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Topp-Leone Generalization of the Generalized Pareto distribution and its Impact on Extreme Value Modelling

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Abstract

Extreme Value theory (EVT) is a phenomenon used to model rare or extreme events and has been useful in well-known areas such as finance, economics, hydrology, insurance, etc. In this paper, we combine EVT and Bayesian statistics to estimate the extreme value index and other distribution parameters. EVT studies the behavior of the tails of the distribution, while Bayesian statistics allows us to incorporate prior knowledge of the parameters. The interdependence between these two statistical branches allows us to account for uncertainty in parameter and tail estimation. Block maxima and Peaks over the Threshold are EVT divisions that are used to model observations. In this paper, we use the Peaks over Threshold approach. The generalized Pareto distribution is a Peaks over Threshold distribution. Existing literature studied the generalizations and extensions of the generalized Pareto distribution. These extensions mostly focus on the positive domain of attraction. We contribute to the study of EVT by considering both the negative and positive domains of attraction. We consider the (Topp and Leone, 1955) generalization for the generalized Pareto distribution. We show, using a simulation study, that this distribution can effectively estimate the extreme value index and that it is less sensitive to threshold selection than the normal generalized Pareto distribution.

Key Words: Generalized Pareto distribution; Extreme value Theory; Extended distribution; Threshold; Extreme value index; Bayesian estimation; Baseline distribution; Markov chain Monte Carlo.

1 Introduction

Extreme Value Theory (EVT) is related to The Central Limit Theory. The Central Limit Theory focuses on the limiting behavior of partial sums $X_1 + X_2 + \cdots + X_n$ as $n \to \infty$, while EVT describes the limiting behavior of the extremes : $\min(X_1, X_2, \ldots X_n)$ or $\max(X_1, X_2, \ldots X_n)$ as $n \to \infty$ (de Haan and Ferreira, 2006). Suppose that a random sample $X_1, X_2, \ldots, X_n$ is i.i.d random variables from an unknown distribution F. Let $X_{n,n} = \max\{X_1, X_2, \ldots, X_n\}$ signify the maximum of the sample. In EVT, there exist a sequence of numbers $a_n \geq 0$ and b_n such that

$$P\left(\frac{X_{n,n} - b_n}{a_n}\right) \to \mathrm{G}(x), \ \mathrm{n} \to \infty \quad (1)$$

where G is a non-degenerate distribution function, see Gnedenko (1943), and Fisher and Tippett (1928). G belongs to the Generalised Extreme Value (GEV) family of distributions and is stated in the form:

$$G(x|\,\mu,\sigma,\gamma)=\begin{cases}\exp(-\left(1+\gamma\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\gamma}},1+\gamma\frac{x-\mu}{\sigma}>0,\gamma\neq 0\\ \exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right),\ x\in\ \mathbb{R}\ ,\gamma=0\end{cases} \quad (2)$$

where $\sigma > 0$ is the scale parameter, $\mu \in \mathbb{R}$ is the location parameter and γ is the extreme value index (EVI), also defined as the shape parameter (Fisher and Tippet, 1928 and Gnedenko, 1943). The GEV can be characterized into three families conditional on γ: The Gumbel family ($\gamma = 0$), the Fréchet family ($\gamma > 0$) and, the Weibull family ($\gamma < 0$). EVT comprises two methods to model observations namely: Block Maxima (BM) and Peaks Over Threshold (POT). The BM approach takes into consideration the distribution of the maximum order statistics (Bommier, 2014).It involves dividing the observations into blocks, and models all the maximum values found in each block as a collection (Gumbel, 1958). With no overlapping data, it is assumed that these maximums follow a GEV distribution (Coles et al., 2001). Various estimation techniques, such as the method of moments, maximum likelihood, and Bayesian estimation, can be used to estimate the parameters of the GEV distribution. The drawback of this approach is that the maximum observation found in one block might be lower than the second or even the third highest observation from the other blocks. This might result in most of the highest observations not been taken into consideration, since they are not the highest in the block of interest. This leads to the loss of valuable information. The POT method seems to be the answer to the BM drawback. A suitable threshold is chosen in POT models. The threshold will separate values considered as extremes from the rest of the data. Only extreme values above, or below, the threshold, depending on the extremes of interest, will be modelled and the chosen extremes have a probability distribution that follows a POT model. The method was introduced by Goda (1988). There are quite a few benefits for using this approach, one being that it provides us with tail flexibility since it only models the exceedances. In this paper, we only consider the POT approach to model observations. The generalized Pareto (GP) distribution was first introduced by Picklands III (1975) as a POT model. The well-known Pickhands-Balkema-de Haan theorem (Balkema and De Haan, 1974) describes the tail distribution of the random variable X as follows: Let X be a random variable with a distribution function F. Then for a large enough threshold u, the conditional excess distribution function of $Y = X - u$ is

$$F(y)=P(Y\leq y\,|X>u)=\frac{F(x+u)-F(u)}{1-F(u)} \quad (3)$$

for $0 \leq x \leq x_F$ where x_F is the right endpoint of the distribution F. For a large threshold, u, $F(y)$ can be approximated by the GP distribution with probability density function and cumulative distribution function respectively as

$$f(y)=\begin{cases}\frac{1}{\sigma}\left(1+\frac{\gamma y}{\sigma}\right)^{-\frac{1}{\gamma-1}},\gamma\neq 0\\ \frac{1}{\sigma}\exp\left(-\frac{y}{\sigma}\right),\ \ \gamma=0\end{cases} \quad (4)$$

and

$$F(y)=\begin{cases}1-\left(1+\frac{\gamma y}{\sigma}\right)^{-\frac{1}{\gamma}},\gamma\neq 0\\ 1-\exp\left(-y/\sigma\right),\ \ \gamma=0\end{cases} \quad (5)$$

where $\sigma > 0$, and $y \geq 0$ when $\gamma \geq 0$ and $0 \leq y \leq -\sigma/\gamma$ when $\gamma < 0$.

From (3), the excesses of the conditional distribution of $Y - u$, given $Y > u$ can be modelled through the GP distribution $F(y)$ in (5). The following papers discuss some applications of the GP distribution Gupta et al., (1998), Davison and Smith (1990), Hosking and Wallis (1987), Davison (1984), Castillo (2005) and Kotz and Nadarajah (2000); the list is not exhaustive.

Sometimes the GP distribution fails to provide an insightful input in the analysis of extreme events and tail behavior, this is due to the ever-arising complexities of real-world situations. For this reason, various literatures extended the GP distribution to improve its flexibility and applicability in different scenarios. Papastathopoulos and Tawn (2013) proposed an extension of the GP distribution that allows an addition of a shape parameter without impacting the tail behavior and allows a lower threshold to be chosen. They emphasize that the importance of the extension is that the scale parameter becomes more stable, providing an extra structure to the body of the distribution. Pimentel et al. (2014) extended the GP distribution with the aim to organize data using Novelty Detection to build a model that distinguishes normal data from abnormal data. Gamet and Jalbert (2022) provided an extension of the GP distribution that is suitable to model at a high threshold. However, choosing an appropriate threshold, when modelling with the GP distribution, remains a problem (Davison and Smith, 1990). The threshold should be chosen where the trade-off between variance and biasness is balanced, and such threshold is often hard to find. A lower threshold choice can make the study bias, whereas too high thresholds will increase the variance of the EVI. To our knowledge, recent literatures that provides GP extensions estimate the parameters using the classical approach and only considers the positive domain of attraction, where $\gamma > 0$. In this paper, we extend the GP distribution with a Topp-Leone extension (which has not been shown before) using a Bayesian approach. We also consider the positive as well as the negative domains of attraction.

The outline of the study is as follows: The Topp-Leone extended GP distribution is derived in Section 2. Since the study considers a Bayesian approach, a prior is introduced, and the posterior distribution of the Topp-Leone extended GP distribution is derived in Section 3. Section 4 conducts a simulation study to investigate how well the Bayesian method performs in estimating the new distribution's parameters. In Section 5, the Topp-Leone extended GP distribution is compared to the baseline (GP) distribution in terms of threshold sensitivity. Observations are simulated from three well-known extreme value distributions. We show that the Topp-Leone GP distribution is not sensitive to the threshold choice.

2 Topp Leone extended (generalized) GP distribution.

This section introduces the extended Topp-Leone generalized Pareto (TL-GP) distribution that has three parameters. The PDF, CDF, quantile, likelihood, and posterior functions of the TL-GP distribution have been derived. Rezaei et al. (2017) generalized the distribution of Topp and Leone (1955). The PDF and CDF of the generalized distribution are given respectively as

$$f(x:a) = 2ag(x)[1-G(x)]\{1-[1-G(x)]^2\}^{a-1} \tag{6}$$

and

$$F(x:a) = \{G(x)[2-G(x)]\}^a = \{1-[1-G(x)]^2\}^a \tag{7}$$

where $G(x)$ and $g(x)$ are the CDF and PDF of any baseline distribution (Rezaei et al., 2017).

The GP distribution is considered as the baseline distribution. By substituting (4) and (5) into (6) and (7), we obtain the TL-GP distribution. The PDF and CDF of the TL-GP are given respectively in (8) and (9) as

$$f(x;a,\gamma,\sigma) = 2a\left[\frac{1}{\sigma}\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}-1}\right]\left[1+\frac{\gamma x}{\sigma}\right]^{-\frac{1}{\gamma}}\left[1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right]^{a-1} \tag{8}$$

and

$$F(x;a,\gamma,\sigma) = \left[1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right]^{a} \tag{9}$$

where $\gamma \neq 0$, $\sigma > 0$ and $a > 0$. The range of x is $x > 0$ if $\gamma \geq 0$ and $0 < x \leq -\sigma\gamma$ for $\gamma < 0$. If $a = 1$ (8) and (9) reduces to the GP distribution. Nadarajah and Kotz (2003) provided the moments of the j-shaped distribution defined by Topp and Leone (1955) by deriving the first inter-order moments. Sindhu et al. (2013) gives a Bayesian approach of the Topp-Leone generated family of distributions without generalizing. It involves showing the properness of the posterior distribution using various priors such as: the uniform, Jeffreys, Exponential and gamma priors. They also present the maximum likelihood estimates. The paper of Rezaei et al. (2017) studies the properties and applications of the Topp-Leone generated family of distributions. Also refer to Ghitany et al. (2005), Zhou et al. (2006), Nadarajah (2017), Sakthive and Dhivakar (2021) and Verster and Raubenheimer (2020) for more information on the Topp-Leone distribution.

The quantile function can be obtained as the inverse of (7). The quantile function is derived as

$$Q(u) = \frac{\sigma}{\gamma}\left[\left(1-u^{\frac{1}{a}}\right)^{-\frac{\gamma}{2}}-1\right]. \tag{10}$$

Figure 1 shows the shape of the density functions of the TL-GP for different parameter values.

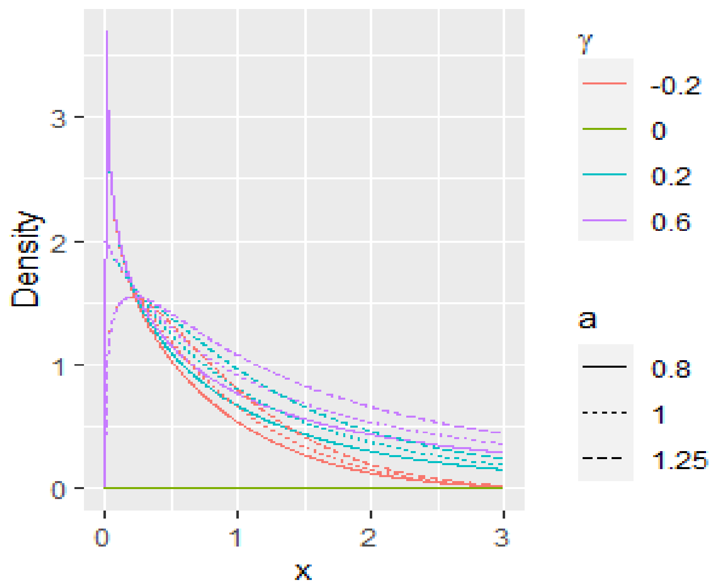

Figure 1: The density function of the TL-GP for different γ and a values.

3 Posterior distribution of the TL-GP

A Bayesian approach is considered to estimate the three parameters of the TL-GP. The likelihood function is:

$$l(x) = [2a]^n \sigma^{-n} e^{\left(-\frac{1}{\gamma}-1\right)\sum \log\left(1+\frac{\gamma x_i}{\sigma}\right)} e^{\left(-\frac{1}{\gamma}\right)\sum \log\left(1+\frac{\gamma x_i}{\sigma}\right)} e^{(a-1)\sum \log\left(1-\left(1+\frac{\gamma x_i}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}. \quad (11)$$

Bayesian statistic involves choosing prior distributions for the unknown parameters. The prior distribution varies in nature, some can be informative, and others can be non-informative. For this study, we consider an independent joint prior, $\pi(a, \xi, \sigma) = \pi(a)\pi(\xi, \sigma)$. The product of the individual priors results in the joint prior. We assume an exponential prior on a , $\pi(a) \propto \lambda e^{-\lambda a}$ (with $\lambda = 0.0001$). The small λ value corresponds to a vague prior with a large variance. The joint maximum data information (MDI) prior of Zellner (1977) is assumed for σ and γ of the GP distribution. The joint MDI prior for the GP distribution was derived in Beirlant et al. (2004) as $\pi(\sigma, \gamma) \propto \sigma^{-1} e^{-\gamma}$. Northrop and Attalides (2016) showed that in the GP distribution case, the truncated MDI prior, where $\gamma > -1$, yields a proper posterior. For this reason, we will consider the truncated MDI prior for the GP distribution. The joint prior is given as

$$\pi(a, \sigma, \gamma) = \pi(a)\pi(\gamma, \sigma) \propto \lambda e^{-\lambda a} \sigma^{-1} e^{-\gamma}, a > 0, \sigma > 0, \gamma > -1. \quad (12)$$

The joint posterior is then obtained as $\pi(a, \sigma, \gamma|x) \propto \pi(a, \sigma, \gamma) l(x)$, thus the joint posterior distributions for the TL-GP is

$$\pi(a, \gamma, \sigma|x)_{TL-GP} \propto \lambda e^{-\lambda a} \sigma^{-1} e^{-\gamma} [2a]^n \sigma^{-n} e^{\left(-\frac{1}{\gamma}-1\right)\sum \log\left(1+\frac{\gamma x_i}{\sigma}\right)} e^{\left(-\frac{1}{\gamma}\right)\sum \log\left(1+\frac{\gamma x_i}{\sigma}\right)} e^{(a-1)\sum \log\left(1-\left(1+\frac{\gamma x_i}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}. \quad (13)$$

Appendix A shows that the posterior is proper.

It can also be shown that the conditional posterior of $\pi(a|\sigma, \gamma, x)$ in (14) follows a Gamma distribution with parameters $(n + 1)$ and $\lambda - \sum \log\left(1 - \left(1 + \frac{\gamma x_i}{\sigma}\right)^{-\frac{2}{\gamma}}\right)$ where n is the number of observations above the threshold,

$$\pi(a|\sigma, \gamma, x) \propto e^{-\lambda a} [a]^n e^{a\sum \log\left(1-\left(1+\frac{\gamma x_i}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}. \quad (14)$$

Equation (14) is useful in the simulation process.

4 Simulation study

In this section, we consider a simulation study to test the performance of the Bayesian estimation method. We simulate $n = 200$ observations from the TL-GP distribution using (10). In this simulation study we consider all domains of attraction, thus γ can take on both negative and positive values. We use the truncated MDI prior of Northrop and Attalides (2016), thus $\gamma \in (-1, \infty)$. To estimate the three parameters, we apply a Bayesian process using a Gibbs

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sampling technique in R. The simulation process is repeated 1000 times (nsim=1000). For each simulation we draw 4000 values from the posteriors, (13) and (14). Since the simulation study is time consuming, we only considered the case $n = 200$ and nsim=1000. The mean over the posterior draws as well as over the nsims denotes the parameter estimates. The mean square error (for example, $\frac{\sum(\gamma-\hat{\gamma})^2}{nsim}$) is calculated to assess the parameter estimates. The 95% credibility interval for each parameter is also calculated. Eight simulation scenarios are considered. For convenience σ is fixed at 1. a takes on various positive values such as 1, 2, 0.5 and 0.8 respectively. Since we consider both the positive and negative domains of attraction, γ were chosen as the following spectrum of positive and negative values; -0.4,-0.2, 0.2, 0.5, 0.6, 1, 2. Results for the following sets of combinations are shown below $[a, \gamma, \sigma] \rightarrow$ [1, -0.2, 1], [1, 0.2, 1], [1, 2, 1], [2, 0.2, 1], [2, -0.4, 1], [2, 0.5, 1], [0.5, 1, 1], [0.8, 0.6, 1]. Other combinations were also considered but excluded here to spare space.

Scenario 1: The parameter estimates, mean squared error and 95% credibility interval for the ZB-GP, TL-GP, PT-GP, and RB-GP, when $\sigma = 1, \gamma = 0.2, a = 1$

Parameter estimates	Mean Squared Error	Credibility Interval
$\hat{\gamma}_{\text{TL-GP}} = 0.198$	$\text{MSE}_{\text{TL-GP}} = 0.04$	(-0.2062; 0.5879)
$\hat{\sigma}_{\text{TL-GP}} = 1.0188$	$\text{MSE}_{\text{TL-GP}} = 0.0355$	(0.6881; 1.4205)
$\hat{a}_{\text{TL-GP}} = 1.0324$	$\text{MSE}_{\text{TL-GP}} = 0.0194$	(0.8192; 1.3443)

Scenario 2: The parameter estimates, mean squared error and 95% credibility interval for the ZB-GP, TL-GP, PT-GP, and RB-GP, when $\sigma = 1, \gamma = -0.2, a = 1$

Parameter estimates	Mean Squared Error	Credibility Interval
$\hat{\gamma}_{\text{TL-GP}} = -0.1735$	$\text{MSE}_{\text{TL-GP}} = 0.0195$	(-0.3857; 0.139)
$\hat{\sigma}_{\text{TL-GP}} = 0.9929$	$\text{MSE}_{\text{TL-GP}} = 0.0166$	(0.7262; 1.2282)
$\hat{a}_{\text{TL-GP}} = 1.0392$	$\text{MSE}_{\text{TL-GP}} = 0.0149$	(0.8599; 1.3033)

Scenario 3: The parameter estimates, mean squared error and 95% credibility interval for the ZB-GP, TL-GP, PT-GP, and RB-GP, when $\sigma = 1, \gamma = 0.2, a = 2$

Parameter estimates	Mean Squared Error	Credibility Interval
$\hat{\gamma}_{\text{TL-GP}} = 0.2192$	$\text{MSE}_{\text{TL-GP}} = 0.0325$	(-0.1608; 0.5617)
$\hat{\sigma}_{\text{TL-GP}} = 0.9971$	$\text{MSE}_{\text{TL-GP}} = 0.0365$	(0.6352; 1.33802)
$\hat{a}_{\text{TL-GP}} = 2.1559$	$\text{MSE}_{\text{TL-GP}} = 0.2104$	(1.551; 3.151)

Scenario 4: The parameter estimates, mean squared error and 95% credibility interval for the ZB-GP, TL-GP, PT-GP, and RB-GP, when $\sigma = 1, \gamma = -0.4, a = 2$

Parameter estimates	Mean Squared Error	Credibility Interval
$\hat{\gamma}_{\text{TL-GP}} = -0.3279$	$\text{MSE}_{\text{TL-}GP} = 0.0115$	(-0.4338; -0.1335)
$\hat{\sigma}_{\text{TL-GP}} = 0.9409$	$\text{MSE}_{\text{TL-GP}} = 0.0106$	(0.7551; 1.0938)
$\hat{a}_{\text{TL-GP}} = 2.1747$	$\text{MSE}_{\text{TL-GP}} = 0.1012$	(1.743; 2.806)

Scenario 5: The parameter estimates, mean squared error and 95% credibility interval for the ZB-GP, TL-GP, PT-GP, and RB-GP, when $\sigma = 1, \gamma = 0.5\ a = 2$

Parameter estimates	Mean Squared Error	Credibility Interval
$\hat{\gamma}_{\text{TL-GP}} = 0.5307$	$\text{MSE}_{\text{TL-GP}} = 0.0357$	(0.1616; 0.9037)
$\hat{\sigma}_{\text{TL-GP}} = 0.9852$	$\text{MSE}_{\text{TL-GP}} = 0.0453$	(0.5772; 1.429)
$\hat{a}_{\text{TL-GP}} = 2.2179$	$\text{MSE}_{\text{TL-GP}} = 0.3247$	(1.5435; 3.6049)

Scenario 6: The parameter estimates, mean squared error and 95% credibility interval for the ZB-GP, TL-GP, PT-GP, and RB-GP, when $\sigma = 1, \gamma = 1, a = 0.5$

Parameter estimates	Mean Squared Error	Credibility Interval
$\hat{\gamma}_{\text{TL-GP}} = 0.9738$	$\text{MSE}_{\text{TL-GP}} = 0.0984$	(0.3773; 1.5986)

$\hat{\sigma}_{\text{TL-GP}} = 1.0649$	$\text{MSE}_{\text{TL-GP}}$ = 0.0693	(0.6330; 1.635)
$\hat{a}_{\text{TL-GP}} = 0.5093$	$\text{MSE}_{\text{TL-GP}}$ = 0.0035	(0.4111; 0.6397)

Scenario 7: The parameter estimates, mean squared error and 95% credibility interval for the ZB-GP, TL-GP, PT-GP, and RB-GP, when $\sigma = 1, \gamma = 0.6, \alpha = 0.8$

Parameter estimates	Mean Squared Error	Credibility Interval
$\hat{\gamma}_{\text{TL-GP}} = 0.5881$	$\text{MSE}_{\text{TL-GP}} = 0.0587$	(0.0871; 1.0487)
$\hat{\sigma}_{TL-GP} = 1.0363$	MSE_{TL-GP} = 0.0464	(0.6535; 1.4805)
$\hat{a}_{\text{TL-GP}} = 0.8232$	$\text{MSE}_{\text{TL-GP}}$ = 0.0115	(0.6588; 1.0666)

Scenario 8: The parameter estimates, mean squared error and 95% credibility interval for the ZB-GP, TL-GP, PT-GP, and RB-GP, when $\sigma = 1, \gamma = 2, \alpha = 1$

Parameter estimates	Mean Squared Error	Credibility Interval
$\hat{\gamma}_{\text{TL-GP}} = 1.9944$	$\text{MSE}_{\text{TL-GP}} = 0.1029$	(1.3412; 2.5865)
$\hat{\sigma}_{\text{TL-GP}} = 1.0604$	$\text{MSE}_{\text{TL-GP}}$ = 0.122	(0.4989; 1.788)
$\hat{a}_{\text{TL-GP}} = 1.0825$	$\text{MSE}_{\text{TL-GP}}$ = 0.1447	(0.7654; 1.6125)

The simulation study shows that the TL-GP performs well in estimating the three parameters of the TL-GP, this is evident from the small mean square error values and the estimates lying within the 95% credibility intervals. The negative domain, where $\gamma < 0$ is mostly ignored in existing literature. In this simulation study we have shown that the TL-GP can successfully estimate $\gamma < 0$ with a Bayesian approach. The last scenario considers a very extreme case with $\gamma = 2$. An extreme case like this will rarely happen in real life, but even for this extreme case the TL-GP performs relatively well for all 3 parameters.

5 Performance of the TL-GP in comparison to the GP distribution

In this section we investigate how the TL-GP distribution performs when fitted to data that is simulated from some well-known extreme value distributions. The values were simulated from three distributions, the Fréchet, extreme Weibull and Exponential distributions. These three distributions cover the three families mentioned in Section 1. The performance of the distributions was compared to the GP distribution in terms of EVI estimation. This section also includes a real data analysis in Section 5.2. Appendix B shows that the EVI value for the TL-GP distribution is $\frac{1}{\gamma}$. Appendix C proves that the parameter a approaches infinity as the TL-GP distribution approaches the GP distribution. Thus, if the threshold is chosen too high the parameter a becomes unstable. To avoid this from happening, we consider the smallest observation of the data as the threshold. Thus, we fit the TL-GP to all the data above the smallest observation (and no additional threshold choice is made). $n = 200$ values are simulated from the three extreme value distributions (Fréchet, extreme value Weibull and Exponential distributions) with different tail behavior ($\gamma > 0, \gamma < 0$ and $\gamma = 0$). The first extreme value distribution is the Fréchet distribution which forms part of the Fréchet family with a positive EVI. Beirlant et al. (2004) showed that the EVI estimate of the Fréchet distribution is $\frac{1}{\alpha}$. The CDF of the Fréchet distribution is defined as:

$$F(x) = e^{-x^{-\alpha}}, x > 0, \alpha > 0. \quad (15)$$

The extreme value Weibull distribution is part of the Weibull family with negative EVI. Beirlant et al. (2004) showed that the EVI of the extreme value Weibull distribution is $-\frac{1}{\alpha}$. The CDF of the extreme value Weibull is defined as:

$$F(x) = e^{-x^{-\alpha}}, x > 0, \alpha > 0. \quad (16)$$

The exponential distribution forms part of the Gumbel family with EVI of 0. The CDF of the exponential distribution is defined as:

$$F(x) = 1 - e^{-\lambda x}, x > 0. \quad (17)$$

As previously stated, the excesses above an appropriate threshold, that have a healthy relationship between the variance and bias trade-off will follow a GP distribution. Therefore, the Fréchet, extreme value Weibull and Exponential distributions will follow a GP distribution at a suitable threshold with their respective EVI's, which corresponds to the EVI of the GP distribution (γ). The TL-GP are also a POT distribution with EVI derived in Appendix B. In EVT we often prefer an EVI estimate that is less sensitive to the threshold choice, as this will guarantee that the EVI estimates will remain applicable for statistical computations, even if the "wrong" threshold was chosen.

5.1 Simulation

We simulate $n = 200$ values from the three extreme value distributions (mentioned earlier). For each simulation case, the simulation is repeated nsim=1000 times, 4000 values are drawn from the TL-GP posteriors for each repetition. The parameter estimates are the mean over the posterior draws and as well as the mean over nsims. An empirical Bayes method is used to estimate the GP distribution parameters, see Zhang (2010). We report the mean square error and 95% credibility interval of the TL-GP EVI estimate for all the different scenarios. Quantile-quantile (Q-Q) plots are also given to see how well the TL-GP distribution, with the estimated parameters, fits the simulated data.

Tables 1 and 2, show the EVI estimates for the TL-GP distribution when $n = 200$ values are simulated from a Fréchet distribution (EVI > 0) with two different values of α. Table 1 shows the first scenario with $\alpha = 2$, where the true EVI value is 0.5. The EVI estimate for the TL-GP distribution is relatively good at 0.386, compared to the GP distribution's EVI estimate of 0.175. The MSE for the EVI estimate of the TL-GP is much smaller than the MSE of the EVI for the GP distribution. Table 2 shows the second scenario, $\alpha = 5$, with true EVI value 0.2. The EVI estimates the TL-GP distribution is 0.119 compared to the EVI estimate of the GP distribution, -0.145. Again, the EVI MSE for the TL-GP is smaller than in the GP case. Figure 2 shows the Q-Q plots for the two scenarios. In each case a simulated data example was taken of the sorted observations and plotted against the theoretical quantiles from the TL-GP distribution with the estimated parameter values. The Q-Q plots are given as an illustration of how well the TL-GP fits the data. The Q-Q plots for both cases are decent and follows the 45^{O} line closely, indicating that the TL-GP fits the data well. Overall, the TL-GP distribution performs well for the positive domain.

Tables 3 and 4 show the EVI estimates for the TL-GP distribution when $n = 200$ values are simulated from an extreme value Weibull distribution (EVI < 0) with two different α values. Table 3 shows the EVI estimate for the observations simulated with $\alpha = 2$. Thus, the true EVI value is -0.5. The EVI estimate the TL-GP distribution at -0.244 compared to the GP distribution with EVI estimate of -0.474. In this case the EVI MSE for the GP distribution is smaller than the EVI MSE for the TL-GP. Note that the MSE for the TL-GP is still small with a value of 0.0712. Table 4 shows the EVI estimate with $\alpha = 5$. The true EVI value -0.2. The TL-GP gave an EVI estimate of -0.406 compared to the GP distribution's EVI estimate of -0.878. In the case the EVI MSE for the TL-GP is much smaller than the EVI MSE of the GP distribution. In more extreme negative cases, when the EVI is for example -0.5, the GP seems to perform slightly better that the TL-GP. Figure 3 shows the Q-Q plots with the estimated TL-GP distribution parameters. The Q-Q plots follows the 45^{o} line closely. Thus, the TL-GP distribution with the estimated parameters, fits the data well in the negative domain.

Tables 5 and 6 show the EVI estimates for the TL-GP distribution when $n = 200$ values are simulated from an exponential distribution (EVI $= 0$) with two distinct λ values. Table 5 shows the EVI estimate with $\lambda = 2$. The EVI estimate for the TL-GP distribution is 0.012 which is close to 0. The EVI estimate of the GP distribution is 0.0002, even closer to 0. Table 6 shows the EVI estimate with $\lambda = 5$. The TL-GP distribution estimated the EVI as 0.0119 while the GP distribution estimated the EVI as -0.0002. For both cases the EVI MSE values for the GP-TL and GP distributions are very small. Figure 4 show that the Q-Q plots lies close to the 45^{o} line, indicating that the TL-GP distribution is a good fit to the data when the true EVI is 0.

Table 1: Estimated parameter of the TL-GP distribution and GP distribution when $n = 200$ values are simulated from the Fréchet distribution with $\alpha = 2$. Thus EVI = 0.5.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	[95%CI_EVI_TL]	MSE_$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	MSE_$\widehat{EVI}_{GP}$
2.87	0.845	0.773	0.386	[0.2216, 0.4648]	0.0171	0.175	0.116

Table 2: Estimated parameter of the TL-GP distribution and GP distribution when $n = 200$ values are simulated from the Fréchet distribution with $\alpha = 5$. Thus EVI = 0.2.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	95%CI_EVI_TL	MSE_$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	MSE_$\widehat{EVI}_{GP}$
3.39	0.433	0.237	0.119	[-0.062, 0.2753]	0.0135	-0.145	0.128

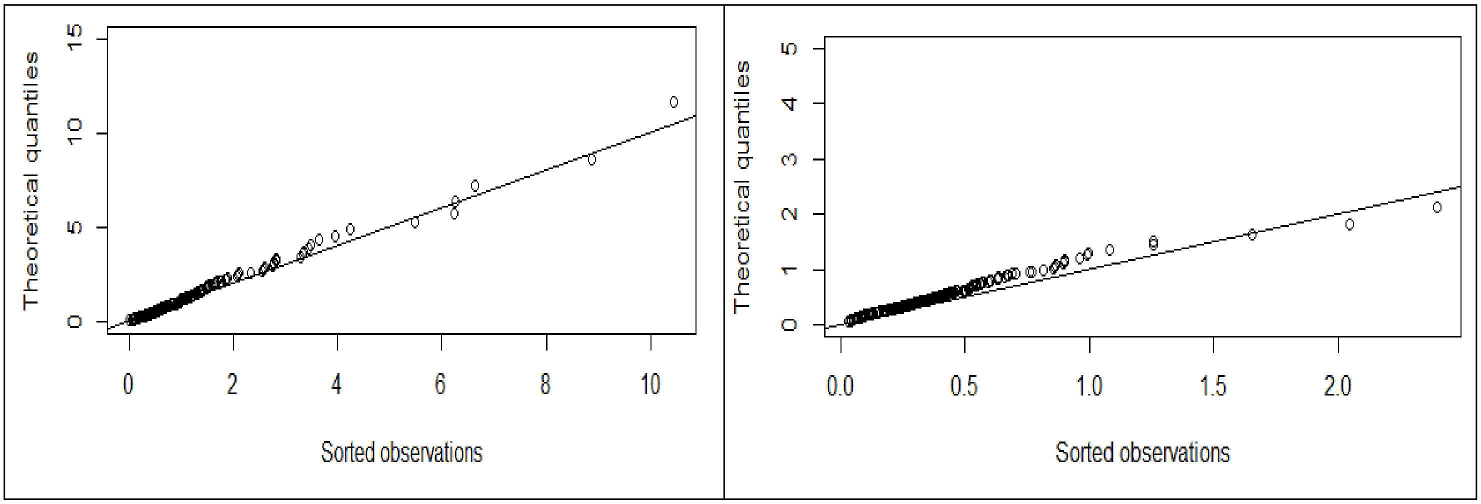

Figure 2: The Q-Q plot for a Fréchet generated data set with $\alpha = 2$ (left) and $\alpha = 5$ (right), against the TL-GP distribution with estimated parameters from Tables 1 and 2 respectively.

Table 3: Estimated parameter of the TL-GP distribution and GP distribution when $n = 200$ values are simulated from the Extreme Weilbull distribution with $\alpha = 2$. Thus EVI = -0.5.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	95%CI_EVI_TL	MSE_$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	MSE_$\widehat{EVI}_{GP}$
2.15	1.41	-0.488	-0.244	[-0.3987, -0.1106]	0.0712	-0.474	0.0068

Table 4: Estimated parameter of the TL-GP distribution and GP distribution when $n = 200$ values are simulated from the Extreme Weilbull distribution with $\alpha = 5$. Thus EVI = -0.2.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	95%CI_EVI_TL	MSE_$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	MSE_$\widehat{EVI}_{GP}$
3.98	0.934	-0.812	-0.406	[-0.4793, -0.2948]	0.0451	-0.878	0.47

Table 5: Estimated parameter of the TL-GP distribution and GP distribution when $n = 200$ values are simulated from the Exponential distribution with $\lambda = 2$. Thus EVI = 0.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	95%CI_EVI_TL	MSE_$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	MSE_$\widehat{EVI}_{GP}$
1.05	0.997	0.0241	0.012	[-0.1798,0.1903]	0.00967	-0.0002	0.0052

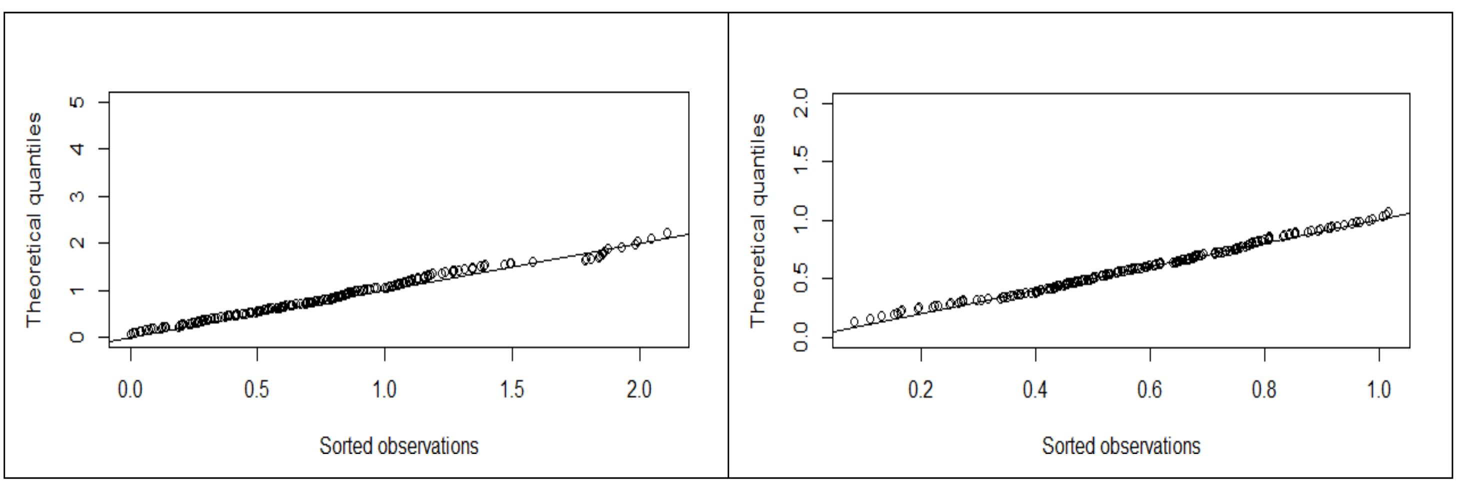

Figure 3: The Q-Q plot for a Weibull generated data set with $\alpha = 2$ (left) and $\alpha = 5$ (right), against the TL-GP distribution with estimated parameters from Tables 3 and 4 respectively.

Table 6: Estimated parameter of the TL-GP distribution and GP distribution when $n = 200$ values are simulated from the Exponential distribution with $\lambda = 5$. Thus EVI = 0.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	95%CI_EVI_TL	MSE_$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	MSE_$\widehat{EVI}_{GP}$
1.05	0.399	0.0298	0.0119	[-0.1884,0.186]	0.0096	-0.0002	0.0052

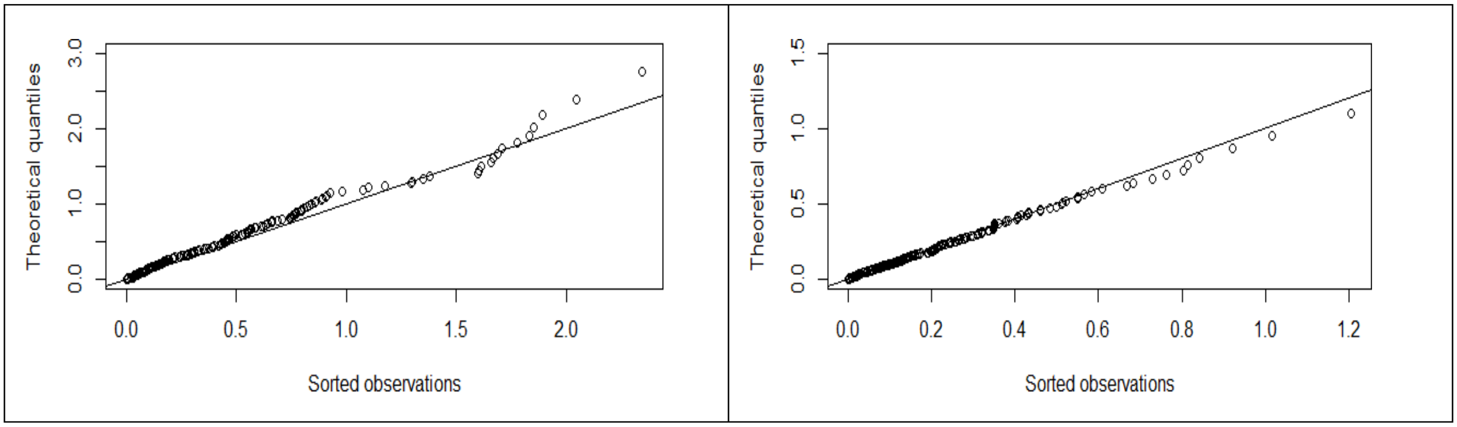

Figure 4: The Q-Q plot for an Exponential generated data set with $\lambda = 2$ (left) and $\lambda = 5$ (right), against the TL-GP distribution with estimated parameters from Tables 5 and 6 respectively.

The simulations in this section have shown that the EVI estimates of the TL-GP distribution is close to the true EVI when all the observations (above the smallest value) were considered. This is evident from the small MSE values. In some of the scenarios the EVI MSEs of the GP distribution were much larger than in the TL-GP case. This is not unexpended since we know that the optimum threshold has not been chosen. Thus, the EVI estimates will not be as accurate. However, we note from these simulations, that the TL-GP distribution is not sensitive to the correct threshold choice. Even when the EVI estimates were slightly over or underestimated, the Q-Q plots indicated a good fit. Thus, we conclude that the parameter a adjusts the distribution appropriately to fit the data. This is beneficial in real life studies where the threshold is unknown.

5.2 Application to Real data

In this section we fit the TL-GP distribution to real world data to assess the performance of the models in terms of parameter estimation and data modelling. The real datasets are introduced below. All the datasets were obtained from ismev package (Gilleland, 2018).

Wavesurge data: The wavesurge data set contains 2894 observations. The first column represents wave height in metres at a location of South-West England. The second column represents surge heights. We only consider the wave heights in this application. More information about the data can be found in Coles et al. (2001).

Exchange data: The exchange data set contains 975 observations. The first column represents the daily exchange rate of UK sterling against the US dollar. The second column represents the daily exchange rate of UK sterling against the Canadian dollar. We only consider the second column in this application. More information about the data can be found in Coles et al. (2001).

Engine data: The engine data set contains 32 observations. The first column represents the corrosion level, and the second column represents the engine failure time. We only consider the second column in this application. More information can be found in the ismev package in R.

Table 8 shows the parameter estimates of the TL-GP and GP distributions fitted to the wave height data. The distributions were fitted to all the data above the smallest observation. We did not consider investigating an optimum threshold choice since the TL-GP has shown (in the simulations) that it can be fitted to all the data. Another reason for not considering a high threshold is because the parameter a becomes too large if the threshold is chosen too high (see Appendix C). Figure 5a shows the Q-Q plot of the theoretical quantiles of the TL-GP quantiles (in black) and GP quantiles (in blue) against the sorted wave height observations. The TL-GP quantiles lies closer to the 45º line than the GP quantiles (except for the top quantiles). In a real data case, we do not know where the threshold should be chosen. Figure 5a shows that the TL-GP is not a bad fit when fitted to all the data. It might however slightly overestimate large quantiles.

Table 9 shows the parameter estimates of the TL-GP and GP distributions fitted to the data of daily exchange rate of UK sterling against the Canadian dollar. The distributions were fitted to all the data. Figure 5b shows the Q-Q plot of the theoretical quantiles of the TL-GP quantiles (in black) and GP quantiles (in blue) against the sorted daily exchange rate observations. The TL-GP quantiles lies closer to the 45º line than the GP quantiles (especially for the lower quantiles). Figure 5b shows that the TL-GP is a good fit when fitted to all the data.

Table 10 shows the parameter estimates of the TL-GP and GP distribution fitted to the data of engine failure time. The distributions were fitted to all the data. Figure 5c shows the Q-Q plot of the theoretical quantiles of the TL-GP quantiles (in black) and GP quantiles (in blue) against the sorted engine failure time observations. The TL-GP quantiles lies much closer to the 45º line than the GP. Figure 5c shows that the TL-GP is a good fit when fitted to all the data.

Table 8: Parameter estimates of the TL-GP and GP fitted to the wave height data.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	$\hat{\sigma}$
3.03	2.78	-0.00821	-0.0041	-0.291	3.171

Table 9: Parameter estimates of the TL-GP and GP fitted to the daily exchange rate of UK sterling against the Canadian dollar.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	$\hat{\sigma}$
3.23	0.416	-0.644	-0.322	-0.733	0.4282

Table 10: Parameter estimates of the TL-GP and GP fitted to the data of engine failure time.

$\hat{a}_{TL}$	$\hat{\sigma}_{TL}$	$\hat{\gamma}_{TL}$	$\widehat{EVI}_{TL}$	$\hat{\gamma}_{GP}$	$\hat{\sigma}$
1.74	0.777	1.01	0.507	0.345	0.609

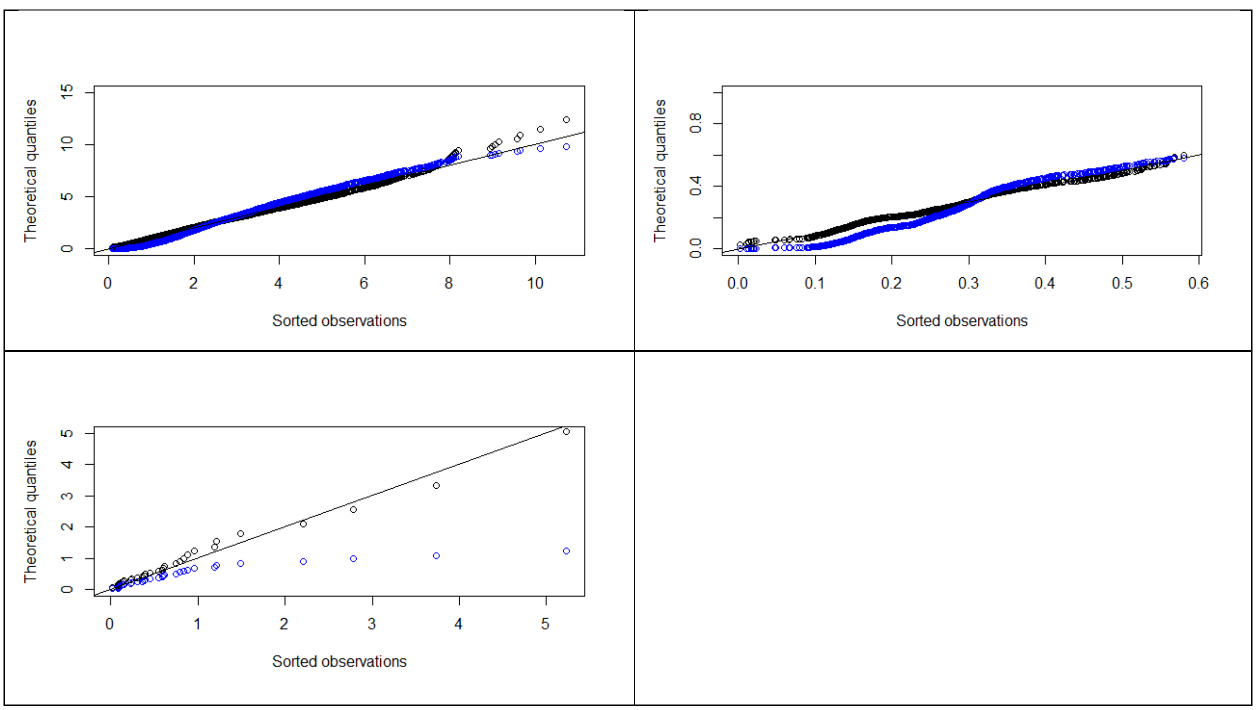

Figure 5: a) Top-left: Q-Q plot of the TL-GP quantiles (in black) and GP quantiles (in blue) against the sorted wave height observations. b) Top-right: Q-Q plot of the TL-GP quantiles (in black) and GP quantiles (in blue) against the sorted daily exchange rate of UK sterling against the Canadian dollar observations. c) Bottom-left: Q-Q plot of the TL-GP quantiles (in black) and GP quantiles (in blue) against the sorted engine failure time observations.

6 Conclusion

This paper considered an extended generalized Pareto distribution, the TL-GP. The aim of the study was to investigate this extended distribution in terms of parameter estimation through a Bayesian approach. We also inspected how well this distribution fits extreme value data. The Bayesian estimation method was assessed through a simulation study. The various parameters of the TL-GP were estimated, and the mean square errors were calculated. The deviance of the estimates over the nsim=1000 was shown by the 95% credibility intervals. A second simulation study was considered to compare the TL-GP with the GP in terms of EVI estimation and data fit. Samples were simulated from three well-known Extreme value distributions. The advantage of this simulation study is that the true EVI is known. This knowledge helped us to inspect how well the TL-GP distribution was able to estimate the EVI. The TL-GP showed not to be sensitive to the threshold choice in estimating the EVI. In fact, it worked well when fitted over all the data above the smallest observation. The TL-GP was also fitted so some real data examples. The goodness of the fits was tested through Q-Q plots. In conclusion: This study contributes to EVT by showing that extreme value observations can be modelled successfully through the TL-GP distribution. In particular, the TL-GP has shown to be unsensitive to threshold choice and is therefore a convenient POT model to work with.

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Appendix A

Consider the posterior distribution for the TL-GP case:

$$\boldsymbol{\pi(a,\sigma,\gamma|x) \propto \lambda e^{-\lambda a}\frac{e^{-\gamma}}{\sigma}[2\alpha]^n\sigma^{-n}e^{\left(-\frac{1}{\gamma}-1\right)\sum\log\left(1+\frac{\gamma x_i}{\sigma}\right)}e^{\left(-\frac{1}{\gamma}\right)\sum\log\left(1+\frac{\gamma x_i}{\sigma}\right)}e^{(a-1)\sum\log\left(1-\left(1+\frac{\gamma x_i}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}, a>0, \sigma>0, \gamma>-1}.$$

Suppose $\boldsymbol{n=1}$ with observation $\boldsymbol{x}$, then

$$\boldsymbol{\pi(a,\sigma,\xi|x) \propto \lambda e^{-\lambda a}\frac{e^{-\gamma}}{\sigma}\,2\alpha\sigma^{-1}e^{\left(-\frac{1}{\gamma}-1\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{\left(-\frac{1}{\gamma}\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{(a-1)\log\left(1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}, a>0, \sigma>0, \gamma>-1}.$$

If c in A1 (below) is finite, it shows that the posterior is proper.

$$\boldsymbol{c=\int_0^\infty\int_{-1}^0\int_{-\gamma x}^\infty \lambda e^{-\lambda a}\frac{e^{-\gamma}}{\sigma}2\alpha\sigma^{-1}e^{\left(-\frac{1}{\gamma}-1\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{\left(-\frac{1}{\gamma}\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{(a-1)\log\left(1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}d\sigma d\gamma da +}$$

$$\boldsymbol{\int_0^\infty\int_0^\infty\int_0^\infty \lambda e^{-\lambda a}\frac{e^{-\gamma}}{\sigma}2\alpha\sigma^{-1}e^{\left(-\frac{1}{\gamma}-1\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{\left(-\frac{1}{\gamma}\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{(a-1)\log\left(1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}d\sigma d\gamma da} \quad \text{(A1)}$$

$$\boldsymbol{=\int_0^\infty \lambda e^{-\lambda a}2a\int_{-1}^0 e^{-\gamma}\int_{-\gamma x}^\infty \sigma^{-2}e^{\left(-\frac{1}{\gamma}-1\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{\left(-\frac{1}{\gamma}\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{(a-1)\log\left(1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}d\sigma d\gamma da}$$

$$\boldsymbol{+\int_0^\infty \lambda e^{-\lambda a}2a\int_0^\infty e^{-\gamma}\int_0^\infty \sigma^{-2}e^{\left(-\frac{1}{\gamma}-1\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{\left(-\frac{1}{\gamma}\right)\log\left(1+\frac{\gamma x}{\sigma}\right)}e^{(a-1)\log\left(1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right)}d\sigma d\gamma da}\ .$$

Let $\boldsymbol{A} = \int_{-\gamma x}^{\infty} \boldsymbol{\sigma^{-2} e^{\left(-\frac{1}{\gamma}-1\right)\log\left(1+\frac{\gamma x}{\sigma}\right)} e^{\left(-\frac{1}{\gamma}\right)\log\left(1+\frac{\gamma x}{\sigma}\right)} e^{(a-1)\log\left(1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right)} d\sigma}$.

If $\boldsymbol{u} = \left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}$ then $\boldsymbol{A} = \frac{1}{x}\int_0^1 \boldsymbol{u.(1-u^2)^{a-1}du} = \frac{1}{2a}$.

Let $\boldsymbol{B} = \int_{0}^{\infty} \boldsymbol{\sigma^{-2} e^{\left(-\frac{1}{\gamma}-1\right)\log\left(1+\frac{\gamma x}{\sigma}\right)} e^{\left(-\frac{1}{\gamma}\right)\log\left(1+\frac{\gamma x}{\sigma}\right)} e^{(a-1)\log\left(1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right)} d\sigma}$.

If $\boldsymbol{u} = \left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}$ then $\boldsymbol{B} = \frac{1}{x}\int_0^1 \boldsymbol{u.(1-u^2)^{a-1}du} = \frac{1}{2a}$.

Therefore: $\boldsymbol{c} = \frac{1}{x}\int_0^{\infty} \boldsymbol{\lambda e^{-\lambda a} 2a.} \frac{1}{2a}\int_{-1}^{\infty} \boldsymbol{e^{-\gamma} d\gamma da} = \frac{1}{x}\int_0^{\infty} \boldsymbol{\lambda e^{-\lambda a} da} \approx \frac{2.7182}{x}$.

Appendix B

The EVI of the TL-GP distribution is $\frac{\gamma}{2}$.

The CDF of the TL-GP is:

$$\boldsymbol{F(x;a,\gamma,\sigma)} = \left[\boldsymbol{1}-\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right]^{a}$$

We can now use the Binomial expansion, as in Beirlant et al. (2004), to find the survival function in terms of a slowly varying function. The Binomial Theorem is expressed as follows: $\boldsymbol{(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \cdots}$

Applying the Binomial theorem to the cdf of the TL-GP results in:

$$\boldsymbol{F(x;a,\gamma,\sigma) = 1 - a\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}} - \frac{a(a-1)}{2!}\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{4}{\gamma}} + \cdots}.$$

The survival function is then expressed as:

$$\boldsymbol{1 - F(x;a,\gamma,\sigma) = a\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}} - \frac{a(a-1)}{2!}\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{4}{\gamma}} + \cdots = \left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\left[a + \frac{a(a-1)}{2}\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}} + \cdots\right]} \qquad \text{(B.1)}$$

Equation (B.1) follows a GP-type distribution with slowly varying function given in B.1 the EVI equal to $\frac{1}{\frac{2}{\gamma}} = \frac{\gamma}{2}$. See Beirlant et al. (2004) p. 59 for more information.

Appendix C

The $\boldsymbol{a}$ value for TL-GP distribution goes to ∞ as the TL-GP distribution becomes a GP distribution.

The CDF of the Topp-Leone generated family of distributions is defined as $\boldsymbol{F(x;a) = \{G(x)[2-G(x)]\}^a}$ (Rezaei et al., 2017), where $\boldsymbol{G(x)}$ is the baseline distribution. In this study the GP distribution is chosen as the baseline distribution.

We can now ask the question: For what value of $\boldsymbol{a}$ will the TL-GP be equal to its baseline (GP) distribution. Thus, we can solve the value of $\boldsymbol{a}$ for which

$\boldsymbol{\{G(x)[2-G(x)]\}^a = G(x)}$ as follows:

$$\boldsymbol{a} = \frac{\ln\{\boldsymbol{G(x)}\}}{\ln\{\boldsymbol{G(x)[2-G(x)]}\}}$$

$$\boldsymbol{a} = \frac{\ln\left\{\boldsymbol{1}-\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}}{\ln\left\{\left(\boldsymbol{1}-\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right)\left[\boldsymbol{2-1}-\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right]\right\}} = \frac{\ln\left\{\boldsymbol{1}-\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}}{\ln\left\{\left(\boldsymbol{1}-\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right)\left[\boldsymbol{1}+\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right]\right\}}$$

$$= \frac{\ln\left\{\boldsymbol{1}-\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}}{\ln\left\{\left[\boldsymbol{1}+\left(\boldsymbol{1}+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}\right]\right\}}$$

$$\approx \frac{-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}}{-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{2}{\gamma}}} = \left(1+\frac{\gamma x}{\sigma}\right)^{\frac{1}{\gamma}}$$

$$\frac{1}{a} = \frac{\ln\left\{1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}+\ln\left\{1+\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}}{\ln\left\{1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}} = 1+\frac{\ln\left\{1+\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}}{\ln\left\{1-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}} \approx 1+\frac{\left\{\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}}{\left\{-\left(1+\frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}\right\}} = 1-\left(1+\frac{\gamma x}{\sigma}\right)^{0}$$

$$\frac{1}{a} = 1 - 1 = 0$$

Therefore $a = \frac{1}{0} \to \infty$. Thus, a will become very large as it strives to the GP distribution, in other words, as the threshold increases.