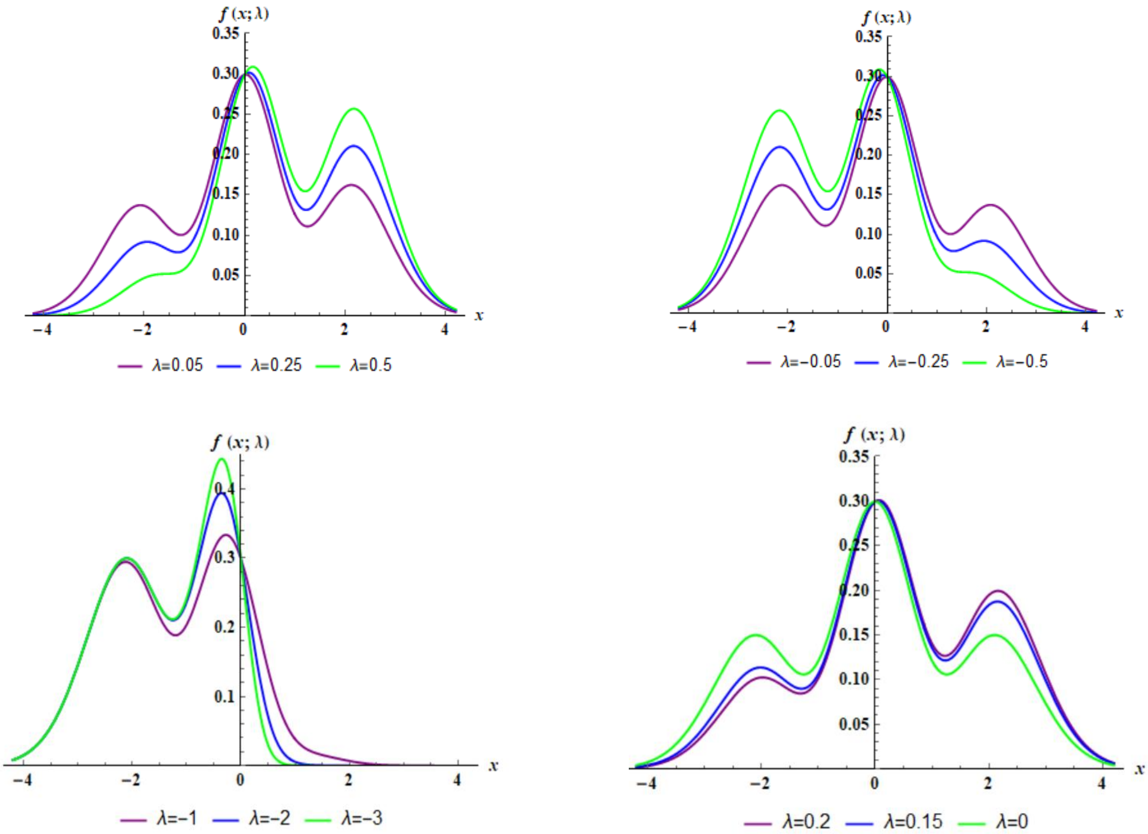

Figure 1: Plots of the pdf of $TSN(\lambda)$ for different choices of λ

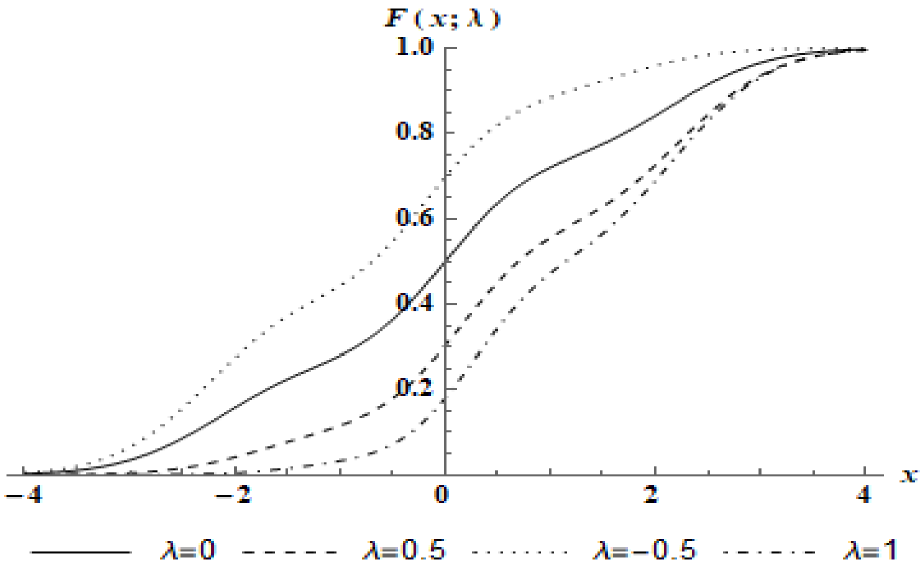

Figure 2: Plot of the cdf of $TSN(\lambda)$ for different choice of λ.

3.2. Moment Generating Function (mgf) and Moments

Theorem 2: *The moment generating function of the* $TSN(\lambda)$ *distribution is expressed as follows*

$$M_X(t)=\frac{1}{4}\left[\begin{array}{l}\left\{\frac{b\lambda t\left(3+t^2+3\lambda^2\right)}{\left(\sqrt{1+\lambda^2}\right)^7}+\frac{b\lambda t\left(1+t^2+\lambda^2\right)}{\left(\sqrt{1+\lambda^2}\right)^5}-\frac{b\lambda t}{\left(\sqrt{1+\lambda^2}\right)^3}+\frac{b\lambda t(2\lambda-1)}{\left(\sqrt{1+\lambda^2}\right)}\right\}\\ \exp\left(\frac{t^2}{2\left(1+\lambda^2\right)}\right)+M_Y(t)\left(t^2\left((2\lambda-1)+2\right)\right)\end{array}\right] \tag{8}$$

with $M_Y(t)=E\left[\exp(tY)\right]=2\Phi\left(\frac{\lambda t}{\sqrt{1+\lambda^2}}\right)\exp\left(\frac{t^2}{2}\right)$ *and* $b=\sqrt{\frac{2}{\pi}}$.

Proof:

$$\begin{aligned}M_X(t)=E\left(e^{tx}\right)&=\int_{-\infty}^{\infty}e^{tx}\left(\frac{\left(x^2-1\right)^2+2}{2}\right)\phi(x)\,\Phi(\lambda x)dx\\&=\frac{1}{2}\int_{-\infty}^{\infty}e^{tx}\left(x^4-2x^2+3\right)\phi(x)\,\Phi(\lambda x)dx\\&=\frac{1}{4}\left[2\int_{-\infty}^{\infty}e^{tx}x^4\,\phi(x)\,\Phi(\lambda x)dx-4\int_{-\infty}^{\infty}e^{tx}x^2\,\phi(x)\,\Phi(\lambda x)dx+6\int_{-\infty}^{\infty}e^{tx}\,\phi(x)\,\Phi(\lambda x)dx\right]\\&=\frac{1}{4}\left[I_4-2I_5+3I_6\right]\end{aligned} \tag{9}$$

Equation (9) indicates that the terms I_5 and I_6 represent the mgf of the one parameter bimodal skew normal distribution (Elal-Olivero *et al.*, 2020)and the skew-normal distribution (Azzalini, 1985), respectively. The remaining term was computed using the integration by parts method. By combining these components, the mgf of the studied distribution was derived as follows:

$$M_X(t)=\frac{1}{4}\left[\begin{array}{l}\left\{\frac{b\lambda t\left(3+t^2+3\lambda^2\right)}{\left(\sqrt{1+\lambda^2}\right)^7}+\frac{b\lambda t\left(1+t^2+\lambda^2\right)}{\left(\sqrt{1+\lambda^2}\right)^5}-\frac{b\lambda t}{\left(\sqrt{1+\lambda^2}\right)^3}+\frac{b\lambda t(2\lambda-1)}{\left(\sqrt{1+\lambda^2}\right)}\right\}\\ \exp\left(\frac{t^2}{2\left(1+\lambda^2\right)}\right)+M_Y(t)\left(t^2\left((2\lambda-1)+2\right)\right)\end{array}\right]$$

Remark 1: The n^{th} moment of $TSN(\lambda)$ distribution is of the form

$$E\left(X^k\right)=\frac{1}{2}\left[E\left(Z_\lambda^{k+4}\right)-2E\left(Z_\lambda^{k+2}\right)+3E\left(Z_\lambda^{k}\right)\right]$$

where, $E\left(Z_\lambda^{k+4}\right)$, $E\left(Z_\lambda^{k+2}\right)$ and $E\left(Z_\lambda^{k}\right)$ are the $(k+4)^{th}, (k+2)^{th}$ and k^{th} moments of standard $SN(\lambda)$ distribution (Azzalini, 1985). One have,

$$\begin{aligned}E\left(X^k\right)&=\int_{-\infty}^{\infty}x^k\left(\frac{\left(x^2-1\right)^2+2}{2}\right)\phi(x)\Phi(\lambda x)dx\\&=\frac{1}{2}\left[\int_{-\infty}^{\infty}x^k\left(x^4-2x^2+3\right)\phi(x)\Phi(\lambda x)dx\right]\\&=\frac{1}{2}\left[E\left(Z_\lambda^{k+4}\right)-2E\left(Z_\lambda^{k+2}\right)+3E\left(Z_\lambda^{k}\right)\right]\end{aligned}$$

By considering the moments of the skew-normal distribution (Henze, 1986), one can derive the first four raw moments of the $TSN(\lambda)$ distribution as follows:

$$E(X)=\frac{b\lambda\left(12+16\lambda^2+7\lambda^4\right)}{4\left(\sqrt{1+\lambda^2}\right)^5};\quad E\left(X^2\right)=3;\quad E\left(X^3\right)=\frac{b\lambda\left(84+164\lambda^2+133\lambda^4+38\lambda^6\right)}{4\left(\sqrt{1+\lambda^2}\right)^7};\quad E\left(X^4\right)=21 \text{ and}$$

variance is given by $Var(x)=E\left(X^2\right)-\left\{E(X)\right\}^2=3-\dfrac{b^2\lambda^2\left(12+16\lambda^2+7\lambda^4\right)^2}{16\left(1+\lambda^2\right)^5}$.

Through numerical optimization methods, the bounds of the mean ($E(X)$) and variance ($Var(X)$) were computed. The analysis indicated that the mean of the distribution ranged from -1.3963 to 1.3963, while the variance was estimated to be between 1.05 and 3.00. To provide a visual representation of these findings, the $E(X)$ and $Var(X)$ were plotted in Figure 3, clearly illustrating the estimated bounds of the distribution.

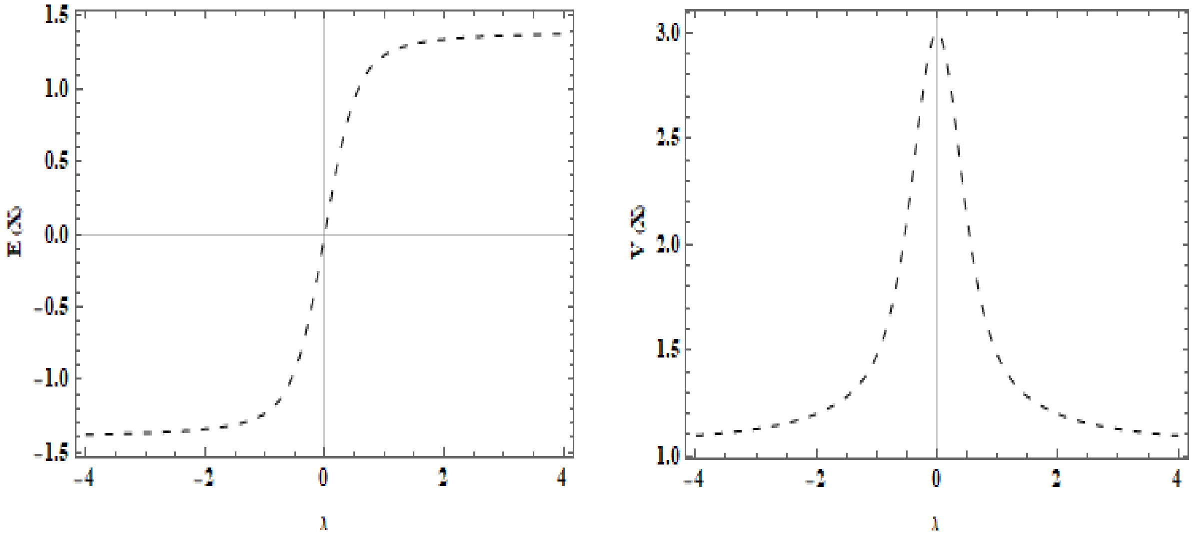

Figure 3: Plot of the mean and variance of $TSN(\lambda)$ distribution

3.3. Skewness and Kurtosis

The skewness and kurtosis of the respective distribution is given by

$$\beta_1=\frac{b^2\lambda^2\left(8\left(1+\lambda^2\right)^4 d_1-72\left(1+\lambda^2\right)^5 d_2+b^2\lambda^2 d_2^{\,3}\right)^2}{27648\left(1+\lambda^2\right)^{15}} \text{ and}$$

$$\beta_2=\frac{7}{3}-\frac{b^4\lambda^4 d_2^{\,4}}{768\left(1+\lambda^2\right)^{10}}+\frac{b^2\lambda^2 d_2 d_3}{72\left(1+\lambda^2\right)^6}$$

where, $d_1=\left(84+164\lambda^2+133\lambda^4+38\lambda^6\right)$, $d_2=\left(12+16\lambda^2+7\lambda^4\right)$ and $d_3=\left(60+76\lambda^2+59\lambda^4+13\lambda^6\right)$

Using numerical optimization methods, the bounds for the parameters β_1 and β_2 were identified. The analysis indicates that the value of β_1 falls within the range of 0 to 0.029, while β_2 was estimated to be between 0.262 and 2.333. To provide a clear understanding of the shape of the skewness and kurtosis, these characteristics were represented graphically in Figure 4, which visually illustrates how the skewness and kurtosis values vary within their respective ranges.

3.4. Order Statistics of $TSN(\lambda)$ distribution

This section is responsible for the results of order statistics of $TSN(\lambda)$ distribution.

Theorem 4: *The cdf of k^{th} order statistics for $TSN(\lambda)$ distribution is given as*

$$F_{i:n}(x)=\sum_{k=i}^{n}\sum_{i=0}^{n-k}\sum_{m=0}^{k+l}\sum_{p=0}^{m}(-1)^{l}\binom{n}{k}\binom{n-k}{l}\binom{k+l}{m}\binom{m}{p}F_{y}(x)^{k+l-m}A_{1}(x)^{m-p}A_{2}(x)^{m}.$$

where, $A_1(x)=\frac{1}{4}x(1+x^2)f_y(x)$, and $A_2(x)=\frac{\lambda b}{4(1+\lambda^2)}\left\{2+(1+\lambda^2)(1+x^2)\right\}\phi\left(x\sqrt{1+\lambda^2}\right)$

Proof: Let, $X_1, X_2, ...X_n$ is a random sample which is drawn from (5). If the corresponding order statistic be $X_{1:n} < X_{2:n} < ... < X_{n:n}$ then the cdf of k^{th} order statistic is defined as

$$F_{i:n}(x)=\sum_{k=i}^{n}\binom{n}{k}F(x)^{k}\left[1-F(x)\right]^{k}$$
$$=\sum_{k=i}^{n}\sum_{i=0}^{n-k}(-1)^{l}\binom{n}{k}\binom{n-k}{l}F(x)^{k+l}$$
$$=\sum_{k=i}^{n}\sum_{i=0}^{n-k}(-1)^{l}\binom{n}{k}\binom{n-k}{l}\left[F_y(x)-A_1(x)-A_2(x)\right]^{k+l},$$

Where, $A_1(x)=\frac{1}{4}x(1+x^2)$, and $A_2(x)=\frac{\lambda b}{4(1+\lambda^2)}\left\{2+(1+\lambda^2)(1+x^2)\right\}\varphi\left(x\sqrt{1+\lambda^2}\right)$

Now,

$$\left[F_y(x)-A_1(x)-A_2(x)\right]^{k+l}=\sum_{m=0}^{k+l}\binom{k+l}{m}F_y(x)^{k+l-m}\left(A_1(x)-A_1(x)\right)^{m}$$

Hence,

$$\sum_{m=0}^{k+l}\binom{k+l}{m}F_y(x)^{k+l-m}\left(A_1(x)-A_1(x)\right)^{m}=\sum_{m=0}^{k+l}\sum_{p=0}^{m}\binom{k+l}{m}\binom{m}{p}F_y(x)^{k+l-m}A_1(x)^{m-p}A_2(x)^{m}.$$

Hence, final expression for the cdf order statistic is given as

$$F_{i:n}(x)=\sum_{k=i}^{n}\sum_{i=0}^{n-k}\sum_{m=0}^{k+l}\sum_{p=0}^{m}(-1)^{l}\binom{n}{k}\binom{n-k}{l}\binom{k+l}{m}\binom{m}{p}F_{y}(x)^{k+l-m}A_{1}(x)^{m-p}A_{2}(x)^{m}.$$

Again, pdf of the same can be obtained as

$$f_{i:n}(x)=\frac{d\,F_{i:n}(x)}{dx}$$
$$=\sum_{k=i}^{n}\sum_{i=0}^{n-k}\sum_{m=0}^{k+l}\sum_{p=0}^{m}(-1)^{l}\binom{n}{k}\binom{n-k}{l}\binom{k+l}{m}\binom{m}{p}\frac{d}{dx}\left[F_{y}(x)^{k+l-m}A_{1}(x)^{m-p}A_{2}(x)^{m}\right]$$
$$=\sum_{k=i}^{n}\sum_{i=0}^{n-k}\sum_{m=0}^{k+l}\sum_{p=0}^{m}(-1)^{l}\binom{n}{k}\binom{n-k}{l}\binom{k+l}{m}\binom{m}{p}B(x).$$

Where,

$$B(x)=\frac{d}{dx}\left[F_y(x)^{k+l-m}A_1(x)^{m-p}A_2(x)^{m}\right]$$
$$=2^{\frac{1}{2}(-7k-7l+6m+p)}\pi^{\frac{1}{2}(-k-l+p)}\left(\frac{be^{-\frac{1}{2}x^2(1+\lambda^2)}\lambda\left(2+(1+x^2)(1+\lambda^2)\right)}{1+\lambda^2}\right)^{k+l-m}\left(e^{-\frac{x^2}{2}}x(1+x^2)\mathrm{Erfc}\left[-\frac{x\lambda}{\sqrt{2}}\right]\right)^{m-p}$$
$$\times\left(-A_3(x)+A_4(x)+A_5(x)\right)\left(\mathrm{Erfc}[-\tfrac{x}{\sqrt{2}}]-4\mathrm{OwenT}[x,\lambda]^{k+l-m}\right)$$

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Where, $A_3(x)=\frac{(k+l-m)(x+x^3)(1+\lambda^2)^2}{3+\lambda^2+x^2(1+\lambda^2)}$, $A_4(x)=\frac{e^{-\frac{1}{2}x^2\lambda^2}(m-p)(\sqrt{\frac{2}{\pi}}x(1+x^2)\lambda-e^{\frac{x^2\lambda^2}{2}}(-1-2x^2+x^4)\text{Erfc}[-\frac{x\lambda}{\sqrt{2}}])}{x(1+x^2)\text{Erfc}[-\frac{x\lambda}{\sqrt{2}}]}$

and $A_4(x)=\frac{e^{-\frac{x^2}{2}}(k+l-m)\sqrt{\frac{2}{\pi}}(1+\text{Erf}[\frac{x\lambda}{\sqrt{2}}])}{\text{Erfc}[-\frac{x}{\sqrt{2}}]-4\text{OwenT}[x,\lambda]}$.

Here, $Erfc[.]$gives the complementary error function while $OwenT[x,\lambda]$ is the Owen T function which is defined as. $T(x,\alpha)=\frac{1}{2\pi}\int_0^a \exp\left(-x^2(1+t^2)/2\right)/((1+t^2))dt$

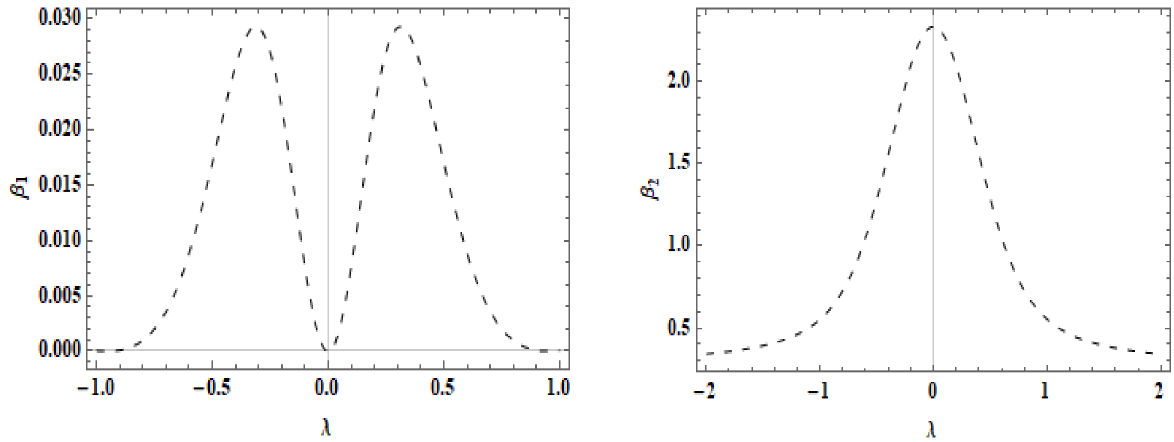

Figure 4: Plot of Skewness$\left(\beta_1\right)$ and Kurtosis $\left(\beta_2\right)$ with different choices of parameter λ

3.5. Mode of $TSN(\lambda)$ Distribution:

In this section, the mode of the TSN distribution has been obtained through graphical methods, following the approach introduced by Behboodian (1970).

Theorem 4:*The tri-modal skew normal distribution has atmost three modes*.

Proof:

From (4) and (5) one can obtain $TSN(\lambda)$ distribution as

$$f_X(x;\lambda)=2f(x)\Phi(\lambda x) \tag{10}$$

Differentiation (10) with respet to *x*, one get

$$f'_X(x;\lambda)=2\left[f'(x)\Phi(\lambda x)+\lambda f(x)\phi(\lambda x)\right] \tag{11}$$

If equation (11) has five roots, equation (10) can exhibit a maximum of three modes. The application of a graphical method has been utilized for the computation of this relationship. As a result,

$$f'_X(x;\lambda)=F_1(x)-F_2(x) \tag{12}$$

where,

$$F_1(x)=2f'(x)\Phi(\lambda x) \tag{13}$$

and

$$F_2(x)=2\lambda f(x)\phi(\lambda x) \tag{14}$$

Now, if one take

$$f_X(x;\lambda)=0, \tag{15}$$

then one get

$$F_1(x)=F_2(x)$$

Figure 5 illustrates the graphical representation of the preceding analysis. The figure reveals that the model displays a range of four to five real zeros, occurring at the points where the *x*-axis intersects. The distribution exhibits two modes when the function (15) possesses four roots. Conversely, the distribution shows three modes if the equation (15) has five roots.

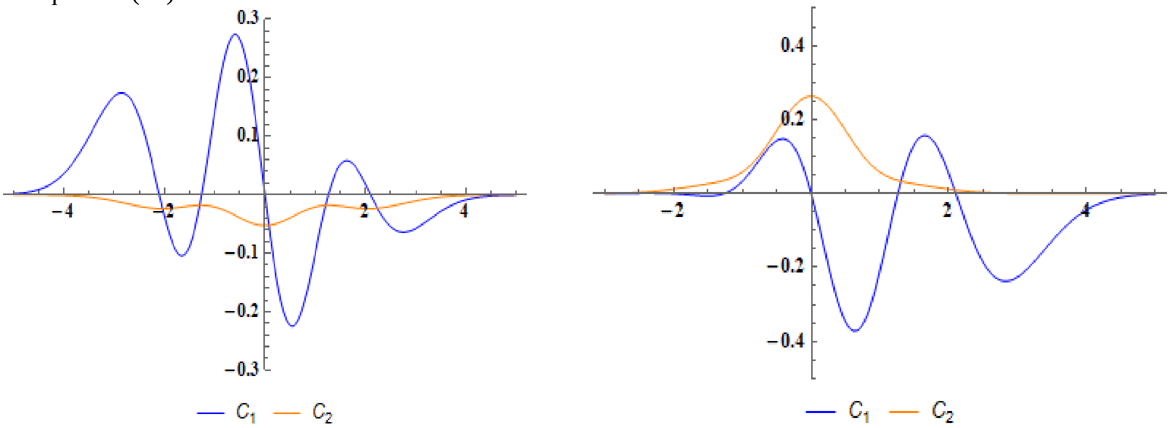

Figure 5: (a) The plot of C1 and C2 for $\lambda = 0.22$ (b) The plot of C1 and C2 for $\lambda = 1.1$

4. Characterization Results

This section is based on the characterization of the TSN distribution by employing two truncated moments. Notably, a closed-form expression for the cumulative distribution function (cdf) is not a prerequisite for this characterization.

4.1. Characterizations based on two truncated moments

This subsection presents a comprehensive characterization of the *TSN* distribution based on the concept of two truncated moments. This characterization is derived from Theorem 4, as proposed by Glänzel (1987), and remains valid even if the interval HHH is not closed. Furthermore, the characterization remains stable under weak convergence (see Glänzel, 1990).

Theorem 5: Consider a given probability space, denoted as (Ω, F, P), and let *H* represent an interval within the range of [*d*, *e*], where $d = -\infty$, $e = \infty$ are allowable values and $d < e$. Further, let $X : \Omega \to H$ be a continuous random variable characterized by the distribution function *F*. Additionally, *k* and *h* are two real functions defined on the interval *H*such that

$$E\left[k(X) \mid X \geq x\right] = E\left[h(X) \mid X \geq x\right]\eta(x), \quad x \in H,$$

is defined with some real function η. Assuming $k,\ h \in C^1(H),\ \eta \in C^2(H)$ and *F* is twice continuously differentiable and strictly monotone functions defined on the interval *H*, and further assuming that the equation $\eta h = k$ has no real solution within the interior of *H*, then it can be concluded that *F* is uniquely determined by the functions *k*, *h*and η, particularly,

$$F(x) = \int_d^x C\left|\frac{\eta'(u)}{\eta(u)h(u) - k(u)}\right|\exp(-(s(u)))du,$$

where, the function s is a solution of the differential equation $s' = \dfrac{\eta' h}{\eta h - k}$ and C is the normalizing constant, such that $\int_H dF = 1$.

Proposition 1: Let the random variable $X : \Omega \to R$ be continuous, and let $h(x) = \left[\left(x^4 - 2x^2 + 3\right)\Phi(\lambda x)\right]^{-1}$ and $k(x) = h(x)\Phi(x)$ for $x \in R$. Then, the density of *X* is given in (5) if and only if the function η defined in Theorem 4 is

$$\eta(x) = \frac{1}{2}\{1 + \Phi(x)\}, \qquad x \in R.$$

Proof: If *X* has a probability density function given by (5), then

$$\left(1-F\left(x\right)\right)E\left[h\left(X\right)\middle|X\geq x\right]=\frac{1}{2}\int_{x}^{\infty}\phi\left(u\right)du=\frac{1}{2}\left\{1-\Phi\left(x\right)\right\},\ x\in R,$$

and $\left(1-F\left(x\right)\right)E\left[k\left(X\right)\middle|X\geq x\right]=\frac{1}{2}\int_{x}^{\infty}\phi\left(u\right)\Phi\left(u\right)du=\frac{1}{4}\left\{1-\left(\Phi\left(x\right)\right)^{2}\right\},\ x\in R,$

and hence $\eta\left(x\right)=\frac{E\left[k\left(X\right)\middle|X\geq x\right]}{E\left[h\left(X\right)\middle|X\geq x\right]}=\frac{1}{2}\left\{1+\Phi\left(x\right)\right\},\ x\in R,$

and finally, $\eta\left(x\right)h\left(x\right)-k\left(x\right)=\frac{1}{2}h\left(x\right)\left\{1-\Phi\left(x\right)\right\}>0\quad for\ \ x\in R.$

On the other hand, if η exhibits the above form, then

$$s'\left(x\right)=\frac{\eta'\left(x\right)h\left(x\right)}{\eta\left(x\right)h\left(x\right)-k\left(x\right)}=\frac{\phi\left(x\right)}{1-\Phi\left(x\right)},$$

$$s\left(x\right)=-\log\left\{1-\Phi\left(x\right)\right\},\quad x\in R$$

Considering Theorem 5, it can be concluded that *X* possesses the probability density function described by (5).

4.2. Related characterization

A characterization based on a first-order differential equation in terms of the function $\eta\left(x\right)$ is stated here.

Corollary 1: Suppose $X:\Omega\rightarrow R$ is a continuous random variable and $h\left(X\right)$ conforms to Proposition 1. In that case, *X* exhibits the probability density function described by (5) if and only if there exist functions k and η , as defined in Theorem 5, that satisfy the following first-order differential equation.

$$\frac{\eta'\left(x\right)h\left(x\right)}{\eta\left(x\right)h\left(x\right)-k\left(x\right)}=\frac{\phi\left(x\right)}{1-\Phi\left(x\right)}$$

Proof: The process is straightforward and is therefore omitted.

Corollary 2: The general solution of the above differential equation in Corollary 1 is

$$\eta\left(x\right)=\left\{1-\Phi\left(x\right)\right\}^{-1}\left[-\int\phi\left(x\right)\left(h\left(x\right)\right)^{-1}k\left(x\right)+D\right],$$

Here, D is treated as a constant. A set of functions that satisfy this differential equation is introduced in Proposition 1 with $D=\frac{1}{2}$. It is evident that other sets of triplets $\left(h,k,\xi\right)$ meet the conditions outlined in Theorem 5.

5. Parameter Estimation of $TSN(\lambda)$ Distribution

5.1. Location and Scale Extension:

Location and scale extension of trimodal skew normal distribution is succeeding as. If $X\sim TSN\left(\lambda\right)$ then $Z=\mu+\sigma X$ is the location (μ) and scale (σ) extension of trimodal skew normal distribution and has the density function is given as

$$f\left(z;\mu,\sigma,\lambda\right)=\frac{1}{2}\left\{\left(\left(\frac{z_i-\mu}{\sigma}\right)^{2}-1\right)^{2}+2\right\}\phi\left(\frac{z_i-\mu}{\sigma}\right)\Phi\left(\lambda\frac{z_i-\mu}{\sigma}\right);Z\in R,\mu\in R,\lambda\in R,\sigma>0\qquad(16)$$

It is denoted as $Z\sim TSN(\mu,\sigma,\lambda)$.

5.2. Maximum Likelihood Estimation

If $z_1, z_2, ..., z_n$ was random sample drawn from the trimodal skew normal distribution, then log-likelihood function for $\theta = (\mu, \sigma, \lambda)$ was obtained as

$$l(\theta) = \sum_{i=1}^{n} \log\left(\left(\left(\frac{z_i-\mu}{\sigma}\right)^2 - 1\right)^2 + 2\right) - n\log 2 - \frac{n}{2}\log(2\,pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i-\mu)^2 + \sum_{i=1}^{n}\log\Phi\left(\lambda\frac{z_i-\mu}{\sigma}\right) \quad (17)$$

Differentiate equation (17) with respect to the parameters $\theta = (\mu, \sigma, \lambda)$, the likelihood equation becomes

$$\frac{\partial l(\theta)}{\partial\mu} = -\frac{4(u_i^2-\sigma^2)u_i^2}{\left\{(u_i^2-\sigma^2)^2 + 2\sigma^4\right\}} + \frac{1}{\sigma^2}\sum_{i=1}^{n} u_i - \frac{1}{\sigma^2}\sum_{i=1}^{n}\zeta\left(\lambda\frac{u_i}{\sigma}\right)$$

$$\frac{\partial l(\theta)}{\partial\sigma} = \frac{4(u_i^2-\sigma^2)u_i^2}{\left\{(u_i^2-\sigma^2)^2 + 2\sigma^4\right\}} - \frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n} u_i^2 - \frac{1}{\sigma^2}\sum_{i=1}^{n} u_i\,\zeta\left(\lambda\frac{u_i}{\sigma}\right)$$

$$\frac{\partial l(\theta)}{\partial\lambda} = \frac{1}{\sigma}\sum_{i=1}^{n} u_i\,\zeta\left(\lambda\frac{u_i}{\sigma}\right)$$

where, $u_i = z_i - \mu$ and $\zeta(.) = \frac{\phi(.)}{\Phi(.)}$. Numerical optimization routines were used to achieve a simultaneous solution for estimating the desired parameters. These routines employed computational algorithms to identify the optimal values that best fit the given data and model. By applying these methods, the parameters of interest were efficiently estimated, providing a solution that satisfied the desired criteria.

5.3. Fisher's Information Matrix

The following section presented the Fisher information matrix for the random variable $Z \sim TSN(\lambda)$ which could be expressed as

$$I = \left[E\left(-\frac{\partial^2 \log L}{\partial\theta_i\partial\theta_j}\right)\right], \quad i, \mathrm{j} = 1, 2, 3$$

The elements of the Fisher information matrix were calculated from the model using the set of parameters $(\theta_1, \theta_2, \theta_3) = (\mu, \sigma, \lambda)$, as

$$\frac{\partial^2 \log L}{\partial\mu^2} = \sum_{i=1}^{n}\left[\frac{8u_i^2}{a(\mu,\sigma)} - \frac{16u_i^2(u_i^2-\sigma^2)^2}{(a(\mu,\sigma))^2} - \frac{1}{\sigma^2} + \frac{4(u_i^2-\sigma^2)}{a(\mu,\sigma)} - \frac{\lambda^2}{\sigma^2}\zeta\left(\lambda\frac{u_i}{\sigma}\right) + \frac{\lambda^2}{\sigma^2}u_i\,\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial\sigma^2} = \sum_{i=1}^{n}\left[\begin{array}{r}\frac{8u_i^4}{\sigma^2 a(\mu,\sigma)} - \frac{16u_i^4(u_i^2-\sigma^2)^2}{\sigma^2(a(\mu,\sigma))^2} - \frac{3u_i^2}{\sigma^4} + \frac{12u_i^2(u_i^2-\sigma^2)}{\sigma^2 a(\mu,\sigma)} + \frac{n}{\sigma^2} + \frac{2\lambda}{\sigma^3}u_i\,\zeta\left(\lambda\frac{u_i}{\sigma}\right)\\ -\frac{\lambda^2}{\sigma^3}u_i^2\left(\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right)^2 + \frac{\lambda^2}{\sigma^4}u_i^2\zeta\left(\lambda\frac{u_i}{\sigma}\right)\end{array}\right]$$

$$\frac{\partial^2 \log L}{\partial\lambda^2} = \sum_{i=1}^{n}\left[-\frac{1}{\sigma^2}u_i^2\left(\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right)^2 + \frac{1}{\sigma^2}u_i^2\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial\mu\,\partial\sigma} = \sum_{i=1}^{n}\left[\begin{array}{r}\sum_{i=1}^{n}\frac{8u_i^3}{\sigma a(\mu,\sigma)} - \frac{16u_i^3(u_i^2-\sigma^2)^2}{\sigma(a(\mu,\sigma))^2} - \frac{u_i}{\sigma^3} + \frac{8u_i(u_i^2-\sigma^2)}{\sigma\,a(\mu,\sigma)} + \frac{\lambda}{\sigma^2}\zeta\left(\lambda\frac{u_i}{\sigma}\right) + \frac{\lambda^2}{\sigma^3}u_i\left(\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right)^2\\ + \frac{\lambda^2}{\sigma^3}u_i\,\zeta\left(\lambda\frac{u_i}{\sigma}\right)\end{array}\right]$$

$$\frac{\partial^2 \log L}{\partial\mu\,\partial\lambda} = \sum_{i=1}^{n}\left[-\frac{1}{\sigma}\zeta\left(\lambda\frac{u_i}{\sigma}\right) + \frac{\lambda}{\sigma^2}u_i\left(\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right)^2 - \frac{\lambda}{\sigma^3}u_i\,\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial\sigma\,\partial\lambda} = \sum_{i=1}^{n}\left[-\frac{1}{\sigma^2}u_i\,\zeta\left(\lambda\frac{u_i}{\sigma}\right) + \frac{\lambda}{\sigma^3}u_i^2\left(\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right)^2 - \frac{\lambda}{\sigma^3}u_i^2\zeta\left(\lambda\frac{u_i}{\sigma}\right)\right]$$

where, $u_i = z_i - \mu$ and $a(\mu,\sigma) = \left(\left((z_i-\mu)^2 - \sigma^2\right)^2 + 2\sigma^4\right)$

Since obtaining the closed-form expression for the elements of I was not tractable, these elements were approximated using numerical methods.

$$E\left(-\frac{\partial^2 \log L}{\partial\mu^2}\right) = \left(-\frac{\partial^2 \log L}{\partial\mu^2}\right)\Bigg|_{\mu=\hat{\mu},\sigma=\hat{\sigma},\lambda=\hat{\lambda}},$$

$$E\left(-\frac{\partial^2 \log L}{\partial\mu\,\partial\sigma}\right) = \left(-\frac{\partial^2 \log L}{\partial\mu\,\partial\sigma}\right)\Bigg|_{\mu=\hat{\mu},\sigma=\hat{\sigma},\lambda=\hat{\lambda}}$$

5.4. Simulation Study of the $TSN(\lambda)$

To evaluate the performance of the maximum-likelihood estimates for the parameters of the $TSN(\lambda)$ model, a simulation study was conducted using the Markov Chain Monte Carlo *(MCMC)* method. Specifically, the Metropolis-Hastings (M-H) algorithm, a widely used *MCMC* approach, was applied. Ten combinations of parameters were considered, and the process was repeated 1000 times for three different sample sizes, including $n = 100, 300$ and 500. For each generated sample, the likelihood function was optimized using the *GenSA*package (Version 4.2.0) in the R software, allowing the estimation of the model parameters.To assess the accuracy of these estimates, two statistical measures were calculated: biases and mean square errors (MSEs). These measures provided information on how closely the estimates approximated the true values and the degree of variability in the estimates. The formulas for calculating the biases and MSEs of the estimates are presented below

$$Bias(\hat{\theta}) = \mathrm{E}(\hat{\theta}) - \theta \quad \text{and} \quad MSE(\hat{\theta}) = V(\hat{\theta}) + \left\{Bias(\hat{\theta})\right\}^2$$

where, $\hat{\theta} = (\hat{\mu}, \hat{\sigma}, \hat{\lambda})$.The obtained results of the estimates were seen in Tables 1-2.

An examination of the data presented in Tables 1 and 2 indicated that the maximum likelihood estimators (MLEs) performed effectively in accurately estimating the model parameters. Furthermore, as the sample size increased, both the bias and mean square error of the MLEs decreased, as expected. These findings suggested that the MLEs provided consistent and reliable estimates for moderate and large sample sizes.

6. Real-Life Application

This section evaluated the performance of the tri-modal skew-normal distribution in fitting two real-life datasets. Its performance was compared with other distributions, including the normal, skew-normal, and alpha skew-normal distributions. The maximum likelihood estimation (MLE) method, implemented using the *GenSA* package in the R software, was applied to estimate the parameters of each distribution. To compare model performance, the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) were used as analytical measures. These criteria enabled the identification of the distribution that provided the best fit for the given datasets. This analysis assessed the ability of the tri-modal skew-normal distribution to capture the characteristics of real-life data relative to the other considered distributions.

6.1. Illustration I

Initially, the Environmental Performance Index (EPI) dataset was examined to assess the suitability of the proposed distribution. This dataset, which provided information on the environmental performance index of 163 countries in 2010, was obtained from the website http://epi.yale.edu/.

Table 3 presents the maximum likelihood estimates (MLEs), log-likelihood values, and AIC and BIC values for the different distributions considered. These measures were used to evaluate how closely each distribution fit the EPI dataset. Additionally, graphical representations of the results were provided in Figure 5, offering a visual comparison of how well the distributions captured the characteristics of the data.

Table 1: Results of simulation

$\mu = 0,\ \sigma = 1$							
		$\hat{\mu}$		$\hat{\sigma}$		$\hat{\lambda}$	
$\lambda\downarrow$	**n↓**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
	100	0.1056	0.0991	0.091	0.0752	-0.0789	0.0665
-1	300	-0.0921	0.0730	-0.0904	0.0453	-0.0653	0.0590
	500	0.0873	0.0640	0.0673	0.0399	0.0421	0.0323
	100	-0.0994	0.0546	0.1000	0.0733	0.0993	0.0756
-0.5	300	-0.0853	0.0621	-0.0906	0.0696	-0.0858	0.0743
	500	-0.0721	0.0432	-0.0434	0.0527	-0.0601	0.0693
	100	0.0821	0.0521	0.0661	0.0396	0.0656	0.0552
0	300	0.0354	0.0126	-0.0404	0.0421	-0.0444	0.0368
	500	-0.0296	0.0096	0.0301	0.0169	0.0134	0.0276
	100	-0.1300	0.0941	0.1401	0.0934	0.0856	0.0953
0.5	300	-0.0956	0.0521	-0.0921	0.0773	0.0621	0.0808
	500	0.0721	0.0421	-0.0667	0.0421	-0.0476	0.0757
	100	0.0920	0.0556	0.1050	0.0867	-0.0993	0.0866
1	300	0.0873	0.0421	0.0903	0.0725	0.0721	0.0727
	500	-0.0521	0.0301	-0.0672	0.0610	0.0560	0.0370

Table 2: Results of simulation

$\mu = 1,\ \sigma = 2$							
		$\hat{\mu}$		$\hat{\sigma}$		$\hat{\lambda}$	
$\lambda\downarrow$	**n↓**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
	100	0.1730	0.1312	-0.2192	0.1598	-0.1933	0.1461
-1	300	-0.9560	0.0993	0.1319	0.1212	-0.1651	0.1090
	500	-0.0922	0.0896	-0.1003	0.0901	-0.1131	0.0981
	100	0.1251	0.1310	-0.1250	0.1111	0.2152	0.1753
-0.5	300	-0.1191	0.0999	0.0970	0.0921	-0.1330	0.0956
	500	0.0943	0.0901	0.0921	0.0807	-0.1096	0.0881
	100	-0.1010	0.1210	0.2050	0.1993	0.1051	0.1651
0	300	0.0954	0.0903	-0.1531	0.1521	-0.0931	0.1059
	500	-0.0723	0.0665	-0.0940	0.1346	0.0931	0.1003
	100	0.1636	0.1219	-0.1616	0.1331	0.1999	0.2133
0.5	300	-0.1521	0.1310	0.1259	0.0953	-0.2051	0.1096
	500	0.0990	0.0413	-0.1032	0.0888	0.1010	0.1101
	100	0.2126	0.1139	-0.3301	0.2152	0.2031	0.1994
1	300	-0.1053	0.1410	-0.1682	0.1659	-0.1940	0.1531
	500	0.1064	0.9290	0.1093	0.0893	-0.1216	0.0949

Based on the results presented in Table 3, onehas determined that the studied distribution best fits the dataset when considering the AIC and BIC. Additionally, the density plots in Figure 6 visually supported our findings, showing that the studied distribution closely captured the patterns and characteristics observed in the dataset.Overall, this indicates that the studied distribution was the most suitable choice among the compared options for accurately representing the dataset.

Table 3:MLE's, log likelihood, AIC and BIC for EPI data

Distribution	Parameters				$logL$	AIC	BIC
	α	σ	μ	λ			
$N(\mu,\sigma)$,	--	12.371	58.371	--	-641.284	1286.569	1292.756
$SN(\mu,\sigma,\lambda)$,	--	14.086	51.633	0.749	- 641.217	1288.435	1297.716
$ASN(\mu,\sigma,\alpha)$	0.243	12.360	61.292	--	-641.275	1288.551	1297.832
$TSN(\mu,\sigma,\lambda)$	**--**	**9.298**	**67.633**	**-0.643**	**-637.394**	**1280.78**	**1290.06**

6.2. Illustration II

In the second example, a dataset originally presented by Frost (2002) and Bro *et al.* (2008) was examined. This dataset focused on nine different types of cream cheese and aimed to analyze various aspects related to sensory properties. The study investigated how changes in fat levels affected sensory attributes, compared two types of fat mimetics (protein-based and carbohydrate-based), and explored the impact of adding cream aroma on sensory properties beyond aroma and flavor. For this analysis, the E-shiny variable, consisting of 240 observations, was considered. The dataset is available at http://www.models.kvl.dk/Cream.

Table 4 presents the maximum likelihood estimates (MLEs), log-likelihood values, and AIC and BIC values for the studied distribution. These statistical measures were used to evaluate how well the distribution fit the cream cheese dataset. Additionally, Figure 7 provides visual representations of the results, offering a clearer understanding of the distribution's performance in capturing the characteristics of the dataset.

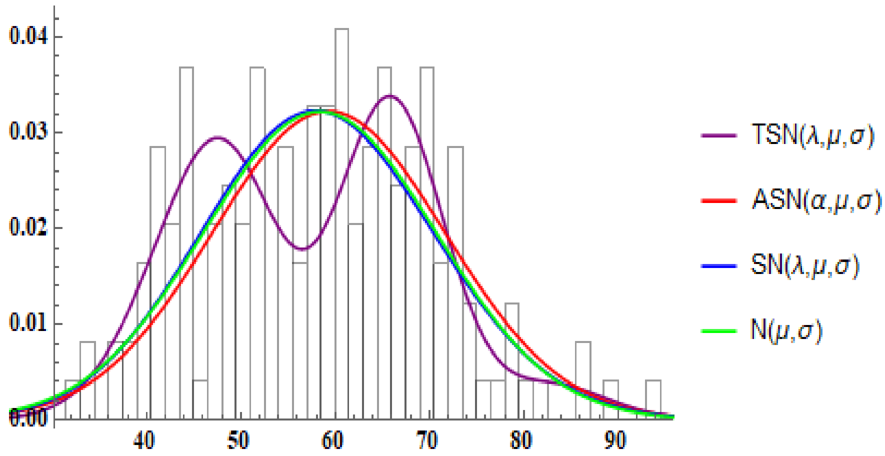

Figure 6: Plot of the fitted densities for EIP data of 163 observations

The results presented in Table 4 indicated that the studied distribution provided a better fit for the data compared to the other three distributions considered in this study. This conclusion was further supported by the findings in Figure 7, which visually demonstrated the superior performance of the studied distribution in capturing the characteristics of the dataset. Overall, these results provided strong evidence that the studied distribution accurately represented the data and outperformed the other distributions in terms of goodness-of-fit.

Table 4: MLE's, log likelihood, AIC and BIC for E- shiny data

Distribution	Parameters				$logL$	AIC	BIC
	α	σ	μ	λ			
$N(\mu,\sigma)$	--	2.411	8.129	--	-549.46	1102.94	1109.90
$SN(\mu,\sigma,\lambda)$	--	3.143	10.142	-1.346	-548.678	1103.36	1113.80
$ASN(\mu,\sigma,\alpha)$	-0.428	2.393	7.193	--	-549.305	1104.61	1115.05
$TSN(\mu,\sigma,\lambda)$	**--**	**1.765**	**6.217**	**0.639**	**-543.015**	**1092.03**	**1102.47**

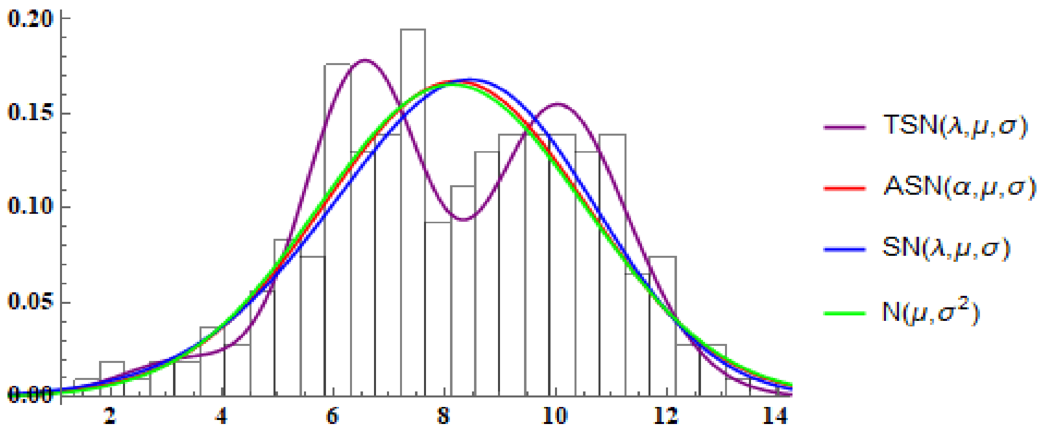

Figure 7: Plot of the fitted densities for E-shiny data of 240 observations

7. Conclusion

Here, we introduced a new type of distribution, referred to as the tri-modal skew-normal distribution. The estimation of this distribution was thoroughly investigated using the maximum likelihood method. Additionally, key characteristics and properties of the distribution were explored. The characterization of the proposed distribution was also discussed using the two truncated moment's method. The tri-modal skew-normal distribution exhibited greater flexibility and was capable of accommodating varying numbers of modes, as illustrated in Figure 1. To evaluate its performance, the distribution was applied to two real-life datasets and compared against other competing models. The results indicated that the tri-modal skew-normal distribution provided a superior fit to the data. Model comparison was conducted using two statistical criteria; the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC), both of consistently supportedthe tri-modal skew-normal distribution. Overall, this research highlighted the versatility of the tri-modal skew-normal distribution, examined its properties, and demonstrated its superior performance in modeling real-world data compared to alternative models.

The TSN distribution exhibited strong empirical and theoretical properties, providing a flexible framework for statistical modeling, especially when addressing skewed or multimodal data. This study examined the properties and applications of a newly proposed probability distribution. However, its inferential aspects were not addressed, providing an opportunity for future research to explore these features. Additionally, there is potential to investigate the logarithmic extension of this distribution, along with a detailed analysis of its properties and applications.

Appendix A

Data Set I: Environmental Performance Index

Data Set I provide the Environmental Performance Index (EPI) data for 163 countries, which serves as the basis for the analysis presented in Illustration I.

93.5, 89.1, 86.4, 86.0, 81.1, 80.6, 78.2, 78.1, 78.1, 76.8, 76.3, 74.7, 74.5, 74.2,73.4, 73.3, 73.2, 73.1, 73.0, 72.5, 72.5, 71.6, 71.4, 71.4, 70.6, 69.9, 69.8, 69.6, 69.4, 69.3, 69.3, 69.2, 69.1, 69.1, 68.7, 68.4, 68.3, 68.2, 68.2, 68.0, 67.8, 67.4,67.3, 67.1, 67.0, 66.4, 66.4, 65.9, 65.9, 65.7, 65.7, 65.6, 65.4, 65.0, 65.0, 64.6, 63.8, 63.7, 63.6, 63.5, 63.5, 63.4, 63.1, 62.9, 62.5, 62.4, 62.2, 62.0, 61.2, 61.0, 60.9, 60.8, 60.6, 60.6, 60.5, 60.4, 60.4, 60.0, 59.7, 59.6, 59.3, 59.2, 59.1, 59.1, 59.0, 58.8, 58.2, 58.1, 58.0, 57.9, 57.3, 57.3, 57.1, 57.0, 56.4, 56.3, 56.1, 55.9,55.3, 54.6, 54.4, 54.3, 54.2, 54.0, 54.0, 51.6, 51.4, 51.4, 51.3, 51.3, 51.3, 51.2,51.1, 51.1, 50.8, 50.3, 50.1, 49.9, 49.8, 49.2, 49.0, 48.9, 48.3, 48.3, 48.0, 47.9, 47.8, 47.3, 47.1, 47.0, 45.9, 44.7, 44.6, 44.6, 44.6, 44.4, 44.3, 44.3, 44.0, 43.9, 43.1, 42.8, 42.3, 42.3, 42.0, 41.9, 41.8, 41.7, 41.3, 41.0, 40.8, 40.7, 40.2, 39.6, 39.5, 39.4, 38.4, 37.6, 36.4, 36.3, 33.7, 33.3, 32.1

Appendix B

Data Set II: E-Shiny Data

The E-shiny dataset comprising 240 observations, which forms the basis for the analysis presented in Illustration II.

12.75, 12.9, 9, 10.35, 11.1, 12.3, 9.15, 5.85, 13.65, 10.5, 11.4, 10.95, 10.5, 10.95, 9.9, 6, 13.05, 12.3, 11.4, 12.6,10.95, 11.85, 10.5, 6.75, 10.05, 10.95, 10.8, 10.2, 8.1, 7.95, 7.5, 4.95, 7.95, 10.8, 8.4, 9.3, 5.7, 10.8, 6.15, 7.35, 6.6, 6, 7.2,8.55,7.05, 9.75, 8.1, 6, 6.45,6.6, 3.63, 5.25, 4.05, 5.25, 3.3, 6.9, 7.05, 4.95, 1.8, 4.5, 5.85, 4.8, 4.35, 2.85, 5.85, 5.4, 5.55, 4.65, 5.25, 4.05, 4.8, 6.45, 8.4, 6.6, 9.45, 6.9, 6.75, 6.3, 3.75, 7.95, 7.8, 5.85, 9.15, 6.15, 9.15, 5.1, 5.1, 6, 9.15, 6.3, 6.45, 6.9, 8.55, 7.2, 5.55, 7.2, 9.9, 7.35, 9.3, 6.3, 7.2, 5.85, 8.7, 8.25, 7.35, 7.2, 8.1, 6.6, 9.3, 6.45, 7.35, 6.9, 7.95, 7.0, 9.45, 7.5, 7.2, 7.05, 6.75, 10.5, 10.35, 7.2, 9, 8.7, 10.65, 6.6, 5.25, 9.3, 9.45, 8.85, 8.4, 8.55, 9.6, 7.2, 5.7, 11.1, 7.35, 8.85, 7.95, 7.2, 9.9, 8.25, 7.35, 10.95, 11.1, 9, 10.35, 9.3, 10.2, 8.1, 6.3, 10.2, 11.85, 8.4, 10.2, 7.65, 10.35, 7.95, 6.1, 9.45, 10.5, 10.2, 9.75, 11.25, 10.5, 7.05, 6.15, 6.9, 8.7, 5.85, 1.65, 4.5, 7.05, 4.8, 1.95, 6.15, 8.7, 6.15, 2.4, 3.9, 6.15, 7.2, 5.1, 5.7, 6.75, 5.85, 3.15, 5.4, 5.7, 5.85, 3, 10.95, 11.85, 9.15, 11.1, 9.3, 11.4, 8.7, 7.2, 12, 11.7, 8.55, 10.5, 10.5, 11.85, 9.45, 9.9, 11.1, 11.55, 9.6, 8.85, 9.75, 12, 6.6, 9.15, 12.15, 11.85,7.8, 11.25, 9.6, 9.6, 8.4, 7.5, 10.5, 11.55, 8.7, 9.9, 9.9, 9.6, 7.8, 9.45, 10.8, 9.9, 10.05, 8.85, 9.45, 10.05, 8.4, 7.5

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