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Characterizations of the Recently Introduced Discrete Distributions

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Abstract

Certain characterizations of 26 recently introduced discrete distributions are presented in three directions: (i) based on an appropriate function of the random variable; (ii) in terms of the reverse hazard function and (iii) in terms of the hazard function.

Key Words: Characterizations; Conditional expectation; Discrete distributions; Hazard function; Reverse hazard function.

1. Introduction

As we mentioned in our previous works, sometimes in real life cases, it is very difficult to obtain samples from a continuous distribution. The observed values are generally discrete due to the fact that they are not measured in continuum. In some cases, it may be possible to measure the observations via a continuous scale, however, they may be recorded in a manner in which a discrete model seems more suitable. Consequently, the discrete models are appearing quite frequently in applied fields and have attracted the attention of many researchers.

Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution.

We list below the cumulative distribution functions and probability mass functions of 26 new discrete distributions which have been introduced after our fifth monograph (Hamedani and Roshani, 2024) was published. Section 2 provides the characterizations of these distributions based on conditional expectations of certain functions of the random variables. Section 3 takes up the characterizations of 7 of the 26 distributions in terms of its reverse hazard function and Section 4 deals with the characterizations of 2 of the 26 distributions in terms of the hazard function.

1.1. Preliminaries

The cumulative distribution (cdf) and the corresponding probability mass function (pmf) of each of the 26 distributions are listed below in (1)-(26).

(1) The cdf, pmf, reverse hazard and hazard functions of Bernoulli-Poisson-Lindley (BPL) distribution of Bakouch

et al. (2023), are given, respectively, by

$$F\left(x;\alpha,\theta\right)=\begin{cases}\frac{(1-\alpha)\theta^2(\theta+2)}{(\theta+1)^3} & ,x=0\\ C\sum_{u=1}^{x}\left(\theta+1\right)^{-u}P\left(u\right) & ,x\in\mathbb{N}\end{cases}, \tag{1}$$

$$f\left(x;\alpha,\theta\right)=\begin{cases}\frac{(1-\alpha)\theta^2(\theta+2)}{(\theta+1)^3} & ,x=0\\ C\left(\theta+1\right)^{-x}P\left(x\right) & ,x\in\mathbb{N}\end{cases}, \tag{2}$$

$$r_F\left(x\right)=\begin{cases}1 & ,x=0\\ \frac{(\theta+1)^{-x}P(x)}{\sum_{u=1}^{x}(\theta+1)^{-u}P(u)} & ,x\in\mathbb{N}\end{cases}, \tag{3}$$

$$h_F\left(x\right)=\begin{cases}\frac{(1-\alpha)\theta^2(\theta+2)}{(\theta+1)^3-(1-\alpha)\theta^2(\theta+2)} & ,x=0\\ \frac{C(\theta+1)^{-x}P(x)}{1-C\sum_{u=1}^{x}(\theta+1)^{-u}P(u)} & ,x\in\mathbb{N}\end{cases}, \tag{4}$$

where $0<\alpha<1,\theta>0$ are parameters, $C=\frac{\theta^2}{(\theta+1)^3}$, $P\left(x\right)=\left(1+\alpha\theta\right)\left(x+\theta+1\right)+\left(1-\alpha\right)$ and $\mathbb{N}$ is the set of all positive integers.

(2) The cdf, pmf, reverse hazard and hazard functions of Discrete Cos-Lindley (DCL) distribution are given, respectively, by

$$F\left(x;\alpha,\beta,\theta\right)=CP\left(x+1\right)e^{-\theta(x+1)},\quad x\in\mathbb{N}^*, \tag{5}$$

$$f\left(x;\alpha,\beta,\theta\right)=C\left\{P\left(x+1\right)e^{-\theta(x+1)}-P\left(x\right)e^{-\theta x}\right\},\quad x\in\mathbb{N}^*, \tag{6}$$

$$r_F\left(x\right)=1-\frac{P\left(x\right)e^{-\theta x}}{P\left(x+1\right)e^{-\theta(x+1)}},\quad x\in\mathbb{N}, \tag{7}$$

$$h_F\left(x\right)=\frac{C\left\{P\left(x+1\right)e^{-\theta(x+1)}-P\left(x\right)e^{-\theta x}\right\}}{1-CP\left(x+1\right)e^{-\theta(x+1)}},\quad x\in\mathbb{N}^*, \tag{8}$$

where $\alpha\in\left[-1,1\right],\beta>0,\theta>0$ are parameters, $C=\left\{\frac{1+\theta}{\theta^2}-\frac{\alpha\theta}{\theta^2+\beta^2}-\frac{\alpha\left(\theta^2-\beta^2\right)}{\left(\theta^2+\beta^2\right)^2}\right\}^{-1}$, and $P\left(x\right)=1-\alpha\cos\left(\beta x\right)\left(1+x\right)$.

Remark 1.1. *The cdf* (5) *is the discrete version of the Cos-Lindley (CL) distribution of Chesneau et al. (2023a). We take* $\alpha=1$ *to simplify the calculations.*

(3) The cdf, pmf, reverse hazard and hazard functions of Discrete Gamma Power Lomax (DGPL) distribution are given, respectively, by

$$F\left(x;\alpha,\lambda,\theta,\upsilon\right)=\gamma\left(\upsilon,-\log\left(1+\frac{\left(x+1\right)^{\lambda}}{\theta}\right)^{-\alpha}\right),\quad x\in\mathbb{N}^*, \tag{9}$$

$$f\left(x;\alpha,\lambda,\theta,\upsilon\right)=\gamma\left(\upsilon,-\log\left(1+\frac{\left(x+1\right)^{\lambda}}{\theta}\right)^{-\alpha}\right)-\gamma\left(\upsilon,-\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right),\quad x\in\mathbb{N}^*, \tag{10}$$

$$r_F\left(x\right)=1-\frac{\gamma\left(\upsilon,-\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon,-\log\left(1+\frac{(x+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)},\quad x\in\mathbb{N}^*, \tag{11}$$

$$h_F(x) = \frac{\gamma\left(\upsilon, -\log\left(1+\frac{(x+1)^\lambda}{\theta}\right)^{-\alpha}\right) - \gamma\left(\upsilon, -\log\left(1+\frac{x^\lambda}{\theta}\right)^{-\alpha}\right)}{1-\gamma\left(\upsilon, -\log\left(1+\frac{(x+1)^\lambda}{\theta}\right)^{-\alpha}\right)}, \quad x \in \mathbb{N}^*, \tag{12}$$

where $\alpha, \lambda, \theta, \upsilon$ are all positive parameters.

Remark 1.2. *The cdf* (9) *is the discrete version of the Gamma Power Lomax (GPL) distribution of Ogunde et al. (2023).*

(4) The cdf, pmf, reverse hazard and hazard functions of New Lomax Rayleigh Discrete (NLRD) distribution are given, respectively, by

$$F(x;\theta,\lambda,\sigma) = 1-\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}, \quad x \in \mathbb{N}^*, \tag{13}$$

$$f(x;\theta,\lambda,\sigma) = \left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-\theta} - \left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}, \quad x \in \mathbb{N}^*, \tag{14}$$

$$r_F(x) = \frac{\left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-\theta} - \left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}}{1-\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}}, \quad x \in \mathbb{N}^*, \tag{15}$$

$$h_F(x) = \frac{\left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}} - 1, \quad x \in \mathbb{N}^*, \tag{16}$$

where θ, λ, σ are all positive parameters.

Remark 1.3. *The cdf* (13) *is the discrete version of the New Lomax Rayleigh (NLR) distribution of Saritha et al. (2023).*

(5) The cdf, pmf, reverse hazard and hazard functions of Discrete Harris Extended-Exponential (DHEE) distribution are given, respectively, by

$$F(x;\alpha,\beta,\theta) = 1-\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)\,e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}, \quad x \in \mathbb{N}^*, \tag{17}$$

$$f(x;\alpha,\beta,\theta) = \left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)\,e^{-\alpha\beta x}}\right]^{1/\alpha} - \left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)\,e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}, \quad x \in \mathbb{N}^* \tag{18}$$

$$r_F(x) = \frac{\left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)e^{-\alpha\beta x}}\right]^{1/\alpha} - \left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}}{1-\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}}, \quad x \in \mathbb{N}^*, \tag{19}$$

$$h_F(x) = \frac{\left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)e^{-\alpha\beta x}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}} - 1, \quad x \in \mathbb{N}^*, \tag{20}$$

where α, β, θ are all positive parameters.

Remark 1.4. *The cdf* (17) *is the discrete version of the Harris Extended-Exponentia (HEE) distribution of Mohammed et al. (2023).*

(6) The cdf, pmf, reverse hazard and hazard functions of Lagrangian Intervened Poisson (LIP) distribution of Irshad et al. (2023), are given, respectively, by

$$F\left(x;\lambda,\rho,\mu\right)=C\sum_{u=1}^{x}e^{-\mu u}P\left(u\right),\quad x\in\mathbb{N}, \tag{21}$$

$$f\left(x;\lambda,\rho,\mu\right)=Ce^{-\mu x}P\left(x\right),\quad x\in\mathbb{N}, \tag{22}$$

$$r_F\left(x\right)=\frac{e^{-\mu x}P\left(x\right)}{\sum_{u=1}^{x}e^{-\mu u}P\left(u\right)},\quad x\in\mathbb{N}, \tag{23}$$

$$h_F\left(x\right)=\frac{Ce^{-\mu x}P\left(x\right)}{1-C\sum_{u=1}^{x}e^{-\mu u}P\left(u\right)},\quad x\in\mathbb{N}, \tag{24}$$

where $\lambda>0,\rho\geq 0,\mu\in(0,1)$ are parameters, $C=\frac{(1-\mu)e^{-\lambda\rho}}{e^{\lambda}-1}$ and $P\left(x\right)=\frac{1}{x!}\left\{(x\mu+\lambda+\lambda\rho)^{x}-(x\mu+\lambda\rho)^{x}\right\}$.

(7) The cdf, pmf, reverse hazard and hazard functions of Lagrangian Zero-Truncated Negative Binomial (LZTNB) distribution of Monisha et al. (2023), are given, respectively, by

$$F\left(x;\lambda,p,\mu\right)=C\sum_{u=1}^{x}p^{u}P\left(u\right),\quad x\in\mathbb{N}, \tag{25}$$

$$f\left(x;\lambda,p,\mu\right)=Cp^{x}P\left(x\right),\quad x\in\mathbb{N}, \tag{26}$$

$$r_F\left(x\right)=\frac{p^{x}P\left(x\right)}{\sum_{u=1}^{x}p^{u}P\left(u\right)},\quad x\in\mathbb{N}, \tag{27}$$

$$h_F\left(x\right)=\frac{Cp^{x}P\left(x\right)}{1-C\sum_{u=1}^{x}p^{u}P\left(u\right)},\quad x\in\mathbb{N}, \tag{28}$$

where $\lambda<qp^{-1},p\in(0,1),q=1-p,\mu>0$ are parameters, $C=\frac{(q-\lambda p)q^{\mu-1}}{(1-q^{\mu})}$ and $P\left(x\right)=q^{\lambda x}\left\{\binom{\mu+\lambda x+x-1}{x}-\binom{\lambda x+x-1}{x}\right\}$.

Remark 1.5. *The pmf* (26) *is similar to the pmf* (22).

(8) The cdf, pmf, reverse hazard and hazard functions of Poisson 2S-Lindley (P2S-L) distribution of Chesneau et al. (2023b), are given, respectively, by

$$F\left(x;\theta\right)=C\sum_{u=0}^{x}\left(\frac{1}{1+\theta}\right)^{u}P\left(u\right),\quad x\in\mathbb{N}^{*}, \tag{29}$$

$$f\left(x;\theta\right)=C\left(\frac{1}{1+\theta}\right)^{x}P\left(x\right),\quad x\in\mathbb{N}^{*}, \tag{30}$$

$$r_F\left(x\right)=\frac{\left(\frac{1}{1+\theta}\right)^{x}P\left(x\right)}{\sum_{u=0}^{x}\left(\frac{1}{1+\theta}\right)^{u}P\left(u\right)},\quad x\in\mathbb{N}^{*}, \tag{31}$$

$$h_F\left(x\right)=\frac{C\left(\frac{1}{1+\theta}\right)^{x}P\left(x\right)}{1-C\sum_{u=0}^{x}\left(\frac{1}{1+\theta}\right)^{u}P\left(u\right)},\quad x\in\mathbb{N}^{*}, \tag{32}$$

where $\theta>0$ is a parameter, $C=\frac{\theta^{4}}{6(1+\theta)^{6}}$ and $P\left(x\right)=x^{2}+6\left(\theta+2\right)^{2}+x\left(11+6\theta\right)$.

Remark 1.6. *The pmf* (30) *is similar to the pmf* (26) *for* $p=\frac{1}{1+\theta}$.

(9) The cdf, pmf, reverse hazard and hazard functions of Zero-and-Plus/Minus-One Inflated Extended Poisson (ZMOIEP) distribution of Kachour and Chesneau (2023), for $p = 1$, are given, respectively, by

$$F\left(x;\alpha_{-1},\alpha_0,\alpha_1,\beta,\lambda\right)=\begin{cases}\alpha_{-1} & ,x=-1\\ \alpha_{-1}+\alpha_0+(1-\beta)e^{-\lambda} & ,x=0\\ \alpha_{-1}+\alpha_0+(1-\beta)e^{-\lambda}+\alpha_1+(1-\beta)\lambda e^{-\lambda} & ,x=1\\ F(1;\alpha_{-1},\alpha_0,\alpha_1,\beta,\lambda)+(1-\beta)e^{-\lambda}\sum_{u=2}^{x}\frac{\lambda^u}{u!} & ,x\geq 2\end{cases}, \tag{33}$$

$$f\left(x;\alpha_{-1},\alpha_0,\alpha_1,\beta,\lambda\right)=\begin{cases}\alpha_{-1} & ,x=-1\\ \alpha_0+(1-\beta)e^{-\lambda} & ,x=0\\ \alpha_1+(1-\beta)\lambda e^{-\lambda} & ,x=1\\ (1-\beta)e^{-\lambda}\frac{\lambda^x}{x!} & ,x\geq 2\end{cases}, \tag{34}$$

$$r_F\left(x\right)=\frac{f\left(x;\alpha_{-1},\alpha_0,\alpha_1,\beta,\lambda\right)}{F\left(x;\alpha_{-1},\alpha_0,\alpha_1,\beta,\lambda\right)},\quad x\in\mathbb{N}^*, \tag{35}$$

$$h_F\left(x\right)=\frac{f\left(x;\alpha_{-1},\alpha_0,\alpha_1,\beta,\lambda\right)}{1-F\left(x;\alpha_{-1},\alpha_0,\alpha_1,\beta,\lambda\right)},\quad x\in\mathbb{N}^*, \tag{36}$$

where $0\leq\alpha_{-1}<1, 0\leq\alpha_0<1, 0<\alpha_1<1, 0\leq 1-\beta=1-(\alpha_{-1}+\alpha_0+\alpha_1)<1, \lambda\geq 0$ are parameters.

Remark 1.7. *As far as characterizations are concerned, the pmf* (34)*, for $x\geq 2$, can be written as $f\left(x;\alpha_{-1},\alpha_0,\alpha_1,\beta,\lambda\right)=\left(1-\beta\right)e^{-\lambda}\lambda^x P\left(x\right)$, where $P\left(x\right)=\frac{1}{x!}$. This kind of the pmf was characterized before.*

(10) The cdf, pmf, reverse hazard and hazard functions of Discretized Cubic Transmuted Ailamujia (DCTA) distribution of Ademuyiwa et al. (2023) are given, respectively, by

$$F\left(x;\beta,c,k\right)=\sum_{u=0}^{x}f\left(u;\beta,c,k\right),\quad x\in\mathbb{N}^*, \tag{37}$$

$$\begin{aligned}f\left(x;\beta,c,k\right)=&\left\{cG\left(x+1\right)+\left(k-c\right)G^2\left(x+1\right)+\left(1-k\right)G^3\left(x+1\right)\right\}-\\&\left\{cG\left(x\right)+\left(k-c\right)G^2\left(x\right)+\left(1-k\right)G^3\left(x\right)\right\},\quad x\in\mathbb{N}^*,\end{aligned} \tag{38}$$

$$r_F\left(x\right)=\frac{f\left(x;\beta,c,k\right)}{F\left(x;\beta,c,k\right)},\quad x\in\mathbb{N}^*, \tag{39}$$

$$h_F\left(x\right)=\frac{f\left(x;\beta,c,k\right)}{1-F\left(x;\beta,c,k\right)},\quad x\in\mathbb{N}^*, \tag{40}$$

where $\beta>0, c\in\left[0,1\right], k\in\left[-1,1\right]$ are parameters and $G\left(x\right)=1-\left(1+\beta x\right)e^{-\beta x}$, $x\in\mathbb{N}^*$.

Remark 1.8. *Characterizations similar to those for the pmf* (18) *can be stated for the pmf* (38).

(11) The cdf, pmf, reverse hazard and hazard functions of Discrete Logistic Exponential (DLE) distribution of Al-Bossly et al. (2023) are given, respectively, by

$$F\left(x;\alpha,\beta\right)=\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}},\quad x\in\mathbb{N}^*, \tag{41}$$

$$f\left(x;\alpha,\beta\right)=\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}}-\frac{\left(e^{\beta x}-1\right)^{\alpha}}{1+\left(e^{\beta x}-1\right)^{\alpha}},\quad x\in\mathbb{N}^*, \tag{42}$$

$$r_F(x) = 1 - \frac{\frac{\left(e^{\beta x}-1\right)^{\alpha}}{1+\left(e^{\beta x}-1\right)^{\alpha}}}{\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}}}, \quad x \in \mathbb{N}^*, \tag{43}$$

$$h_F(x) = \frac{\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}} - \frac{\left(e^{\beta x}-1\right)^{\alpha}}{1+\left(e^{\beta x}-1\right)^{\alpha}}}{1 - \frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}}}, \quad x \in \mathbb{N}^*, \tag{44}$$

where $\alpha > 0, \beta > 0$ are parameters.

(12) The cdf, pmf, reverse hazard and hazard functions of Discrete Odd Inverse Pareto Exponential (DOIPEx) distribution of Saengthong and Senoi (2023) are given, respectively, by

$$F(x;\alpha,\beta,\lambda) = \left(\frac{1-e^{-\lambda(x+1)}}{1-(1-\beta)\,e^{-\lambda(x+1)}}\right)^{\alpha}, \quad x \in \mathbb{N}^*, \tag{45}$$

$$f(x;\alpha,\beta,\lambda) = \left(\frac{1-e^{-\lambda(x+1)}}{1-(1-\beta)\,e^{-\lambda(x+1)}}\right)^{\alpha} - \left(\frac{1-e^{-\lambda x}}{1-(1-\beta)\,e^{-\lambda x}}\right)^{\alpha}, \quad x \in \mathbb{N}^*, \tag{46}$$

$$r_F(x) = 1 - \frac{\left(\frac{1-e^{-\lambda x}}{1-(1-\beta)e^{-\lambda x}}\right)^{\alpha}}{\left(\frac{1-e^{-\lambda(x+1)}}{1-(1-\beta)e^{-\lambda(x+1)}}\right)^{\alpha}}, \quad x \in \mathbb{N}^*, \tag{47}$$

$$h_F(x) = \frac{\left(\frac{1-e^{-\lambda(x+1)}}{1-(1-\beta)e^{-\lambda(x+1)}}\right)^{\alpha} - \left(\frac{1-e^{-\lambda x}}{1-(1-\beta)e^{-\lambda x}}\right)^{\alpha}}{1 - \left(\frac{1-e^{-\lambda(x+1)}}{1-(1-\beta)e^{-\lambda(x+1)}}\right)^{\alpha}}, \quad x \in \mathbb{N}^*, \tag{48}$$

where α, β, λ are all positive parameters.

Remark 1.9. *Characterizations similar to those for the pmf* (42) *can be stated for the pmf* (46).

(13) The cdf, pmf, reverse hazard and hazard functions of New Discrete Rayleigh Type I (NDRTI) distribution of Krishnakumari and George (2023) are given, respectively, by

$$F(x;\sigma) = \sum_{u=1}^{x} f(u;\sigma), \quad x \in \mathbb{N}, \tag{49}$$

$$f(x;\sigma) = \left(e^{-\frac{1}{2\sigma^2}}\right)^{x^2} P(x), \quad x \in \mathbb{N}, \tag{50}$$

$$r_F(x) = \frac{\left(e^{-\frac{1}{2\sigma^2}}\right)^{x^2} P(x)}{\sum_{u=1}^{x}\left(e^{-\frac{1}{2\sigma^2}}\right)^{u^2} P(u)}, \quad x \in \mathbb{N}, \tag{51}$$

$$h_F(x) = \frac{\left(e^{-\frac{1}{2\sigma^2}}\right)^{x^2} P(x)}{1 - \sum_{u=1}^{x}\left(e^{-\frac{1}{2\sigma^2}}\right)^{u^2} P(u)}, \quad x \in \mathbb{N}, \tag{52}$$

where $\sigma > 0$ is a parameter and $P(x) = \exp\left(\sum_{j=2}^{\infty}(-1)^j\left(\frac{x^2}{2\sigma^2}\right)^j\right)\left(\Pi_{i=1}^{x-1}\left(\exp\left\{e^{-\frac{x^2}{2\sigma^2}}-1\right\}\right)\right)$.

Remarks 1.10. (a) *The pmf* (50) *is similar to the pmf* (22). (b) *The same can be said for NDRTII and NDRTIII mentioned on pages 859-860 of the authors paper.*

(14) The cdf, pmf, reverse hazard and hazard functions of Poisson-Samade (PS) distribution of Aderoju et al. (2023) are given, respectively, by

$$F\left(x;\alpha,\theta\right)=\sum_{u=0}^{x}f\left(u;\alpha,\theta\right),\quad x\in\mathbb{N}^{*}, \tag{53}$$

$$f\left(x;\alpha,\theta\right)=C\left(\theta+1\right)^{-x}P\left(x\right),\quad x\in\mathbb{N}^{*}, \tag{54}$$

$$r_{F}\left(x\right)=\frac{\left(\theta+1\right)^{-x}P\left(x\right)}{\sum_{u=0}^{x}\left(\theta+1\right)^{-u}P\left(u\right)},\quad x\in\mathbb{N}^{*}, \tag{55}$$

$$h_{F}\left(x\right)=\frac{C\left(\theta+1\right)^{-x}P\left(x\right)}{1-C\sum_{u=0}^{x}\left(\theta+1\right)^{-u}P\left(u\right)},\quad x\in\mathbb{N}^{*}, \tag{56}$$

where $\alpha\geq0,\theta>0$ are parameters, $C=\frac{\theta^{4}}{(\theta^{4}+6\alpha)(\theta+1)^{4}}$ and $P\left(x\right)=\theta\left(\theta+1\right)^{3}+\alpha\left(x^{3}+6x^{2}+11x+6\right)$.

Remark 1.11. *The pmf* (54) *is similar to the pmf* (30).

(15) The cdf, pmf, reverse hazard and hazard functions of Discrete Alpha Log-Transformation of Inverse Exponential (DALTIE) distribution of Ahmad and Almetwally (2023) are given, respectively, by

$$F\left(x;\alpha,\theta\right)=1-\frac{\log\left(\alpha-\left(\alpha-1\right)e^{-\theta/(x+1)}\right)}{\log\left(\alpha\right)},\quad x\in\mathbb{N}^{*}, \tag{57}$$

$$f\left(x;\alpha,\theta\right)=\frac{1}{\log\left(\alpha\right)}\left\{\log\left(\alpha-\left(\alpha-1\right)e^{-\theta/x}\right)-\log\left(\alpha-\left(\alpha-1\right)e^{-\theta/(x+1)}\right)\right\},\quad x\in\mathbb{N}^{*}, \tag{58}$$

$$r_{F}\left(x\right)=\frac{\log\left(\alpha-\left(\alpha-1\right)e^{-\theta/x}\right)-\log\left(\alpha-\left(\alpha-1\right)e^{-\theta/(x+1)}\right)}{\log\left(\alpha\right)-\log\left(\alpha-\left(\alpha-1\right)e^{-\theta/(x+1)}\right)},\quad x\in\mathbb{N}^{*}, \tag{59}$$

$$h_{F}\left(x\right)=\frac{\log\left(\alpha-\left(\alpha-1\right)e^{-\theta/x}\right)}{\log\left(\alpha-\left(\alpha-1\right)e^{-\theta/(x+1)}\right)}-1,\quad x\in\mathbb{N}^{*}, \tag{60}$$

where $\alpha>0\left(\alpha\neq1\right),\theta>0$ are parameters.

Remark 1.12. *Characterizations similar to those for the pmf* (42) *can be stated for the pmf* (58).

(16) The cdf, pmf, reverse hazard and hazard functions of Modified Discrete Version of the Length Biased Tornumonkpe (MDVLBT) distribution of Saraja et al. (2023) are given, respectively, by

$$F\left(x;\theta\right)=\sum_{u=1}^{x}f\left(u;\theta\right),\quad x\in\mathbb{N}, \tag{61}$$

$$f\left(x;\theta\right)=Ce^{-\theta x}P\left(x\right),\quad x\in\mathbb{N}, \tag{62}$$

$$r_{F}\left(x\right)=\frac{e^{-\theta x}P\left(x\right)}{\sum_{u=1}^{x}e^{-\theta u}P\left(u\right)},\quad x\in\mathbb{N}, \tag{63}$$

$$h_{F}\left(x\right)=\frac{Ce^{-\theta x}P\left(x\right)}{1-C\sum_{u=1}^{x}e^{-\theta u}P\left(u\right)},\quad x\in\mathbb{N}, \tag{64}$$

where $\theta>0$ is a parameter, $C=\frac{\left(e^{\theta}-1\right)^{3}}{e^{\theta}[(1+\theta)e^{\theta}+(1-\theta)]}$ and $P\left(x\right)=x^{2}+\theta x$.

Remark 1.13. *Characterizations similar to those for the pmf* (22) *can be stated for the pmf* (62).

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(17) The cdf, pmf, reverse hazard and hazard functions of Discrete Exponentiated-G (DE-G) family of distributions of EL-Hady et al. (2023) are given, respectively, by

$$F\left(x;\alpha\right)=\left[G\left(x+1\right)\right]^{\alpha},\quad x\in\mathbb{N}^{*}, \tag{65}$$

$$f\left(x;\alpha\right)=\left[G\left(x+1\right)\right]^{\alpha}-\left[G\left(x\right)\right]^{\alpha},\quad x\in\mathbb{N}^{*}, \tag{66}$$

$$r_{F}\left(x\right)=1-\frac{\left[G\left(x\right)\right]^{\alpha}}{\left[G\left(x+1\right)\right]^{\alpha}},\quad x\in\mathbb{N}^{*}, \tag{67}$$

$$h_{F}\left(x\right)=\frac{\left[G\left(x+1\right)\right]^{\alpha}-\left[G\left(x\right)\right]^{\alpha}}{1-\left[G\left(x+1\right)\right]^{\alpha}},\quad x\in\mathbb{N}^{*}, \tag{68}$$

where $\alpha>0$ is a parameter and $G\left(x\right)$ is a baseline cdf.

Remark 1.14. *The pmf* (66) *can be considered as a special case of the pmf* (38).

(18) The cdf, pmf, reverse hazard and hazard functions of Alternative Discrete Analogue of Half-Logistic (ADAHL) of distribution by Barbiero and Hitaj (2023) are given, respectively, by

$$F\left(x;\theta\right)=\frac{2}{\theta}\log\left(\frac{1+e^{\theta(x+1)}}{1+e^{\theta x}}\right)-1,\quad x\in\mathbb{N}^{*}, \tag{69}$$

$$f\left(x;\theta\right)=\frac{2}{\theta}\left\{\log\left(\frac{1+e^{\theta(x+1)}}{1+e^{\theta x}}\right)-\log\left(\frac{1+e^{\theta x}}{1+e^{\theta(x-1)}}\right)\right\},\ x\in\mathbb{N}^{*}, \tag{70}$$

$$r_{F}\left(x\right)=1-\frac{\frac{2}{\theta}\log\left(\frac{1+e^{\theta x}}{1+e^{\theta(x-1)}}\right)-1}{\frac{2}{\theta}\log\left(\frac{1+e^{\theta(x+1)}}{1+e^{\theta x}}\right)-1},\quad x\in\mathbb{N}^{*}, \tag{71}$$

$$h_{F}\left(x\right)=\frac{\frac{2}{\theta}\left\{\log\left(\frac{1+e^{\theta(x+1)}}{1+e^{\theta x}}\right)-\log\left(\frac{1+e^{\theta x}}{1+e^{\theta(x-1)}}\right)\right\}}{2-\frac{2}{\theta}\log\left(\frac{1+e^{\theta(x+1)}}{1+e^{\theta x}}\right)},\ x\in\mathbb{N}^{*}, \tag{72}$$

where $\theta>0$ is a parameter and $f\left(0;\theta\right)=\frac{4}{\theta}\log\left(\frac{1+e^{\theta}}{2}\right)-2$.

Remark 1.15. *The pmf* (70) *is similar to the pmf* (58).

(19) The cdf, pmf, reverse hazard and hazard functions of Zero-Inflated Poisson Moment Exponential (ZIPMEx) distribution of Skinder et al. (2023), are given, respectively, by

$$F\left(x;p,\delta\right)=\begin{cases}p+\left(1-p\right)\frac{1}{(1+\delta)^{2}} & ,x=0\\ C\sum_{u=1}^{x}\left(1+\delta\right)^{-u}P\left(u\right) & ,x\in\mathbb{N}\end{cases}, \tag{73}$$

$$f\left(x;p,\delta\right)=\begin{cases}p+\left(1-p\right)\frac{1}{(1+\delta)^{2}} & ,x=0\\ C\left(1+\delta\right)^{-x}P\left(x\right) & ,x\in\mathbb{N}\end{cases}, \tag{74}$$

$$r_{F}\left(x\right)=\begin{cases}1 & ,x=0\\ \dfrac{\left(1+\delta\right)^{-x}P\left(x\right)}{\sum_{u=1}^{x}\left(1+\delta\right)^{-u}P\left(u\right)} & ,x\in\mathbb{N}\end{cases}, \tag{75}$$

$$h_F(x) = \begin{cases} \dfrac{p + (1-p)\frac{1}{(1+\delta)^2}}{1 - p - (1-p)\frac{1}{(1+\delta)^2}} & , x = 0 \\ \dfrac{C(1+\delta)^{-x} P(x)}{1 - C\sum_{u=1}^{x}(1+\delta)^{-u} P(u)} & , x \in \mathbb{N} \end{cases}, \tag{76}$$

where $0 < p < 1, \delta > 0$ are parameters, $C = \frac{(1-p)}{(1+\delta)^2}$ and $P(x) = \delta^x (1+x)$.

Remark 1.16. *The cdf* (73) *is similar to the cdf* (1).

(20) The cdf, pmf, reverse hazard and hazard functions of Zero-Inflated Poisson-Akash (ZIPA) distribution of Wani and Ahmad (2023), are given, respectively, by

$$F(x;\theta,\psi) = \begin{cases} \psi + (1-\psi)\frac{\theta^5+2\theta^4+3\theta^3}{(\theta^2+2)(1+\theta)^3} & , x = 0 \\ C\ \sum_{u=1}^{x}(1+\theta)^{-u} P(u) & , x \in \mathbb{N} \end{cases}, \tag{77}$$

$$f(x;\theta,\psi) = \begin{cases} \psi + (1-\psi)\frac{\theta^5+2\theta^4+3\theta^3}{(\theta^2+2)(1+\theta)^3} & , x = 0 \\ C(1+\theta)^{-x} P(x) & , x \in \mathbb{N} \end{cases}, \tag{78}$$

$$r_F(x) = \begin{cases} 1 & , x = 0 \\ \frac{(1+\theta)^{-x} P(x)}{\sum_{u=1}^{x}(1+\theta)^{-u} P(u)} & , x \in \mathbb{N} \end{cases}, \tag{79}$$

$$h_F(x) = \begin{cases} \frac{\psi + (1-\psi)\frac{\theta^5+2\theta^4+3\theta^3}{(\theta^2+2)(1+\theta)^3}}{1 - \psi - (1-\psi)\frac{\theta^5+2\theta^4+3\theta^3}{(\theta^2+2)(1+\theta)^3}} & , x = 0 \\ \frac{C(1+\theta)^{-x} P(x)}{1 - C\sum_{u=1}^{x}(1+\theta)^{-u} P(u)} & , x \in \mathbb{N} \end{cases}, \tag{80}$$

where $0 < \psi < 1, \theta > 0$ are parameters, $C = \frac{(1-\psi)\theta^3}{(\theta^2+2)(1+\theta)^3}$ and $P(x) = x^2 + 3x + \theta^2 + 2\theta + 3$.

Remark 1.17. *The cdf* (77) *is similar to the cdf* (73).

(21) The cdf, pmf, reverse hazard and hazard functions of Two Parameter Poisson-XGamma (TPPXG) distribution of Wani et al. (2023) are given, respectively, by

$$F(x;\theta) = \sum_{u=0}^{x} f(x;\theta), \quad x \in \mathbb{N}^*, \tag{81}$$

$$f(x;\theta) = C(1+\theta)^{-x} P(x), \quad x \in \mathbb{N}^*, \tag{82}$$

$$r_F(x) = \frac{(1+\theta)^{-x} P(x)}{\sum_{u=0}^{x}(1+\theta)^{-u} P(u)}, \quad x \in \mathbb{N}^*, \tag{83}$$

$$h_F(x) = \frac{C(1+\theta)^{-x} P(x)}{1 - C\sum_{u=0}^{x}(1+\theta)^{-u} P(u)}, \quad x \in \mathbb{N}^*, \tag{84}$$

where $\theta > 0$ is a parameter, $C = \frac{\theta^2}{(\alpha+\theta)(1+\theta)^3}$ and $P(x) = (1+\theta)^2 + \frac{\alpha\theta}{2}(x+1)(x+2)$.

Remark 1.18. *The cdf* (82) *is similar to the cdf* (53).

(22) The cdf, pmf, reverse hazard and hazard functions of Discretized Fréchet-Weibull (DFW) distribution of Das and Das (2023) are given, respectively, by

$$F\left(x;\alpha,\beta,m,k\right)=\exp\left\{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha k}\right\},\quad x\in\mathbb{N}^{*},\tag{85}$$

$$f\left(x;\alpha,\beta,m,k\right)=\exp\left\{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha k}\right\}-\exp\left\{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}\right\},\quad x\in\mathbb{R},\tag{86}$$

$$r_{F}\left(x\right)=1-\frac{\exp\left\{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha k}\right\}},\quad x\in\mathbb{N}^{*},\tag{87}$$

$$h_{F}\left(x\right)=\frac{\exp\left\{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha k}\right\}-\exp\left\{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}\right\}}{1-\exp\left\{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha k}\right\}},\quad x\in\mathbb{N}^{*},\tag{88}$$

where α,β,m,k are all positive parameters.

(23) The cdf, pmf, reverse hazard and hazard functions of Double Poisson (DP) distribution of Kokonendji et al. (2023) are given, respectively, by

$$F\left(x;\gamma,\lambda\right)=\sum_{u=1}^{x}f\left(u;\gamma,\lambda\right),\quad x\in\mathbb{N}^{*},\tag{89}$$

$$f\left(x;\gamma,\lambda\right)=Ce^{-x}P\left(x\right),\quad x\in\mathbb{N}^{*},\tag{90}$$

$$r_{F}\left(x\right)=\frac{e^{-x}P\left(x\right)}{\sum_{u=1}^{x}e^{-u}P\left(u\right)},\quad x\in\mathbb{N}^{*},\tag{91}$$

$$h_{F}\left(x\right)=\frac{Ce^{-x}P\left(x\right)}{1-C\sum_{u=1}^{x}e^{-u}P\left(u\right)},\quad x\in\mathbb{N}^{*},\tag{92}$$

where $\gamma>0,\lambda>0$ are parameters, $C=k\left(\gamma,\lambda\right)\gamma^{1/2}\lambda^{\gamma\lambda}, k\left(\gamma,\lambda\right)$ is the normalizing constant and $P\left(x\right)=\frac{x^{x-\gamma\lambda}}{x!}$.

Remark 1.19. *The pmf* (90) *is similar to the pmf* (62).

(24) The cdf, pmf, reverse hazard and hazard functions of Gamma-Count (GC) distribution of Kokonendji et al. (2023) are given, respectively, by

$$F\left(x;\alpha,\beta\right)=1-G\left(\alpha(x+1),\beta T\right),\quad x\in\mathbb{N}^{*},\tag{93}$$

$$f\left(x;\alpha,\beta\right)=G(\alpha x,\beta T)-G\left(\alpha(x+1),\beta T\right),\quad x\in\mathbb{N}^{*},\tag{94}$$

$$r_{F}\left(x\right)=\frac{G(\alpha x,\beta T)-G\left(\alpha(x+1),\beta T\right)}{1-G\left(\alpha(x+1),\beta T\right)},\quad x\in\mathbb{N}^{*},\tag{95}$$

$$h_{F}\left(x\right)=\frac{G(\alpha x,\beta T)}{G\left(\alpha(x+1),\beta T\right)}-1,\quad x\in\mathbb{N}^{*},\tag{96}$$

where $\alpha>0,\beta>0$ are parameters and $G(\alpha x,\beta T)=\frac{1}{\Gamma(\alpha x)}\int_{0}^{\beta T}u^{\alpha x-1}e^{-u}du$ is a baseline cdf.

Remark 1.20. *The pmf* (94) *is similar to the pmf* (66).

(25) The cdf, pmf, reverse hazard and hazard functions of Poisson Epanechnikov-Exponential (PEE) distribution of Karakaya (2023) are given, respectively, by

$$F(x;\beta)=\sum_{u=0}^{x} f(u;\beta), \quad x\in\mathbb{N}^{*}, \tag{97}$$

$$f(x;\beta)=C(2\beta+1)^{-x}P(x), \quad x\in\mathbb{N}^{*}, \tag{98}$$

$$r_F(x)=\frac{(2\beta+1)^{-x}P(x)}{\sum_{u=0}^{x}(2\beta+1)^{-u}P(u)}, \quad x\in\mathbb{N}^{*}, \tag{99}$$

$$h_F(x)=\frac{C(2\beta+1)^{-x}P(x)}{1-C\sum_{u=0}^{x}(2\beta+1)^{-u}P(u)}, \quad x\in\mathbb{N}^{*}, \tag{100}$$

where $\beta>0$ is a parameter, $C=\frac{3\beta}{2\beta+1}$ and $P(x)=2-\left(\frac{3\beta+1}{2\beta+1}\right)^{-(x+1)}$.

Remark 1.21. *The pmf* (98) *is similar to the pmf* (82).

(26) The cdf, pmf, reverse hazard and hazard functions of Discrete Odd Exponentiated Half-Logistic Inverse Exponential (DOEHLIEx) distribution are given, respectively, by

$$F(x;\lambda,\beta,\alpha)=\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}, \quad x\in\mathbb{N}^{*}, \tag{101}$$

$$f(x;\lambda,\beta,\alpha)=\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}-\left(\frac{1-e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}\right)^{\alpha}, \quad x\in\mathbb{N}^{*}, \tag{102}$$

$$r_F(x)=1-\frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}\right)^{\alpha}}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}}, \quad x\in\mathbb{N}^{*}, \tag{103}$$

$$h_F(x)=\frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}-\left(\frac{1-e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}\right)^{\alpha}}{1-\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}}, \quad x\in\mathbb{N}^{*}, \tag{104}$$

where λ,β,α are all positive parameters.

Remark 1.22. *The cdf* (101) *is the discrete version of the cdf OEHLIEx of Eliwa et al. (2021).*

2. Characterizations Based on Conditional Expectation

In this Section, we present our characterizations of all the distributions listed in Section 1 in terms of the conditional expectations of certain functions of the random variables. The choice of each function depends on the form of the pmf. Most of the proofs follow the same scheme, we will give all of them for the sake of completeness.

2.1. Burnoulli-Poisson-Lindley (BPL) Distribution

Proposition 2.1. *Let* $X:\Omega\to\mathbb{N}^{*}=\mathbb{N}\cup\{0\}$ *be a random variable. The pmf of* X *is* (2) *if and only if*

$$E\left\{[P(X)]^{-1}\mid X\le k\right\}=\frac{1-(\theta+1)^{-(k+1)}}{\theta\sum_{x=1}^{k}(\theta+1)^{-x}P(x)}. \tag{105}$$

Proof. If X has pmf (2), then for $k \in \mathbb{N}$, the left-hand side of (105), using finite geometric sum formula, will be

$$\begin{aligned}(F(k))^{-1} \sum_{x=1}^{k} \left\{ C(\theta+1)^{-x} \right\} &= \left(C \sum_{x=1}^{k} (\theta+1)^{-x} P(x) \right)^{-1} \sum_{x=1}^{k} \left\{ C(\theta+1)^{-x} \right\} \\ &= \frac{1-(\theta+1)^{-(k+1)}}{\theta \sum_{x=1}^{k} (\theta+1)^{-x} P(x)} .\end{aligned}$$

Conversely, if (105) holds, then

$$\sum_{x=1}^{k} \left\{ [P(x)]^{-1} \ f(x) \right\} = F(k) \left(\frac{1-(\theta+1)^{-(k+1)}}{\theta \sum_{x=1}^{k} (\theta+1)^{-x} P(x)} \right). \tag{106}$$

From (106), we also have

$$\begin{aligned}\sum_{x=1}^{k-1} \left\{ [P(x)]^{-1} \ f(x) \right\} &= F(k-1) \left(\frac{1-(\theta+1)^{-k}}{\theta \sum_{u=1}^{k-1} (\theta+1)^{-u} P(u)} \right) \\ &= (F(k) - f(k)) \left(\frac{1-(\theta+1)^{-k}}{\theta \sum_{u=1}^{k-1} (\theta+1)^{-u} P(u)} \right).\end{aligned} \tag{107}$$

Now, subtracting (107) from (106), yields

$$\begin{aligned}\frac{1}{P(k)} f(k) =& F(k) \left\{ \left(\frac{1-(\theta+1)^{-(k+1)}}{\theta \sum_{x=1}^{k} (\theta+1)^{-x} P(x)} \right) - \left(\frac{1-(\theta+1)^{-k}}{\theta \sum_{x=1}^{k-1} (\theta+1)^{-x} P(x)} \right) \right\} \\ &+ f(k) \left(\frac{1-(\theta+1)^{-k}}{\theta \sum_{x=1}^{k-1} (\theta+1)^{-x} P(x)} \right).\end{aligned}$$

From the above equality, after some algebra, we have

$$\begin{aligned}\frac{f(k)}{F(k)} &= \frac{\left(\frac{1-(\theta+1)^{-(k+1)}}{\theta \sum_{x=1}^{k} (\theta+1)^{-x} P(x)} \right) - \left(\frac{1-(\theta+1)^{-k}}{\theta \sum_{x=1}^{k-1} (\theta+1)^{-x} P(x)} \right)}{\frac{1}{P(k)} - \left(\frac{1-(\theta+1)^{-k}}{\theta \sum_{x=1}^{k-1} (\theta+1)^{-x} P(x)} \right)} \\ &= \frac{P(k)(\theta+1)^{-k} \left\{ \sum_{x=1}^{k-1} (\theta+1)^{-x} P(x) - P(k) + (\theta+1)^{-k} P(k) \right\}}{\sum_{x=1}^{k} (\theta+1)^{-x} P(x) \left\{ \sum_{x=1}^{k-1} (\theta+1)^{-x} P(x) - P(k) + (\theta+1)^{-k} P(k) \right\}} \\ &= \frac{P(k)(\theta+1)^{-k}}{\sum_{x=1}^{k} (\theta+1)^{-x} P(x)},\end{aligned}$$

which is the reverse hazard function corresponding to the pmf (2), so X has pmf (2). □

2.2. Discrete Cos-Lindley (DCL) Distribution

Proposition 2.2. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (6) *if and only if*

$$E \left\{ \left[P(X+1) e^{-(X+1)\theta} + P(X) e^{-\theta X} \right] \mid X \leq k \right\} = P(k+1) e^{-\theta(k+1)}. \tag{108}$$

Proof. If X has pmf (6), then for $k \in \mathbb{N}$, the left-hand side of (108), using telescoping sum formula, will be

$$(F(k))^{-1}\sum_{x=0}^{k}C\left[P^{2}(x+1)e^{-2(x+1)\theta}-P^{2}(x)e^{-2\theta x}\right]=\left(CP(k+1)e^{-\theta(k+1)}\right)^{-1}C\left(P^{2}(k+1)e^{-2\theta(k+1)}\right)$$
$$=P(k+1)e^{-\theta(k+1)}.$$

Conversely, if (108) holds, then

$$\sum_{x=0}^{k}\left\{[P(x)]^{-1}f(x)\right\}=F(k)\left(P(k+1)e^{-\theta(k+1)}\right). \tag{109}$$

From (109), we also have

$$\sum_{x=0}^{k-1}\left\{[P(x)]^{-1}f(x)\right\}=F(k-1)\left(P(k)e^{-\theta k}\right)=(F(k)-f(k))\left(P(k)e^{-\theta k}\right). \tag{110}$$

Now, subtracting (110) from (109), yields

$$\left\{\frac{1}{P(k)}-\left(P(k)e^{-\theta k}\right)\right\}f(k)=F(k)\left\{\left(P(k+1)e^{-\theta(k+1)}\right)-\left(P(k)e^{-\theta k}\right)\right\}.$$

From the above equality, we have

$$\frac{f(k)}{F(k)}=\frac{\left(P(k+1)e^{-\theta(k+1)}\right)-\left(P(k)e^{-\theta k}\right)}{\frac{1}{P(k)}-\left(P(k)e^{-\theta k}\right)},$$

which after some computations

$$\frac{f(k)}{F(k)}=1-\frac{P(k)e^{-\theta k}}{P(k+1)e^{-\theta(k+1)}}$$

which is the reverse hazard function corresponding to the pmf (6), so X has pmf (6). □

2.3. Discrete Gamma Power Lomax (DGPL) Distribution

Proposition 2.3. *Let* $X:\Omega\to\mathbb{N}^{*}$ *be a random variable. The pmf of* X *is* (10) *if and only if*

$$E\left\{\left[\gamma\left(\upsilon,-\log\left(1+\frac{(X+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)+\gamma\left(\upsilon,-\log\left(1+\frac{X^{\lambda}}{\theta}\right)^{-\alpha}\right)\right]\mid X\leq k\right\}$$
$$=\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)}{\theta}\right)^{-\alpha}\right). \tag{111}$$

Proof. If X has pmf (10), then for $k\in\mathbb{N}$, the left-hand side of (111), using telescoping sum formula, will be

$$(F(k))^{-1}\sum_{x=0}^{k}\frac{1}{\Gamma(\upsilon)}\left\{\gamma^{2}\left(\upsilon,-\log\left(1+\frac{(x+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)-\gamma^{2}\left(\upsilon,-\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right)\right\}$$
$$=\left(\frac{1}{\Gamma(\upsilon)}\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)}{\theta}\right)^{-\alpha}\right)\right)^{-1}\frac{1}{\Gamma(\upsilon)}\gamma^{2}\left(\upsilon,-\log\left(1+\frac{(k+1)}{\theta}\right)^{-\alpha}\right)$$
$$=\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)}{\theta}\right)^{-\alpha}\right).$$

Conversely, if (111) holds, then

$$\sum_{x=0}^{k}\left\{\gamma\left(\upsilon,-\log\left(1+\frac{(x+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)+\gamma\left(\upsilon,-\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right)\right\}$$
$$=F(k)\,\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)^{\lambda}}{\theta}\right)^{-\alpha}\right). \tag{112}$$

From (112), we also have

$$\sum_{x=0}^{k-1}\left\{\gamma\left(\upsilon,-\log\left(1+\frac{(x+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)+\gamma\left(\upsilon,-\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right)\right\}$$
$$=F(k-1)\,\gamma\left(\upsilon,-\log\left(1+\frac{k^{\lambda}}{\theta}\right)^{-\alpha}\right)$$
$$=(F(k)-f(k))\,\gamma\left(\upsilon,-\log\left(1+\frac{k^{\lambda}}{\theta}\right)^{-\alpha}\right). \tag{113}$$

Now, subtracting (113) from (112), yields

$$\left\{\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)\right\}f(k)$$
$$=F(k)\left\{\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)-\gamma\left(\upsilon,-\log\left(1+\frac{k^{\lambda}}{\theta}\right)^{-\alpha}\right)\right\}.$$

From the above equality, we have

$$\frac{f(k)}{F(k)}=\frac{\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)-\gamma\left(\upsilon,-\log\left(1+\frac{k^{\lambda}}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)}=1-\frac{\gamma\left(\upsilon,-\log\left(1+\frac{k^{\lambda}}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon,-\log\left(1+\frac{(k+1)^{\lambda}}{\theta}\right)^{-\alpha}\right)},$$

which is the reverse hazard function corresponding to the pmf (10), so X has pmf (10). □

2.4. New Lomax Rayleigh Discrete (NLRD) Distribution

Proposition 2.4. *Let* $X:\Omega\rightarrow\mathbb{N}^{*}$ *be a random variable. The pmf of* X *is* (14) *if and only if*

$$E\left\{\left[\left[1+\frac{X^{2}}{2\lambda\sigma^{2}}\right]^{-\theta}+\left[1+\frac{(X+1)^{2}}{2\lambda\sigma^{2}}\right]^{-\theta}\right]\mid X>k\right\}=\left[1+\frac{(k+1)^{2}}{2\lambda\sigma^{2}}\right]^{-\theta}. \tag{114}$$

Proof. If X has pmf (14), then for $k\in\mathbb{N}$, the left-hand side of (114), using telescoping sum formula, will be

$$(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-2\theta}-\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-2\theta}\right\}=\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{\theta}\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-2\theta}$$
$$=\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}.$$

Conversely, if (114) holds, then

$$\begin{aligned}&\sum_{x=k+1}^{\infty}\left\{\left[\left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-\theta}+\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}\right]f(x)\right\}\\&=(1-F(k))\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}\\&=(1-F(k+1)+f(k+1))\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}.\end{aligned}\tag{115}$$

From (115), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-\theta}+\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}\right]f(x)\right\}=(1-F(k+1))\left[1+\frac{(k+2)^2}{2\lambda\sigma^2}\right]^{-\theta}.\tag{116}$$

Now, subtracting (116) from (115), yields

$$\begin{aligned}&f(k+1)\left[\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}+\left[1+\frac{(k+2)^2}{2\lambda\sigma^2}\right]^{-\theta}\right]\\&=(1-F(k+1))\left\{\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}-\left[1+\frac{(k+2)^2}{2\lambda\sigma^2}\right]^{-\theta}\right\}+f(k+1)\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}.\end{aligned}$$

From the above equality, we have

$$\frac{f(k+1)}{1-F(k+1)}=\frac{\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}-\left[1+\frac{(k+2)^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(k+2)^2}{2\lambda\sigma^2}\right]^{-\theta}}=\frac{\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(k+2)^2}{2\lambda\sigma^2}\right]^{-\theta}}-1,$$

which is the hazard function corresponding to the pmf (14), so X has pmf (14). □

2.5. Discrete Harris Extended-Exponential (DHEE) Distribution

Proposition 2.5. *Let* $X:\Omega\to\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (18) *if and only if*

$$\begin{aligned}&E\left\{\left[\left[\frac{\theta e^{-\alpha\beta X}}{1-(1-\theta)e^{-\alpha\beta X}}\right]^{1/\alpha}+\left[\frac{\theta e^{-\alpha\beta(X+1)}}{1-(1-\theta)e^{-\alpha\beta(X+1)}}\right]^{1/\alpha}\right]\mid X>k\right\}\\&=\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}.\end{aligned}\tag{117}$$

Proof. If X has pmf (18), then for $k \in \mathbb{N}$, the left-hand side of (117), using telescoping sum formula, will be

$$\begin{aligned}
&(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)e^{-\alpha\beta x}}\right]^{2/\alpha}-\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{2/\alpha}\right\}\\
&=\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{-1/\alpha}\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{2/\alpha}\\
&=\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}.
\end{aligned}$$

Conversely, if (117) holds, then

$$\begin{aligned}
&\sum_{x=k+1}^{\infty}\left\{\left[\left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)e^{-\alpha\beta x}}\right]^{1/\alpha}+\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}\right]f(x)\right\}\\
&=(1-F(k))\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}\\
&=(1-F(k+1)+f(k+1))\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}.
\end{aligned} \tag{118}$$

From (118), we also have

$$\begin{aligned}
&\sum_{x=k+2}^{\infty}\left\{\left[\left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)e^{-\alpha\beta x}}\right]^{1/\alpha}+\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}\right]f(x)\right\}\\
&=(1-F(k+1))\left[\frac{\theta e^{-\alpha\beta(k+2)}}{1-(1-\theta)e^{-\alpha\beta(k+2)}}\right]^{1/\alpha}.
\end{aligned} \tag{119}$$

Now, subtracting (119) from (118), yields

$$\begin{aligned}
&f(k+1)\left[\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}+\left[\frac{\theta e^{-\alpha\beta(k+2)}}{1-(1-\theta)e^{-\alpha\beta(k+2)}}\right]^{1/\alpha}\right]\\
&=(1-F(k+1))\left\{\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}-\left[\frac{\theta e^{-\alpha\beta(k+2)}}{1-(1-\theta)e^{-\alpha\beta(k+2)}}\right]^{1/\alpha}\right\}\\
&+f(k+1)\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}.
\end{aligned}$$

From the above equality, we have

$$\begin{aligned}
\frac{f(k+1)}{1-F(k+1)}&=\frac{\left\{\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}-\left[\frac{\theta e^{-\alpha\beta(k+2)}}{1-(1-\theta)e^{-\alpha\beta(k+2)}}\right]^{1/\alpha}\right\}}{\left[\frac{\theta e^{-\alpha\beta(k+2)}}{1-(1-\theta)e^{-\alpha\beta(k+2)}}\right]^{1/\alpha}}\\
&=\frac{\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(k+2)}}{1-(1-\theta)e^{-\alpha\beta(k+2)}}\right]^{1/\alpha}}-1,
\end{aligned}$$

which is the hazard function corresponding to the pmf (18), so X has pmf (18). □

2.6. Lagrangian Intervend Poisson (LIP) Distribution

Proposition 2.6. *Let $X:\Omega\to\mathbb{N}$ be a random variable. The pmf of X is* (22) *if and only if*

$$E\left\{[P(x)]^{-1}\mid X\leq k\right\}=\frac{1-e^{-\mu k}}{(e^{\mu}-1)\sum_{x=1}^{k}e^{-\mu x}P(x)}. \tag{120}$$

Proof. If X has pmf (22), then for $k\in\mathbb{N}$, the left-hand side of (120), using finite geometric sum formula, will be

$$(F(k))^{-1}\sum_{x=1}^{k}C\left\{e^{-\mu x}\right\}=\frac{\frac{e^{-\mu}\left(e^{-\mu k}-1\right)}{e^{-\mu}-1}}{\sum_{x=1}^{k}e^{-\mu x}P(x)}=\frac{1-e^{-\mu k}}{(e^{\mu}-1)\sum_{x=1}^{k}e^{-\mu x}P(x)}.$$

Conversely, if (120) holds, then

$$\sum_{x=1}^{k}\left\{[P(x)]^{-1}f(x)\right\}=F(k)\left(\frac{1-e^{-\mu k}}{(e^{\mu}-1)\sum_{x=1}^{k}e^{-\mu x}P(x)}\right). \tag{121}$$

From (121), we also have

$$\begin{aligned}\sum_{x=1}^{k-1}\left\{[P(x)]^{-1}f(x)\right\}&=F(k-1)\left(\frac{1-e^{-\mu(k-1)}}{(e^{\mu}-1)\sum_{x=1}^{k-1}e^{-\mu x}P(x)}\right)\\&=(F(k)-f(k))\left(\frac{1-e^{-\mu(k-1)}}{(e^{\mu}-1)\sum_{x=1}^{k-1}e^{-\mu x}P(x)}\right).\end{aligned} \tag{122}$$

Now, subtracting (122) from (121), yields

$$\begin{aligned}f(k)[P(k)]^{-1}=&F(k)\left\{\left(\frac{1-e^{-\mu k}}{(e^{\mu}-1)\sum_{x=1}^{k}e^{-\mu x}P(x)}\right)-\left(\frac{1-e^{-\mu(k-1)}}{(e^{\mu}-1)\sum_{x=1}^{k-1}e^{-\mu x}P(x)}\right)\right\}\\&+f(k)\left(\frac{1-e^{-\mu(k-1)}}{(e^{\mu}-1)\sum_{x=1}^{k-1}e^{-\mu x}P(x)}\right).\end{aligned}$$

From the above equality, we have

$$\frac{f(k)}{F(k)}=\frac{\left(\frac{1-e^{-\mu k}}{(e^{\mu}-1)\sum_{x=1}^{k}e^{-\mu x}P(x)}\right)-\left(\frac{1-e^{-\mu(k-1)}}{(e^{\mu}-1)\sum_{x=1}^{k-1}e^{-\mu x}P(x)}\right)}{\frac{1}{P(k)}-\left(\frac{1-e^{-\mu(k-1)}}{(e^{\mu}-1)\sum_{x=1}^{k-1}e^{-\mu x}P(x)}\right)},$$

from which, after some long manipulations, we arrive at

$$\frac{f(k)}{F(k)}=\frac{e^{-\mu k}}{\sum_{x=1}^{k}e^{-\mu x}P(x)},$$

which is the reverse hazard function corresponding to the pmf (22), so X has pmf (22). □

2.7. Discrete Logistic Exponential (DLE) Distribution

Proposition 2.7. *Let $X:\Omega\to\mathbb{N}^{*}$ be a random variable. The pmf of X is* (42) *if and only if*

$$E\left\{\left[\frac{\left(e^{\beta(X+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(X+1)}-1\right)^{\alpha}}+\frac{\left(e^{\beta X}-1\right)^{\alpha}}{1+\left(e^{\beta X}-1\right)^{\alpha}}\right]\mid X\leq k\right\}=\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right). \tag{123}$$

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Proof. If X has pmf (42), then for $k \in \mathbb{N}^*$, the left-hand side of (123), using telescoping sum formula, will be

$$\begin{aligned}
&(F(k))^{-1}\sum_{x0}^{k}\left\{\left(\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}}\right)^{2}-\left(\frac{\left(e^{\beta x}-1\right)^{\alpha}}{1+\left(e^{\beta x}-1\right)^{\alpha}}\right)^{2}\right\}\\
&=\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)^{-1}\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)^{2}\\
&=\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right).
\end{aligned}$$

Conversely, if (123) holds, then

$$\sum_{x=0}^{k}\left\{\left[\left(\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}}\right)+\left(\frac{\left(e^{\beta x}-1\right)^{\alpha}}{1+\left(e^{\beta x}-1\right)^{\alpha}}\right)\right]f(x)\right\}=F(k)\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right). \tag{124}$$

From (124), we also have

$$\begin{aligned}
\sum_{x=0}^{k-1}\left\{\left[\left(\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}}\right)+\left(\frac{\left(e^{\beta x}-1\right)^{\alpha}}{1+\left(e^{\beta x}-1\right)^{\alpha}}\right)\right]f(x)\right\}&=F(k-1)\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right)\\
&=\left(F(k)-f(k)\right)\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right).
\end{aligned} \tag{125}$$

Now, subtracting (125) from (124), yields

$$\begin{aligned}
&f(k)\left[\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)+\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right)\right]\\
&=F(k)\left\{\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)-\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right)\right\}+f(k)\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right).
\end{aligned}$$

From the above equality, we have

$$\frac{f(k)}{F(k)}=\frac{\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)-\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right)}{\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)},$$

from which, after some long manipulations, we arrive at

$$\frac{f(k)}{F(k)}=1-\frac{\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right)}{\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)},$$

which is the reverse hazard function corresponding to the pmf (42), so X has pmf (42). □

2.8. Discretized Fréchet-Weibull (DFW) Distribution

Proposition 2.8. *Let* $X:\Omega\to\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (86) *if and only if*

$$E\left\{\left[\exp\left\{-\beta^{\alpha}\left(\frac{m}{X+1}\right)^{\alpha k}\right\}+\exp\left\{-\beta^{\alpha}\left(\frac{m}{X}\right)^{\alpha k}\right\}\right]\mid X\leq k\right\}=\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}. \tag{126}$$

Proof. If X has pmf (86), then for $k\in\mathbb{N}^{*}$, the left-hand side of (126), using telescoping sum formula, will be

$$\begin{aligned}
&(F(k))^{-1}\sum_{x=0}^{k}\left\{\exp\left\{-2\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha k}\right\}-\exp\left\{-2\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}\right\}\right\}\\
&=\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}^{-1}\exp\left\{-2\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}\\
&=\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}.
\end{aligned}$$

Conversely, if (126) holds, then

$$\sum_{x=0}^{k}\left\{\left[\exp\left\{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha k}\right\}+\exp\left\{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}\right\}\right]f(x)\right\}=F(k)\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}. \tag{127}$$

From (127), we also have

$$\begin{aligned}
&\sum_{x=0}^{k-1}\left\{\left[\exp\left\{-\beta^{\alpha}\left(\frac{m}{x+1}\right)^{\alpha k}\right\}+\exp\left\{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}\right\}\right]f(x)\right\}\\
&=F(k-1)\exp\left\{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha k}\right\}\\
&=(F(k)-f(k))\exp\left\{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha k}\right\}.
\end{aligned} \tag{128}$$

Now, subtracting (128) from (127), yields

$$\begin{aligned}
&f(k)\left[\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}+\exp\left\{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha k}\right\}\right]\\
&=F(k)\left\{\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}-\exp\left\{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha k}\right\}\right\}+f(k)\exp\left\{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha k}\right\}.
\end{aligned}$$

From the above equality, we have

$$\frac{f(k)}{F(k)}=\frac{\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}-\exp\left\{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}},$$

from which, after some long manipulations, we arrive at

$$\frac{f(k)}{F(k)}=1-\frac{\exp\left\{-\beta^{\alpha}\left(\frac{m}{k}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^{\alpha}\left(\frac{m}{k+1}\right)^{\alpha k}\right\}},$$

which is the reverse hazard function corresponding to the pmf (86), so X has pmf (86). □

2.9. Discrete Odd Exponentiated Half-Logistic Inverse Exponential (DOEHLIEx) Distribution

Proposition 2.9. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (102) *if and only if*

$$E\left\{\left[\left(\frac{1-e^{-\lambda\left(e^{\beta/(X+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(X+1)}-1\right)^{-1}}}\right)^{\alpha}+\left(\frac{1-e^{-\lambda\left(e^{\beta/X}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/X}-1\right)^{-1}}}\right)^{\alpha}\right] \mid X\leq k\right\}$$
$$=\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{\alpha}. \tag{129}$$

Proof. If X has pmf (102), then for $k \in \mathbb{N}^*$, the left-hand side of (129), using telescoping sum formula, will be

$$(F(k))^{-1}\sum_{x=0}^{k}\left\{\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{2\alpha}-\left(\frac{1-e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}\right)^{2\alpha}\right\}$$
$$=\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{-\alpha}\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{2\alpha}$$
$$=\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{\alpha}.$$

Conversely, if (129) holds, then

$$\sum_{x=0}^{k}\left\{\left[\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}+\left(\frac{1-e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}\right)^{\alpha}\right]f(x)\right\}$$
$$=F(k)\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{\alpha}. \tag{130}$$

From (130), we also have

$$\sum_{x=0}^{k-1}\left\{\left[\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}+\left(\frac{1-e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}\right)^{\alpha}\right]f(x)\right\}$$
$$=F(k-1)\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^{\alpha}$$
$$=(F(k)-f(k))\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^{\alpha}. \tag{131}$$

Now, subtracting (131) from (130), yields

$$f(k)\left[\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{\alpha}+\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^{\alpha}\right]$$
$$=F(k)\left\{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{\alpha}-\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^{\alpha}\right\}+f(k)\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^{\alpha}.$$

From the above equality, we have

$$\frac{f(k)}{F(k)}=\frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{\alpha}-\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^{\alpha}}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{\alpha}}=1-\frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^{\alpha}}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^{\alpha}},$$

which is the reverse hazard function corresponding to the pmf (102), so X has pmf (102). □

3. Characterizations of distributions based on reverse hazard

This chapter deals with 7 distributions listed in Section 1. The characterizations presented here are in terms of the reverse hazard function.

3.1. Burnoulli-Poisson-Lindley (BPL) Distribution

Proposition 3.1. *Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is* (2) *if and only if its reverse hazard function, r_F, satisfies the difference equation*

$$r_F(k+1)-r_F(k)=\frac{P(k+1)(\theta+1)^{-(k+1)}}{\sum_{x=1}^{k+1}(\theta+1)^{-x}P(x)}-\frac{P(k)(\theta+1)^{-k}}{\sum_{x=1}^{k}(\theta+1)^{-x}P(x)},\quad k\in\mathbb{N},\tag{132}$$

with the initial condition $r_F(1)=1$.

Proof. If X has pmf (2), then clearly (132) holds. Now, if (132) holds, then for every $x\in\mathbb{N}$, we have

$$\sum_{k=1}^{x-1}\{r_F(k+1)-r_F(k)\}=\sum_{k=1}^{x-1}\left\{\frac{P(k+1)(\theta+1)^{-(k+1)}}{\sum_{x=1}^{k+1}(\theta+1)^{-x}P(x)}-\frac{P(k)(\theta+1)^{-k}}{\sum_{x=1}^{k}(\theta+1)^{-x}P(x)}\right\},$$

or, using telescoping sum

$$r_F(x)-r_F(0)=\frac{P(x)(\theta+1)^{-x}}{\sum_{u=1}^{x}(\theta+1)^{-u}P(u)}-1,$$

or in view of the initial condition

$$r_F(x)=\frac{P(x)(\theta+1)^{-x}}{\sum_{u=1}^{x}(\theta+1)^{-u}P(u)},\quad x\in\mathbb{N},$$

which is the reverse hazard function corresponding to the pmf (2). □

3.2. Discrete Cos-Lindley (DCL) Distribution

Proposition 3.2. *Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is* (6) *if and only if its reverse hazard function, r_F, satisfies the difference equation*

$$r_F(k+1)-r_F(k)=\frac{P(k)e^{-\theta k}}{P(k+1)e^{-\theta(k+1)}}-\frac{P(k+1)e^{-\theta(k+1)}}{P(k+2)e^{-\theta(k+2)}},\quad k\in\mathbb{N}^*,\tag{133}$$

with the initial condition $r_F(0) = 1$.

Proof. If X has pmf (6), then clearly (133) holds. Now, if (133) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1}\{r_F(k+1) - r_F(k)\} = \sum_{k=0}^{x-1}\left\{\frac{P(k)\,e^{-\theta k}}{P(k+1)\,e^{-\theta(k+1)}} - \frac{P(k+1)\,e^{-\theta(k+1)}}{P(k+2)\,e^{-\theta(k+2)}}\right\},$$

or, using telescoping sum

$$r_F(x) - r_F(0) = -\frac{P(x)\,e^{-\theta x}}{P(x+1)\,e^{-\theta(x+1)}},$$

or in view of the initial condition

$$r_F(x) = 1 - \frac{P(x)\,e^{-\theta x}}{P(x+1)\,e^{-\theta(x+1)}}, \quad x \in \mathbb{N}^*,$$

which is the reverse hazard function corresponding to the pmf (6). □

3.3. Discrete Gamma Power Lomax (DGPL) Distribution

Proposition 3.3. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (10) *if and only if its reverse hazard function,* r_F*, satisfies the difference equation*

$$r_F(k+1) - r_F(k) = \frac{\gamma\left(\upsilon, -\log\left(1+\frac{k^\lambda}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon, -\log\left(1+\frac{(k+1)^\lambda}{\theta}\right)^{-\alpha}\right)} - \frac{\gamma\left(\upsilon, -\log\left(1+\frac{(k+1)^\lambda}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon, -\log\left(1+\frac{(k+2)^\lambda}{\theta}\right)^{-\alpha}\right)}, \quad k \in \mathbb{N}^*, \tag{134}$$

with the initial condition $r_F(0) = 1$.

Proof. If X has pmf (10), then clearly (134) holds. Now, if (134) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1}\{r_F(k+1) - r_F(k)\} = \sum_{k=0}^{x-1}\left\{\frac{\gamma\left(\upsilon, -\log\left(1+\frac{k^\lambda}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon, -\log\left(1+\frac{(k+1)^\lambda}{\theta}\right)^{-\alpha}\right)} - \frac{\gamma\left(\upsilon, -\log\left(1+\frac{(k+1)^\lambda}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon, -\log\left(1+\frac{(k+2)^\lambda}{\theta}\right)^{-\alpha}\right)}\right\},$$

or, using telescoping sum

$$r_F(x) - r_F(0) = -\frac{\gamma\left(\upsilon, -\log\left(1+\frac{x^\lambda}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon, -\log\left(1+\frac{(x+1)^\lambda}{\theta}\right)^{-\alpha}\right)},$$

or in view of the initial condition

$$r_F(x) = 1 - \frac{\gamma\left(\upsilon, -\log\left(1+\frac{x^\lambda}{\theta}\right)^{-\alpha}\right)}{\gamma\left(\upsilon, -\log\left(1+\frac{(x+1)^\lambda}{\theta}\right)^{-\alpha}\right)}, \quad x \in \mathbb{N}^*,$$

which is the reverse hazard function corresponding to the pmf (10). □

3.4. Lagrangian Intervend Poisson (LIP) Distribution

Proposition 3.4. *Let* $X : \Omega \to \mathbb{N}$ *be a random variable. The pmf of* X *is* (22) *if and only if its reverse hazard function,* r_F*, satisfies the difference equation*

$$r_F\left(k+1\right)-r_F\left(k\right)=\frac{P(k+1)e^{-\mu(k+1)}}{\sum_{x=1}^{k+1}e^{-\mu x}P\left(x\right)}-\frac{P(k)e^{-\mu k}}{\sum_{x=1}^{k}e^{-\mu x}P\left(x\right)},\quad k\in\mathbb{N},\tag{135}$$

with the initial condition $r_F\left(0\right)=1.$

Proof. If X has pmf (22), then clearly (135) holds. Now, if (135) holds, then for every $x\in\mathbb{N}$, we have

$$\sum_{k=1}^{x-1}\left\{r_F\left(k+1\right)-r_F\left(k\right)\right\}=\sum_{k=1}^{x-1}\left\{\frac{P(k+1)e^{-\mu(k+1)}}{\sum_{x=1}^{k+1}e^{-\mu x}P\left(x\right)}-\frac{P(k)e^{-\mu k}}{\sum_{x=1}^{k}e^{-\mu x}P\left(x\right)}\right\},$$

or, using telescoping sum

$$r_F\left(x\right)-r_F\left(0\right)=\frac{P(x)e^{-\mu x}}{\sum_{u=1}^{x}e^{-\mu u}P\left(u\right)}-1,$$

or in view of the initial condition

$$r_F\left(x\right)=\frac{P(x)e^{-\mu x}}{\sum_{u=1}^{x}e^{-\mu u}P\left(u\right)},\quad x\in\mathbb{N},$$

which is the reverse hazard function corresponding to the pmf (22). □

3.5. Discrete Logistic Exponential (DLE) Distribution

Proposition 3.5. *Let* $X:\Omega\to\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (42) *if and only if its reverse hazard function,* r_F*, satisfies the difference equation*

$$r_F\left(k+1\right)-r_F\left(k\right)=\frac{\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right)}{\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)}-\frac{\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)}{\left(\frac{\left(e^{\beta(k+2)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+2)}-1\right)^{\alpha}}\right)},\quad k\in\mathbb{N}^*,\tag{136}$$

with the initial condition $r_F\left(0\right)=1.$

Proof. If X has pmf (42), then clearly (136) holds. Now, if (136) holds, then for every $x\in\mathbb{N}$, we have

$$\sum_{k=0}^{x-1}\left\{r_F\left(k+1\right)-r_F\left(k\right)\right\}=\sum_{k=0}^{x-1}\left\{\frac{\left(\frac{\left(e^{\beta k}-1\right)^{\alpha}}{1+\left(e^{\beta k}-1\right)^{\alpha}}\right)}{\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)}-\frac{\left(\frac{\left(e^{\beta(k+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+1)}-1\right)^{\alpha}}\right)}{\left(\frac{\left(e^{\beta(k+2)}-1\right)^{\alpha}}{1+\left(e^{\beta(k+2)}-1\right)^{\alpha}}\right)}\right\},$$

or, using telescoping sum

$$r_F\left(x\right)-r_F\left(0\right)=-\frac{\left(\frac{\left(e^{\beta x}-1\right)^{\alpha}}{1+\left(e^{\beta x}-1\right)^{\alpha}}\right)}{\left(\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}}\right)},$$

or in view of the initial condition

$$r_F\left(x\right)=1-\frac{\left(\frac{\left(e^{\beta x}-1\right)^{\alpha}}{1+\left(e^{\beta x}-1\right)^{\alpha}}\right)}{\left(\frac{\left(e^{\beta(x+1)}-1\right)^{\alpha}}{1+\left(e^{\beta(x+1)}-1\right)^{\alpha}}\right)},\quad x\in\mathbb{N}^*,$$

which is the reverse hazard function corresponding to the pmf (42). □

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3.6. Discretized Fréchet-Weibull (DFW) Distribution

Proposition 3.6. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (86) *if and only if its reverse hazard function,* r_F*, satisfies the difference equation*

$$r_F(k+1) - r_F(k) = \frac{\exp\left\{-\beta^\alpha \left(\frac{m}{k}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^\alpha \left(\frac{m}{k+1}\right)^{\alpha k}\right\}} - \frac{\exp\left\{-\beta^\alpha \left(\frac{m}{k+1}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^\alpha \left(\frac{m}{k+2}\right)^{\alpha k}\right\}}, \quad k \in \mathbb{N}^*, \tag{137}$$

with the initial condition $r_F(0) = 1$.

Proof. If X has pmf (86), then clearly (137) holds. Now, if (137) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1}\{r_F(k+1) - r_F(k)\} = \sum_{k=0}^{x-1}\left\{\frac{\exp\left\{-\beta^\alpha \left(\frac{m}{k}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^\alpha \left(\frac{m}{k+1}\right)^{\alpha k}\right\}} - \frac{\exp\left\{-\beta^\alpha \left(\frac{m}{k+1}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^\alpha \left(\frac{m}{k+2}\right)^{\alpha k}\right\}}\right\},$$

or, using telescoping sum

$$r_F(x) - r_F(0) = -\frac{\exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^\alpha \left(\frac{m}{x+1}\right)^{\alpha k}\right\}},$$

or in view of the initial condition

$$r_F(x) = 1 - \frac{\exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}}{\exp\left\{-\beta^\alpha \left(\frac{m}{x+1}\right)^{\alpha k}\right\}}, \quad x \in \mathbb{N}^*,$$

which is the reverse hazard function corresponding to the pmf (86). □

3.7. Discrete Odd Exponentiated Half-Logistics Inverse Exponential (DOEHLIEx) Distribution

Proposition 3.7. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (102) *if and only if its reverse hazard function,* r_F*, satisfies the difference equation*

$$r_F(k+1) - r_F(k) = \frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^\alpha}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^\alpha} - \frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^\alpha}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+2)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+2)}-1\right)^{-1}}}\right)^\alpha}, \quad k \in \mathbb{N}^*, \tag{138}$$

with the initial condition $r_F(0) = 1$.

Proof. If X has pmf (102), then clearly (138) holds. Now, if (138) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1}\{r_F(k+1) - r_F(k)\} = \sum_{k=0}^{x-1}\left\{\frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/k}-1\right)^{-1}}}\right)^\alpha}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^\alpha} - \frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+1)}-1\right)^{-1}}}\right)^\alpha}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(k+2)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(k+2)}-1\right)^{-1}}}\right)^\alpha}\right\},$$

or, using telescoping sum

$$r_F(x) - r_F(0) = -\frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}\right)^{\alpha}}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}},$$

or in view of the initial condition

$$r_F(x) = 1 - \frac{\left(\frac{1-e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/x}-1\right)^{-1}}}\right)^{\alpha}}{\left(\frac{1-e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}{1+e^{-\lambda\left(e^{\beta/(x+1)}-1\right)^{-1}}}\right)^{\alpha}}, \quad x \in \mathbb{N}^*,$$

which is the reverse hazard function corresponding to the pmf (102). □

4. Characterizations of distributions based on hazard function

This subsection is devoted to 2 distribution listed in the Introduction. The characterizations presented here are in terms of the hazard function.

4.1. New Lomax Rayleigh Discrete (NLRD) Distribution

Proposition 4.1. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (14) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \frac{\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(k+2)^2}{2\lambda\sigma^2}\right]^{-\theta}} - \frac{\left[1+\frac{k^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}}, \tag{139}$$

with the initial condition $h_F(0) = \frac{1}{\left[1+\frac{1}{2\lambda\sigma^2}\right]^{-\theta}} - 1.$

Proof. If X has pmf (14), then clearly (139) holds. Now, if (139) holds, then for every $x \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k=0}^{x-1}\{h_F(k+1) - h_F(k)\} &= \sum_{k=0}^{x-1}\left\{\frac{\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(k+2)^2}{2\lambda\sigma^2}\right]^{-\theta}} - \frac{\left[1+\frac{k^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(k+1)^2}{2\lambda\sigma^2}\right]^{-\theta}}\right\} \\ &= \frac{\left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}} - \frac{1}{\left[1+\frac{1}{2\lambda\sigma^2}\right]^{-\theta}},\end{aligned}$$

or

$$h_F(x) - h_F(0) = \frac{\left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}} - \frac{1}{\left[1+\frac{1}{2\lambda\sigma^2}\right]^{-\theta}},$$

or, in view of the initial condition

$$h_F(x) = \frac{\left[1+\frac{x^2}{2\lambda\sigma^2}\right]^{-\theta}}{\left[1+\frac{(x+1)^2}{2\lambda\sigma^2}\right]^{-\theta}} - 1, \quad x \in \mathbb{N}^*,$$

which is the hazard function corresponding to the pmf (14). □

4.2. Discrete Harris Extended-Exponential (DHEE) Distribution

Proposition 4.2. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (18) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \frac{\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(k+2)}}{1-(1-\theta)e^{-\alpha\beta(k+2)}}\right]^{1/\alpha}} - \frac{\left[\frac{\theta e^{-\alpha\beta k}}{1-(1-\theta)e^{-\alpha\beta k}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}}, \tag{140}$$

with the initial condition $h_F(0) = \frac{1}{\left[\frac{\theta e^{-\alpha\beta}}{1-(1-\theta)e^{-\alpha\beta}}\right]} - 1.$

Proof. If X has pmf (18), then clearly (140) holds. Now, if (140) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1}\{h_F(k+1) - h_F(k)\} = \sum_{k=0}^{x-1}\left\{\frac{\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(k+2)}}{1-(1-\theta)e^{-\alpha\beta(k+2)}}\right]^{1/\alpha}} - \frac{\left[\frac{\theta e^{-\alpha\beta k}}{1-(1-\theta)e^{-\alpha\beta k}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(k+1)}}{1-(1-\theta)e^{-\alpha\beta(k+1)}}\right]^{1/\alpha}}\right\}$$

$$= \frac{\left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)e^{-\alpha\beta x}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}} - \frac{1}{\left[\frac{\theta e^{-\alpha\beta}}{1-(1-\theta)e^{-\alpha\beta}}\right]},$$

or

$$h_F(x) - h_F(0) = \frac{\left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)e^{-\alpha\beta x}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}} - \frac{1}{\left[\frac{\theta e^{-\alpha\beta}}{1-(1-\theta)e^{-\alpha\beta}}\right]},$$

or, in view of the initial condition

$$h_F(x) = \frac{\left[\frac{\theta e^{-\alpha\beta x}}{1-(1-\theta)e^{-\alpha\beta x}}\right]^{1/\alpha}}{\left[\frac{\theta e^{-\alpha\beta(x+1)}}{1-(1-\theta)e^{-\alpha\beta(x+1)}}\right]^{1/\alpha}} - 1, \quad x \in \mathbb{N}^*,$$

which is the hazard function corresponding to the pmf (18). □

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