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The Beta-Weibull-X Family of Distributions with some Properties and Applications to Engineering and Health Data when X $\sim$ Rayleigh Distribution

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Abstract

Generating new statistical distributions that provide sufficient characterization for real-life phenomena such as those in reliability engineering, meteorology, and health sciences is an important concern for researchers. Many complex real-life phenomena have yet to be optimally characterized by some existing method, and this study has proposed the Beta Weibull-X (BWei-X) family. The Beta-Weibull-Rayleigh (BWR) distribution was developed by taking X as a random variable from the classical Rayleigh distribution; some notable distributions in the literature were obtained as special cases. Moments and some basic statistical properties, including the numerical computations for mean, variance, skewness, and kurtosis of the BWR distribution, were investigated with an extension to the statistical property for order statistics. Distribution parameters were estimated using the maximum likelihood estimation method and the procedure was evaluated by a simulation study. Graphical reports show that failure rates can be decreasing or increasing, J-shaped, bathtub, and inverted bathtub, making it an exciting tool in diverse areas of applications for modeling noisy phenomena with left-skewed, right-skewed, and approximately symmetric features. Engineering data and systolic blood pressure data sets were applied to investigate the performance of the model, the findings of the study including the results of data analysis using R software justified the importance of the investigation.

Key Words: Beta-Weibull-X Family; Beta-Weibull-Rayleigh Distribution; Data Analysis; Inverted Bathtub Shape; Simulations; Order Statistics; Statistical Property for Order Statistics; Systolic Blood Pressure.

1. Introduction

Generating a new family of distributions is a continuous and important area of research in probability and statistics, and this has recently been reiterated in the work of Gómez-Déniz et al. (2023), and more than two decades ago Eugene et al. (2002) used the beta distribution as a generator to propose a generalized family of distributions, and therein introduced the beta-normal distribution as a special case, and Famoye et al. (2004) extended the study by investigating the bimodality properties of the beta-normal distribution.

Many important distributions with various applications were developed including beta-Weibull (BW) and beta-Gumbel by (Famoye et al. 2005) and (Nadarajah and Kotz 2004) respectively. Some other contributions to the area are; (see

Nadarajah and Kotz (2006), Akinsete et al. (2008), Barreto-Souza et al. (2010), Cordeiro et al. (2013),Shittu and Adepoju (2013)). More compound distributions have been proposed; Handique and Chakraborty (2016) introduced the beta-generated Kumaraswamy-G, from which Madaki et al. (2022) developed the beta-Kumaraswamy-Burr type X distribution. Alzaatreh et al. (2013b) introduced the beta exponential-G (BEx-G) family from which the beta exponential-Fréchet (BExFr) was developed by Mead et al. (2017). The beta power muth (BPM) was proposed by Benkhelifa (2022) Rashwan and Kamel (2020) generalized exponential Pareto and introduced beta-exponential Pareto (BEP), but Adeyemi et al. (2023) generalized exponential Pareto and introduced the Weibull-Exponential Pareto (WEP) distribution using the Weibull-X family. The cumulative distribution (CDF) and probability density function (PDF) of the Weibull-X family introduced by Alzaatreh and Ghosh (2015) is given by

$$G(x) = 1 - exp\left(-\left(-\frac{log(1-F_1(x;\psi))}{\gamma}\right)^{\alpha}\right), \quad \alpha,\gamma > 0, x > 0 \tag{1}$$

$$g(x) = \frac{\alpha f_1(x;\psi)}{\gamma(1-F_1(x;\psi))}\left(-\frac{log(1-F_1(x;\psi))}{\gamma}\right)^{\alpha-1} exp\left(-\left(-\frac{log(1-F_1(x;\psi))}{\gamma}\right)^{\alpha}\right), \tag{2}$$

Where $\alpha > 0$ is shape parameter, and $\gamma > 0$ a scale parameter to add flexibility to the new model. ψ is the parameter vector of baseline cdf $F_1(x;\psi)$.

The Weibull-Pareto by Alzaatreh et al. (2013a) and Weibull-Rayleigh by Akarawak et al. (2013) belong to the Weibull-X. When X is the logistic distribution, Alzaatreh and Ghosh (2015) developed the Weibull-Logistics distribution, and by taking X to be the Rayleigh distribution, Ahmad et al. (2017) developed the Weibull-Rayleigh distribution. Concerning application to real-life datasets, the lifetime dataset from Aarset (2009) has been analyzed by several authors in an attempt to provide a suitable model, including using the beta-linear failure rate (BLFR) distribution by Jafari and Mahmoudi (2012), beta Gompertz (BGo) by Jafari et al. (2014), beta generalized Gompertz (BGGo) by Benkhelifa (2017), and the beta-exponential Pareto distribution Rashwan and Kamel(2020).

The objective of this study is to provide alternative and an improved solutions to some problems associated with analysis of complex phenomena exhibited in form of skewness, kurtosis, and tail weights of some lifetime datasets. To develop a family of distribution that can be explored to characterize order statistics of many lifetime distributions, and to derive the Beta Weibull-Rayleigh (BWR) distribution that can describe events having different shapes including bathtubs and inverted bathtubs for modeling lifetime datasets with noisy behaviors. The beta distribution has parameters that can moderate the tail weights, thus adding flexibility; the Weibull and Rayleigh distributions are useful in analyzing lifetime data in survival and reliability analysis, hydrology, and clinical research.

The remaining part of the study is organized as follows; the Beta-Weibull-X family is introduced in Section 2, and by taking X as the Rayleigh random variable, the Beta-Weibull-Rayleigh is developed and the special cases are stated. Section 3 contains some properties of the proposed model, the method of estimation is given in Section 4; two applications to real-life datasets are presented in Section 5; Section 6 is for discussion, and Section 7 is the concluding remark.

2. Material and method

Here are some important sub-sections:

2.1. Beta-Generalized Method

Eugene et al. (2002) used the beta distribution as a generator and proposed the Beta-generated technique (Beta-G) with cdf defined for a random variable X as

$$\mathscr{F}(x;a,b) = \frac{1}{B(a,b)}\int_0^{G(x)} b(t;a,b)dt, \qquad a > 0, b > 0 \tag{3}$$

$G(x)$ is the cdf of any continuous distribution and $b(t;a,b)$ is the density function of the beta distribution with the parameters a and b. Using the quantile transformation $F^{-1}(u) = t$ and by replacing t with $G(x)$, the pdf of the Beta-G family of distribution is given by

$$f(x;a,b)=\frac{1}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1}g(x), \qquad x>0; a>0, b>0 \tag{4}$$

$B(.)$ is the Beta function while a and b are additional shape parameters that can add flexibility to the new distributions and $g(x)$ is the pdf of the baseline distribution.

2.2. Beta-Weibull-X (BWei-X) Family of Distribution

The cdf and pdf of Weibull distribution for a random variable $T>0$ and positive parameters α,β can be defined respectively as

$$G_{Weibull}(t;\alpha,\beta)=1-exp(-\beta t^{\alpha}); \qquad g_{Weibull}(t;\alpha,\beta)=\alpha\beta t^{\alpha-1}exp(-\beta t^{\alpha}) \tag{5}$$

Bourguignon et al. (2014) developed the Weibull-G family of distribution using the cdf in equation (5) as the generator by replacing t with $\frac{G(x)}{1-G(x)}$. Merovci and Elbatal (2015) employed equation (5) to generate the Weibull-Rayleigh distribution, and the three-parameter Weibull-Pareto distribution was derived using the equation (5) taking $\beta=1$ by Tahir et al. (2016).

By using the Weibull cdf in equation (5) in this research as the generator, and replacing t by $-log(1-F_1(x))$ as defined by Alzaatreh and Ghosh (2015); the cdf and pdf of Weibull-X is hereby derived and given respectively by

$$G(x)=1-exp\left(-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right) \tag{6}$$

$$g(x)=\frac{\alpha\beta f_1(x;\psi)}{(1-F_1(x;\psi))}\left(-log(1-F_1(x;\psi))\right)^{\alpha-1}exp\left(-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right) \tag{7}$$

where $\alpha>0$ is the shape parameter, $\beta>0$ is the scale parameter and $F_1(x;\psi)$ is the cdf of a baseline distribution with the parameter vector ψ

Remark: By setting the parameter $\beta=\frac{1}{\gamma^{\alpha}}$, the new result obtained in equation (6) reduces to the Weibull-X in equation (1) derived and studied by Alzaatreh and Ghosh (2015).

A new family of distribution called Beta-Weibull-X is developed using the beta distribution to generalize the Weibull-X derived in (6). The cdf of BWei-X is constructed and given by

$$\begin{aligned}\mathscr{F}(x;a,b,\alpha,\beta)&=\frac{1}{B(a,b)}\int_0^{1-exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\}}t^{a-1}(1-t)^{b-1}dt\\&=\frac{1}{B(a,b)}B\left(1-exp\left(-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right);a,b\right)\end{aligned} \tag{8}$$

The corresponding pdf of BWei-X is given by

$$f(x)=\begin{cases}\frac{1}{B(a,b)}\frac{\alpha\beta f_1(x;\psi)}{(1-F_1(x;\psi))}\left[1-exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\}\right]^{a-1}\\ \times\left[exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\}\right]^{b}\left(-log(1-F_1(x;\psi))\right)^{\alpha-1}\end{cases} \tag{9}$$

Where $x>0$ and $a,b,\alpha>0$ are the shape parameters, and $\beta>0$ the scale parameter of the new model; $F_1(x;\psi)$ and $f_1(x;\psi)$ are the cdf and pdf of the baseline distribution respectively with parameter vector ψ. If X is a random variable having the pdf in equation (9), we write $X\sim$ BWei-X(a,b,α,β,ψ). The incomplete beta function ratio is $I_z(a,b)=B_z(a,b)/B(a,b)$ and the incomplete beta function is defined by

$$B_z(a,b)=\int_0^z t^{a-1}(1-t)^{b-1}dt$$

The survival function of the BWei-X is

$$S(x)=\frac{B(a,b)-B\left(1-exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\};a,b\right)}{B(a,b)},\quad \alpha,\beta,x>0 \tag{10}$$

The hazard rate, $h(x)$ function of the BWei-X is

$$\frac{\left[1-exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\}\right]^{a-1}\left[exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\}\right]^{b}h_{Wei-X}(x)}{B(a,b)-B\left(1-exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\};a,b\right)} \tag{11}$$

The reversed hazard rate $rh(x)$ for the BWei-X$(a,b,\alpha,\beta,\psi^T)$ is

$$\frac{\left[1-exp\left\{-\beta\left(-log(1-F(x;\psi))\right)^{\alpha}\right\}\right]^{a-1}\left[exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\}\right]^{b}h_{Wei-X}(x)}{B\left(1-exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\};a,b\right)} \tag{12}$$

where $h_{Wei-X}(x)$ is the hazard function of the Weibull-X distribution in (6) given by,

$$h_{Wei-X}(x)=\frac{G(x)}{1-G(x)}=\frac{\alpha\beta f_1(x;\psi)}{(1-F_1(x;\psi))}\left(-log(1-F_1(x;\psi))\right)^{\alpha-1}$$

$a,b>0$ are beta parameters and $\alpha,\beta>0$ are the Weibull-X parameters, ψ denotes the parameter space of the baseline model for for $X>0$

The Quantile function of random variable X from the BWei-X can be obtained by inverting the cdf of BWei-X where $Q(u)=\mathscr{F}^{-1}(u)$, the cdf in (8) can be written as

$$\mathscr{F}(x;a,b,\alpha,\beta,\psi)=\frac{1}{B(a,b)}B\left(z;a,b\right),\quad a,b,\alpha,\beta,\psi;x>0 \tag{13}$$

where $z=\left(1-exp\left\{-\beta\left(-log(1-F_1(x;\psi))\right)^{\alpha}\right\}\right)$.

Let $Q_{a,b}(u)$ be the quantile function of the beta distribution given that U is a uniform distribution random variable on the unit interval; the BWwei-X quantile function can be written as

$$Q(u)=\mathscr{F}^{-1}\left(1-e^{-[-\frac{1}{\beta}log(1-Q_{a,b}(u))]^{\frac{1}{\alpha}}}\right),0<u<1 \tag{14}$$

Many important lifetime distributions can be constructed and investigated with application to different real-life datasets from equation (8) such as the Beta-Weibull Rayleigh (BWR), Beta-Weibull-Exponential (BWE), Beta-Weibull-Pareto (BWP), Beta-Weibull-Weibull (BWW), Beta-Weibull-Inverse Exponential (BWIE), by taking X to be a random varible with the baseline cdf $F_1(x;\psi)$.

The BWE distribution for example can be constructed by taking X to be the exponential distribution random variable with the cdf given by$F_1(x;\lambda)=1-e^{-\lambda x}$ where $\lambda>0$ is the exponential rate parameter.

2.3. The Five-Parameter Beta-Weibull-Rayleigh (BWR) Distribution

The random variable X that follows the Rayleigh distribution with parameter $\sigma>0$ has the pdf and cdf defined by $f_1(x;\psi)=\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$ and $F_1(x;\psi)=1-e^{-\frac{x^2}{2\sigma^2}}$ respectively, the cdf of the BWR is constructed by substituting the cdf of the Rayleigh distribution into equation (8) and is given by

$$\begin{aligned}\mathscr{F}(x;a,b,\alpha,\beta,\sigma) &= \frac{1}{B(a,b)}\int_0^{1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}} t^{a-1}(1-t)^{b-1}dt \\ &= \frac{1}{B(a,b)}B\left(1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\};a,b\right),\quad a,b,\sigma,x>0\end{aligned} \tag{15}$$

The corresponding pdf of BWR is given by

$$f(x) = \frac{1}{B(a,b)}\left[1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}\right]^{a-1}\left[exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}\right]^b\frac{\alpha\beta x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1} \tag{16}$$

where $a,b,\alpha>0$ are shape parameters; $\sigma,\beta,>0$ are scale parameters and $x>0$. The random variable X that follows the BWR with the pdf in equation (16) is denoted by $X\sim \text{BWR}(a,b,\alpha,\beta,\sigma)$.

The survival function of the $\text{BWR}(a,b,\alpha,\beta,\sigma)$ is

$$S(x) = 1-\mathscr{F}(x) = \frac{B(a,b)-B\left(1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\};a,b\right)}{B(a,b)},\ a,b,\alpha,\beta,\sigma>0 \tag{17}$$

The hazard rate, $h(x)$ function of the $\text{BWR}(a,b,\alpha,\beta,\sigma)$ is $h(x)=\frac{f(x)}{1-\mathscr{F}(x)}$,

$$h(x) = \frac{\left[1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}\right]^{a-1}\left[exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}\right]^b\frac{\alpha\beta x}{\sigma^2}\left(\frac{x^2}{2\sigma 2}\right)^{\alpha-1}}{B(a,b)-B\left(1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\};a,b\right)} \tag{18}$$

The reversed hazard rate $rh(x)$ function of the BWR is $rh(x)=\frac{f(x)}{\mathscr{F}(x)}$,

$$rh(x) = \frac{\left[1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}\right]^{a-1}\left[exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}\right]^b\frac{\alpha\beta x}{\sigma^2}\left(\frac{x^2}{2\sigma 2}\right)^{\alpha-1}}{B\left(1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\};a,b\right)} \tag{19}$$

2.4. The Four-Parameter Beta-Weibull-Rayleigh Distribution

Let T be a Weibull random variable defined in equation (5) with the scale parameter $\beta=1$, and shape parameter $\alpha>0$, the cdf is $1-exp(-t^\alpha)$, and the cdf of the $\text{BWR}(a,b,\alpha,\beta,\sigma)$ in equation (15) reduces to the cdf of $\text{BWR}(a,b,\alpha,\sigma)$ given by

$$\mathscr{F}(x;a,b,\alpha,\sigma) = \frac{1}{B(a,b)}\int_0^{1-exp\left\{-\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}} t^{a-1}(1-t)^{b-1}dt,\qquad a,b,\alpha,\sigma>0;x>0 \tag{20}$$

The corresponding pdf of the $\text{BWR}(a,b,\alpha,\sigma)$ is given by

$$f(x) = \frac{1}{B(a,b)}\left[1-exp\left\{-\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}\right]^{a-1}\left[exp\left\{-\left(\frac{x^2}{2\sigma^2}\right)^\alpha\right\}\right]^b\frac{\alpha x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1} \tag{21}$$

where $a,b,\alpha>0$ are shape parameters; $\sigma>0$ is a scale parameter, and $x>0$.

The random variable X that follows the BWR with the pdf as defined in equation (21) is denoted by $X\sim \text{BWR}(a,b,\alpha,\sigma)$.

The density function of the $\text{BWR}(a,b,\alpha,\sigma)$ can be written in another form using appropriate mathematical expansion for the expression.

The popular binomial series expansion is given by;

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

, and by using the gamma function for positive integers;

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}.$$

Equation (21) can be simplified using a combination of techniques(see (page 14, Ademola et al. 2021) and (page 44, Mead et al. 2017.)
If b is a positive real non-integer and $|z| < 1$, then authors in Gradshteyn and Ryzhik (2014) established that

$$\left(1-z\right)^{b-1} = \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} z^i, |z| < 1, b > 0 \tag{22}$$

If b is a positive real integer, then the upper limit of the summation stops at $b-1$.
where

$$\binom{b-1}{i} = \frac{\Gamma(b)}{\Gamma(i+1)\Gamma(b-i)}.$$

After the implementation of (22) in (21), the pdf becomes

$$f(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty}(-1)^i \binom{a-1}{i} exp\left\{-(b+i)\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\} \frac{\alpha x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1} \tag{23}$$

By applying exponential power series,

$$f(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{j!} \binom{a-1}{i} \left(b+i\right)^j \left(\frac{x^2}{2\sigma^2}\right)^{\alpha j} \frac{\alpha x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1} \tag{24}$$

$$f(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{j!} \binom{a-1}{i} \left(b+i\right)^j \frac{\alpha x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha(j+1)-1} \tag{25}$$

Equation (23) can be expressed as;

$$f(x) = \Psi_i h_{b+i}(x) = \Psi_i \frac{\alpha(b+i)x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1} exp\left\{-(b+i)\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\} \tag{26}$$

where

$$\Psi_i = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{(b+i)} \binom{a-1}{i}, \quad and \quad h_{b+i}(x) = \frac{\alpha(b+i)x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1} exp\left\{-(b+i)\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}$$

The result in (26) is an indication that the pdfs of the BWR can be expressed as a mixture of linear combinations of the Weibull-Rayleigh densities; and when $\alpha = 1$, as Rayleigh densities where $h_{b+i}(x)$ is the pdf of the Rayleigh distribution that has a scale parameter $\sigma^{-1}(b+i)$.

The possible shapes of the BWR distribution are visualized from the plots of the pdfs for different arbitrary values of the parameters (see Figure 1).
The Figures showed that the BWR has a variety of shapes, making it an exciting tool for application to lifetime datasets that are positively or negatively skewed and some natural phenomena that are approximately symmetric. Actuarial and insurance phenomena that are heavy-tailed can be fitted to the distribution. Other areas of important applications can be found in engineering, meteorology, hydrology, and health sciences.

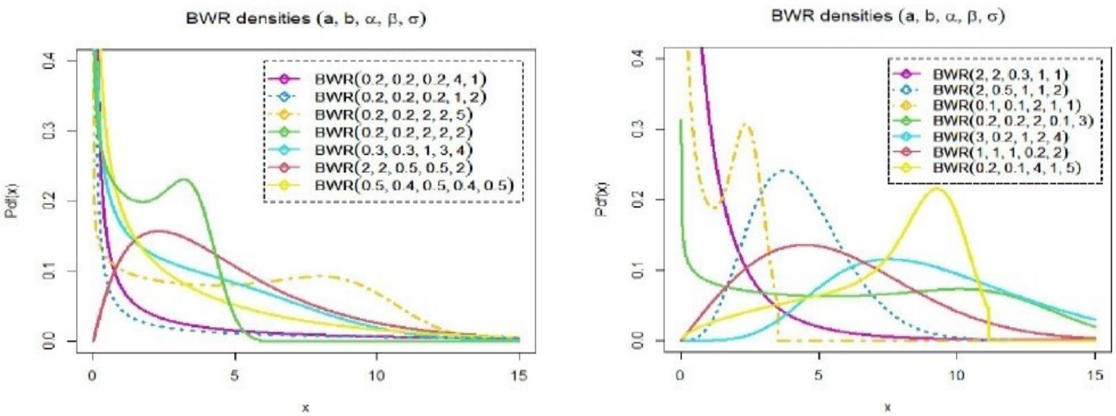

Figure 1: Plots for the Beta-Weibull-Rayleigh density for some arbitrary values of the parameters.

The graphical structure of the BWR hazard rate function is investigated and the plots is displayed in Figure 2, which established that the BWR distribution has constant, decreasing and increasing failure rate, the new distribution also has the J-shape and reversed J-shape, bathtub, and inverted bathtub shapes.

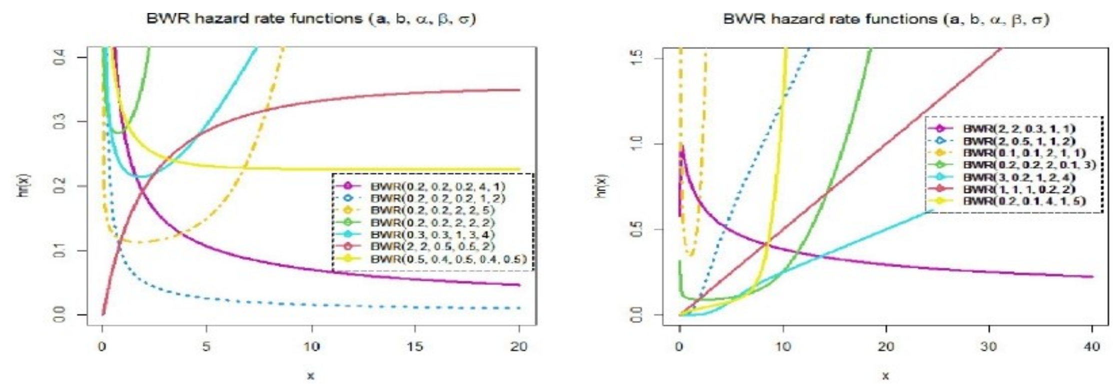

Figure 2: Plots for the BWR hazard rate function for arbitrary values of the parameters.

Figure 3 is the plots for the cdfs of the BWR for different values of the parameters, which shows that the characteristics of the cdf has distinct identifiable parameters (Hennig 2024; Maclaren and Nicholson 2019).

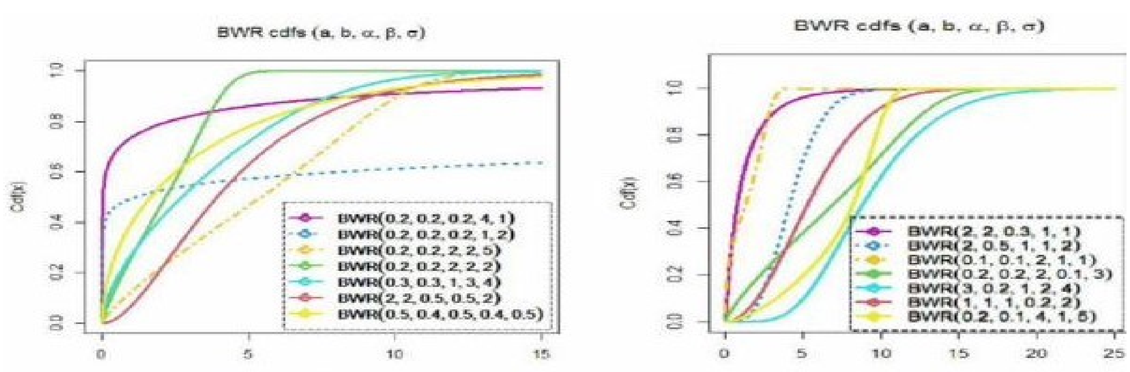

Figure 3: Plots for the Beta-Weibull-Rayleigh cdfs for some arbitrary values of the parameters.

The new five-parameter distribution generalized some lifetime distributions existing in the literature as sub-models of the Beta-Weibull-Rayleigh distribution and reduced to a four-parameter BWR distribution by setting the parameter $\beta = 1$. Some notable probability distributions from the BWR model are presented in Table 1.

Table 1: Sub-models of the BWR Distribution.

Sub-Model	The Parameters a	b	α	β	σ	Author
Beta-Weibull-Rayleigh	–	–	–	1	–	new model (2025)
Generalized Rayleigh	–	1	1	1	$\sqrt{2\lambda}$	Shen et al. (2022)
Weibull-Rayleigh	1	1	–	–	–	Ahmad et al. (2017)
Exponentiated-Rayleigh	–	1	1	$\frac{\gamma^{\alpha}}{2}$	1	Mahmoud and Ghazal 2017
Beta-Rayleigh	–	–	1	1	–	Akinsete and Lowe(2009)
Exponentiated-Exponential	–	1	$1/2$	$(2\lambda^2)^{\alpha}$	1	Nadarajah (2011)
Beta exponential	–	–	$1/2$	$(2/\lambda)^{\alpha}$	1	Nadarajah and Kotz (2006)
Exponentiated-Weibull	–	1	$\frac{c}{2}$	$(2/\lambda^2)^{\alpha}$	1	Pal et al. 2006
Beta-Weibull	–	–	$\frac{c}{2}$	$\sqrt{2\gamma}$	1	Famoye et al. (2005)

The table revealed that the BWR has the potential that can be used to characterize the distributions in Table 1

3. Basic Statistical Properties of BWR Distribution

Some statistical properties of the BWR distribution, such as the moment generating function, mean, variance, and order statistics, are investigated and presented in this section.

3.1. Mathematical Expansion of the Density Function

Theorem 3.1. *The cdf of the BWR distribution can be expressed as a linear combination of the Weibull-Rayleigh model given by*

$$\mathscr{F}(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{a+i} \binom{b-1}{i} \binom{a+i}{j} exp\left\{-\beta j \left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\} \tag{27}$$

Proof of Theorem 1. The BWR cdf is given by

$$\mathscr{F}(x;a,b,\alpha,\beta,\sigma) = \frac{1}{B(a,b)} \int_0^z t^{a-1}(1-t)^{b-1} dt \tag{28}$$

Using the expansion process in (22) $\forall\, b > 0$, stated as

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} z^i, |z| < 1, b > 0$$

$$\mathscr{F}(x;a,b,\alpha,\beta,\sigma) = \frac{1}{B(a,b)} \sum_{i=0}^{b-1} (-1)^i \binom{b-1}{i} \int_0^z t^{a+i-1} dt \tag{29}$$

for $z = 1 - exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}$; then

$$\int_0^z t^{a+i-1} dt = \frac{\left[1 - exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}\right]^{a+i}}{a+i} \tag{30}$$

substitute equation (30) into equation (29) to get

$$\mathscr{F}(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{a+i} \binom{b-1}{i} \left(1 - exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}\right)^{a+i} \tag{31}$$

If $a > 0$, and equation (22) is satisfied, the expansion of (31) gives

$$\mathscr{F}(x) = \frac{1}{B(a,b)} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{a+i} \binom{b-1}{i} \binom{a+i}{j} exp\left\{ -j\beta \left(\frac{x^2}{2\sigma^2} \right)^{\alpha} \right\} \tag{32}$$

equation (32) completes the proof of Theorem 1. □

Theorem 2. *The pdf of the five-parameter BWR distribution can be expressed as a linear combination of the Weibull-Rayleigh distribution studied by [4], if* $\beta = \frac{1}{\gamma^\alpha}$ *and the expression is given by*

$$f_{WR}(x) = \Psi_i h_{b+i}(x) \tag{33}$$

where $h_{b+i}(x)$ is the Weibull-Rayleigh density function derived as

$$h_{b+i}(x) = \frac{\alpha\beta(b+i)x}{\sigma^2} \left(\frac{x^2}{2\sigma^2} \right)^{\alpha-1} exp\left\{ -\beta(b+i) \left(\frac{x^2}{2\sigma^2} \right)^{\alpha} \right\}; b,\alpha,\beta,\sigma > 0 \tag{34}$$

and

$$\Psi_i = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{(b+i)} \binom{a-1}{i};$$

$h_{b+i}(x)$ has α as a shape parameter, and the scale parameter is $\beta(b+1)$ and σ.
Proof of Theorem 2: The solution can be obtained by following the procedure in Subsection (2.4).

3.2. Quantile Function and Moments of BWR Distribution

The quantile function (QF) of a distribution is the Inverse Distribution Function (IDF).
Lemma 1. *T*he quantile function of the beta generalized class of distribution (BG) with a baseline distribution having cdf denoted by $G(x)$ is given by

$$Q_{BG}(p) = Q_G(I_p^{-1}(a,b)) \tag{35}$$

where $Q_G(.)$ is the QF of the baseline distribution.
Proof of Lemma 1. Let $b(t;a,b)$ be the beta distribution with parameters $a > 0$ and $b > 0$, the cdf is

$$F_{beta}(x) = I_t(a,b) = \frac{\int_0^x t^{a-1}(1-t)^{b-1}}{B(a,b)} dt,$$

and the pdf is given by

$$f_{beta}(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$$

By replacing t with an arbitrary cdf, (see (Eugene et al. 2002) and (Sharma and Chakrabarty 2017)); the pdf of BG class is

$$f_{BG}(x) = \frac{1}{B(a,b)} [G(x)]^{a-1} [1-G(x)]^{b-1} gx; \quad 0 \leq G(x) \leq 1$$

The cdf of BG for the baseline distribution with cdf G(x) is

$$F_{BG}(x) = I_{G(x)}(a,b) \tag{36}$$

Sharma and Chakrabarty (2017) established the quantile function for the BG class in equation (36), where $Q_G(.)$ is the QF of $G(.)$ and is given by (37).

$$Q_{BG}(p) = Q_G(I_p^{-1}(a,b)) \tag{37}$$

and $I_p^{-1}(a,b) = Q_{a,b}(p)$ is the QF of the beta random variable with parameters $a,b > 0$ and $0 < p < 1$.

Theorem 3. *Let X be a random variable of the BWR distribution, the QF denoted $Q_{BWR}(p)$ is derived and given by*

$$Q_{BWR}(p) = \sigma\sqrt{2}\left[-\frac{1}{\beta}log\left(1-I_p^{-1}(a,b)\right)\right]^{\frac{1}{2\alpha}} \tag{38}$$

Proof. The proof is obtained by taking the IDF of the Weibull-Rayleigh distribution and then applying Lemma 1. The cdf of WR is given by

$$G(x) = 1-exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}; \qquad \alpha,\beta,\sigma>0 \tag{39}$$

Let $u = G(x)$ such that $x = G^{-1}(u)$, then by inverse transformation of variable; the QF of WR can be derived from (39), and is given by

$$Q_{WR}(u) = \sigma\left[2\left(-\frac{1}{\beta}log(1-u)\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{2}} \tag{40}$$

By using equation (40), the QF of BWR is

$$Q_{BWR}(p) = Q_{WR}(I_p^{-1}(a,b)) \tag{41}$$

And the proof is completed by application of Lemma 1 to get

$$Q_{BWR}(p) = \sigma\left[2\left(-\frac{1}{\beta}log(1-I_p^{-1}(a,b)\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{2}} \tag{42}$$

□

Suppose T is a beta random variable with parameters a and b in $b(t;a,b)$, the BWR variate can be generated using the quantile function for BWR variable X given by

$$X = \sigma\left[2\left(-\frac{1}{\beta}log(1-T)\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{2}} \tag{43}$$

The moment of BWR is investigated starting with Theorem 4

Theorem 4. *Let X be a random variable from the BWR distribution, the t^{th} non-central moment denoted by $\mu^{(t)}$ is given by*

$$\mu^{(t)} = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\sigma^t(2)^{\frac{t}{2}}\left(\frac{1}{\beta(b+i)}\right)^{\frac{t}{2\alpha}}\Gamma\left(\frac{t}{2\alpha}+1\right) \tag{44}$$

where Γ is the gamma function; and $a,b,\alpha,\sigma,\beta>0$ are the parameters.

Proof of Theorem 4: The t^{th} non- central moment of a random variable X is defined by

$$\begin{aligned}\mu^{(t)} &= \int_0^{\infty} x^t f(x)dx \\ &= \int_0^{\infty}\frac{x^t}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\left[exp\left\{-\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}\right]^{b+i}\frac{\alpha\beta x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1}dx\end{aligned} \tag{45}$$

Let $y = \beta(b+i)\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}$ by the transformation of the variable and after simplification

$$\begin{aligned}\mu^{(t)} &= \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\int_0^{\infty}\left(\sigma\sqrt{2}\left(\frac{1}{\beta(b+i)}\right)^{\frac{1}{2\alpha}}y^{\frac{1}{2\alpha}}\right)^t e^{-y}dy \\ &= \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\sigma^t(2)^{\frac{t}{2}}\left(\frac{1}{\beta(b+i)}\right)^{\frac{t}{2\alpha}}\int_0^{\infty}y^{\frac{t}{2\alpha}}e^{-y}dy\end{aligned} \tag{46}$$

Using the identity $\int_0^\infty y^t e^{-y} = \Gamma(t+1)$ completes the proof □

Corollary 1. The result of Theorem 4 can be used to deduce the expected value of the Weibull-Rayleigh distribution studied by Ahmad et al. (2017).
Proof of Corollary 1. The result can be easily obtained by relaxing the beta distribution parameters a,b by setting $a = b = 1$ and $\beta = \frac{1}{\gamma^\alpha}$. The expected value of the Weibull-Rayleigh distribution by Ahmad et al. (2017) is derived and given by

$$\mu = \sigma\sqrt{2\gamma}\Gamma\left(\frac{1}{2\alpha}+1\right) \tag{47}$$

Note that $t = 1$, and other expressions for higher moments can be obtained for $t = 2,3,4$. The first four moments of the BWR distribution are

$$\mu_1^{'} = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\sigma\sqrt{2}\left(\frac{1}{\beta(b+i)}\right)^{\frac{1}{2\alpha}}\Gamma\left(\frac{1}{2\alpha}+1\right) \tag{48}$$

$$\mu_2^{'} = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}2\sigma^2\left(\frac{1}{\beta(b+i)}\right)^{\frac{1}{\alpha}}\Gamma\left(\frac{1}{\alpha}+1\right) \tag{49}$$

$$\mu_3^{'} = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\sigma^3(2)^{\frac{3}{2}}\left(\frac{1}{\beta(b+i)}\right)^{\frac{3}{2\alpha}}\Gamma\left(\frac{3}{2\alpha}+1\right) \tag{50}$$

$$\mu_4^{'} = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}4\sigma^4\left(\frac{1}{\beta(b+i)}\right)^{\frac{2}{\alpha}}\Gamma\left(\frac{2}{\alpha}+1\right) \tag{51}$$

The results in equations (48)-(51) are useful for deriving some other properties of the BWR such as skewness, kurtosis, and coefficient of variation. The mean and variance of the BWR distribution are presented as follows;
The mean is given by

$$\mu = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\sigma\sqrt{2}\left(\frac{1}{\beta(b+i)}\right)^{\frac{1}{2\alpha}}\Gamma\left(\frac{1}{2\alpha}+1\right) \tag{52}$$

The variance is obtained using the expression;
$Var(X) = E(X^2) - E(X)^2$

$$= \left\{\begin{array}{l} \left\{\frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}2\sigma^2\left(\frac{1}{\beta(b+i)}\right)^{(\frac{1}{\alpha})}\Gamma\left(\frac{1}{\alpha}+1\right)\right\} \\ -\left\{\frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\sigma\sqrt{2}\left(\frac{1}{\beta(b+i)}\right)^{(\frac{1}{2\alpha})}\Gamma\left(\frac{1}{2\alpha}+1\right)\right\}^2 \end{array}\right. \tag{53}$$

Table 2 represents the computational values for the mean, variance, skewness, and kurtosis for some arbitrary values of the parameters of the four-parameter BWR(a,b,α,σ) distribution.
Table 2 shows that the BWR has a negative (platykurtic) and positive (leptokurtic) kurtosis, and the distribution is positively and negatively skewed. It is observed that

- As the beta parameter b increases for $0 < b < 1$, the skewness and kurtosis increase, but the mean and variance decrease.
- As the beta parameter b increases for $b > 1, a < 1$, the skewness and kurtosis increase, while the mean and variance continue to decrease
- As the beta parameter b increases for $b > 1, a > 1$, the skewness and kurtosis decrease, while the mean and variance continue to decrease
- An increase in the parameter α increases the mean and the skewness, but the variance and kurtosis decrease.
- An increase in the parameter σ increases the mean and the skewness, but the variance and kurtosis decrease.

Table 2: Numerical results for basic statistical measures for parameters BWR(a,b,α,σ).

a,b,α,σ	μ	$\mu_2^{'}$	$\mu_3^{'}$	$\mu_4^{'}$	σ^2	skew.	Kurt.
$0.2,0.2,1.1,0.5$	0.5949	0.8687	1.5614	3.1876	0.5148	1.1698	3.5506
$0.2,0.5,1.1,0.5$	0.3037	0.2611	0.2922	0.3811	0.1688	1.5918	5.0916
$0.5,0.2,1.1,0.5$	0.8419	1.2908	2.3503	4.8187	0.5821	0.6380	2.8180
$0.5,2,1.1,0.5$	0.2394	0.1121	0.0650	0.0436	0.0548	0.9293	3.3481
$0.5,5,1.1,0.5$	0.1529	0.0458	0.0171	0.0074	0.0224	0.9696	3.4346
$2,1,1.1,0.5$	0.6028	0.5365	0.5239	0.5536	0.1731	-1.4922	2.1317
$2,2,1.1,0.5$	0.4629	0.3148	2.2336	0.1868	0.1005	-0.1931	2.1008
$0.5,5,1.1,1.2$	0.5211	0.3750	2.3352	0.3471	0.1034	0.9640	3.5622
$0.5,5,1.1,1.4$	0.8055	0.7041	0.7484	0.9099	0.0553	7.0849	-7.5932
$0.5,5,1.2,1.2$	0.6104	0.4572	0.4166	0.4342	0.0846	1.3920	3.1606

The skewness (Skew) and kurtosis(Kurt) are obtained by using the expressions;

$$Skew = \frac{\mu_3^{'} - 3\,\mu_1^{'}\sigma^2 - \mu^3}{\sigma^3}; \qquad Kurt = \frac{\mu_4^{'} - 4\,\mu\,\mu_3^{'} + 6\mu_2^{'}\sigma^2 + 3\mu^4}{\sigma^4}$$

3.3. Moment Generating Function of the BWR Distribution

The moment generating function is approached using Theorem 5.

Theorem 5. *The moment generating function of a random variable X from the BWR distribution with parameter $a,b,\alpha,\beta,\sigma > 0$, denoted $M_X(t)$ is given by*

$$\frac{1}{B(a,b)}\sum_{s=0}^{\infty}\frac{t^s}{s!}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\sigma^s\left(2\right)^{\frac{s}{2}}\left(\frac{1}{\beta(b+i)}\right)^{\left(\frac{s}{2\alpha}\right)}\Gamma\left(\frac{s}{2\alpha}+1\right) \tag{54}$$

Proof of Theorem 5.

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx, \tag{55}$$

$$= \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\int_0^{\infty} e^{tx}\left[exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}\right]^{b+i}\frac{\alpha\beta x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1}dx, \tag{56}$$

Using the Taylor series expansion for e^{tx}

$$E(e^{tx}) = \sum_{s=0}^{\infty}\frac{t^s}{s!}\int_0^{\infty} x^s f(x)dx, \tag{57}$$

$$\int_0^{\infty} x^s f(x)dx = \mu^s \tag{58}$$

The desired result is obtained by substituting equation (44) into equation (57) □

Corollary 2. If $a = b = 1$, the moment generating function of the BWR distribution is reduced to that of the WR distribution given by

$$\sum_{s=0}^{\infty}\frac{t^s}{s!}\sigma^s\left(2\right)^{\frac{s}{2}}\left(\frac{1}{\beta}\right)^{\frac{s}{2\alpha}}\Gamma\left(\frac{s}{2\alpha}+1\right) \tag{59}$$

The result of Corollary 2 agrees with the mgf of the Weibull-Rayleigh distribution studied by Ahmad et al. (2017), which further strengthens the validity of our results.

3.4. Order Statistics of Beta-Weibull-Rayleigh Distribution

Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the order statistics of the sample size n from the BWR, the pdf of the order statistics Arnold et al. (1992) is given by

$$f_{r:n}(x) = C_{r:n} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \tag{60}$$

The cdf of the r^{th} order statistics (see page 12, David and Nagaraja (2004)) where n is the number of trials and $F(x)$ is the probability of success can also be written in terms of the binomial distribution given as

$$F_{r:n}(x) = \sum_{k=r}^{n} \binom{n}{k} [F(x)]^k [1-F(x)]^{n-k}, \qquad -\infty < x < \infty \tag{61}$$

The cdf of $b(t;a,b)$ can be expressed in terms of the cdf for the binomial distribution Binom(n,p) denoted by $I_p(a,b)$; (see (Aludaat, 2018), and (Arnold et al. 1992) using the parameterization relation $a = r$ and $b = n-r+1$,, and $p = F(x)$ is the probability of success in equation (61).
The cdf $\mathscr{F}_{BWR}(x)$ of the BWR has the identity

$$I_z(a,b) = \frac{B(z;a,b)}{B(a,b)}$$

This can be obtained by substituting $r = a$ and $n = r+b-1$ into equation (61) and is given by

$$\mathscr{F}_{BWR}(x) = I_z(a,b) = \sum_{k=a}^{a+b-1} \binom{a+b-1}{k} [z]^k [1-z]^{a+b-1-k}, \tag{62}$$

where z is the Weibull-Rayleigh cdf given by $G_{WR}(x) = 1 - exp\{-(\frac{x^2}{2\gamma\sigma^2})^{\alpha}\}$.
Equation (60) can be written for BWR as

$$f_{r:n}(x) = C_{r:n} f(x) \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \left[F_{BWR}(x)\right]^{r-1+j} \tag{63}$$

Substitute the cdf of BWR in equation (62) into (63) to get equation (64),

$$f_{r:n}(x) = C_{r:n} f(x) \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \left[\sum_{k=a}^{a+b-1} \binom{a+b-1}{k} [z]^k [1-z]^{a+b-1-k}\right]^{r-1+j} \tag{64}$$

In (64), let $I = \left[\sum_{k=a}^{a+b-1} \binom{a+b-1}{k} [z]^k [1-z]^{a+b-1-k}\right]^{r-1+j}, (n = a+b-1)$.
Substitute the quantity z to obtain

$$\begin{aligned}
I &= \left[\sum_{k=a}^{n} \binom{n}{k} \left[1 - exp\{-\beta(\frac{x^2}{2\sigma^2})^{\alpha}\}\right]^k \left[exp\{-\beta(\frac{x^2}{2\sigma^2})^{\alpha}\}\right]^{n-k}\right]^{r-1+j} \\
&= \left[\sum_{k=a}^{n} \binom{n}{k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \left[exp\{-\beta(\frac{x^2}{2\sigma^2})^{\alpha}\}\right]^l \left[exp\{-\beta(\frac{x^2}{2\sigma^2})^{\alpha}\}\right]^{n-k}\right]^{r-1+j} \\
&= \left[\sum_{k=a}^{n} \binom{n}{k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \left[exp\{-\beta(\frac{x^2}{2\sigma^2})^{\alpha}\}\right]^{n-k+l}\right]^{r-1+j} \\
&= \left[\sum_{k=a}^{n} \sum_{l=0}^{k} \binom{n}{k} \binom{k}{l} (-1)^l exp\left\{-\beta m_1 (\frac{x^2}{2\sigma^2})^{\alpha}\right\}\right]^{m_2}
\end{aligned} \tag{65}$$

where $n = (a+b-1)$, $m_1 = (n-k+l)$, and $m_2 = (r-1+j)$

Substitute the result in (65) into (64) to get the desired pdf of the r^{th} order statistics of $X_{(r:n)}$ for the BWR given by,

$$f_{r:n}(x) = C_{r:n} f(x) \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \left[\sum_{k=a}^{n} \sum_{l=0}^{k} \binom{n}{k} \binom{k}{l} (-1)^l exp\left\{ -\beta m_1 \left(\frac{x^2}{2\sigma^2} \right)^{\alpha} \right\} \right]^{m_2} \tag{66}$$

The minimum and maximum order statistics for a random sample of size n is derived as follows;
For $r = 1$, the *minimum order statistics* of Beta-Weibull-Rayleigh is

$$f_{1:n}(x) = n f(x) \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \left[\sum_{k=a}^{n} \sum_{l=0}^{k} \binom{n}{k} \binom{k}{l} (-1)^l exp\left\{ -\beta m_1 \left(\frac{x^2}{2\sigma^2} \right)^{\alpha} \right\} \right]^{j} \tag{67}$$

For $r = n$, the *maximum order statistics* of Beta-Weibull-Rayleigh is given by

$$f_{n:n}(x) = n f(x) \left[\sum_{k=a}^{n} \sum_{l=0}^{k} \binom{n}{k} \binom{k}{l} (-1)^l exp\left\{ -\beta m_1 \left(\frac{x^2}{2\sigma^2} \right)^{\alpha} \right\} \right]^{n-1} \tag{68}$$

$$f(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{a-1}{i} \left[exp\left\{ -\beta \left(\frac{x^2}{2\sigma^2} \right)^{\alpha} \right\} \right]^{b+i} \frac{\alpha\beta x}{\sigma^2} \left(\frac{x^2}{2\sigma^2} \right)^{\alpha-1}$$

The moment of order statistics is defined for a random sample of size n denoted $X_1, X_2, ..., X_n$ for a distribution with the cdf and pdf denoted by $F(x)$ and $f(x)$ respectively, if the corresponding order statistics are $X_{(1)} \leq X_{(2)} \leq, ... \leq X_{(n)}$. Then the t^{th} moment of order statistics is of $X_{(r:n)}$ is

$$E[X^t_{(r:n)}] = \int_0^{\infty} x^t f_{r,n}(x) dx \tag{69}$$

Table 3 represents the computational values for the mean, variance, skewness and kurtosis of order statistics for some arbitrary values of the parameters of the four-parameter BWR(a, b, α, σ) distribution.

Table 3: Estimated order statistical measures for the parameter values $a = 1.0, b = 2.5, \alpha = 0.5, \sigma = 1.5$.

n	r	*mean*	*variance*	*skewness*	*kurtosis.*
1	1	0.8485	0.7200	15.7285	-45.9113
2	1	0.2121	0.1350	2.6946	13.0009
2	2	0.6364	0.8550	1.9561	8.0358
3	1	0.0943	0.0444	3.3990	18.6454
3	2	0.1178	0.1128	3.7562	20.5098
3	3	0.5185	0.8644	2.1997	8.8067
4	1	0.0530	0.0197	3.9851	24.5409
4	2	0.0413	0.0291	5.4842	40.5071
4	3	0.0766	0.0899	4.8893	31.5416
4	4	0.4419	0.8422	2.4544	9.9232

The mean of order statistics based on the underlining probability distribution shows that $E(x_{(1:n)})$ and $E(x_{(n:n)})$ decrease and increase, respectively, as the sample size increases in (Adeyemi et al. 2024,) and (Saqib and Memon, 2025). In Saqib and Memon (2025), the variances of $x_{(1:n)}$, and $x_{(n:n)}$ decrease and increase, respectively, as the sample size increases. The moment property of order statistics of the BWR distribution is established and given by

- The expectation of the maximum sample of order statistics denoted $\mu_{(n:n)}$ decreases as the sample size increases
- The expectation of the minimum order statistics denoted $\mu_{(1:n)}$ decreases as sample size increases
- The variance of the minimum sample of order statistics $Var(X_{(1:n)})$ decreases as the sample size increases
- The kurtosis of BWR maximum and minimum order statistics increases as the sample size increases
- The skewness of $x_{(n:n)}$ for the BWR distribution increases with increasing sample size n

This study established from Table 3 that $E[x_{(1:n)}] \geq E[x_{(1:n+1)}]$ and $E[x_{(n:n)}] \geq E[x_{(n+1:n+1)}]$, which is an indication that the BWR distribution is influenced by the tail-weight of the beta distribution denoted by the normalized constant B(a,b) that causes a decreasing order of expectation of the minimum sample of order statistics. The mean of order statistics of the BWR distribution is summarized for $1 \leq n \leq 4$ by the property defined with the equation (70)

$$E[X_{(1:1)}] = \mu = E[X_{(1:n)}] + E[X_{(n:n)}] + (n-1)\sum_{k=2}^{n-1} E[X_{(k:n)}] \tag{70}$$

4. Estimation of the BWR Parameters, and Simulation

Let $X_1, X_2, ..., X_n$ be random sample of size n from the BWR distribution, and let $\Psi = (a, b, \alpha, \beta, \sigma)^T$ be the parameter vector. The likelihood function denoted by $\mathscr{L}(\Psi)$ is

$$\mathscr{L}(\Psi/x) = \prod_{i=1}^{n} f(x_i; \Psi) \tag{71}$$

The density function is

$$f(x) = \frac{1}{B(a,b)}\left[1 - exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}\right]^{a-1}\left[exp\left\{-\beta\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right\}\right]^{b}\frac{\alpha\beta x}{\sigma^2}\left(\frac{x^2}{2\sigma^2}\right)^{\alpha-1} \tag{72}$$

The log-likelihood ($loglik$) function is derived as

$$\begin{aligned} LL = &\sum log(x) + nlog(\beta) - 2nlog(\sigma) - nlogB(a,b) - b\beta\sum\left(\frac{x^2}{2\sigma^2}\right)^{\alpha} \\ &+ (a-1)\sum log\left(1 - exp\left(-\beta\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}\right)\right) + (\alpha-1)\sum log\left(\frac{x^2}{2\sigma^2}\right) + nlog(\alpha) \end{aligned} \tag{73}$$

By taking the first derivative of the $loglik$ function on the parameters, the score vector is, $\frac{dLL}{d\Psi} = (\frac{dLL}{da}, \frac{dLL}{db}, \frac{dLL}{d\alpha}, \frac{dLL}{d\beta}, \frac{dLL}{d\sigma})^T$.
Let $s_i = \frac{x_i^2}{2\sigma^2}$, then

$$\frac{dLL}{da} = n\psi(a,b) - n\psi(a) + \sum log\left(1 - exp(-\beta(s_i)^{\alpha})\right) = 0$$

$$\frac{dLL}{db} = n\psi(a,b) - n\psi(b) - \beta\sum(s_i)^{\alpha} = 0$$

$$\frac{dLL}{d\alpha} = \frac{n}{\alpha} + (a-1)\sum\frac{exp(-\beta(s_i)^{\alpha})log(s_i)(s_i)^{\alpha}}{1 - exp(-\beta(s_i)^{\alpha})} - b\beta\sum(s_i)^{\alpha}log(s_i) + \sum log(s_i) = 0$$

$$\frac{dLL}{d\beta} = \frac{n}{\beta} - b\sum(s_i)^{\alpha} + (a-1)\sum\frac{(s_i)^{\alpha}exp(-\beta(s_i)^{\alpha})}{1 - exp(-\beta(s_i)^{\alpha})} = 0$$

$$\frac{dLL}{d\sigma} = \frac{-2n\alpha}{\sigma} + \frac{2b\beta\alpha}{\sigma}\sum(s_i)^{\alpha} - \frac{2\beta\alpha(a-1)}{\sigma}\sum\frac{exp(-\beta(s_i)^{\alpha}(s_i)^{\alpha}}{1 - exp(-\beta(s_i)^{\alpha}} = 0$$

where $\Gamma(.)$ is the gamma function and the digamma function which is the derivative of $log\Gamma(.)$ is denoted, $\psi(.)$ The simultaneous non-linear equation to the score vector set at zero is the solution for the estimates of the parameters denoted as $\hat{\Psi} = (\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\sigma})$. The numerical solution to the equations can be obtained by using the computer R software. The expected information matrix can be obtained from the second derivatives where

$$\frac{dLL^2}{dij} = \frac{dLL^2}{dji} = V_{ij}, \quad and \quad \frac{dLL^2}{di^2} = V_{ii}, \quad \frac{dLL^2}{dj^2} = V_{jj}$$

The Information matrix is given by

$$J(\Psi) = -\begin{bmatrix} Vaa & Vab & Va\alpha & Va\sigma \\ Vba & Vbb & Vb\alpha & Vb\sigma \\ V\alpha a & V\alpha b & V\alpha\alpha & V\alpha\sigma \\ V\sigma a & V\sigma b & V\sigma\alpha & V\sigma\sigma \end{bmatrix}$$

The normal approximation distribution $N(0, J^{-1}(\Psi))$ for the model parameters following the normal approximation distribution $N(0, J^{-1}(\Psi))$ is given by

$$\hat{a} \pm Z_{\frac{p}{2}}\sqrt{var(\hat{a})}, \quad \hat{b} \pm Z_{\frac{p}{2}}\sqrt{var(\hat{b})}, \quad \hat{\alpha} \pm Z_{\frac{p}{2}}\sqrt{var(\hat{\alpha})}, \quad and \quad \hat{\sigma} \pm Z_{\frac{p}{2}}\sqrt{var(\hat{\sigma})}.$$

$var(.)$ are the diagonal elements of the inverse information matrix $J^{-1}(\Psi)$ for each respective $\frac{p}{2}$ represent the one hundred percent quantile $100(1-\frac{p}{2})$ of the standard normal distribution.

4.1. Simulation study to Assess the Behavior of the Model Parameters

Assessment of the model parameters is investigated by conducting a simulation study using some arbitrary set of the parameters of the BWR distribution. The biases and mean square errors (MSE) are computed to evaluate the performance of the parameters, the simulation involved the stated procedure

- Define the quantile function (QF_{BWR}), and simulate random samples of size n
- Define two set of parameters $A = (a_1, b_1, \alpha_1, \sigma_1)$ and $B = (a_2, b_2, \alpha_2, \sigma_2)$
- Perform a replication for N=1000 for each sample size n
- Calculate the bias and the MSE
- Repeat the process for all the biases and MSE for each parameter and the sample size

The computational results is presented in Table 4 for the first set of values of the parameters;
$A(a = 0.2, b = 0.1, \alpha = 1.6, \sigma = 0.8)$

Table 4: Mean, bias and MSE for the parameters $A(a = 0.2, b = 0.1, \alpha = 1.6, \sigma = 0.8)$

n	parameter	mean	bias	MSE
20	a	0.3387	0.1388	1.2201
	b	0.4983	0.3983	1.9338
	α	2.4656	0.8656	2.0297
	σ	1.2789	0.4789	0.3664
50	a	0.1833	−0.0167	0.0021
	b	0.1707	0.0706	0.0182
	α	2.0334	0.4335	0.5318
	σ	1.0357	0.2358	0.1308
150	a	0.1953	-0.0047	0.0010
	b	0.1315	0.0316	0.0028
	α	1.7532	0.1532	0.1373
	σ	0.8949	0.0949	0.0496
400	a	0.1999	-0.0000	0.0000
	b	0.1124	0.0124	0.0002
	α	1.6006	0.0006	0.0000
	σ	0.8012	0.0012	0.0000

The result for the second set of the parameters $B(a=1.5, b=0.5, \alpha=1.2, \sigma=0.1)$ are presnted in Table 5

Table 5: Mean, bias and MSE for the parameters $B(a=1.5, b=0.5, \alpha=1.2, \sigma=0.1)$

n	parameter	mean	bias	MSE
20	a	1.8585	0.3585	32.1894
	b	1.9004	1.4004	9.4253
	α	2.1438	0.9437	3.2298
	σ	0.1535	0.0535	0.0065
50	a	1.4405	-0.0595	0.0021
	b	1.5071	1.0072	7.4137
	α	1.5976	0.3979	0.7886
	σ	0.1402	0.0401	0.0046
150	a	1.5139	0.0139	1.6755
	b	1.1037	0.6036	3.2996
	α	1.2847	0.0845	0.0801
	σ	0.1257	0.0259	0.0023
400	a	1.4978	-0.0022	0.1156
	b	0.7848	0.2846	0.6151
	α	1.2206	0.0207	0.0175
	σ	0.1142	0.0142	0.0011

The simulation results revealed that the biases decreased consistently as the sample size increased and the MSE decreased as the sample size increased for both sets of parameters. It was discovered that the estimated parameters were close to the true parameters, which is an indication that the estimation procedure is adequate.

5. Application to Real-Life Datasets

The section investigates the performance of the proposed BWei-X family of distributions when the baseline distribution of X, is the Rayleigh distribution. The maximum likelihood estimates of parameters are estimated using the PSO algorithm in the AdequacyModel package in the R statistical software Marinho et al. (2019) to maximize the log-likelihood function. The goodness-of-fit criteria for the selection of appropriate distribution in the study are log-likelihood (Logll), Akaike Information Criterion (AIC), Bayesian Information Criteria (BIC), Anderson Darling (AD) statistics, the Kolmogorov-Smirnov (KS) test statistics, and the corresponding p-value. AIC and BIC are reliable and acceptable goodness-of-fit statistics considered, and they are defined as follows;

$$AIC = -2 \times Logll + 2 \times p, \qquad BIC = -2 \times Logll + p \times log(n)$$

Where p is the number of parameters in the model, n is the sample size and $Logll$ is the negative log-likelihood function. The goodness-of-fit criteria are defined in the literature, including Chen and Balakrishnan (1995). The model with the smallest goodness-of-fit criteria or the corresponding highest p-value is considered to be the better fit for the data. The descriptive statistics of the datasets are given in Table 6.

Table 6: Summary statistics of the two datasets.

Data	n	Min	Max	Q_1	Q_3	Mean	Median	Sd	Skewness	Kurtosis
Data I	50	0.10	86	13.50	81.25	45.69	48.50	32.8353	-0.1334	-1.6421
Data II	30	122	198	146.8	170.8	158.6	156.5	322.654	0.1046	-0.6131

n=sample size; Q_1=first quartile; Q_3=third quartile; Sd=standard deviation..

Table 6 showed that data I is skewed to the left and have a negative kurtosis with an average lifetime of $\overline{X} = 45.69$. The descriptive statistics of (data II), systolic blood pressure is skewed to the right with negative kurtosis and mean systolic blood pressure of $\overline{X} = 158.60$ mmHg, indicating patients with high blood pressure

5.1. Application to Lifetimes of 50 Devices (Data I)

The data set is given by Aarset (2009) and represents the failure times of 50 devices (in weeks). This popular data set was applied to test the performance of important distributions developed by some notable authors (see Benkhelifa 2017, Jafari and Mahmoudi 2012, Jafari et al. 2014, and Rashwan and Kamel 2020). The datasets are:

$$0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50,$$

$$55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.$$

Table 7 contains the estimated values of the parameters obtained from data analysis with the aid of the R statistical software for the BWR and some existing distributions that have been investigated using the data set.

Table 7: Maximum likelihood estimates of the model parameters for the lifetime data.

Distribution	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
BWR	0.0798	0.0675	0.0025	–	34.7849
BGGo	0.1465	0.0939	2.2369	0.0063	0.0631
BGo	0.2635	0.1823	0.0022	0.0675	–
BEP	0.1214	0.0628	2.9342	0.0148	6.3899
BLFR	0.2752	0.0911	0.0043	0.0050	–

Table 8 contains results from data analysis showing that the BWR has the lowest model selection criteria with the corresponding highest p-value compared to the notable existing distributions previously applied to the datasets. The result shows that the BWR distribution is the best candidate among the competing distributions.

Table 8: Goodness-of-fit measures for BWR and other competitive models for the data.

Distribution	$-Logll$	AIC	BIC	KS	$p-value$
BWR	435.7064	443.7063	451.3544	0.1185	0.4835
BGGo	440.2050	450.2051	459.7652	0.1295	0.3717
BGo	442.1111	450.1111	457.7592	0.1339	0.3316
BEP	449.7481	459.7489	469.3090	0.1341	0.3295
BLFR	460.7157	468.7158	476.3638	0.1371	0.3038

AIC=Akaike information criteria; BIC=Bayesian information criteria, KS=Kolmogorov-Simonrov.

Figure 4 shows the plots of the data fitted to the distributions, the histogram overlaid with the densities on the left of the panel, and the empirical cdfs is on the right side of the panel.

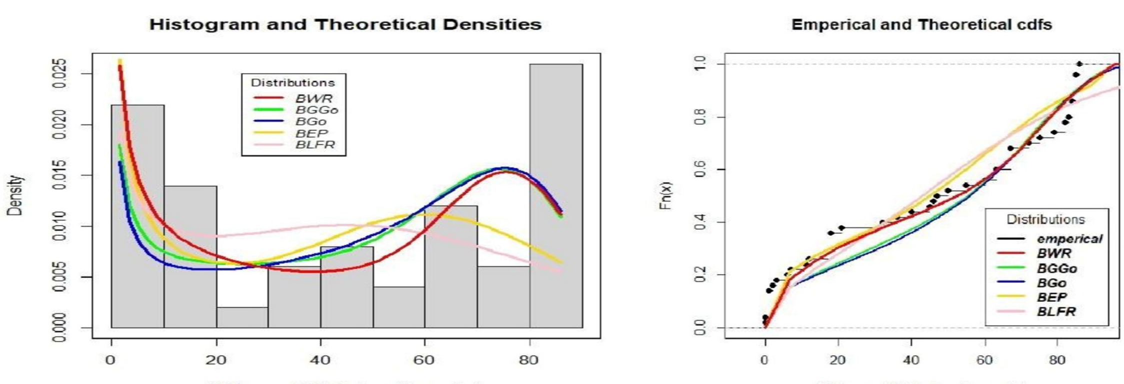

Figure 4: Performance of competing distributions with pdfs (left) and cdfs (right).

The figure supports the results in Table 8 and established that the BWR model in the red curve is the best fit to the lifetime data set.

5.2. Application to High Blood Pressure Data (Data II)

The medical data obtained from one of the Hospitals in Lagos state, Nigeria, represent the maximum daily measurement of systolic blood pressure (SBP) in millimeters of mercury (mmHg) from 500 patients for the period that covers 30 days from November 16 to December 15, 2022. The datasets fitted to the distributions are:
$170, 155, 156, 135, 137, 170, 170, 198, 132, 165, 161, 172, 183, 149, 171, 146, 157, 143, 154, 172, 139, 156, 155, 172, 138, 122, 184, 158, 189, 150.$
The results of the data analysis using the R-software for the estimated values of the BWR parameters and other competing distributions are tabulated in Table 9

Table 9: Maximum likelihood estimates of the model parameters for the lifetime data.

Distribution	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
BWR	5.7377	15.7894	1.8971	–	154.3795
WR	–	–	4.7879	17.1593	28.4579
BGGo	1.7681	3.1515	4.7694	0.0029	0.0138
BEP	30.3452	2.1710	1.7140	0.0036	3.2393
BGo	7.8084	0.5657	0.0031	–	0.0201

Table 10 is the result of the model selection criteria based on the goodness-of-fit statistics and shows that BWR has the lowest AIC and BIC for model selection with the corresponding highest p-value compared to the other notable distributions previously applied to the data.

Table 10: Goodness-of-fit measures for BWR and other competitive models for the data.

Distribution	$-Logll$	AIC	BIC	KS	$p-value$
BWR	256.5058	265.5058	271.1106	0.0987	0.9320
WR	259.7353	265.7342	269.9389	0.1256	0.7313
BGGo	260.4407	270.4407	277.4467	0.1349	0.6454
BEP	260.7672	270.7672	277.7732	0.1384	0.6133
BGo	262.5271	270.5271	276.1319	0.1438	0.5646

Figure 5 is the graphical view of the fitted BWR density with the red curve together with other competing distributions on the left side, the estimated cdfs is on the right side.

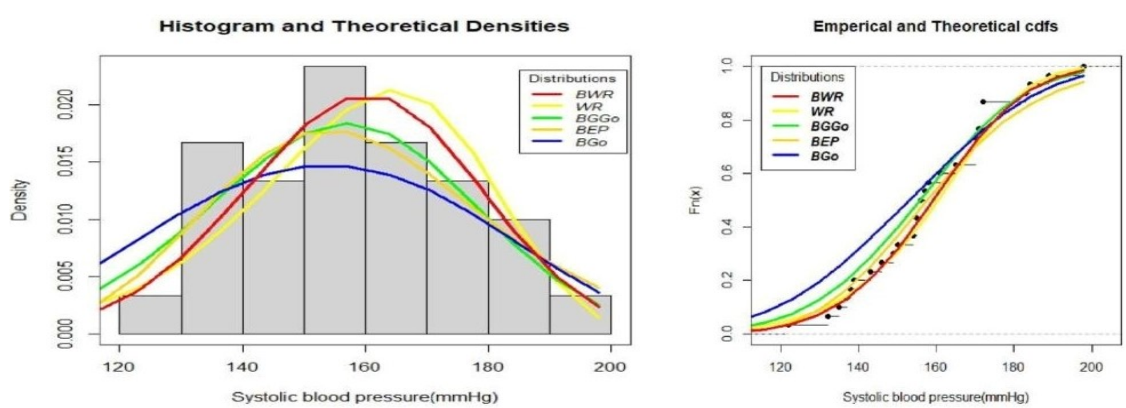

Figure 5: Plots of BWR and competing models fitted to the systolic blood pressure data.

The Figure supported the choice of BWR as the model with the best performance for modeling the systolic blood pressure data.

6. Discussion

A new family of distributions called BWei-X was developed. The distributions from the family are flexible and have the potential to capture tail characteristics of random events, and it can provide an improvement on existing results from statistical applications to some complex real-life datasets. The Beta-Weibull-Rayleigh generated from the family characterized some important existing lifetime distributions of notable researchers stated in Table 1 when some of the parameters were relaxed. Some properties of the distribution were investigated with an extension to some properties of order statistics, and the maximum likelihood method was proposed for the estimation of parameters. The lifetime of 50 devices data reported by Aarset (2009) which was analyzed using BLFR, BGo, BGGo, and BEP distributions received a fresh analysis in this study to evaluate the performance of the new model. In addition, the performance of the BWR was evaluated by applying it to systolic blood pressure data from the medical field. The results of the analysis revealed that the goodness-of-fit criteria for model selection were smaller for BWR than for the competing models.

7. Concluding Remarks

The BWei-X is a new family of distributions for constructing new convoluted distributions that have the capacity to model real-life complex phenomena exhibited in the form of their kurtosis and skewness. When X is the Rayleigh distribution, the BWei-X family produced a better fit to the lifetime data of 50 devices than BLFR, BGo, BGGo and BEP distributions previously proposed in the literature. The result shows that the BWR distribution is more attractive with application to health datasets, as indicated by a stronger modeling capacity than the competing distributions. The BWei-X family of distributions can provide useful applications in medical science, hydrology, reliability engineering, meteorology, and environmental engineering. Future research will incorporate the theory of order statistics with real-life application to climate change and some extreme phenomena in actuarial science and finance.

References

1. Aarset, M. V. (2009). How to identify a bathtub hazard rate. *IEEE transactions on reliability*, 36(1):106–108.
2. Ademola, A., Adeyeye, J., Khaleel, M., and Aako, O. (2021). Exponentiated gompertz exponential (egoe) distribution: Derivation, properties and applications. *Istatistik Journal of the Turkish Statistical Association*, 13(1):12–28.
3. Adeyemi, A. O., Adeleke, I. A., and Akarawak, E. E. (2023). Extension of exponential pareto distribution with the order statistics: Some properties and application to lifetime dataset. *Science and Technology Indonesia*, 8(2):265–279.
4. Adeyemi, A. O., Adeleke, I. A., and Akarawak, E. E. (2024). Weibull-exponential pareto distribution: Order statistics and their properties and application to nigeria covid-19 active cases. *Electronic Journal of Applied Statistical Analysis*, 17(3).
5. Ahmad, A., Ahmad, S., and Ahmed, A. (2017). Characterization and estimation of weibull-rayleigh distribution with applications to life time data. *Appl. Math. Inf. Sci. Lett*, 5:71–79.
6. Akarawak, E., Adeleke, I., and Okafor, R. (2013). The weibull-rayleigh distribution and its properties. *Journal of Engineering Research*, 18(1):61–72.
7. Akinsete, A., Famoye, F., and Lee, C. (2008). The beta-pareto distribution. *Statistics*, 42(6):547–563.
8. Akinsete, A. and Lowe, C. (2009). Beta-rayleigh distribution in reliability measure. section on physical and engineering sciences. In *Proceedings of the American Statistical Association*, volume 1, pages 3103–3107.
9. Aludaat, K. M. (2018). On the beta cumulative distribution function. *Applied Mathematical Sciences*, 12(10):461–466.
10. Alzaatreh, A., Famoye, F., and Lee, C. (2013a). Weibull-pareto distribution and its applications. *Communications in Statistics-Theory and Methods*, 42(9):1673–1691.
11. Alzaatreh, A. and Ghosh, I. (2015). On the weibull-x family of distributions. *Journal of statistical theory and applications*, 14(2):169–183.
12. Alzaatreh, A., Lee, C., and Famoye, F. (2013b). A new method for generating families of continuous distributions. *Metron*, 71(1):63–79.
13. Arnold, B. C., Balakrishnan, N., and Nagaraja, H. N. (1992). *A first course in order statistics*, volume 54. SIAM, Philadelphia.

14. Barreto-Souza, W., Santos, A. H., and Cordeiro, G. M. (2010). The beta generalized exponential distribution. *Journal of statistical Computation and Simulation*, 80(2):159–172.
15. Benkhelifa, L. (2017). The beta generalized gompertz distribution. *Applied Mathematical Modelling*, 52:341–357.
16. Benkhelifa, L. (2022). The beta power muth distribution: Regression modeling, properties and data analysis. *Pakistan Journal of Statistics and Operation Research*, pages 225–243.
17. Bourguignon, M., Silva, R. B., and Cordeiro, G. M. (2014). The weibull-g family of probability distributions. *Journal of data science*, 12(1):53–68.
18. Chen, G. and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, 27(2):154–161.
19. Cordeiro, G. M., Cristino, C. T., Hashimoto, E. M., and Ortega, E. M. (2013). The beta generalized rayleigh distribution with applications to lifetime data. *Statistical papers*, 54(1):133–161.
20. David, H. A. and Nagaraja, H. N. (2004). *Order statistics*. John Wiley & Sons.
21. Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and methods*, 31(4):497–512.
22. Famoye, F., Lee, C., and Eugene, N. (2004). Beta-normal distribution: Bimodality properties and application. *Journal of Modern Applied Statistical Methods*, 3(1):10.
23. Famoye, F., Lee, C., and Olumolade, O. (2005). The beta-weibull distribution. *Journal of Statistical Theory and Applications*, 4(2):121–136.
24. Gómez-Déniz, E., Venegas, O., and Gómez, H. W. (2023). Beyond the lognormal distribution with properties and applications. *Brazilian Journal of Probability and Statistics*, 37(2):263–281.
25. Gradshteyn, I. S. and Ryzhik, I. M. (2014). *Table of integrals, series, and products*. Academic press.
26. Handique, L. and Chakraborty, S. (2016). Beta generated kumaraswamy-g and other new families of distributions. *arXiv preprint arXiv:1603.00634*.
27. Hennig, C. (2024). Parameters not empirically identifiable or distinguishable, including correlation between gaussian observations. *Statistical Papers*, 65(2):771–794.
28. Jafari, A. A. and Mahmoudi, E. (2012). Beta-linear failure rate distribution and its applications. *arXiv preprint arXiv:1212.5615*.
29. Jafari, A. A., Tahmasebi, S., and Alizadeh, M. (2014). The beta-gompertz distribution. *Revista Colombiana de Estadística*, 37(1):141–158.
30. Maclaren, O. J. and Nicholson, R. (2019). What can be estimated? identifiability, estimability, causal inference and ill-posed inverse problems. *arXiv preprint arXiv:1904.02826*.
31. Madaki, U. Y., Bakar, M. R. A., and Handique, L. (2022). Beta kumaraswamy burr type x distribution and its properties. *ASEANA Journal of Science and education*, 2(1):9–36.
32. Mahmoud, M. and Ghazal, M. (2017). Estimations from the exponentiated rayleigh distribution based on generalized type-ii hybrid censored data. *Journal of the Egyptian Mathematical Society*, 25(1):71–78.
33. Marinho, P. R. D., Silva, R. B., Bourguignon, M., Cordeiro, G. M., and Nadarajah, S. (2019). Adequacymodel: An r package for probability distributions and general purpose optimization. *PloS one*, 14(8):e0221487.
34. Mead, M. E., Afify, A. Z., Hamedani, G., and Ghosh, I. (2017). The beta exponential fréchet distribution with applications. *Austrian Journal of Statistics*, 46(1):41–63.
35. Merovci, F. and Elbatal, I. (2015). Weibull rayleigh distribution: Theory and applications. *Applied Mathematics & Information Sciences*, 9(4):1–11.
36. Nadarajah, S. (2011). The exponentiated exponential distribution: a survey. *AStA Advances in Statistical Analysis*, 95(3):219–251.
37. Nadarajah, S. and Kotz, S. (2004). The beta gumbel distribution. *Mathematical Problems in engineering*, 2004(4):323–332.
38. Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. *Reliability engineering & system safety*, 91(6):689–697.
39. Pal, M., Ali, M. M., and Woo, J. (2006). Exponentiated weibull distribution. *Statistica*, 66(2):139–147.
40. Rashwan, N. I. and Kamel, M. M. (2020). The beta exponential pareto distribution. *Far East Journal of Theoretical Statistics*, 58(2):91–113.
41. Saqib, M. and Memon, A. Z. (2025). Some remarks on extreme sample order statistics relating to power lomax geometric distribution. *Scientific African*, page e03008.

42. Sharma, D. and Chakrabarty, T. K. (2017). Some general results on quantile functions for the generalized beta family. *Statistics, Optimization & Information Computing*, 5(4):360–377.
43. Shen, Z., Alrumayh, A., Ahmad, Z., Abu-Shanab, R., Al-Mutairi, M., and Aldallal, R. (2022). A new generalized rayleigh distribution with analysis to big data of an online community. *Alexandria Engineering Journal*, 61(12):11523–11535.
44. Shittu, O. I. and Adepoju, K. A. (2013). On the beta-nakagami distribution. *Progress in Applied Mathematics*, 5(1):49–58.
45. Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M., and Zubair, M. (2016). A new weibull–pareto distribution: properties and applications. *Communications in Statistics-Simulation and Computation*, 45(10):3548–3567.