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Characterizations of Certain (2023-2024) Introduced Univariate Continuous Distributions

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Abstract

This paper deals with various characterizations of certain univariate continuous distributions proposed in (2023-2024). These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) reverse hazard function and (iv) conditional expectation of a single function of the random variable. It should be mentioned that for the characterization (i) the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

Key Words: Characterizations; Conditional expectation; Continuous distributions; Hazard function; Reverse hazard function.

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with certain characterizations of each of the following distributions: 1) Gamma Zero-Truncated Poisson (GZTP) distribution of Niyomdecha et al. (2023); 2) New Transmuted Logistic-Exponential (NTLE) distribution of Adesegun et al. (2023); 3) Gull Alpha Power Transformed Log-Logistic (GAPLOL) distribution of Alrajhi and Almarzouki (2023); 4) Complementary Gamma Zero-Truncated Poisson (CGZTP) distribution of Niyomdecha and Srisuradetchai (2023); 5) Weibull Additive Hazard (WAH) distribution of Suresh et al. (2023); 6) Kumaraswamy Bell Exponential (KwBE) distribution of Imran et al. (2023); 7) New Kumaraswamy Exponential (NKwE) distribution of Naz et al. (2023); 8) Truncated Inverse Power Lindley (TIPL) distribution of Elgarhy et al. (2023); 9) Modified Alpha Power Transformed Weibull (MAPTW) distribution of Alotaibi et al. (2023); 10) Generalized Rayleigh (GRa) distribution of Bashiru et al. (2023); 11) Sine-Lomax (SLom) distribution of Mustapha et al. (2023); 12) New Lehmann Type II Lomax (NLTIIL) distribution of Isa et al. (2023); 13) New Exponentiated Extended Inverse Exponential (NEtEIEx) distribution of Bashiru (2023); 14) Half-Cauchy Chen (HCC) distribution of Chaudhary et al. (2023); 15) Quasi-Xgamma Frailty (QXgF) distribution of Loubna et al.

(2023); 16) Transmuted Record Type Lindley (TRTL) distribution of Taniş (2023); 17) Inverse Unit Gompertz (IUG) distribution of Bashir et al. (2023); 18) Modified Flexible Weibull (MFW) distribution of Al-Marzouki et al. (2023);19) Topp-Leone Odd Burr X-G (TLOBX-G) family of distributions of Oluyede et al. (2023); 20) Power Lambert Uniform (PLU) distribution of Gemeay et al. (2023); 21) Topp-Leone Cauchy Rayleigh (TLCAR) distribution of Atchadé et al. (2023); 22) Length-Biased Truncated Lomax-Generated (LBTLo-G) family of distributions of Hassan et al. (2023); 23) Truncated Topp-Leone Inverse Lomax (TTLILo) distribution of Alyami et al. (2023); 24) Power Unit Burr-Hatke (PUBH) distribution of Abdulrahman et al. (2023); 25) Marshall-Olkin Pareto Type-I (MOPTI) distribution of Aldahan et al. (2023); 26) Quartic Transmuted Weibull (QTW) distribution of Moloy et al. (2023); 27) Geometric Generated Rayleigh (GCGR-G) family of distributions of Abdullah and Masmoudi (2023); 28) Cubic Transmuted Weighted Exponential (CTWE) distribution of Sabri and Adetunji (2023); 29) Right truncated Xgamma-G (RXg-G) family of distributions of Al-Abbasi et al. (2023); 30) [0,1] Truncated Inverse Weibull Exponential ([0,1] TIWE) distribution of Khubbaz and Khaleel (2023); 31) Bimodal Extension of Suja (BES) distribution of Enogwe et al. (2023); 32) Slash Lomax-Rayleigh (SLR) distribution of Santoro et al. (2023); 33) Type I Half Logistic-Topp-Leone-G (TIHLTL-G) family of distributions of Adepoju et al. (2023); 34) New Quasi Aradhana (NQA) distribution of Shanker and Soni (2023); 35) Generalized Gamma Weibull (GGW) distribution of Dauda et al. (2023); 36) Power Chris-Jerry (PCJ) distribution of Ezeilo et al. (2023); 37) Odd Gompertz-G (OG-G) family of distributions of Kajuru et al. (2023); 38) Topp-Leone Generated q-Weibull (TLqW) distribution of Sebastian et al. (2023); 39) New Alpha Power Inverse Weibull (NAPIW) distribution of Omar (2023); 40) Exponential Pareto-Fréchet (EPF) distribution of Hadi and Nasser (2023); 41) Gompertz Chen (GOCH) distribution of Yusur and Khaleel (2023); 42) Extended Exponentiated Gamma-Lindley (EEGL) distribution of Masmoudi et al. (2023); 43) Lehmann Type II Teissier (LTII-T) distribution of Kumaran and Jha (2023); 44) Type II Exponentiated Half-Logistic Gompertz-G (TIIEHLGom-G) family of distributions of Moakofi and Oluyede (2023);

We list below the cumulative distribution function (cdf) and probability density function (pdf) of each one of these distributions in the same order as listed above. We will be employing the same notation for the parameters as chosen by the original authors.

1) The cdf and pdf of (GZTP) are given, respectively, by

$$F\left(x;\lambda,\alpha,\beta\right)=\frac{1-e^{-\lambda+\lambda\left(\frac{\Gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)}}{1-e^{-\lambda}},\quad x\geq 0, \tag{1}$$

and

$$f\left(x;\lambda,\alpha,\beta\right)=Ce^{-\beta x}P\left(x\right),\quad x>0, \tag{2}$$

where λ,α,β are all positive parameters, $C=\frac{\lambda\beta^{\alpha}e^{-\lambda}}{(1-e^{-\lambda})\Gamma(\alpha)}$, $P\left(x\right)=x^{\alpha-1}e^{\lambda\left(\frac{\Gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)}$ and $\Gamma\left(\alpha,\beta x\right)=\int_{\beta x}^{\infty}t^{\alpha-1}e^{-t}dt$.

2) The cdf and pdf of (NTLE) are given, respectively, by

$$F\left(x;\beta,\delta,\phi\right)=\frac{\left(e^{\phi x}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi x}-1\right)^{\beta}\right)}{\left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^{2}},\quad x\geq 0, \tag{3}$$

and

$$f\left(x;\beta,\delta,\phi\right)=\beta\phi e^{\phi x}\left(e^{\phi x}-1\right)^{\beta-1}\left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^{-3}P\left(x\right),\quad x>0, \tag{4}$$

where $\beta\geq 0,\phi\geq 0,\delta\in[-1,1]$ are parameters and $P\left(x\right)=1+\left(e^{\phi x}-1\right)^{\beta}+\delta\left(1-\left(e^{\phi x}-1\right)^{\beta}\right)$.

3) The cdf and pdf of (GAPLOL) are given, respectively, by

$$F\left(x;\lambda,\xi,\theta\right)=\left[1-\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}\right]\lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}},\quad x\geq 0, \tag{5}$$

and

$$f(x;\lambda,\xi,\theta)=\frac{\theta}{\xi^{\theta}}x^{\theta-1}\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-2}\lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}}P(x),\quad x>0, \tag{6}$$

where $\lambda\,(\neq 1)>0,\xi>0,\theta>0$ are parameters and $P(x)=1-\left(1-\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}\right)\log\lambda.$

4) The cdf and pdf of (CGZTP) are given, respectively, by

$$F(x;\lambda,\alpha,\beta)=\frac{e^{-\lambda\left(\frac{\Gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)}-e^{-\lambda}}{1-e^{-\lambda}},\quad x\geq 0, \tag{7}$$

and

$$f(x;\lambda,\alpha,\beta)=\frac{d}{dx}F(x;\lambda,\alpha,\beta),\quad x>0, \tag{8}$$

where λ,α,β are all positive parameters.

Remark 1.1. *The cdf* (7) *can be written as*

$$F(x;\lambda,\alpha,\beta)=\frac{e^{\lambda}e^{-\lambda\left(\frac{\Gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)}-1}{e^{\lambda}-1}=\frac{e^{\lambda\left[1-\left(\frac{\Gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right]}-1}{e^{\lambda}-1}=\frac{e^{\lambda G(x)}-1}{e^{\lambda}-1},\quad x\geq 0,$$

where $G(x)=1-\left(\frac{\Gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)$, $x\geq 0$. *This cdf, however, has been characterized in Hamedani (2021).*

5) The cdf and pdf of (WAH) are given, respectively, by

$$\begin{aligned}F(x;\lambda,\alpha,\beta,\theta)&=1-\exp\left[-\theta x-\frac{\lambda^{\alpha\beta}\beta^{\alpha}}{\alpha(\beta-1)+1}x^{\alpha(\beta-1)+1}\right]\\&=1-\exp\left[-\theta x-Cx^{\alpha(\beta-1)+1}\right],\quad x\geq 0,\end{aligned} \tag{9}$$

and

$$f(x;\lambda,\alpha,\beta,\theta)=\left[\theta+C\left(\alpha(\beta-1)+1\right)x^{\alpha(\beta-1)}\right]\exp\left[-\theta x-Cx^{\alpha(\beta-1)+1}\right],\quad x>0, \tag{10}$$

where $\lambda,\alpha,\beta,\theta$ are all positive parameters and $C=\frac{\lambda^{\alpha\beta}\beta^{\alpha}}{\alpha(\beta-1)+1}$.

Remark 1.2. *The characterizations stated in Section 2 for the cdf* (9) *can be stated for the Gompertz Additive Hazard (GAH) and Negative Gompertz Additive Hazard (NGAH) distributions mentioned by Suresh et al. (2023).*

6) The cdf and pdf of (KwBE) are given, respectively, by

$$F(x;a,b,\omega,\theta)=1-\left[1-\left\{\frac{1-\exp\left(-e^{\omega}\left[1-e^{-\omega\left(1-e^{-\theta x}\right)}\right]\right)}{1-\exp\left(1-e^{\omega}\right)}\right\}^{a}\right]^{b},\quad x\geq 0, \tag{11}$$

and

$$f(x;a,b,\omega,\theta)=\frac{d}{dx}F(x;a,b,\omega,\theta),\quad x>0, \tag{12}$$

where a,b,ω,θ are all positive parameters.

Remark 1.3. *Taking*

$$G(x) = \frac{1-\exp\left(-e^{\omega}\left[1-e^{-\omega\left(1-e^{-\theta x}\right)}\right]\right)}{1-\exp\left(1-e^{\omega}\right)}, \quad x \geq 0,$$

the cdf (11) *can be written as*

$$F(x;a,b,\omega,\theta) = 1-\left[1-\{G(x)\}^{a}\right]^{b}, \quad x \geq 0,$$

which was mentioned in Remark 1.39 in Hamedani (2023).

7) The cdf and pdf of (NKwE) are given, respectively, by

$$F(x;\alpha,s,\omega,\theta) = 1-\left[1-\left\{1-e^{1-\theta x-e^{-\theta x}}\right\}^{\alpha}\right]^{s}, \quad x \geq 0, \tag{13}$$

and

$$f(x;\alpha,s,\omega,\theta) = \frac{d}{dx}F(x;\alpha,s,\omega,\theta), \quad x > 0, \tag{14}$$

where $\alpha, s, \omega, \theta$ are all positive parameters.

Remark 1.4. *Taking* $G(x) = 1-e^{1-\theta x-e^{-\theta x}}$, $x \geq 0$, *the cdf* (13) *can be written as*

$$F(x;\alpha,b,\omega,\theta) = 1-\left[1-\{G(x)\}^{\alpha}\right]^{s}, \quad x \geq 0,$$

which was mentioned in Remark 1.39 in Hamedani (2023).

8) The cdf and pdf of (TIPL) are given, respectively, by

$$F(x;\psi,\xi) = \left(\frac{1+\xi}{1+2\xi}\right)e^{\xi}\left(1+\frac{\xi x^{-\psi}}{1+\xi}\right)e^{-\xi x^{-\psi}}, \quad 0 \leq x \leq 1, \tag{15}$$

and

$$f(x;\psi,\xi) = Cx^{-\psi-1}e^{-\xi x^{-\psi}}P(x), \quad 0 < x < 1, \tag{16}$$

where $\psi > 0, \xi > 0$ are parameters, $C = \frac{\psi\xi^{2}e^{\xi}}{1+2\xi}$ and $P(x) = x^{-\psi}\left(1+x^{\psi}\right)$.

9) The cdf and pdf of (MAPTW) are given, respectively, by

$$F(x;\alpha,\lambda,\theta) = \frac{\alpha^{1-e^{-\lambda x^{\theta}}}-1}{(\alpha-1)\left(1+\alpha-\alpha^{1-e^{-\lambda x^{\theta}}}\right)}, \quad x \geq 0, \tag{17}$$

and

$$f(x;\alpha,\lambda,\theta) = \frac{d}{dx}F(x;\alpha,\lambda,\theta), \quad x > 0, \tag{18}$$

where $\alpha > 0\ (\neq 1), \lambda > 0, \theta > 0$ are parameters.

Remark 1.5. *Taking* $G(x) = \frac{\alpha^{1-e^{-\lambda x^{\theta}}}-1}{\alpha-1}$, $x \geq 0$, *the cdf* (17) *can be expressed as*

$$F(x;\alpha,\lambda,\theta) = \frac{G(x)}{\alpha+(1-\alpha)G(x)}, \quad x \geq 0,$$

which is a special case of the cdf (1.1591) *in Hamedani (2023).*

10) The cdf and pdf of (GRa) are given, respectively, by

$$F(x;\lambda,\alpha,\theta)=\frac{1-\left[1-\left[1-e^{-(\theta/2)x^2}\right]\right]^{\lambda}}{1+\left[1-\left[1-e^{-(\theta/2)x^2}\right]\right]^{\lambda}},\quad x\geq 0, \tag{19}$$

and

$$f(x;\lambda,\alpha,\theta)=\frac{d}{dx}F(x;\lambda,\alpha,\theta),\quad x>0, \tag{20}$$

where λ,α,θ are all positive parameters.

Remark 1.6. *Taking* $G(x)=1-\left[1-\left[1-e^{-(\theta/2)x^2}\right]\right]^{\lambda}$, $x\geq 0$, *the cdf* (19) *can be written as*

$$F(x;\lambda,\alpha,\theta)=\frac{G(x)}{2-G(x)},\quad x\geq 0,$$

which is a special case of the cdf (1.1591) *in Hamedani (2023).*

11) The cdf and pdf of (SLom) are given, respectively, by

$$F(x;\alpha,\lambda)=\sin\left\{\frac{\pi}{2}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]\right\},\quad x\geq 0, \tag{21}$$

and

$$f(x;\alpha,\lambda)=\frac{d}{dx}F(x;\alpha,\lambda),\quad x>0, \tag{22}$$

where $\alpha>0,\lambda>0$ are parameters.

Remark 1.7. *Taking* $G(x)=1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}$, $x\geq 0$, *the cdf* (21) *can be expressed as*

$$F(x;\alpha,\lambda)=\sin\left\{\frac{\pi}{2}G(x)\right\},\quad x\geq 0,$$

which is a special case of the cdf (1.43) *in Hamedani (2023).*

12) The cdf and pdf of (NLTIIL) are given, respectively, by

$$F(x;\gamma,\alpha,\lambda)=1-\left\{1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]\right\}^{\gamma},\quad x\geq 0, \tag{23}$$

and

$$f(x;\gamma,\alpha,\lambda)=\frac{d}{dx}F(x;\gamma,\alpha,\lambda),\quad x>0, \tag{24}$$

where γ,α,λ are all positive parameters.

Remark 1.8. *Taking* $G(x)=1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}$, $x\geq 0$, *the cdf* (23) *can be expressed as*

$$F(x;\gamma,\alpha,\lambda)=1-\{1-G(x)\}^{\gamma},\quad x\geq 0,$$

which was mentioned in Remark 1.39 in Hamedani (2023).

13) The cdf and pdf of (NEtEIEx) are given, respectively, by

$$F(x;\alpha,\beta,\lambda.\theta)=\left[1-\left[1-\left[e^{-\beta/x}\right]\right]^{\alpha\lambda}\right]^{\theta},\quad x\geq 0, \tag{25}$$

and

$$f(x;\alpha,\beta,\lambda.\theta)=\frac{d}{dx}F(x;\alpha,\beta,\lambda.\theta), \quad x>0, \tag{26}$$

where $\alpha,\beta,\lambda.\theta$ are all positive parameters.

Remark 1.9. *Taking $G(x)=e^{-\beta/x}$, $x\geq 0$, the cdf* (25) *can be expressed as*

$$F(x;\alpha,\beta,\lambda.\theta)=\left[1-[1-G(x)]^{\alpha\lambda}\right]^{\theta}, \quad x\geq 0,$$

which was mentioned in Remark 1.39 in Hamedani (2023).

14) The cdf and pdf of (HCC) are given, respectively, by

$$F(x;\beta,\lambda.\theta)=\frac{2}{\pi}\arctan\left\{-\frac{\lambda}{\theta}\left(1-e^{x^{\beta}}\right)\right\}, \quad x\geq 0, \tag{27}$$

and

$$f(x;\beta,\lambda.\theta)=Cx^{\beta-1}e^{-x^{\beta}}P(x), \quad x>0, \tag{28}$$

where $\beta,\lambda.\theta$ are all positive parameters, $C=\frac{2\beta\lambda\theta}{\pi}$ and $P(x)=\left\{e^{-2x^{\beta}}\left[\theta^2+\left[-\lambda\left(1-e^{x^{\beta}}\right)\right]^2\right]\right\}^{-1}$.

15) The cdf and pdf of (QXgF) are given, respectively, by

$$F(x;\zeta)=\int_0^x f(u;\zeta)\,du, \quad x\geq 0, \tag{29}$$

and

$$f(x;\zeta)=C\zeta^{x}\ln(\zeta)(1+\zeta^{x})^{-2}\exp\left(-\frac{3+\zeta}{1+\zeta^{x}}\right)P(x), \quad x>0, \tag{30}$$

where $\zeta>0$ is a parameter, $C=\frac{(3+\zeta)}{(1+\zeta)^2}$ and $P(x)=\zeta^{-1x}(\ln(\zeta))^{-1}(1+\zeta^{x})^2\left(\zeta+\left(\frac{3+\zeta}{1+\zeta}\right)^2x^2\right)$.

16) The cdf and pdf of (TRTL) are given, respectively, by

$$F(x;\theta,p)=\int_0^x f(u;\theta,p)\,du, \quad x\geq 0, \tag{31}$$

and

$$f(x;\theta,p)=Ce^{-\theta x}P(x), \quad x>0, \tag{32}$$

where $\theta>0, p\in(0,1)$ are parameters, $C=\frac{\theta^2}{\theta+1}$ and $P(x)=(1+x)\left[1+p\left\{\theta x-\log\left(\frac{\theta+1+\theta x}{\theta+1}\right)-1\right\}\right]$.

Remark 1.10. *The pdf* (32) *is similar to the pdf* (2).

17) The cdf and pdf of (IUG) are given, respectively, by

$$F(x;\alpha,\beta)=1-\exp\left[-\alpha\left(x^{\beta}-1\right)\right]=1-\exp\left[-\alpha\left(\frac{1-x^{-\beta}}{x^{-\beta}}\right)\right], \quad x\geq 1, \tag{33}$$

and

$$f(x;\alpha,\beta)=\frac{d}{dx}F(x;\alpha,\beta), \quad x>1, \tag{34}$$

where $\alpha>0,\beta>0$ are parameters.

Remark 1.11. *Taking $G(x) = 1 - x^{-\beta}$, $x \geq 1$, the cdf* (33) *can be written as*

$$F(x;\alpha,\beta) = 1 - \exp\left[-\alpha\left(\frac{G(x)}{1-G(x)}\right)\right], \quad x \geq 1,$$

which is a special case of the cdf mentioned in Remark 1.2 in Hamedani (2023).

18) The cdf and pdf of (MFW) are given, respectively, by

$$F(x;\tau,\lambda_1,\lambda_2) = 1 - \frac{\tau e^{-e^{\lambda_1 x - \frac{\lambda_2}{x}}}}{\tau - 1 + e^{-e^{\lambda_1 x - \frac{\lambda_2}{x}}}}, \quad x \geq 0, \tag{35}$$

and

$$f(x;\tau,\lambda_1,\lambda_2) = \frac{d}{dx}F(x;\tau,\lambda_1,\lambda_2), \quad x > 0, \tag{36}$$

where $\tau > 1, \lambda_1 > 0, \lambda_2 > 0$ are parameters.

Remark 1.12. *Taking $G(x) = 1 - e^{-e^{\lambda_1 x - \frac{\lambda_2}{x}}}$, $x \geq 0$, the cdf* (35) *can be expressed as*

$$F(x;\tau,\lambda_1,\lambda_2) = \frac{(\tau-1)G(x)}{\tau - G(x)} = \frac{G(x)}{\frac{\tau}{(\tau-1)} - \left(\frac{1}{\tau-1}\right)G(x)} = \frac{G(x)}{\gamma + (1-\gamma)G(x)}, \quad x \geq 0,$$

for $\gamma = \frac{\tau}{(\tau-1)}$, which is a special case of the cdf (1.1591) *in Hamedani (2023).*

19) The cdf and pdf of (TLOBX-G) are given, respectively, by

$$F(x;\alpha,\theta,\xi) = \left[1 - \left[1 - \left\{1 - \exp\left[-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^2\right]\right\}^{\theta}\right]^2\right]^{\alpha}, \quad x \geq 0, \tag{37}$$

and

$$f(x;\alpha,\theta,\xi) = \frac{d}{dx}F(x;\alpha,\theta,\xi), \quad x > 0, \tag{38}$$

where $\alpha > 0, \theta > 0$ are parameters and $G(x;\xi)$ is a baseline cdf which depends on the parameter vector ξ.

Remark 1.13. *Taking $K(x) = 1 - \exp\left[-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^2\right]$, $x \geq 0$, the cdf* (37) *can be written as*

$$F(x;\alpha,\theta,\xi) = \left[1 - \left[1 - \{G(x)\}^{\theta}\right]^2\right]^{\alpha}, \quad x \geq 0,$$

which was mentioned in Remark1.39 in Hamedani (2023).

20) The cdf and pdf of (PLU) are given, respectively, by

$$F(x;\alpha,\beta) = 1 - \left(1 - x^{\beta}\right)\alpha^{x^{\beta}}, \quad 0 \leq x \leq 1, \tag{39}$$

and

$$f(x;\alpha,\beta) = \beta x^{\beta-1}\alpha^{x^{\beta}}P(x), \quad 0 < x < 1, \tag{40}$$

where $\alpha \in (0,1) \cup (1,e), \beta > 0$ are parameters and $P(x) = \log(\alpha)\left(1 - x^{\beta}\right) + 1$.

21) The cdf and pdf of (TLCAR) are given, respectively, by

$$F\left(x;\alpha,a,\theta,m\right)=\int_{-\infty}^{x}f\left(u;\alpha,a,\theta,m\right)du,\quad x\in\mathbb{R},\tag{41}$$

and

$$f\left(x;\alpha,a,\theta,m\right)=\frac{2\alpha}{\pi a}\left(\frac{1+\left(\frac{x^2}{2\theta^2}-1\right)e^{-\frac{x^2}{2\theta^2}}+m}{1+\left(\frac{x\left(\left(\frac{x^2}{2\theta^2}-1\right)e^{-\frac{x^2}{2\theta^2}}\right)-b}{a}\right)^2}\right)\times$$
$$\left(\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right)P\left(x\right),\quad x\in\mathbb{R},\tag{42}$$

where α,a,θ,m are all positive parameters and $P\left(x\right)=\left\{1-\left[\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right]\right\}^{\alpha-1}$.

Remark 1.14. *The formula given in the original paper as the cdf was not a cdf, which was pointed out by I. I subsequently suggested a correct cdf to the authors which they used.*

22) The cdf and pdf of (LBTLo-G) are given, respectively, by

$$F\left(x;\alpha,\zeta\right)=\int_{-\infty}^{x}f\left(u;\alpha,\zeta\right)du,\quad x\in\mathbb{R},\tag{43}$$

and

$$f\left(x;\alpha,\zeta\right)=Cg\left(x;\zeta\right)G\left(x;\zeta\right)P\left(x\right),\quad x\in\mathbb{R},\tag{44}$$

where $\alpha>0$ is a parameter, $C=\alpha\left(1-\alpha\right)\left[2^{-\alpha}\left(1+\alpha\right)-1\right]^{-1}$ and $P\left(x\right)=\left(1+G\left(x;\zeta\right)\right)^{-\alpha-1}$ is a baseline cdf with the corresponding pdf $g\left(x;\zeta\right)$, which depends on the parameter vector ζ.

23) The cdf and pdf of (TTLILo) are given, respectively, by

$$F\left(x;b,\omega,\rho,c\right)=\left(1+c\right)G\left(x\right)-cG\left(x\right)^2,\quad x\geq 0,\tag{45}$$

and

$$f\left(x;b,\omega,\rho,c\right)=\frac{d}{dx}F\left(x;b,\omega,\rho,c\right),\quad x>0,\tag{46}$$

where $b>0,\omega>0,\rho>0,c\in[-1,1]$ are parameters and $G\left(x\right)=\left\{1-\left[1-\left(1+\rho/x\right)^{-\omega}\right]^2\right\}^b$, $x\geq 0$.

Remark 1.15. *The cdf* (45) *is a special case of the cdf mentioned in Remark 1.247 in Hamedani (2023).*

24) The cdf and pdf of (PUBH) are given, respectively, by

$$F\left(x;k,\beta\right)=\frac{x^{\beta k}}{1-\log\left(x^k\right)}=\frac{x^{\beta k}}{1-k\log\left(x\right)},\quad 0\leq x\leq 1,\tag{47}$$

and

$$f(x;k,\beta)=\frac{kx^{\beta k-1}\left(\beta\left(1-k\log\left(x\right)\right)+1\right)}{\left(1-k\log\left(x\right)\right)^{2}}=\frac{kx^{-1}}{\left(1-k\log\left(x\right)\right)^{2}}P\left(x\right),\quad 0<x<1, \tag{48}$$

where $k>0,\beta>0$ are parameters and $P(x)=(\beta(1-k\log(x))+1)$.

25) The cdf and pdf of (MOPTI) are given, respectively, by

$$F(x;\alpha,\beta,\delta)=\frac{1-\left(\frac{\alpha}{x}\right)^{\beta}}{1-(1-\delta)\left(\frac{\alpha}{x}\right)^{\beta}},\quad x\geq\alpha, \tag{49}$$

and

$$f(x;\alpha,\beta,\delta)=\frac{d}{dx}F(x;\alpha,\beta,\delta),\quad x>\alpha, \tag{50}$$

where α,β,δ are all positive parameters.

Remark 1.16. *Taking $G(x)=1-\left(\frac{\alpha}{x}\right)^{\beta}$, $x\geq\alpha$, the cdf (49) can be expressed as*

$$F(x;\alpha,\beta,\delta)=\frac{G(x)}{\delta+(1-\delta)G(x)},\quad x\geq\alpha,$$

which was mentioned in Remark 1.12.

26) The cdf and pdf of (QTW) are given, respectively, by

$$F(x;\alpha,\beta,\lambda_1,\lambda_2,\lambda_3)=\int_0^x f(u;\alpha,\beta,\lambda_1,\lambda_2,\lambda_3)\,du,\quad x\geq 0, \tag{51}$$

and

$$f(x;\alpha,\beta,\lambda_1,\lambda_2,\lambda_3)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}P(x),\quad x>0, \tag{52}$$

where $\alpha>0,\beta>0,\lambda_i\in[0,2], i=1,2,3, 0<\lambda_1+\lambda_2+\lambda_3<2$ are parameters and $P(x)=4a_4+3a_3e^{\left(\frac{x}{\beta}\right)^{\alpha}}+2a_2e^{2\left(\frac{x}{\beta}\right)^{\alpha}}+a_1e^{3\left(\frac{x}{\beta}\right)^{\alpha}}$ with $a_1=4-2\lambda_1-2\lambda_2-2\lambda_3, a_2=-6+3\lambda_1+3\lambda_2+6\lambda_3, a_3=4-2\lambda_1-6\lambda_3, a_4=-1+\lambda_1-\lambda_2+2\lambda_3$.

27) The cdf and pdf of (GCGR-G) are given, respectively, by

$$\begin{aligned}F(x;\gamma,\sigma,\mathbf{V})&=\frac{\gamma-\gamma\exp\left[-\left(\frac{G(x;\mathbf{V})}{1-G(x;\mathbf{V})}\right)^{2}\right]}{1-\left\{1-\exp\left[-\left(\frac{G(x;\mathbf{V})}{1-G(x;\mathbf{V})}\right)^{2}\right]\right\}+\gamma\left\{1-\exp\left[-\left(\frac{G(x;\mathbf{V})}{1-G(x;\mathbf{V})}\right)^{2}\right]\right\}}\\&=\frac{\gamma\left(1-\exp\left[-\left(\frac{G(x;\mathbf{V})}{1-G(x;\mathbf{V})}\right)^{2}\right]\right)}{\gamma+(1-\gamma)\exp\left[-\left(\frac{G(x;\mathbf{V})}{1-G(x;\mathbf{V})}\right)^{2}\right]},\quad x\in\mathbb{R},\end{aligned} \tag{53}$$

and

$$f(x;\gamma,\sigma,\mathbf{V})=\frac{d}{dx}F(x;\gamma,\sigma,\mathbf{V}),\quad x\in\mathbb{R}, \tag{54}$$

where $\gamma>0,\sigma>0$, are parameters and $G(x;\mathbf{V})$ is a baseline cdf which depends on the parameter vector $\mathbf{V}$.

Remark 1.17. *Taking* $K(x) = 1 - \exp\left[-\left(\frac{G(x;\mathbf{V})}{1-G(x;\mathbf{V})}\right)^2\right]$, $x \in \mathbb{R}$, *the cdf* (53) *can be written as*

$$F(x;\gamma,\sigma,\mathbf{V}) = \frac{K(x)}{\frac{1}{\gamma} + \left(1 - \frac{1}{\gamma}\right)K(x)}, \quad x \in \mathbb{R},$$

which was mentioned in Remark 1.12.

28) The cdf and pdf of (CTWE) are given, respectively, by

$$F(x;\alpha,\theta,p) = \int_0^x f(u;\alpha,\theta,p)\,du, \quad x \geq 0, \tag{55}$$

and

$$f(x;\alpha,\theta,p) = Ce^{-\theta x}P(x), \quad x > 0, \tag{56}$$

where $\alpha > 0, \theta > 0, p \in [-1,1]$ are parameters and $P(x) = -6p(2\alpha+3)e^{-2(\alpha+1)\theta x} + 6p(\alpha+1)(\alpha+3)e^{-(\alpha+2)\theta x} + \alpha^2(p-1)e^{-\alpha\theta x} + 6p\alpha(\alpha+2)e^{-(\alpha+1)\theta x} + 6pe^{-3\alpha\theta x} + 6p\alpha e^{-(2\alpha+1)\theta x} - 6p(\alpha+1)^2 e^{-2\theta x} + 6p\alpha(\alpha+1)e^{-\theta x} + \alpha^2(p-1)$.

Remark 1.18. *The pdf* (56) *is a special case of the pdf* (52).

29) The cdf and pdf of (RXg-G) are given, respectively, by

$$F(x;\alpha,\theta,\xi) = \int_0^x f(u;\alpha,\theta,\xi)\,du, \quad x \in \mathbb{R}, \tag{57}$$

and

$$f(x;\alpha,\theta,\xi) = Cg(x;\xi)e^{-\alpha\theta G(x;\xi)}P(x), \quad x \in \mathbb{R}, \tag{58}$$

where $\alpha > 0, \theta > 0$ are parameters, $C = \frac{\alpha\theta^2}{1+\theta-\left(1+\theta+\alpha\theta+\frac{\alpha^2\theta^2}{2}\right)e^{-\alpha\theta}}$, $P(x) = 1 + \frac{\alpha^2\theta}{2}[G(x;\xi)]^2$ and $G(x;\xi)$ is a baseline cdf with the corresponding pdf $g(x;\xi)$ which depends on the parameter vector ξ.

Remark 1.19. *The pdf* (58) *is similar to the pdf* (1.80) *in Hamedani (2023).*

30) The cdf and pdf of ([0,1] TIWE) are given, respectively, by

$$F(x;\omega,\delta,\varepsilon) = \int_0^x f(u;\omega,\delta,\varepsilon)\,du, \quad x \geq 0, \tag{59}$$

and

$$f(x;\omega,\delta,\varepsilon) = Ce^{-\epsilon x}P(x), \quad x > 0, \tag{60}$$

where $\omega, \delta, \varepsilon$ are all positive parameters, $C = \frac{\omega\delta\varepsilon}{e^{-\delta}}$ and $P(x) = \left(1 - e^{-\epsilon x}\right)^{-\omega-1} e^{-\delta\left(1-e^{-\epsilon x}\right)^{-\omega}}$.

Remark 1.20. *The pdf* (60) *is similar to the pdf* (56).

31) The cdf and pdf of (BES) are given, respectively, by

$$F(x;\xi,\lambda) = (1+\lambda)G(x) - \lambda[G(x)]^2, \quad x \geq 0, \tag{61}$$

and

$$f(x;\xi,\lambda) = \frac{d}{dx}F(x;\xi,\lambda), \quad x > 0, \tag{62}$$

where $\xi > 0, \lambda \in [-1, 1]$ are parameters and $G(x) = 1 - \left(1 + \frac{\xi x\left(\xi^3 x^3 + 4\xi^2 x^2 + 12\xi x + 24\right)}{\xi^4 + 24}\right) e^{-\xi x}$.

Remarks 1.21. *(a) $e^{\xi x}$ in the equation* (5) *should be $e^{-\xi x}$. (b)The cdf* (61) *was mentioned in Remark 1.247 in Hamedani (2023).*

32) The cdf and pdf of (SLR) are given, respectively, by

$$F(x; \theta, \alpha) = \int_0^x f(u; \theta, \alpha)\, du, \quad x \geq 0, \tag{63}$$

and

$$f(x; \theta, \alpha) = Cx^{-(\alpha+1)} P(x), \quad x > 0, \tag{64}$$

where $\theta > 0, \alpha > 0$ are parameters, $C = \alpha^2 \theta^{\alpha/2}$, $P(x) = B\left(\frac{x^2}{\theta + x^2}; \frac{\alpha}{2} + 1, \frac{\alpha}{2}\right)$ and $B(x; a, b) = \int_0^x u^{a-1} (1-u)^{b-1}\, du$.

Remark 1.22. *The pdf* (64) *is a special case of the pdf* (52)*.*

33) The cdf and pdf of (TIHLTL-G) are given, respectively, by

$$F(x; \zeta, \theta, \chi) = \frac{K(x)}{2 - K(x)}, \quad x \in \mathbb{R}, \tag{65}$$

and

$$f(x; \zeta, \theta, \chi) = \frac{d}{dx} F(x; \zeta, \theta, \chi), \quad x \in \mathbb{R}, \tag{66}$$

where $\zeta > 0, \theta > 0$ are parameters and $K(x) = 1 - \left[1 - \left\{1 - (1 - G(x; \chi))^2\right\}^\theta\right]^\zeta$.

Remark 1.23. *The cdf* (65) *was mentioned in Remark 1.51 in Hamedani (2023).*

34) The cdf and pdf of (NQA) are given, respectively, by

$$F(x; \theta, \alpha) = \int_0^x f(u; \theta, \alpha)\, du, \quad x \geq 0, \tag{67}$$

and

$$f(x; \theta, \alpha) = Ce^{-\theta x} P(x), \quad x > 0, \tag{68}$$

where $\theta > 0, \alpha > 0$ are parameters, $C = \frac{\theta^3}{\theta^4 + 2\theta^2 \alpha + 2\alpha^2}$ and $P(x) = \theta^2 + 2\theta\alpha x + \alpha^2 x^2$.

Remark 1.24. *The pdf* (68) *is similar to the pdf* (60)*.*

35) The cdf and pdf of (GGW) are given, respectively, by

$$F(x; \delta) = \frac{1}{\Gamma(\delta)} \int_0^{-\log[1 - G(x)]} t^{\delta - 1} e^{-t} dt, \quad x \geq 0, \tag{69}$$

and

$$f(x; \delta) = \frac{d}{dx} F(x; \delta), \quad x > 0, \tag{70}$$

where $\delta > 0$ is a parameters and $G(x)$ is a baseline cdf.

Remark 1.25. *The cdf* (70) *is a special case of the cdf mentioned in Remark 1.130 in Hamedani (2023).*

36) The cdf and pdf of (PCJ) are given, respectively, by

$$F(x;\theta,\alpha)=\int_0^x f(u;\theta,\alpha)\,du,\quad x\geq 0, \tag{71}$$

and

$$f(x;\theta,\alpha)=Cx^{\alpha-1}e^{-\theta x^{\alpha}}P(x),\quad x>0, \tag{72}$$

where $\theta>0,\alpha>0$ are parameters, $C=\frac{\alpha\theta^2}{\theta+2}$ and $P(x)=1+\theta x^{2\alpha}$.

Remark 1.26. *The pdf* (72) *is similar to the pdf* (52).

37) The cdf and pdf of (OG-G) are given, respectively, by

$$F(x;\theta,\gamma,\mathbf{\Phi})=1-\exp\left\{-\frac{\theta}{\gamma}\left[e^{\gamma\frac{G(x;\mathbf{\Phi})}{1-G(x;\mathbf{\Phi})}}-1\right]\right\},\quad x\geq 0, \tag{73}$$

and

$$f(x;\theta,\gamma,\mathbf{\Phi})=\frac{d}{dx}F(x;\theta,\gamma,\mathbf{\Phi}),\quad x>0, \tag{74}$$

where $\theta>0,\gamma>0$ are parameters and $G(x;\mathbf{\Phi})$ is a baseline cdf which depends on the parameter vector $\mathbf{\Phi}$.

Remark 1.27. *Taking* $K(x)=1-\exp\left\{-\gamma\left[e^{\gamma\frac{G(x;\mathbf{\Phi})}{1-G(x;\mathbf{\Phi})}}\right]\right\}$, $x\geq 0$*, the cdf* (73) *can be written as*

$$F(x;\theta,\gamma,\mathbf{\Phi})=1-\exp\left\{-\frac{\theta}{\gamma}\left[\frac{K(x)}{1-K(x)}\right]\right\},\quad x\geq 0,$$

which is a special case of the cdf mentioned in Remark 1.2 in Hamedani (2023).

38) The cdf and pdf of (TLqW) are given, respectively, by

$$F(x;\lambda,\alpha,\gamma,p)=\int_0^x f(u;\lambda,\alpha,\lambda,p)\,du,\quad x\geq 0, \tag{75}$$

and

$$f(x;\theta,\alpha,\gamma,p)=Cx^{\gamma-1}\left[1+(p-1)(\lambda x)^{\gamma}\right]^{\frac{p-3}{p-1}}P(x),\quad x>0, \tag{76}$$

where $\lambda>0,\alpha>0,\gamma>0,p\in(1,2)$ are parameters, $C=2\alpha\gamma\lambda^{\gamma}(2-p)$ and $P(x)=1-\left[\left[1+(p-1)(\lambda x)^{\gamma}\right]^{\frac{2p-4}{p-1}}\right]^{\alpha-1}$.

Remark 1.28. *The pdf* (76) *is similar to the pdf* (6).

39) The cdf and pdf of (NAPIW) are given, respectively, by

$$F(x;\alpha,t,\rho,b)=\frac{\alpha^{G(x)}-1}{\alpha-1},\quad x\geq 0, \tag{77}$$

and

$$f(x;\alpha,t,\rho,b)=\frac{d}{dx}F(x;\alpha,t,\rho,b),\quad x>0, \tag{78}$$

where $\alpha(\neq 1)>0,t>0,\rho>0,b>0$ are parameters and $G(x)=\frac{\exp\left\{-\alpha\left[1-e^{-(tx)^2}\right]^{-\rho b}\right\}}{e^{-\alpha}}$.

Remark 1.29. *The cdf* (77) *was mentioned in Remark 143 in Hamedani (2021).*

40) The cdf and pdf of (EPF) are given, respectively, by

$$F(x;\alpha,\beta,\lambda,\theta,\upsilon,\rho)=\int_0^x f(u;\alpha,\beta,\lambda,\theta,\upsilon,\rho)\,du,\quad x\geq 0, \tag{79}$$

and

$$f(x;\alpha,\beta,\lambda,\theta,\upsilon,\rho)=\frac{1}{\alpha+1}\left(\frac{x}{\rho}\right)^{\theta-1}e^{-\upsilon\left(\frac{x}{\rho}\right)^{\theta}}P(x),\quad x>0, \tag{80}$$

where $\alpha,\beta,\lambda,\theta,\upsilon,\rho$ are all positive parameters and $P(x)=\frac{\theta\upsilon}{\rho}+\frac{\alpha\lambda}{\beta}\left(\frac{x}{\rho}\right)^{1-\theta}\left(\frac{\beta}{x}\right)^{\lambda+1}e^{\upsilon\left(\frac{x}{\rho}\right)^{\theta}-\left(\frac{\beta}{x}\right)^{\lambda}}$.

Remark 1.30. *The pdf* (80) *is similar to the pdf* (6)*.*

41) The cdf and pdf of (GOCH) are given, respectively, by

$$F(x;\theta,\alpha,\lambda,\beta)=1-\exp\left\{-\frac{\theta}{\alpha}\left[e^{-\alpha\lambda\left(1-e^{x^{\beta}}\right)}-1\right]\right\},\quad x\geq 0, \tag{81}$$

and

$$f(x;\theta,\alpha,\lambda,\beta)=\frac{d}{dx}F(x;\theta,\alpha,\lambda,\beta),\quad x>0, \tag{82}$$

where $\theta,\alpha,\lambda,\beta$ are all positive parameters.

Remark 1.31. *Taking* $G(x)=1-e^{\alpha\lambda\left(1-e^{x^{\beta}}\right)}$, $x\geq 0$*, the cdf* (81) *can be expressed as*

$$F(x;\theta,\alpha,\lambda,\beta)=1-\exp\left\{-\frac{\theta}{\alpha}\left[\frac{G(x)}{1-G(x)}\right]\right\},\quad x\geq 0,$$

which is a special case of the cdf mentioned in Remark 1.2 in Hamedani (2023).

42) The cdf and pdf of (EEGL) are given, respectively, by

$$F(x;\theta,\alpha,\beta,\gamma)=\int_0^x f(u;\theta,\alpha,\beta,\gamma)\,du,\quad x\geq 0, \tag{83}$$

and

$$f(x;\theta,\alpha,\beta,\gamma)=C(\theta x)^{\gamma-1}e^{-(\theta x)^{\gamma}}P(x),\quad x>0, \tag{84}$$

where $\theta>0,\alpha<0,\beta>\frac{\theta}{\theta+1},\lambda,\gamma>0$ are parameters and
$P(x)=-\left((\theta\beta+\beta-\theta)(\theta x+1)+\theta\right)^{\alpha-1}\Big[(\theta\beta+\beta-\theta)(\theta x)^{1-\gamma}-\gamma\left((\theta\beta+\beta-\theta)(\theta x+1)+\theta\right)\Big]$.

Remark 1.32. *The pdf* (82) *is similar to the pdf* (6)*.*

43) The cdf and pdf of (LTII-T) are given, respectively, by

$$F(x;\theta,\beta)=1-\exp\left\{\beta\left(\theta x-e^{\theta x}+1\right)\right\},\quad x\geq 0, \tag{85}$$

and

$$f(x;\theta,\beta)=\beta\theta e^{-\theta x}P(x),\quad x>0, \tag{86}$$

where $\theta > 0, \beta > 0$ are parameters and $P(x) = \left(e^{\theta x} - 1\right) \exp\left\{\theta x + \beta\left(\theta x - e^{\theta x} + 1\right)\right\}$.

44) The cdf and pdf of (TIIEHLGom-G) are given, respectively, by

$$F(x;\alpha,\gamma,\psi) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\left\{1 - [1 - G(x;\psi)]^{-\gamma}\right\}\right)}{1 + \left[1 - \exp\left(\frac{1}{\gamma}\left\{1 - [1 - G(x;\psi)]^{-\gamma}\right\}\right)\right]}\right]^{\alpha}, \quad x \in \mathbb{R}, \tag{87}$$

and

$$f(x;\alpha,\gamma,\psi) = \frac{d}{dx} F(x;\alpha,\gamma,\psi), \quad x \in \mathbb{R}, \tag{88}$$

where $\alpha > 0, \gamma > 0$ are parameters and $G(x;\psi)$ is a baseline cdf which depends on the parameter vector ψ.

Remark 1.33. *Taking*

$$K(x) = 1 - \frac{\exp\left(\frac{1}{\gamma}\left\{1 - [1 - G(x;\psi)]^{-\gamma}\right\}\right)}{1 + \left[1 - \exp\left(\frac{1}{\gamma}\left\{1 - [1 - G(x;\psi)]^{-\gamma}\right\}\right)\right]}, \quad x \geq 0,$$

the cdf (87) *can be written as*

$$F(x;\alpha,\gamma,\psi) = 1 - [1 - K(x)]^{\alpha}, \quad x \in \mathbb{R},$$

which is a special case of the cdf mentioned in Remark 1.39 in Hamedani (2023).

2. Characterization of Distributions

As mentioned in the Introduction, characterizations of distributions is an important area of research which has recently attracted the attention of many researchers. This section deals with various characterizations of the distributions listed in the Introduction. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) the reverse hazard function and (iv) conditional expectation of a single function of the random variable. It should be mentioned that for the characterization (i) the cdf need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

2.1. Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of all of the distributions mentioned in the Introduction, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to Glänzel (1987), see Theorem 2.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glanzel (1990) , this characterization is stable in the sense of weak convergence.

Theorem 2.1. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that*

$$\mathbf{E}[q_2(X) \mid X \geq x] = \mathbf{E}[q_1(X) \mid X \geq x]\,\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)\,q_1(u) - q_2(u)}\right| \exp(-s(u))\ du\ ,$$

where the function s *is a solution of the differential equation* $s' = \frac{\eta' \, q_1}{\eta \, q_1 - q_2}$ *and* C *is the normalization constant, such that* $\int_H dF = 1$.

Here is our first characterization.

Proposition 2.1. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable and let* $q_1(x) = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x) e^{-\beta x}$ *for* $x > 0$. *The random variable* X *has pdf* (2) *if and only if the function* η *defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2} e^{-\beta x}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (2), then

$$(1 - F(x)) E[q_1(X) \mid X \geq x] = \int_x^\infty C e^{-\beta u} du = \frac{C}{\beta} e^{-\beta x}, \quad x > 0,$$

and

$$(1 - F(x)) E[q_2(X) \mid X \geq x] = \int_x^\infty C e^{-2\beta u} du = \frac{C}{2\beta} e^{-2\beta x}, \quad x > 0,$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\beta x} < 0 \quad \text{for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \beta, \quad x > 0,$$

and hence

$$s(x) = \beta x, \quad x > 0.$$

Now, in view of Theorem 2.1, X has density (2). □

Corollary 2.1. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition 2.1. The pdf of* X *is* (2) *if and only if there exist functions* q_2 *and* η *defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \beta, \quad x > 0.$$

Corollary 2.2. *The general solution of the differential equation in Corollary 2.1 is*

$$\eta(x) = e^{\beta x} \left[-\int \beta e^{-\beta x} (q_1(x))^{-1} q_2(x) \, dx + D \right],$$

where D *is a constant.*

Proof. If X has pdf (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \beta \eta(x) - \beta (q_1(x))^{-1} q_2(x),$$

or

$$\eta'(x) - \beta \eta(x) = -\beta (q_1(x))^{-1} q_2(x),$$

or

$$e^{-\beta x} \eta'(x) - \beta e^{-\beta x} \eta(x) = -\beta e^{-\beta x} (q_1(x))^{-1} q_2(x),$$

or

$$\frac{d}{dx}\left\{e^{-\beta x}\eta\left(x\right)\right\} = -\beta e^{-\beta x}\left(q_1\left(x\right)\right)^{-1} q_2\left(x\right),$$

from which we arrive at

$$\eta\left(x\right) = e^{\beta x}\left[-\int \beta e^{-\beta x}\left(q_1\left(x\right)\right)^{-1} q_2\left(x\right)dx + D\right].$$

Note that a set of functions satisfying the differential equation in Corollary 2.1, is given in Proposition 2.1 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1. □

Remark 2.1. *Taking $G\left(x\right) = 1 - \frac{\Gamma(\alpha,\beta x)}{\Gamma(\alpha)}$, $x \geq 0$, the cdf (1) can be written as*

$$\frac{1-e^{-\lambda G(x)}}{1-e^{-\lambda}}, \qquad x \geq 0,$$

which has been characterized in Hamedani (2021).

Proposition 2.2. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1\left(x\right) = \left[P\left(x\right)\right]^{-1}$ and $q_2\left(x\right) = q_1\left(x\right)\left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^{-2}$ for $x > 0$. The random variable X has pdf (4) if and only if the function η defined in Theorem 2.1 has the form*

$$\eta\left(x\right) = \frac{1}{2}\left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^{-2}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (4), then

$$\begin{aligned}\left(1-F\left(x\right)\right)E\left[q_1\left(X\right) \mid X \geq x\right] &= \int_x^{\infty} \beta\phi e^{\phi u}\left(e^{\phi u}-1\right)^{\beta-1}\left[1+\left(e^{\phi u}-1\right)^{\beta}\right]^{-3} du \\ &= \frac{1}{2}\left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^{-2}, \quad x > 0,\end{aligned}$$

and

$$\begin{aligned}\left(1-F\left(x\right)\right)E\left[q_2\left(X\right) \mid X \geq x\right] &= \int_x^{\infty} \beta\phi e^{\phi u}\left(e^{\phi u}-1\right)^{\beta-1}\left[1+\left(e^{\phi u}-1\right)^{\beta}\right]^{-5} du \\ &= \frac{1}{4}\left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^{-4}, \quad x > 0,\end{aligned}$$

and finally

$$\eta\left(x\right)q_1\left(x\right) - q_2\left(x\right) = -\frac{q_1\left(x\right)}{2}\left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^{-2} < 0 \quad \text{for} \quad x > 0.$$

Conversely, if η is given as above, then

$$s'\left(x\right) = \frac{\eta'\left(x\right)q_1\left(x\right)}{\eta\left(x\right)q_1\left(x\right) - q_2\left(x\right)} = \frac{2\beta\phi e^{\phi x}\left(e^{\phi x}-1\right)^{\beta-1}}{1+\left(e^{\phi x}-1\right)^{\beta}}, \quad x > 0,$$

and hence

$$s\left(x\right) = 2\ln\left[1+\left(e^{\phi x}-1\right)^{\beta}\right], \quad x > 0.$$

Now, in view of Theorem 2.1, X has density (4). □

Corollary 2.3. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1\left(x\right)$ be as in Proposition 2.2. The pdf of X is (4) if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{2\beta\phi e^{\phi x}\left(e^{\phi x}-1\right)^{\beta-1}}{1+\left(e^{\phi x}-1\right)^{\beta}}, \quad x>0.$$

Corollary 2.4. *The general solution of the differential equation in Corollary 2.3 is*

$$\eta(x) = \left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^2 \left[-\int 2\beta\phi e^{\phi x}\left(e^{\phi x}-1\right)^{\beta-1}\left[1+\left(e^{\phi x}-1\right)^{\beta}\right]^{-3} (q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.2, is given in Proposition 2.2 with $D=0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.3. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, \lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}}$ for $x>0$. The random variable X has pdf (6) if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left[\lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}} + 1\right], \quad x>0.$$

Proof. Let X be a random variable with pdf (6), then

$$\begin{aligned}(1-F(x))\, E\left[q_1(X) \mid X \geq x\right] &= \int_x^{\infty} \frac{\theta}{\xi^{\theta}} u^{\theta-1}\left(1+\left(\frac{u}{\xi}\right)^{\theta}\right)^{-2} \lambda^{\left(1+\left(\frac{u}{\xi}\right)^{\theta}\right)^{-1}} du \\ &= \frac{1}{\log\lambda}\left[\lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}} - 1\right], \quad x>0,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\, E\left[q_2(X) \mid X \geq x\right] &= \int_x^{\infty} \frac{\theta}{\xi^{\theta}} u^{\theta-1}\left(1+\left(\frac{u}{\xi}\right)^{\theta}\right)^{-2} \lambda^{2\left(1+\left(\frac{u}{\xi}\right)^{\theta}\right)^{-1}} du \\ &= \frac{1}{2\log\lambda}\left[\lambda^{2\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}} - 1\right], \quad x>0,\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left[\lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}} - 1\right] < 0 \quad \text{for } x>0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\frac{\theta}{\xi^{\theta}} x^{\theta-1}\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-2} \lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}} \log\lambda}{\lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}} - 1}, \quad x>0,$$

and hence

$$s(x) = -\log\left[\lambda^{\left(1+\left(\frac{x}{\xi}\right)^{\theta}\right)^{-1}} - 1\right], \quad x>0.$$

Now, in view of Theorem 2.1, X has density (6). □

Corollary 2.5. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.3. The pdf of X is* (6) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\frac{\theta}{\xi^\theta}x^{\theta-1}\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-2}\lambda^{\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-1}}\log\lambda}{\lambda^{\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-1}}-1},\quad x>0.$$

Corollary 2.6. *The general solution of the differential equation in Corollary 2.5 is*

$$\eta(x)=\left[\lambda^{\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-1}}-1\right]^{-1}\left[-\int\frac{\theta}{\xi^\theta}x^{\theta-1}\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-2}\lambda^{\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-1}}\log\lambda\,(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.3, is given in Proposition 2.3 with $D=\frac{1}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.4. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)\equiv 1$ and $q_2(x)=q_1(x)\exp\left[-\theta x-Cx^{\alpha(\beta-1)+1}\right]$ for $x>0$. The random variable X has pdf* (10) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}\exp\left[-\theta x-Cx^{\alpha(\beta-1)+1}\right],\quad x>0.$$

Proof. Let X be a random variable with pdf (10), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\int_x^\infty f(x;\lambda,\alpha,\beta,\theta)\,du=\exp\left[-\theta x-Cx^{\alpha(\beta-1)+1}\right],\quad x>0,$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]&=\int_x^\infty f(x;\lambda,\alpha,\beta,\theta)\exp\left[-\theta u-Cu^{\alpha(\beta-1)+1}\right]du\\&=\frac{1}{2}\exp\left[-2\theta x-2Cx^{\alpha(\beta-1)+1}\right],\quad x>0,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\exp\left[-\theta x-Cx^{\alpha(\beta-1)+1}\right]<0\quad\text{for}\quad x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\theta+C\left(\alpha(\beta-1)+1\right)x^{\alpha(\beta-1)},\quad x>0,$$

and hence

$$s(x)=\theta x+Cx^{\alpha(\beta-1)+1},\quad x>0.$$

Now, in view of Theorem 2.1, X has density (10). □

Corollary 2.7. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.4. The pdf of X is* (10) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\theta+C\left(\alpha(\beta-1)+1\right)x^{\alpha(\beta-1)},\quad x>0.$$

Corollary 2.8. *The general solution of the differential equation in Corollary 2.7 is*

$$\eta(x) = \exp\left[\theta x + Cx^{\alpha(\beta-1)+1}\right] \times \\ \left[-\int \left[\theta + C\left(\alpha\left(\beta-1\right)+1\right)x^{\alpha(\beta-1)}\right] \exp\left[-\theta x - Cx^{\alpha(\beta-1)+1}\right] \left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.7, is given in Proposition 2.4 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.5. *Let $X : \Omega \to (0,1)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\xi x^{-\psi}}$ for $0 < x < 1$. The random variable X has pdf (16) if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{e^{-\xi} + e^{-\xi x^{-\psi}}\right\}, \quad 0 < x < 1.$$

Proof. Let X be a random variable with pdf (16), then

$$(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] = \int_x^1 Cu^{-\psi-1}e^{-\xi u^{-\psi}}du = \frac{C}{\xi\psi}\left\{e^{-\xi} - e^{-\xi x^{-\psi}}\right\}, \quad 0 < x < 1,$$

and

$$(1 - F(x))\, E\left[q_2(X) \mid X \geq x\right] = \int_x^1 Cu^{-\psi-1}e^{-2\xi u^{-\psi}}du = \frac{C}{2\xi\psi}\left\{e^{-2\xi} - e^{-2\xi x^{-\psi}}\right\}, \quad 0 < x < 1,$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left\{e^{-\xi} - e^{-\xi x^{-\psi}}\right\} < 0 \quad \text{for} \quad 0 < x < 1.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\xi\psi x^{-\psi-1}e^{-\xi x^{-\psi}}}{e^{-\xi} - e^{-\xi x^{-\psi}}}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\ln\left\{e^{-\xi} - e^{-\xi x^{-\psi}}\right\}, \quad 0 < x < 1.$$

Now, in view of Theorem 2.1, X has density (16). □

Corollary 2.9. *Let $X : \Omega \to (0,1)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.5. The pdf of X is (16) if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\xi\psi x^{-\psi-1}e^{-\xi x^{-\psi}}}{e^{-\xi} - e^{-\xi x^{-\psi}}}, \quad 0 < x < 1.$$

Corollary 2.10. *The general solution of the differential equation in Corollary 2.9 is*

$$\eta(x) = \left\{e^{-\xi} - e^{-\xi x^{-\psi}}\right\}^{-1}\left[-\int \xi\psi x^{-\psi-1}e^{-\xi x^{-\psi}}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.9, is given in Proposition 2.5 with $D = \frac{1}{2}e^{-\xi}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.6. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)e^{-x^\beta}$ for $x > 0$. The random variable X has pdf* (28) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}e^{-x^\beta}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (28), then

$$(1 - F(x)) E[q_1(X) \mid X \geq x] = \int_x^\infty Cu^{\beta-1}e^{-u^\beta}du = \frac{C}{\beta}e^{-x^\beta}, \quad x > 0,$$

and

$$(1 - F(x)) E[q_2(X) \mid X \geq x] = \int_x^\infty Cu^{\beta-1}e^{-2u^\beta}du = \frac{C}{2\beta}e^{-2x^\beta}, \quad x > 0,$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2}e^{-x^\beta} < 0 \quad \text{for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \beta x^{\beta-1}, \quad x > 0,$$

and hence

$$s(x) = x^\beta, \quad x > 0.$$

Now, in view of Theorem 2.1, X has density (28). □

Corollary 2.11. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.6. The pdf of X is* (28) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \beta x^{\beta-1}, \quad x > 0.$$

Corollary 2.12. *The general solution of the differential equation in Corollary 2.11 is*

$$\eta(x) = e^{x^\beta}\left[-\int \beta x^{\beta-1}e^{-x^\beta}(q_1(x))^{-1} q_2(x)\,dx + D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.11, is given in Proposition 2.6 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.7. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)$ for $x > 0$. The random variable X has pdf* (30)*, for $\zeta > 1$, if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{1 + \exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)\right\}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (30), then

$$\begin{aligned}(1-F(x))\,E\left[q_1(X)\mid X\geq x\right] &= \int_x^{\infty} C\zeta^u(\ln(\zeta))(1+\zeta^u)^{-2}\exp\left(-\frac{3+\zeta}{1+\zeta^u}\right)du \\ &= \frac{C}{(3+\zeta)}\left\{1-\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)\right\}, \quad x>0,\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^{\infty} C\zeta^u(\ln(\zeta))(1+\zeta^u)^{-2}\exp\left(-2\frac{3+\zeta}{1+\zeta^u}\right)du \\ &= \frac{C}{2(3+\zeta)}\left\{1-\exp\left(-2\frac{3+\zeta}{1+\zeta^x}\right)\right\}, \quad x>0,\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)\right\}>0 \quad \text{for} \quad x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{(3+\zeta)\,\zeta^x(\ln(\zeta))(1+\zeta^x)^{-2}\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)}{1-\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)}, \quad x>0,$$

and hence

$$s(x)=-\ln\left\{1-\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)\right\}, \quad x>0.$$

Now, in view of Theorem 2.1, X has density (30). □

Corollary 2.13. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.7. The pdf of X is* (30) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{(3+\zeta)\,\zeta^x(\ln(\zeta))(1+\zeta^x)^{-2}\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)}{1-\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)}, \quad x>0.$$

Corollary 2.14. *The general solution of the differential equation in Corollary 2.13 is*

$$\eta(x)=\left\{1-\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)\right\}^{-1}\left[-\int(3+\zeta)\,\zeta^x(\ln(\zeta))(1+\zeta^x)^{-2}\exp\left(-\frac{3+\zeta}{1+\zeta^x}\right)(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.13, is given in Proposition 2.7 with $D=\frac{1}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 2.1.

Proposition 2.8. *Let $X:\Omega\to(0,1)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,\alpha^{x^{\beta}}$ for $0<x<1$. The random variable X has pdf* (40) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left(\alpha + \alpha^{x^{\beta}}\right), \quad 0 < x < 1.$$

Proof. Let X be a random variable with pdf (40), then

$$(1 - F(x))\, E[q_1(X) \mid X \geq x] = \int_x^1 \beta u^{\beta-1} \alpha^{u^{\beta}} du = \frac{1}{\log(\alpha)}\left(\alpha - \alpha^{x^{\beta}}\right), \quad 0 < x < 1,$$

and

$$(1 - F(x))\, E[q_2(X) \mid X \geq x] = \int_x^1 \beta u^{\beta-1} \alpha^{2u^{\beta}} du = \frac{1}{2\log(\alpha)}\left(\alpha^2 - \alpha^{2x^{\beta}}\right), \quad 0 < x < 1,$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left(\alpha - \alpha^{x^{\beta}}\right) > 0 \quad \text{for} \quad 0 < x < 1.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\beta x^{\beta-1} \alpha^{x^{\beta}} \log(\alpha)}{\alpha - \alpha^{x^{\beta}}}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\ln\left\{\alpha - \alpha^{x^{\beta}}\right\}, \quad 0 < x < 1.$$

Now, in view of Theorem 2.1, X has density (40). ☐

Corollary 2.15. *Let $X : \Omega \to (0,1)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.8. The pdf of X is* (40) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\beta x^{\beta-1} \alpha^{x^{\beta}} \log(\alpha)}{\alpha - \alpha^{x^{\beta}}}, \quad 0 < x < 1.$$

Corollary 2.16. *The general solution of the differential equation in Corollary 2.15 is*

$$\eta(x) = \left(\alpha - \alpha^{x^{\beta}}\right)^{-1}\left[-\int \beta x^{\beta-1} \alpha^{x^{\beta}} \log(\alpha)\, (q_1(x))^{-1}\, q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. ☐

Note that a set of functions satisfying the differential equation in Corollary 2.15, is given in Proposition 2.8 with $D = \frac{\alpha^2}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.9. *Let the random variable $X : \Omega \to \mathbb{R}$ be continuous, and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\left[\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right]$ for $x \in \mathbb{R}$. The random variable X has pdf* (42) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{x\left(1 - e^{-\frac{x^2}{2\theta^2}} + m\right) - b}{a}\right)\right\}, \quad x \in \mathbb{R}.$$

Proof. Let X be a random variable with pdf (42), then

$$(1-F(x))\,E\left[q_1(X)\mid X\ge x\right]=\frac{2\alpha}{\pi a}\int_x^{\infty}\left\{\frac{\left(\frac{1+\left(\frac{u^2}{2\theta^2}-1\right)e^{-\frac{u^2}{2\theta^2}}+m}{1+\left(\frac{u\left(\left(\frac{u^2}{2\theta^2}-1\right)e^{-\frac{u^2}{2\theta^2}}\right)-b}{a}\right)^2}\right)\times}{\left(\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{u\left(1-e^{-\frac{u^2}{2\theta^2}}+m\right)-b}{a}\right)\right)}du\right\}$$

$$=2\alpha\left\{\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right\},\quad x\in\mathbb{R},$$

and similarly

$$(1-F(x))\,E\left[q_2(X)\mid X\ge x\right]=\alpha\left\{\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right\},\quad x\in\mathbb{R},$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{1}{2}q_1(x)\left\{\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right\}<0\quad\text{for}\quad x\in\mathbb{R}.$$

Conversely, if η has the above form, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=-\frac{\left(\frac{1+\left(\frac{x^2}{2\theta^2}-1\right)e^{-\frac{x^2}{2\theta^2}}+m}{1+\left(\frac{x\left(\left(\frac{x^2}{2\theta^2}-1\right)e^{-\frac{x^2}{2\theta^2}}\right)-b}{a}\right)^2}\right)}{\left\{\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right\}},$$

and hence

$$s(x)=\log\left\{\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right\}^{-1},\quad x\in\mathbb{R}.$$

In view of Theorem 2.1, X has pdf (42). □

Corollary 2.17. *If $X:\Omega\to\mathbb{R}$ is a continuous random variable and $q_1(x)$ is as in Proposition 2.9. Then, X has pdf* (42) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the following first order differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=-\frac{\left(\frac{1+\left(\frac{x^2}{2\theta^2}-1\right)e^{-\frac{x^2}{2\theta^2}}+m}{1+\left(\frac{x\left(\left(\frac{x^2}{2\theta^2}-1\right)e^{-\frac{x^2}{2\theta^2}}\right)-b}{a}\right)^2}\right)}{\left\{\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right\}}.$$

Corollary 2.18. *The general solution of the differential equation in Corollary 2.17 is*

$$\eta(x)=\left\{\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x\left(1-e^{-\frac{x^2}{2\theta^2}}+m\right)-b}{a}\right)\right\}^{-1}\times$$
$$\left[\int\left(\frac{1+\left(\frac{x^2}{2\theta^2}-1\right)e^{-\frac{x^2}{2\theta^2}}+m}{1+\left(\frac{x\left(\left(\frac{u^2}{2\theta^2}-1\right)e^{-\frac{u^2}{2\theta^2}}\right)-b}{a}\right)^2}\right)(q_1(x))^{-1}\,q_2(x)+D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.17, is given in Proposition 2.9 with $D=0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.10. *Let the random variable $X:\Omega\rightarrow\mathbb{R}$ be continuous, and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,G(x;\zeta)^2$ for $x\in\mathbb{R}$. The random variable X has pdf* (44) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x)=\frac{1}{2}\left\{1+G(x;\zeta)^2\right\},\quad x\in\mathbb{R}.$$

Proof. Let X be a random variable with pdf (44), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=C\int_x^{\infty}g(u;\zeta)\,G(u;\zeta)\,du=\frac{C}{2}\left\{1-G(x;\zeta)^2\right\},\quad x\in\mathbb{R},$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\frac{C}{4}\left\{1-G(x;\zeta)^4\right\},\quad x\in\mathbb{R},$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-G(x;\zeta)^2\right\}>0\quad\text{for}\quad x\in\mathbb{R}.$$

Conversely, if η has the above form, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{2g(x;\zeta)\, G(x;\zeta)}{1 - G(x;\zeta)^2},$$

and hence

$$s(x) = -\log\left\{1 - G(x;\zeta)^2\right\}, \quad x \in \mathbb{R}.$$

In view of Theorem 2.1, X has pdf (44). □

Corollary 2.19. *If $X : \Omega \to \mathbb{R}$ is a continuous random variable and $q_1(x)$ is as in Proposition 2.10. Then, X has pdf* (44) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the following first order differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{2g(x;\zeta)\, G(x;\zeta)}{1 - G(x;\zeta)^2}.$$

Corollary 2.20. *The general solution of the differential equation in Corollary 2.19 is*

$$\eta(x) = \left\{1 - G(x;\zeta)^2\right\}^{-1}\left[-\int 2g(x;\zeta)\, G(x;\zeta)\,(q_1(x))^{-1}\, q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.19, is given in Proposition 2.10 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.11. *Let the random variable $X : \Omega \to (0,1)$ be continuous, and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)(1 - k\log(x))^{-1}$ for $x \in (0,1)$. The random variable X has pdf* (48) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{1 + (1 - k\log(x))^{-1}\right\}, \quad x \in (0,1).$$

Proof. Let X be a random variable with pdf (48), then

$$(1 - F(x))\, E[q_1(X) \mid X \geq x] = \int_x^1 k u^{-1}(1 - k\log(u))^{-2}\, du = 1 - (1 - k\log(x))^{-1}, \quad x \in (0,1),$$

and

$$(1 - F(x))\, E[q_2(X) \mid X \geq x] = \frac{1}{2}\left\{1 - (1 - k\log(x))^{-2}\right\}, \quad x \in (0,1),$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left\{1 - (1 - k\log(x))^{-1}\right\} > 0 \quad \text{for} \quad x \in (0,1).$$

Conversely, if η has the above form, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{k x^{-1}(1 - k\log(x))^{-2}}{1 - (1 - k\log(x))^{-1}},$$

and hence

$$s(x) = -\log\left\{1 - (1 - k\log(x))^{-1}\right\}, \quad x \in (0,1).$$

In view of Theorem 2.1, X has pdf (48). □

Corollary 2.21. *If $X : \Omega \to (0,1)$ is a continuous random variable and $q_1(x)$ is as in Proposition 2.11. Then, X has pdf* (48) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the following first order differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{k x^{-1} (1 - k \log(x))^{-2}}{1 - (1 - k \log(x))^{-1}}.$$

Corollary 2.22. *The general solution of the differential equation in Corollary 2.21 is*

$$\eta(x) = \left\{1 - (1 - k\log(x))^{-1}\right\}^{-1} \left[-\int k x^{-1} (1 - k\log(x))^{-2} (q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.21, is given in Proposition 2.11 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.12. *Let the random variable $X : \Omega \to (0,\infty)$ be continuous, and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}$ for $x > 0$. The random variable X has pdf* (52) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (52), then

$$(1 - F(x))\, E[q_1(X) \mid X \geq x] = \int_x^\infty \frac{\alpha}{\beta} \left(\frac{u}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{u}{\beta}\right)^{\alpha}} du = \frac{1}{4} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x > 0,$$

and

$$(1 - F(x))\, E[q_2(X) \mid X \geq x] = \frac{1}{8} e^{-8\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x > 0,$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}} < 0 \quad \text{for} \quad x > 0.$$

Conversely, if η has the above form, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = 4 \left(\frac{\alpha}{\beta}\right) \left(\frac{x}{\beta}\right)^{\alpha-1},$$

and hence

$$s(x) = 4 \left(\frac{x}{\beta}\right)^{\alpha}, \quad x > 0.$$

In view of Theorem 2.1, X has pdf (52). □

Corollary 2.23. *If $X : \Omega \to (0,\infty)$ is a continuous random variable and $q_1(x)$ is as in Proposition 2.12. Then, X has pdf* (52) *if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the following first order differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = 4 \left(\frac{\alpha}{\beta}\right) \left(\frac{x}{\beta}\right)^{\alpha-1}.$$

Corollary 2.24. *The general solution of the differential equation in Corollary 2.23 is*

$$\eta(x) = e^{4\left(\frac{x}{\beta}\right)^{\alpha}} \left[-\int 4\left(\frac{\alpha}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D \right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.23, is given in Proposition 2.12 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

Proposition 2.13. *Let the random variable $X : \Omega \to (0, \infty)$ be continuous, and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\theta x}$ for $x > 0$. The random variable X has pdf (86) if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{1}{2} e^{-\theta x}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (86), then

$$(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] = \int_x^{\infty} \beta\theta e^{-\theta u} du = \beta e^{-\theta x}, \quad x > 0,$$

and

$$(1 - F(x))\, E\left[q_2(X) \mid X \geq x\right] = \frac{\beta}{2} e^{-2\theta x}, \quad x > 0,$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\theta x} < 0 \quad \text{for } x > 0.$$

Conversely, if η has the above form, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \theta,$$

and hence

$$s(x) = \theta x, \quad x > 0.$$

In view of Theorem 2.1, X has pdf (86). □

Corollary 2.25. *If $X : \Omega \to (0, \infty)$ is a continuous random variable and $q_1(x)$ is as in Proposition 2.13. Then, X has pdf (86) if and only if there exist functions q_2 and η defined in Theorem 2.1 satisfying the following first order differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \theta.$$

Corollary 2.26. *The general solution of the differential equation in Corollary 2.23 is*

$$\eta(x) = e^{\theta x} \left[-\int \theta e^{-\theta x} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D \right],$$

where D is a constant.

Proof. It is similar to the proof of Corollary 2.2. □

Note that a set of functions satisfying the differential equation in Corollary 2.23, is given in Proposition 2.12 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.

2.2. Characterization in Terms of Hazard Function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of two of the new distributions in terms of the hazard function, which are not of the above trivial form.

Proposition 2.14. *Let $X : \Omega \to (0,1)$ be a continuous random variable. The random variable X has pdf* (40) *if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$h'_F(x) - (\beta - 1)\, x^{-1} h_F(x) = \beta x^{\beta-1} \frac{d}{dx}\left\{ \frac{\log(\alpha)\left(1 - x^{\beta}\right) + 1}{1 - x^{\beta}} \right\}, \quad 0 < x < 1,$$

with the boundary condition $\lim_{x \to 0} h_F(x) = 0$ *for* $\beta > 1$.

Proof. Multiplying both sides of the above equation by $x^{1-\beta}$, we have

$$\frac{d}{dx}\left\{x^{1-\beta} h_F(x)\right\} = \beta \frac{d}{dx}\left\{ \frac{\log(\alpha)\left(1 - x^{\beta}\right) + 1}{1 - x^{\beta}} \right\},$$

or

$$x^{1-\beta} h_F(x) = \beta \left\{ \frac{\log(\alpha)\left(1 - x^{\beta}\right) + 1}{1 - x^{\beta}} \right\},$$

or

$$h_F(x) = \beta x^{\beta-1} \left\{ \frac{\log(\alpha)\left(1 - x^{\beta}\right) + 1}{1 - x^{\beta}} \right\},$$

which is the hazard function corresponding to the pdf (40). □

Proposition 2.15. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable. The random variable X has pdf* (86) *if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$h'_F(x) - \theta h_F(x) = \beta \theta^2, \quad x > 0,$$

with the boundary condition $\lim_{x \to 0} h_F(x) = 0$.

Proof. Straightforward and hence omitted. □

2.3. Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \;\; x \in \text{support of } F.$$

In this subsection we present characterizations of two of the new distributions in terms of the reverse hazard function.

Proposition 2.16. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable. The random variable X has pdf* (6) *if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation*

$$r_F'(x)+\frac{2\theta}{\xi^\theta}x^{\theta-1}\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-1}r_F(x)$$
$$=\frac{\theta}{\xi^\theta}\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-2}\frac{d}{dx}\left\{\frac{x^{\theta-1}\left[1-\left(1-\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-1}\right)\log\lambda\right]}{1-\left(1+\left(\frac{x}{\xi}\right)^\theta\right)^{-1}}\right\},\quad x>0,$$

with boundary condition $\lim_{x\to\infty}r_F(x)=0$ *for* $\theta>1$.

Proposition 2.17. *Let* $X:\Omega\to(0,\infty)$ *be a continuous random variable. The random variable* X *has pdf* (10) *if and only if its reverse hazard function* $r_F(x)$ *satisfies the following differential equation*

$$r_F'(x)-r_F(x)=-\theta-C(\alpha(\beta-1)+1)(x-\alpha(\beta-1)),\quad x>0,$$

with boundary condition $\lim_{x\to0}r_F(x)=\theta$ *for* $\beta>1$.

Proposition 2.18. *Let* $X:\Omega\to(0,1)$ *be a continuous random variable. The random variable* X *has pdf* (48) *if and only if its reverse hazard function* $r_F(x)$ *satisfies the following differential equation*

$$r_F'(x)+x^{-1}r_F(x)=kx^{-1}\frac{d}{dx}\left\{\frac{\beta(1-k\log(x))+1}{1-k\log(x)}\right\},\quad x\in(0,1),$$

with boundary condition $\lim_{x\to1}r_F(x)=k(\beta+1)$.

2.4. Characterization Based on the Conditional Expectation of Certain Function of the Random Variable

In this subsection we employ a single function ψ (or ψ_1) of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$ (or $\psi_1(X)$). The following propositions have already appeared in Hamedani's previous work (2013), so we will just state them here which can be used to characterize two of the new distributions listed in Section 1.

Proposition 2.19. *Let* $X:\Omega\to(e,f)$ *be a continuous random variable with cdf* F. *Let* $\psi(x)$ *be a differentiable function on* (e,f) *with* $\lim_{x\to e^+}\psi(x)=1$. *Then for* $\delta\neq1$,

$$E[\psi(X)\mid X\geq x]=\delta\psi(x),\quad x\in(e,f),$$

if and only if

$$\psi(x)=(1-F(x))^{\frac{1}{\delta}-1},\quad x\in(e,f)$$

Proposition 2.20. *Let* $X:\Omega\to(e,f)$ *be a continuous random variable with cdf* F. *Let* $\psi_1(x)$ *be a differentiable function on* (e,f) *with* $\lim_{x\to f^-}\psi_1(x)=1$. *Then for* $\delta_1\neq1$,

$$E[\psi_1(X)\mid X\leq x]=\delta_1\psi_1(x),\quad x\in(e,f)$$

implies

$$\psi_1(x)=(F(x))^{\frac{1}{\delta_1}-1}.\quad x\in(e,f)$$

Remarks 2.2. (A) *For* $(e,f)=(0,\infty)$, $\psi(x)=\exp\left\{-\left[x+\frac{C}{\theta}x^{\alpha(\beta-1)+1}\right]\right\}$ *and* $\delta=\frac{\theta}{\theta+1}$, *Proposition 2.19 provides a characterization of the WAH distribution.*
(B) *For* $(e,f)=(0,\infty)$, $\psi(x)=\exp\left\{\theta x-e^{\theta x}+1\right\}$ *and* $\delta=\frac{\beta}{\beta+1}$, *Proposition 2.19 provides a characterization of the LTII-T distribution.*

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