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Consistency Issues in Skew Random Fields: Investigating Proposed Alternatives and Identifying Persisting Problems

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Abstract

Multiple researchers have proposed skew random fields derived from multivariate skew distributions, yet the consistency of these fields has been questioned. Mahmoudian (2018) and Saber et al. (2018) have put forth alternative suggestions to address these concerns. In our study, we identify that the random fields outlined by Mahmoudian (2018) continue to demonstrate consistency issues, suggesting a flaw in their definition. Finally we propose a skew random field and apply it to spatial prediction.

Key Words: Random field, Multivariate closed skew-normal distribution, Multivariate unified skew-normal distribution, spatial prediction.

1. Introduction

Random fields (RFs) have gained popularity in various scientific fields over the last decade, such as physics (dynamics, ergodic theory), biology (computational molecular, structural, ecology), control theory of complex networks, computer vision, and data science. When we need to specify the joint distribution for an arbitrary number of variables, it is essential to establish a suitable random field. For instance, to calculate the conditional expectations of field variables at the interpolation location of observed data, spatial or temporal interpolation methods are commonly used. To perform these calculations, it is essential to know the common distribution of the observed variables at the desired location. Typically, finite-dimensional distributions form the basis of such required random fields, and multivariate distributions are defined based on this requirement. The Gaussian random field (GRF) is the first and most widely used random field, which can be applied to symmetrically and normally distributed data. However, there are cases where we need a Skew random field (SRF), which requires a multivariate log distribution to be defined.

Kim and Mallick (2004) introduced the skewed normal random field (SNRF) based on the multivariate skewed normal (SN) distribution. Allard and Naveau (2007) developed the closed skew normal (CSN) random field using the CSN multivariate distribution. Zairefard and Khalidi (2013) determined a second-order stationary random field called the Unified Skew-Normal (SUN) based on the multivariate distribution of the SUN. These distributions, including SN, CSN, and SUN, have been widely applied in various fields of applied problems, and their usage is still increasing.
Saber et al. (2018) demonstrated that some SRFs containing SNRF, CSNRF (closed skew-normal random field), and USNRF (Unified Skew-Normal Random Field) are not well-defined. Thus, reconsideration is necessary when using these RFs in spatial analysis. Mahmoudian (2018) proposed a family of random fields called SGRF, which incorporates changes in the SN, CSN, and SUN distributions. However, our study indicates that the SNRF, CSNRF, and USNRF recommended by Mahmoudian (2018) are still not well-defined.

The article is organized as follows: Section 2 introduces the multivariate distributions of SN, CSN, and SUN. In Section 3, we explore the SRF defined by Mahmoudian (2018) and find that it is not well-defined, making it inappropriate for spatial and spatiotemporal data modeling. A well-defined skew random field is proposed in Section 4. This random field is applied to spatial prediction problem.

2. A review of distributions

In this section, as a reminder, we will begin by examining the multivariate distributions of SN, CSN, and SUN, and highlight the characteristics of their joint and marginal distributions.

2.1. Multivariate SN distribution

Let $\boldsymbol{X} = (X_1, \dots, X_p)'$ be a p-dimensional continuous random vector from the distribution of SN, denoted by $\boldsymbol{X} \sim SN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\alpha})$. The probability density function (PDF) of the random variable X is defined as:

$$f(\boldsymbol{x}) = 2\phi_p(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\Phi_1(\boldsymbol{\alpha}'\boldsymbol{\omega}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})),\ \boldsymbol{x} \in \mathcal{R}^p \tag{2.1}$$

Here, $\phi_n(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ represents the pdf of a p-dimensional normal distribution with mean $\boldsymbol{\mu} \in R^p$ and positive definite dispersion matrix $\boldsymbol{\Sigma}$, while $\Phi_1(.)$ denotes the cumulative distribution function (cdf) of the standard normal distribution. The diagonal matrix $\boldsymbol{\omega}$ is formed by the standard deviation of $\boldsymbol{\Sigma}$. Additionally, the moment-generating function can be derived from the following relationship, (Azzalini, A., Dalla-Valle, A. (1996))

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = 2exp\ (\boldsymbol{t}'\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})\Phi_1(\xi^*(\boldsymbol{\alpha})\boldsymbol{\alpha}'\boldsymbol{H}\boldsymbol{\omega}\boldsymbol{t}),\ \boldsymbol{t} \in \mathcal{R}^p \tag{2.2}$$

where $\xi^*(\boldsymbol{\alpha}) = \sqrt{1 + \boldsymbol{\alpha}'\boldsymbol{H}\boldsymbol{\alpha}}$ and $\boldsymbol{H} = \boldsymbol{\omega}^{-1}\boldsymbol{\Sigma}\boldsymbol{\omega}^{-1}$ is the correlation matrix associated with $\boldsymbol{\Sigma}$.

2.2. Multivariate CSN distribution

Let $\boldsymbol{X} = (X_1, \dots, X_p)'$ be a continuous random vector with p dimensions, drawn from the CSN distribution denoted $\boldsymbol{X} \sim CSN_{p,q}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{D}, \boldsymbol{\nu}, \boldsymbol{\Theta})$. The pdf of $\boldsymbol{X}$ is given by:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \phi_p(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\frac{\Phi_q(\boldsymbol{D}(\boldsymbol{x}-\boldsymbol{\mu}); \boldsymbol{\nu}, \boldsymbol{\Theta})}{\Phi_q(\boldsymbol{0}; \boldsymbol{\nu}, \boldsymbol{\Theta} + \boldsymbol{D}\ \boldsymbol{\Sigma}\boldsymbol{D}')},\ \boldsymbol{x} \in R^p, \tag{2.3}$$

Here, p and q are both greater than or equal to 1, $\boldsymbol{D}$ is a $q \times p$ matrix that defines the correlation structure, and $\Phi_q(\boldsymbol{x}; \boldsymbol{\nu}, \boldsymbol{\Theta})$ represents the q -dimensional normal cumulative distribution function with mean $\nu \in \boldsymbol{\nu} \in R^p$ and dispersion matrix Θ. The CSN distribution is frequently used in multivariate data analysis due to its flexible framework for modeling correlation structures. For more information on this distribution and its applications, please refer to Gonzalez-Farias et al. (2004). This CSN distribution is closed under marginalization. Consider the partition $\mathbf{X}' = (\mathbf{X}_1', \mathbf{X}_2')$ with $\dim(\boldsymbol{X}_1) = p_1$, $\dim(\boldsymbol{X}_2) = p_2$, $p_1 + p_2 = p$ and the corresponding partition of the parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{D})$ such that $\boldsymbol{\mu} = \begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}$, $\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$, and $\mathbf{D} = (\mathbf{D}_1 \quad \mathbf{D}_2)$. Then, the marginal distribution of $\boldsymbol{X}_1$ is given by

$$\boldsymbol{X}_1 \sim CSN_{p_1,q}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}, \boldsymbol{D}_1^*, \boldsymbol{\nu}, \boldsymbol{\Theta}_1^*), \tag{2.4}$$

Where $\mathbf{D}_1^* = \mathbf{D}_1 + \mathbf{D}_2\ \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}$, $\mathbf{\Theta}_1^* = \mathbf{\Theta} + \mathbf{D}_2\ \mathbf{\Sigma}_{22.1}\mathbf{D}_2{}'$ and $\mathbf{\Sigma}_{22.1} = \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}$.

For more details on this result, please see González-Farías et al. (2004)

2.3 Multivariate SUN distribution

A random variable $\boldsymbol{X}$ is said to have a SUN distribution with parameters $p, m, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \Delta, \boldsymbol{\nu}$ and $\boldsymbol{\Gamma}$, denoted $\boldsymbol{X} \sim SUN_{p,m}(\boldsymbol{\mu}, \mathbf{\Sigma}, \mathbf{\Delta}, \boldsymbol{\nu}, \boldsymbol{\Gamma})$, if its pdf is given by

$$f(\boldsymbol{x}) = \frac{\phi_p(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})}{\Phi_m(\boldsymbol{0}; \boldsymbol{\nu}, \boldsymbol{\Gamma})}\Phi_m(\boldsymbol{\Delta}'\ \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}); \boldsymbol{\nu}, \boldsymbol{\Gamma} - \boldsymbol{\Delta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}), \tag{2.5}$$

where Δ is a $p \times m$ correlation matrix and $\boldsymbol{\Gamma}$ is an $m \times m$ matrix.

If we let $\mathbf{\Theta} = \boldsymbol{\Gamma} - \boldsymbol{\Delta}'\Sigma\boldsymbol{\Delta}''$ and $\mathbf{\Gamma} = \boldsymbol{\Delta}\ \boldsymbol{\Sigma}\mathbf{\Delta}'$, the SUN and CSN are equivalent.

In practice, the SUN and CSN are used in a variety of statistical models to describe random variables that exhibit certain characteristics. For example, they may be used in models that involve correlated random variables or variables with heavy tails. We can use the same partition as the CSN distribution for $\mathbf{X}' = (\mathbf{X}_1', \mathbf{X}_2')$ with $\dim(\boldsymbol{X}_1) = p_1$, $\dim(\boldsymbol{X}_2) = p_2$, $p_1 + p_2 = p$, along with the corresponding partition of the parameters

$$(\boldsymbol{\mu}, \mathbf{\Sigma}, \mathbf{\Delta}, \boldsymbol{\nu}, \boldsymbol{X}_1 \sim SUN_{p_1,m}(\mu_1, \Sigma_{11}, \mathrm{D}_1, \boldsymbol{\nu}, \mathbf{\Gamma}). \tag{2.6}$$

where μ_1 represents the mean of $\boldsymbol{X}_1$, Σ_{11} represents the covariance matrix of $\boldsymbol{X}_1$, D_1 is a $p_1 \times m$ correlation matrix and $\mathbf{\Gamma}$ is an $m \times m$ matrix. This marginal distribution can be useful in modeling certain types of data, such as correlated variables with heavy tails. (Zareifard, H. and Khaledi, M.J. (2013)).

3. New defined RFs and their problem.

This section demonstrates that the Skew random fields (SRFs) defined in Mahmoudian (2018) are not well-defined.

3.1. The SNRF

Mahmoudian (2018) introduced a skewed normal random field (SNRF) based on the multivariate skewed normal (SN) distribution. This distribution is defined by a novel representation given as:

$$f(\boldsymbol{y}) = 2\phi_n(\boldsymbol{y};\boldsymbol{\mu},\omega^2\boldsymbol{H})\Phi_1\left(\frac{\boldsymbol{\delta}'\boldsymbol{H}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})}{\omega\sqrt{1-\boldsymbol{\delta}'\boldsymbol{H}^{-1}\boldsymbol{\delta}}}\right) \tag{3.1}$$

$$\text{where } \boldsymbol{\delta} = \frac{\boldsymbol{H\alpha}}{\sqrt{1+\boldsymbol{\alpha}'\boldsymbol{H\alpha}}}. \tag{3.2}$$

By substituting (3.2) into the pdf (3.1) and performing some computations, we obtain:

$$\frac{\boldsymbol{\delta}'\boldsymbol{H}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})}{\omega\sqrt{1-\boldsymbol{\delta}'\boldsymbol{H}^{-1}\boldsymbol{\delta}}} = \frac{\boldsymbol{\alpha}'(\boldsymbol{y}-\boldsymbol{\mu})}{\omega}. \tag{3.3}$$

Thus, the finite-dimensional distribution of this SNRF takes the form:

$$f(\boldsymbol{y}) = 2\phi_n(\boldsymbol{y};\boldsymbol{\mu},\omega^2\boldsymbol{H})\Phi_1\left(\frac{\boldsymbol{\alpha}'(\boldsymbol{y}-\boldsymbol{\mu})}{\omega}\right). \tag{3.4}$$

This distribution family represents a wider range of the distribution family used by Kim and Malik in 2004 to construct their field. To be more precise, by replacing the vector $\boldsymbol{\alpha}$ with the vector $\alpha\boldsymbol{1}_n$ in the final equation, we can obtain the same field as the one created by Kim and Malik. However, as emphasized in Mahmoudian's (2018) research, the field constructed by Kim and Malik in 2004 exhibits an inconsistency problem. Therefore, its more comprehensive variation also faces the same issue of inconsistency.

3.2. CSN RF

Mahmoudian (2018) has claimed that a valid closed skew-normal random field (CSNRF) is defined as:

$$CSN_{n,n}\left(\boldsymbol{\mu},\boldsymbol{\Sigma},\frac{\lambda}{\sqrt{1+\lambda^2}}\boldsymbol{\Sigma}^{-\frac{1}{2}},\boldsymbol{0},\frac{1}{1+\lambda^2}\boldsymbol{I}_n\right) \tag{3.5}$$

However, regardless of the parameters of this RF, this definition is incorrect. Suppose $\boldsymbol{X} = (X_1,\dots,X_{10})'$ is a realization of the CSNRF in (3.5). Under this assumption, we have $\boldsymbol{X} \sim CSN_{10,10}(.)$. Consider the partition $\mathbf{X}' = (\mathbf{X}_1', \mathbf{X}_2')$ with $\dim(\boldsymbol{X}_1) = \dim(\boldsymbol{X}_2) = 5$. By using (2.2), we can show that $\boldsymbol{X}_1 \sim CSN_{5,10}(.)$, while according to (3.5), $\boldsymbol{X}_1 \sim CSN_{5,5}(.)$. Furthermore, it is important to note that the definition (3.5) is more general than the definition of CSNRF by Rimstad and Omre (2014). They explored the impact of the spatial correlation structure and demonstrated that the absence of true stationary in the SRF can lead to the parameters of the marginal distribution being influenced by the correlation structure across the entire study area. Hence, the spatial correlation structure can influence the skewness parameters in the marginal distribution.

3.3. SUN RF

Mahmoudian (2018) defined a univariate skew normal random field (SUNRF) as shown below and claimed its validity:

$$SUN_{n,n}(\boldsymbol{\mu},\omega^2\boldsymbol{H},\delta\boldsymbol{H}^{\frac{1}{2}},\boldsymbol{0},\boldsymbol{I}_n). \tag{3.6}$$

However, we reject this claim based on the same reasoning as in section 3.2. To illustrate, let $\mathbf{X}$ be a realization of SUNRF in (2.5) with p = 10, which gives us $\boldsymbol{X} \sim SUN_{10,10}(.)$. Now, let us consider the partition $\mathbf{X}' = (\mathbf{X}_1', \mathbf{X}_2')$ such that $\dim(\boldsymbol{X}_1) = \dim(\boldsymbol{X}_2) = 5$. Using (2.6), we obtain $\boldsymbol{X}_1 \sim SUN_{5,10}(.)$, while according to (3.6), $\boldsymbol{X}_1 \sim SUN_{5,5}(.)$.

4. Application

According to Sahu et al. (2003), the pdf of the GSN distribution is given by

$$f(\boldsymbol{x}) = 2^p\phi_p(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{\Sigma}+\boldsymbol{D}^2)\Phi_p(\boldsymbol{D}(\boldsymbol{\Sigma}+\boldsymbol{D}^2)^{-1}(\boldsymbol{x}-\boldsymbol{\mu});\boldsymbol{0},\Delta), \boldsymbol{x}\in\mathcal{R}^p$$

where $\boldsymbol{D} = diag(\lambda_1,\dots,\lambda_p)$ is the diagonal matrix introducing skewness through

$\boldsymbol{\lambda}\in\mathcal{R}^p, \Delta = I_p - \boldsymbol{D}(\boldsymbol{\Sigma}+\boldsymbol{D}^2)^{-1}\boldsymbol{D}$ and $\Phi_p(.,\boldsymbol{\nu},\boldsymbol{\Delta})$ is the cdf of $N_p(\boldsymbol{\nu},\boldsymbol{\Delta})$.

Unlike other skewed distributions, the skewness in the GSN distribution is introduced through the product of the Gaussian probability density function and the Gaussian cumulative distribution function. When a certain parameter λ is not equal to zero, the variance of the Gaussian pdf increases, and the values are distorted to capture skewness. GSN distribution is denoted by $GSN_p(\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\lambda})$ and its moment-generating function is as follows:

$$M_X(t) = 2^p\exp\left\{\boldsymbol{t\mu}+\frac{1}{2}\boldsymbol{t}'(\boldsymbol{\Sigma}+\boldsymbol{D}^2)\boldsymbol{t}\right\}\Phi_p(\boldsymbol{Dt}) \qquad \boldsymbol{t}\in\mathcal{R}^p,$$

where $\phi_p(.)$ is the cdf of $N_p(,)$. let $\boldsymbol{X} \sim GSN_p(\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\lambda})$ and consider the partition $\boldsymbol{X}^T = (\boldsymbol{X}_1^T,\boldsymbol{X}_2^T)$ with $dim(\boldsymbol{X}_1) = p_1$, $\dim(\boldsymbol{X}_2) = p_2 = p - p_1$ and the corresponding partition of the parameters $(\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\lambda})$. Then $\boldsymbol{X}_1 \sim GSN_{p_1}(\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_{11},\text{q})$.

Let X $(\boldsymbol{t}_1), \dots$ X$(\boldsymbol{t}_n)$ be the observations from a Gaussian Spatial Network (GSN) random field $\{\boldsymbol{X}(\boldsymbol{t}): \boldsymbol{t} \in D \subseteq R^d\}$ at n locations $(\boldsymbol{t}_1, \dots, \boldsymbol{t}_n)$. To predict $X(\boldsymbol{t}_0)$ at a new location $\boldsymbol{t}_0$, based on the observations $\mathbf{X} = (\mathrm{X}(\boldsymbol{t}_1), \dots \mathrm{X}(\boldsymbol{t}_n))$, we define the predictor $\mathbf{X}^* = (\mathrm{X}(\boldsymbol{t}_0), \mathbf{X}^T)^T$ as follows. We have $\mathbf{X}^* \sim \mathrm{GSN}_{n+1}(\boldsymbol{F}^*\boldsymbol{\beta}, \boldsymbol{\Sigma}^*, \boldsymbol{\lambda})$, where $\boldsymbol{F}^* = (\boldsymbol{f}(\boldsymbol{t}_0), \boldsymbol{F}^T)^T$, , $\boldsymbol{F} = [f_j(\boldsymbol{t}_i)]_{n\times r}$ as known regression functions (covariates), $\boldsymbol{\beta}$ as regression coefficients, $\boldsymbol{\Sigma}^* = \frac{C^*}{q} - F^*\beta\,\beta^T F^{*T}$ as the spatial covariance matrix for all observations and predictions, $\boldsymbol{c} = (C_{0i})_{n\times 1}$, C as the covariance matrix, and C_{ij} as the covariance between observations i and j.

We are focusing on a stationary Gaussian process characterized by a stationary spatial covariance function $\mathrm{C(h)} = \sigma^2\rho(h, \boldsymbol{\theta})$. Here, $\rho(., \boldsymbol{\theta})$ represents a known correlation function, while θ denotes the spatial correlation parameter, and σ^2 signifies the variance of the random field. In practical applications, both θ and σ^2 can be determined through variogram estimation based on available data, without making any assumptions about their distributions. Consequently, the sole remaining unknown parameter is $\boldsymbol{\beta}$, which is estimated using the maximum likelihood method.

4.1. Simulation

As there is no closed form for X $(t_0)|\mathbf{X}$, the Metropolis-Hastings algorithm is employed to produce a sample from $X(t_0)|\mathbf{X}$. The proposal distribution $g_{X_0}(\boldsymbol{x}): N(X_0, b_1^2)$ in the Metropolis-Hasting algorithm is applied, resulting in an approximation of $\mathrm{E}(X(\boldsymbol{s}_0)|X) \cong \frac{1}{M}\sum_{j=1}^{M} X_j$. To further analyze this, a simulation study with 100 realizations was conducted. A stationary GSN RF with a power exponential covariance function $C(|\boldsymbol{h}|) = \sigma^2 \exp(-|\boldsymbol{h}|^p/\eta)$ was used on a regular 500×500 lattice with parameters $\boldsymbol{\beta} = (5,2,7)$, $\left(f_1(\boldsymbol{s}_\mathrm{i}), f_2(\boldsymbol{s}_\mathrm{i}), f_3(\boldsymbol{s}_\mathrm{i})\right)^T = \left(\frac{|s_\mathrm{i}|+\sqrt{|s_\mathrm{i}|}}{|s_\mathrm{i}|+1}, \frac{|s_\mathrm{i}|}{|s_\mathrm{i}|+2}, \frac{\sqrt{|s_\mathrm{i}|}}{\sqrt{|s_\mathrm{i}|}+3}\right)^T$, $\sigma^2 = 1$, $\eta = 4$ and $p = 1.35$.

For the parameter $\boldsymbol{\lambda} = \begin{pmatrix}\lambda_1 \boldsymbol{J}_{70} \\ \lambda_2 \boldsymbol{J}_{30}\end{pmatrix}$, values of $\lambda_1 = 2$, $\lambda_2 = -3$ were considered, and $\boldsymbol{J}_m$ is an m-dimensional vector with all components set to 1. This simulation was repeated 500 times, and two data sets were randomly chosen for further analysis. The Histogram and Normal Q-Q plot of the simulated data in Figure 1 reveal a resemblance to the GSN distribution, showcasing characteristics such as skewness, heavy tails, and a departure from a Gaussian distribution.

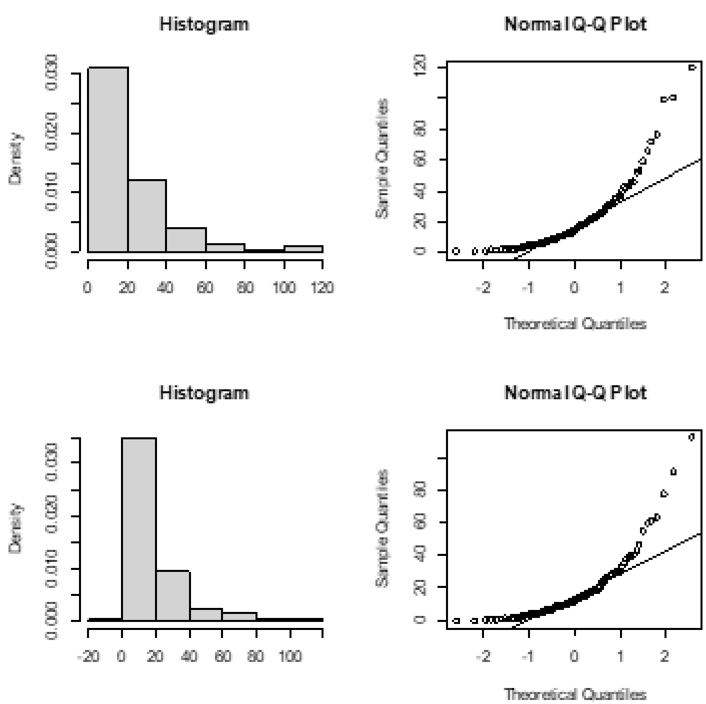

Figure 1. The Histogram and Normal Q-Q plot of the simulated data

To determine the values of $\widehat{\boldsymbol{\beta}}$, $\hat{\sigma}^2$, $\hat{\eta}$, $\hat{p}$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$, parameter estimation has been repeated 15,000 times and the average of these estimates is obtained. The values of the estimates, along with the bias and mean squared error (MSE) of the estimators, are presented in Table 1, where $\boldsymbol{\beta} = (5,2,7)$, $\left(f_1(\boldsymbol{x}_{\mathrm{i}}), f_2(\boldsymbol{x}_{\mathrm{i}}), f_3(\boldsymbol{x}_{\mathrm{i}})\right)^T = \left(\frac{|\boldsymbol{x}_{\mathrm{i}}|+\sqrt{|\boldsymbol{x}_{\mathrm{i}}|}}{|\boldsymbol{x}_{\mathrm{i}}|+1}, \frac{|\boldsymbol{x}_{\mathrm{i}}|}{|\boldsymbol{x}_{\mathrm{i}}|+2}, \frac{\sqrt{|\boldsymbol{x}_{\mathrm{i}}|}}{\sqrt{|\boldsymbol{x}_{\mathrm{i}}|}+3}\right)^T$, $\sigma^2 = 1$, $\eta = 4$ and $p = 1.35$. For parameter of $\boldsymbol{\lambda}$ we consider $\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \boldsymbol{J}_{70} \\ \lambda_2 \boldsymbol{J}_{30} \end{pmatrix}$ where $\lambda_1 = 2$, $\lambda_2 = -3$.

Table 1: Results for parameter estimation

Estimator	value	bias	MSE
$\hat{\beta}_1$	4.90	0.0587	0.0614
$\hat{\beta}_2$	1.85	0.0485	0.0519
$\hat{\beta}_3$	9.05	-0.0478	0.0369
$\hat{\sigma}^2$	1.14	0.0808	0.0578
$\hat{\eta}$	3.98	0.0832	0.0619
$\hat{p}$	1.42	0.0454	0.0069
$\hat{\lambda}_1$	1.80	-0.0848	0.0566
$\hat{\lambda}_2$	-2.74	0.0677	0.0371

5. Discussion and Conclusion

In this article, we demonstrate that certain SRFs defined by Mahmoudian (2018), are not well-defined, despite his claims to the contrary based on Kolmogorov's existence theorem. Although we presented evidence that three SRFs (namely SNRF, CSNRF, and SUNRF) defined by Mahmoudian (2018) are not well-defined, we identified two SRFs that are suitable for valid applications. These are the Generalized Asymmetric Laplace RF (GALRF), which was defined by Saber et al. (2018), and the Generalized Skew-Normal Distributed RF (GSNRF), introduced by Mahmoudian (2018). For heavy-tailed data, we recommend using the GALRF, while for light-tailed data, the GSNRF model is preferable. Finally, we showed that our claim was correct with a simulated example.

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